Stability and Control: Theory, Methods and Applications Volume 15

Almost Periodic Solutions of Differential Equations in Banach Spaces

Y. Hino, T. Naito,
Nguyen Van Minh
and Jong Son Shin

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Almost Periodic Solutions of Differential Equations in Banach Spaces

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Contents

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Introduction to the Series

The problems of modern society are both complex and interdisciplinary. Despite the apparent diversity of problems, tools developed in one context are often adaptable to an entirely different situation. For example, consider the Lyapunov's well known second method. This interesting and fruitful technique has gained increasing significance and has given a decisive impetus for modern development of the stability theory of differential equations. A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in application. It is now well recognized that the concept of Lyapunov-like functions and the theory of differential and integral inequalities can be utilized to investigate qualitative and quantitative properties of nonlinear dynamic systems. Lyapunov-like functions serve as vehicles to transform the given complicated dynamic systems into a relatively simpler system and therefore it is sufficient to study the properties of this simpler dynamic system. It is also being realized that the same versatile tools can be adapted to discuss entirely different nonlinear systems, and that other tools, such as the variation of parameters and the method of upper and lower solutions provide equally effective methods to deal with problems of a similar nature. Moreover, interesting new ideas have been introduced which would seem to hold great potential.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behavior so as to accomplish a desired goal. In order to implement this influence, practitioners build devices that incorporate various mathematical techniques. The study of these devices and their interaction with the object being controlled is the subject of control theory. There have been, roughly speaking, two main lines of work in control theory which are complementary. One is based on the idea that a good model of the object to be controlled is available and that we wish to optimize its behavior, and the other is based on the constraints imposed by uncertainty about the model in which the object operates. The control tool in the latter is the use of feedback in order to correct for deviations from the desired behavior. Mathematically, stability theory, dynamic systems and functional analysis have had a strong influence on this approach.

Volume 1, *Theory of Integro-Differential Equations*, is a joint contribution by V. Lakshmikantham (USA) and M. Rama Mohana Rao (India).

Volume 2, *Stability Analysis: Nonlinear Mechanics Equations*, is by A.A. Martynyuk (Ukraine).

Volume 3, *Stability of Motion of Nonautonomous Systems: The Method of Limiting Equations*, is a collaborative work by J. Kato (Japan), A.A. Martynyuk (Ukraine) and A.A. Shestakov (Russia).

Volume 4, *Control Theory and its Applications*, is by E.O. Roxin (USA).

Volume 5, *Advances in Nonlinear Dynamics*, is edited by S. Sivasundaram (USA) and A.A. Martynyuk (Ukraine) and is a multiauthor volume dedicated to Professor S. Leela.

Volume 6, *Solving Differential Problems by Multistep Initial and Boundary Value Methods*, is a joint contribution by L. Brugnano (Italy) and D. Trigiante (Italy).

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Due to the increased interdependency and cooperation among the mathematical sciences across the traditional boundaries, and the accomplishments thus far achieved in the areas of stability and control, there is every reason to believe that many breakthroughs await us, offering existing prospects for these versatile techniques to advance further. It is in this spirit that we see the importance of the 'Stability and Control' series, and we are immensely thankful to Gordon and Breach Science Publishers for their interest and cooperation in publishing this series.

Preface

Almost periodic solutions of differential equations have been studied since the very beginning of this century. The theory of almost periodic solutions has been developed in connection with problems of differential equations, dynamical systems, stability theory and its applications to control theory and other areas of mathematics. The classical books by C. Corduneanu [50], A.M. Fink [67], T. Yoshizawa [231], L. Amerio and G. Prouse [7], B.M. Levitan and V.V. Zhikov [137] gave a very nice presentation of methods as well as results in the area. In recent years, there has been an increasing interest in extending certain classical results to differential equations in Banach spaces. In this book we will make an attempt to gather systematically certain recent results in this direction.

We outline briefly the contents of our book. The main results presented here are concerned with conditions for the existence of periodic and almost periodic solutions and its connection with stability theory. In the qualitative theory of differential equations there are two classical results which serve as models for many works in the area. Namely,

> **Theorem A** *A periodic inhomogeneous linear equation has a unique periodic solution (vith the same period) if* 1 *is not an eigenvalue of its monodromy operator.*
>
> **Theorem B** *A periodic inhomogeneous linear equation has a periodic solution (uith the same period) if and only if it has a bounded solution.*

In our book, a main part will be devoted to discuss the question as how to extend these results to the case of almost periodic solutions of (linear and nonlinear) equations in Banach spaces. To this end, in the first chapter we present introductions to the theory of semigroups of linear operators (Section 1), its applications to evolution equations (Section 2) and the harmonic analysis of bounded functions on the real line (Section 3). In Chapter 2 we present the results concerned with autonomous as well as periodic evolution equations, extending Theorems A and B to the infinite dimensional case. In contrast to the finite dimensional case, in general one cannot treat periodic evolution equations as autonomous ones. This is due to the fact that in the infinite dimensional case there is no Floquet representation, though one can prove many similar assertions to the autonomous case (see e.g. [78], [90], [131]). Sections 1, 2 of this chapter are devoted to the investigation by means of evolution semigroups in translation invariant subspaces of $BUC(\mathbf{R}, \mathbf{X})$ (of bounded uniformly continuous $\mathbf{X}$-valued functions on the real line). A new technique of spectral decomposition is presented in Section 3. Section 4 presents various results extending Theorem B to periodic solutions of abstract functional differential equations. In Section 5 we prove analogues of results in Sections 1, 2. 3 for discrete systems and discuss an alternative method to extend Theorems A and B to periodic and almost periodic solutions of differential equations. In Sections 6 and 7 we extend the method used in the previous ones to semilinear and fully nonlinear equations. The conditions are given in terms of the dissipativeness of the equations under consideration.

In Chapter 3 we present the existence of almost periodic solutions of almost periodic evolution equations by using stability properties of nonautonomous dynamical systems. Sections 1 and 2 of this chapter extend the concept of skew product flow of processes to a more general concept which is called skew product flow of quasi-processes and investigate the existence of almost periodic integrals for almost periodic quasi-processes. For abstract functional differential equations uith infinite delay, there are three kinds of definitions of stabilities. In Sections 3 and 4. we prove some equivalence of these definitions of stabilities and show that these stabilities fit in with quasiprocesses. By using results in Section 2, we discuss the existence of almost periodic solutions for abstract almost periodic evolution equations in Section 5. Concrete applications for functional partial differential equations are given in Section 6.

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CHAPTER 1

C_0-SEMIGROUPS, WELL POSED EVOLUTION EQUATIONS, SPECTRAL THEORY AND ALMOST PERIODICITY OF FUNCTIONS

1.1. STRONGLY CONTINUOUS SEMIGROUPS OF LINEAR OPERATORS

In this section we collect some well-known facts from the theory of strongly continuous semigroups of operators on a Banach space for the reader's convenience. We will focus the reader's attention on several important classes of semigroups such as analytic and compact semigroups which will be discussed later in the next chapters. Among the basic properties of strongly continuous semigroups we will put emphasis on the spectral mapping theorem. Since the materials of this section as well as of the chapter in the whole can be found in any standard book covering the area, here we aim at freshening up the reader's memory rather than giving a logically self contained account of the theory.

Throughout the book we will denote by $\mathbf{X}$ a complex Banach space. The set of all real numbers and the set of nonnegative real numbers will be denoted by $\mathbf{R}$ and $\mathbf{R}^+$, respectively. $BC(\mathbf{R},\mathbf{X})$, $BUC(\mathbf{R},\mathbf{X})$ stand for the spaces of bounded, continuous functions and bounded, uniformly continuous functions, respectively.

1.1.1. Definition and Basic Properties

Definition 1.1 A family $(T(t))_{t\geq 0}$ of bounded linear operators acting on a Banach space $\mathbf{X}$ is a *strongly continuous semigroup of bounded linear operators*, or briefly, a *C_0-semigroup* if the following three properties are satisfied:

i) $T(0) = I$, the identity operator on $\mathbf{X}$;

ii) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$;

iii) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0$ for all $x \in \mathbf{X}$.

The *infinitesimal generator* of $(T(t))_{t\geq 0}$, or briefly, the *generator*, is the linear operator A with domain $D(A)$ defined by

$$\begin{aligned} D(A) &= \{x \in \mathbf{X} : \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) \text{ exists}\}, \\ Ax &= \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x), \quad x \in D(A). \end{aligned}$$

The generator is always a closed, densely defined operator.

Theorem 1.1 *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \ \forall t \geq 0.$$

Proof. For the proof see e.g. [179, p. 4].

Corollary 1.1 *If $(T(t))_{t\geq 0}$ is a C_0-semigroup, then the mapping $(x,t) \mapsto T(t)x$ is a continuous function from $\mathbf{X} \times \mathbf{R}^+ \to \mathbf{X}$.*

Proof. For any $x, y \in \mathbf{X}$ and $t \leq s \in \mathbf{R}^+ := [0, \infty)$,

$$\begin{aligned} \|T(t)x - T(s)y\| &\leq \|T(t)x - T(s)x\| + \|T(s)x - T(s)y\| \\ &\leq Me^{\omega s}\|x-y\| + \|T(t)\|\|T(s-t)x - x\| \\ &\leq Me^{\omega s}\|x-y\| + Me^{\omega t}\|T(s-t)x - x\|. \end{aligned} \tag{1.1}$$

Hence, for fixed x, t $(t \leq s)$ if $(y,s) \to (x,t)$, then $\|T(t)x - T(s)y\| \to 0$. Similarly, for $s \leq t$

$$\begin{aligned} \|T(t)x - T(s)y\| &\leq \|T(t)x - T(s)x\| + \|T(s)x - T(s)y\| \\ &\leq Me^{\omega s}\|x-y\| + \|T(s)\|\|T(t-s)x - x\| \\ &\leq Me^{\omega s}\|x-y\| + Me^{\omega s}\|T(t-s)x - x\|. \end{aligned} \tag{1.2}$$

Hence, if $(y,s) \to (x,t)$, then $\|T(t)x - T(s)y\| \to 0$.

Other basic properties of a C_0-semigroup and its generator are listed in the following:

Theorem 1.2 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbf{X}$. Then*

i) For $x \in \mathbf{X}$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

ii) For $x \in \mathbf{X}$, $\int_0^t T(s)x ds \in D(A)$ and

$$A\left(\int_0^t T(s)x ds\right) = T(t)x - x.$$

iii) For $x \in D(A), T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

iv) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax d\tau = \int_s^t AT(\tau)x d\tau.$$

Proof. For the proof see e.g. [179, p. 5].

We continue with some useful fact about semigroups that will be used throughout this book. The first of these is the *Hille-Yosida theorem*, which characterizes the generators of C_0-semigroups among the class of all linear operators.

Theorem 1.3 *Let A be a linear operator on a Banach space $\mathbf{X}$, and let $\omega \in \mathbf{R}$ and $M \geq 1$ be constants. Then the following assertions are equivalent:*

i) A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$;

ii) A is closed, densely defined, the half-line (ω, ∞) is contained in the resolvent set $\rho(A)$ of A, and we have the estimates

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \qquad \forall \lambda > \omega, \quad n = 1, 2, ... \tag{1.3}$$

Here, $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent of A at λ. If one of the equivalent assertions of the theorem holds, then actually $\{Re\lambda > \omega\} \subset \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n}, \qquad \forall Re\lambda > \omega, \quad n = 1, 2, ... \tag{1.4}$$

Moreover, for $Re\lambda > \omega$ the resolvent is given explicitly by

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt, \qquad \forall x \in \mathbf{X}. \tag{1.5}$$

We shall mostly need the implication (i)$\Rightarrow$(ii), which is the easy part of the theorem. In fact, one checks directly from the definitions that

$$R_\lambda x := \int_0^\infty e^{-\lambda t} T(t)x\, dt$$

defines a two-sided inverse for $\lambda - A$. The estimate (1.4) and the identity (1.5) follow trivially from this.

A useful consequence of (1.3) is that

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\| = 0, \quad \forall x \in X. \tag{1.6}$$

This is proved as follows. Fix $x \in D(A)$ and $\mu \in \rho(A)$, and let $y \in X$ be such that $x = R(\mu, A)y$. By (1.3) we have $\|R(\lambda, A)\| = O(\lambda^{-1})$ as $\lambda \to \infty$. Therefore, the *resolvent identity*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \tag{1.7}$$

implies that

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda, A)x - x\| = \lim_{\lambda\to\infty} \|R(\lambda, A)(\mu R(\mu, A)y - y)\| = 0.$$

This proves (1.6) for elements $x \in D(A)$. Since $D(A)$ is dense in X and the operators $\lambda R(\lambda, A)$ are uniformly bounded as $\lambda \to \infty$ by (1.3), (1.6) holds for all $x \in \mathbf{X}$.

1.1.2. Compact Semigroups and Analytic Strongly Continuous Semigroups

Definition 1.2 A C_0-semigroup $(T(t))_{t\geq 0}$ is called *compact* for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator. $(T(t))_{t\geq 0}$ is called *compact* if it is compact for $t > 0$.

If a C_0-semigroup $(T(t))_{t\geq 0}$ is compact for $t > t_0$, then it is continuous in the uniform operator topology for $t > t_0$.

Theorem 1.4 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then $(T(t))_{t\geq 0}$ is a compact semigroup if and only if $T(t)$ is continuous in the uniform operator topology for $t > 0$ and $R(\lambda; A)$ is compact for $\lambda \in \rho(A)$.*

Proof. For the proof see e.g. [179, p. 49].

In this book we distinguish the notion of analytic C_0-semigroups from that of analytic semigroups in general. To this end we recall several notions. Let A be a linear operator $D(A) \subset \mathbf{X} \to \mathbf{X}$ with *not necessarily dense domain.*

Definition 1.3 A is said to be *sectorial* if there are constants $\omega \in \mathbf{R}, \theta \in (\pi/2, \pi), M > 0$ such that the following conditions are satisfied:

$$\begin{cases} i) \ \ \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbf{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}, \\ \\ ii) \ \ ||R(\lambda, A)|| \leq M/|\lambda - \omega| \ \forall \lambda \in S_{\theta,\omega}. \end{cases}$$

If we assume in addtion that $\rho(A) \neq \oslash$, then A is closed. Thus, $D(A)$, endowed with the graph norm

$$||x||_{D(A)} := ||x|| + ||Ax||,$$

is a Banach space. For a sectorial operator A, from the definition, we can define a linear bounded operator e^{tA} by means of the Dunford integral

$$e^{tA} := \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, t > 0, \tag{1.8}$$

where $r > 0, \eta \in (\pi/2, \theta)$ and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbf{C} : |arg\lambda| = \eta, |\lambda| \geq r||\} \cup \{\lambda \in \mathbf{C} : |arg\lambda| \leq \eta, |\lambda| = r\},$$

oriented counterclockwise. In addition, set $e^{0A}x = x, \ \forall x \in \mathbf{X}$.

Theorem 1.5 *Under the above notation, for a sectorial operator A the following assertions hold true:*

i) $e^{tA}x \in D(A^k)$ for every $t > 0, x \in \mathbf{X}, k \in \mathbf{N}$. If $x \in D(A^k)$, then

$$A^k e^{tA} x = e^{tA} A^k x, \ \forall t \geq 0;$$

ii) $e^{tA}e^{sA} = e^{(t+s)A}$, $\forall t, s \geq 0$;

iii) There are positive constants $M_0, M_1, M_2, ...,$ *such that*

$$\begin{cases} (a) \quad ||e^{tA}|| \leq M_0 e^{\omega t}, \ t \geq 0, \\ (b) \quad ||t^k(A - \omega I)^k e^{tA}|| \leq M_k e^{\omega t}, \ t \geq 0, \end{cases}$$

where ω *is determined from Definition 1.3. In particular, for every* $\varepsilon > 0$ *and* $k \in \mathbf{N}$ *there is* $C_{k,\varepsilon}$ *such that*

$$||t^k A^k e^{tA}|| \leq C_{k,\varepsilon} e^{(\omega+\varepsilon)t}, \ t > 0;$$

iv) The function $t \mapsto e^{tA}$ *belongs to* $C^\infty((0, +\infty), L(\mathbf{X}))$, *and*

$$\frac{d^k}{dt^k} e^{tA} = A^k e^{tA}, \ t > 0,$$

moreover it has an analytic extension in the sector

$$S = \{\lambda \in \mathbf{C} : |arg\lambda| < \theta - \pi/2\}.$$

Proof. For the proof see [140, pp. 35-37].

Definition 1.4 For every sectorial operator A the semigroup $(e^{tA})_{t\geq 0}$ defined in Theorem 1.5 is called *the analytic semigroup* generated by A in $\mathbf{X}$. An analytic semigroup is said to be an *analytic strongly continuous semigroup* if in addition, it is strongly continuous.

There are analytic semigroups which are not strongly continuous, for instance, the analytic semigroups generated by nondensely defined sectorial operators. From the definition of sectorial operators it is obvious that for a sectorial operator A the intersection of the spectrum $\sigma(A)$ with the imaginary axis is bounded.

1.1.3. Spectral Mapping Theorems

If A is a bounded linear operator on a Banach space $\mathbf{X}$, then by the Dunford Theorem [63] $\sigma(exp(tA)) = exp(t\sigma(A))$, $\forall t \geq 0$. It is natural to expect this relation holds for any C_0-semigroups on a Banach space. However, this is not true in general as shown by the following counterexample

Example 1.1

For $n = 1, 2, 3, ...,$ let A_n be the $n \times n$ matrix acting on $\mathbf{C}^n$ defined by

$$A_n := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Each matrix A_n is nilpotent and therefore $\sigma(A_n) = \{0\}$. Let X be the Hilbert space consisting of all sequences $x = (x_n)_{n\in\mathbf{N}}$ with $x_n \in \mathbf{C}^n$ such that

$$||x|| := \left(\sum_{n=1}^{\infty} ||x_n||^2_{\mathbf{C}^n}\right)^{\frac{1}{2}} < \infty.$$

Let $(T(t))_{t\geq 0}$ be the semigroup on X defined coordinatewise by

$$(T(t)) = (e^{int}e^{tA_n})_{n\in\mathbf{N}}.$$

It is easily checked that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X and that $(T(t))_{t\geq 0}$ extends to a C_0-group. Since $||A_n|| = 1$ for $n \geq 2$, we have $||e^{tA_n}|| \leq e^t$ and hence $||T(t)|| \leq e^t$, so $\omega_0((T(t))_{t\geq 0}) \leq 1$, where

$$\omega_0((T(t))_{t\geq 0}) := \inf\{\alpha : \exists N \geq 1 \text{ such that } ||T(t)|| \leq Ne^{\alpha t},\ \forall t \geq 0\}.$$

First, we show that $s(A) = 0$, where A is the generator of $(T(t))_{t\geq 0}$ and $s(A) := \{\sup Re\lambda, \lambda \in \sigma(A)\}$. To see this, we note that A is defined coordinatewise by

$$A = (in + A_n)_{n\geq 1}.$$

An easy calculation shows that for all $Re\lambda > 0$,

$$\lim_{n\to\infty} ||R(\lambda, A_n + in)||_{\mathbf{C}^n} = 0.$$

It follows that the operator $(R(\lambda, A_n+in))_{n\geq 1}$ defines a bounded operator on X, and clearly this operator is a two-sided inverse of $\lambda - A$. Therefore $\{Re\lambda > 0\} \subset rho(A)$ and $s(A) \leq 0$. On the other hand, $in \in \sigma(in+A_n) \subset \sigma(A)$ for all $n \geq 1$, so $s(A) = 0$.

Next, we show that $\omega_0((T(t))_{t\geq 0}) = 1$. In view of $\omega_0((T(t))_{t\geq 0}) \leq 1$ it suffices to show that $\omega_0((T(t))_{t\geq 0}) \geq 1$. For each n we put

$$x_n := n^{-\frac{1}{2}}(1, 1, ..., 1) \in \mathbf{C}^n.$$

Then, $||x_n||_{\mathbf{C}^n} = 1$ and

$$\begin{aligned}
||e^{tA_n}x_n||^2_{\mathbf{C}^n} &= \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{j=0}^{m}\frac{t^j}{j!}\right)^2 \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\left(\sum_{j,k=0}^{m}\frac{t^{j+k}}{j!k!}\right) \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\sum_{i=0}^{2m} t^i \sum_{j+k=i}\frac{1}{j!k!} \\
&= \frac{1}{n}\sum_{m=0}^{n-1}\sum_{i=0}^{2m}\frac{t^i}{i!}\sum_{j=0}^{i}\frac{i!}{j!(i-j)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{m=0}^{n-1}\sum_{i=0}^{2m}\frac{2^i t^i}{i!} \\
&\geq \frac{1}{n}\sum_{i=0}^{2n-2}\frac{2^i t^i}{i!}.
\end{aligned}
$$

For $0 < q < 1$, we define $x_q \in X$ by $x_q := (n^{\frac{1}{2}} q^n x_n)_{n\geq 1}$. It is easy to check that $x_q \in D(A)$ and

$$
\begin{aligned}
\|T(t)x_q\|^2 &= \sum_{n=1}^{\infty} n q^{2n} \|e^{tA_n} x_n\|^2 \\
&\geq \sum_{n=1}^{\infty} n q^{2n} \left(\frac{1}{n}\sum_{i=0}^{2n-2}\frac{2^i t^i}{i!}\right) \\
&= \sum_{i=0}^{\infty}\frac{2^i t^i}{i!}\sum_{n=\{i/2\}+1}^{\infty} q^{2n} \\
&= \sum_{i=0}^{\infty}\frac{q^{2\{i/2\}+2}}{1-q^2}\frac{2^i t^i}{i!} \\
&\geq \frac{q^3}{1-q^2}e^{2tq}.
\end{aligned}
$$

Here $\{a\}$ denotes the least integer greater than or equal to a; we used that $2\{k/2\} + 2 \leq k+3$ for all $k = 0, 1, \ldots$ Thus, $\omega_0((T(t))_{t\geq 0}) \geq q$ for all $0 < q < 1$, so $\omega_0((T(t))_{t\geq 0}) \geq 1$. Hence, the relation $\sigma(T(t)) = e^{t\sigma(A)}$ does not holds for the semigroup $(T(t))_{t\geq 0}$.

In this section we prove the spectral inclusion theorem:

Theorem 1.6 *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with generator A. Then we have the spectral inclusion relation*

$$\sigma(T(t)) \supset e^{t\sigma(A)}, \ \forall t \geq 0.$$

Proof. By Theorem 1.2 for the semigroup $(T_\lambda(t))_{t\geq 0} := \{e^{-\lambda t}T(t)\}_{t\geq 0}$ generated by $A - \lambda$, for all $\lambda \in \mathbf{C}$ and $t \geq 0$

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)x\,ds = (e^{\lambda t} - T(t))x, \quad \forall x \in X,$$

and

$$\int_0^t e^{\lambda(t-s)}T(s)(\lambda - A)x\,ds = (e^{\lambda t} - T(t))x, \quad \forall x \in D(A). \tag{2.1.1}$$

Suppose $e^{\lambda t} \in \rho(T(t))$ for some $\lambda \in \mathbf{C}$ and $t \geq 0$, and denote the inverse of $e^{\lambda t} - T(t)$ by $Q_{\lambda,t}$. Since $Q_{\lambda,t}$ commutes with $T(t)$ and hence also with A, we have

$$(\lambda - A)\int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}x\,ds = x, \quad \forall x \in X,$$

and

$$\int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}(\lambda - A)x\,ds = x, \quad \forall x \in D(A).$$

This shows the boundedness of the operator B_λ defined by

$$B_\lambda x := \int_0^t e^{\lambda(t-s)}T(s)Q_{\lambda,t}x\,ds$$

is a two-sided inverse of $\lambda - A$. It follows that $\lambda \in \varrho(A)$.

As shown by Example 1.1 the converse inclusion

$$\exp(t\sigma(A)) \supset \sigma(T(t))\backslash\{0\}$$

in general fails. For certain parts of the spectrum, however, the spectral mapping theorem holds true. To make it more clear we recall that for a given closed operator A on a Banach space $\mathbf{X}$ the *point spectrum* $\sigma_p(A)$ is the set of all $\lambda \in \sigma(A)$ for which there exists a non-zero vector $x \in D(A)$ such that $Ax = \lambda x$, or equivalently, for which the operator $\lambda - A$ is not injective; the *residual spectrum* $\sigma_r(A)$ is the set of all $\lambda \in \sigma(A)$ for which $\lambda - A$ does not have dense range; the *approximate point spectrum* $\sigma_a(A)$ is the set of all $\lambda \in \sigma(A)$ for which there exists a sequence (x_n) of norm one vectors in X, $x_n \in D(A)$ for all n, such that

$$\lim_{n\to\infty} ||Ax_n - \lambda x_n|| = 0.$$

Obviously, $\sigma_p(A) \subset \sigma_a(A)$.

Theorem 1.7 *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space $\mathbf{X}$, with generator A. Then*

$$\sigma_p(T(t))\backslash\{0\} = e^{t\sigma_p(A)}, \quad \forall t \geq 0.$$

Proof. For the proof see e.g. [179, p. 46].

Recall that a family of bounded linear operators $(T(t))_{t\in\mathbf{R}}$ is said to be a *strongly continuous group* if it satisfies

i) $T(0) = I$,

ii) $T(t+s) = T(t)T(s), \ \forall t, s \in \mathbf{R}$,

iii) $\lim_{t\to 0} T(t)x = x, \ \forall x \in \mathbf{X}$.

Similarly to C_0-semigroups, the generator of a strongly continuous group $(T(t))_{t\in\mathbf{R}}$ is defined to be the operator

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t},$$

with the domain $D(A)$ consisting of all elements $x \in \mathbf{X}$ such that the above limit exists. For bounded strongly continuous groups of linear operators the following weak spectral mapping theorem holds:

Theorem 1.8 *Let $(T(t))_{t\in\mathbf{R}}$ be a bounded strongly continuous group, i.e., there exists a positive M such that $\|T(t)\| \le M$, $\forall t \in \mathbf{R}$ with generator A. Then*

$$\sigma(T(t)) = \overline{e^{t\sigma(A)}}, \ \forall t \in \mathbf{R}. \tag{1.9}$$

Proof. For the proof see e.g. [163] or [173, Chapter 2].

Example 1.2 *Let $\mathcal{M}$ be a closed translation invariant subspace of the space of $\mathbf{X}$-valued bounded uniformly continuous functions on the real line $BUC(\mathbf{R},\mathbf{X})$, i.e., $\mathcal{M}$ is closed and $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t$, where $(S(t))_{t\in\mathbf{R}}$ is the translation group on $BUC(\mathbf{R},\mathbf{X})$. Then*

$$\sigma(S(t)|_{\mathcal{M}}) = \overline{e^{t\sigma(\mathcal{D}_{\mathcal{M}})}}, \ \forall t \in \mathbf{R},$$

where $\mathcal{D}_{\mathcal{M}}$ is the generator of $(S(t)|_{\mathcal{M}})_{t\in\mathbf{R}}$ (the restriction of the group $(S(t))_{t\in\mathbf{R}}$ to $\mathcal{M}$).

In the next chapter we will again consider situations similar to this example which arise in connection with invariant subspaces of so-called evolution semigroups.

1.2. EVOLUTION EQUATIONS

1.2.1. Well-Posed Evolution Equations

Homogeneous and inhomogeneous equations

For a densely defined linear operator A let us consider the *abstract Cauchy problem*

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \ \forall t > 0, \\ u(0) = x \in D(A). \end{cases} \tag{1.10}$$

The problem (1.10) is called *well posed* if $\rho(A) \neq \oslash$ and for every $x \in D(A)$ there is a unique (classical) solution $u : [0,\infty) \to D(A)$ of (1.10) in $C^1([0,\infty),\mathbf{X})$. The well posedness of (1.10) involves the existence, uniqueness and continuous dependence on the initial data. The following result is fundamental.

Theorem 1.9 *The problem (1.10) is well posed if and only if A generates a C_0-semigroup on $\mathbf{X}$. In this case the solution of (1.10) is given by $u(t) = T(t)x, t > 0$.*

Proof. The detailed proof of this theorem can be found in [71, p. 83].

In connection with the well posed problem (1.10) we consider the following Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \ \forall t > 0, \\ u(0) = u_0. \end{cases} \tag{1.11}$$

Theorem 1.10 *Let the problem (1.10) be well posed and $u_0 \in D(A)$. Assume either*

i) $f \in C([0,\infty), \mathbf{X})$ takes values in $D(A)$ and $Af(\cdot) \in C([0,\infty), \mathbf{X})$, or

ii) $f \in C^1([0,\infty), \mathbf{X})$.

Then the problem (1.11) has a unique solution $u \in C^1([0,\infty), \mathbf{X})$ with values in $D(A)$.

Proof. The detailed proof of this theorem can be found in [71, pp. 84-85].

Even when the conditions of Theorem 1.10 are not satisfied we can speak of *mild solutions* by which we mean continuous solutions of the equation

$$\begin{cases} u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)f(\xi)d\xi, \ \forall t \geq s \geq 0 \\ u(0) = u_0, \ u_0 \in \mathbf{X}, \end{cases} \tag{1.12}$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by A and f is assumed to be continuous. It is easy to see that there exists a unique mild solution of Eq.(1.12) for every $x \in \mathbf{X}$.

Nonautonomous equations

To a time-dependent equation

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t), \ \forall t \geq s \geq 0, \\ u(s) = x, \end{cases} \tag{1.13}$$

where $A(t)$ is in general unbounded linear operator, the notion of well posedness can be extended, roughly speaking, as follows: if the initial data x is in a dense set of the phase space $\mathbf{X}$, then there exists a unique (classical) solution of (1.13) which depends continuously on the initial data. Let us denote by $U(t,s)x$ the solution of (1.13). By the uniqueness we see that $(U(t,s))_{t\geq s\geq 0}$ is a family of bounded linear operators on $\mathbf{X}$ with the properties

i) $U(t,s)U(s,r) = U(t,r), \ \forall t \geq s \geq r \geq 0$;

ii) $U(t,t) = I, \ \forall t \geq 0$;

iii) $U(\cdot,\cdot)x$ is continuous for every fixed $x \in \mathbf{X}$.

In the next chapter we will deal with families $(U(t,s))_{t\geq s\geq 0}$ rather than with the equations of the form (1.13) which generate such families. This general setting enables us to avoid stating complicated sets of conditions imposed on the coefficient-operators $A(t)$. We refer the reader to [71, pp. 140-147] and [179, Chapter 5] for more information on this subject.

Semilinear evolution equations

The notion of well posedness discussed above can be extended to semilinear equations of the form

$$\frac{dx}{dt} = Ax + Bx \ , \quad x \in \mathbf{X} \tag{1.14}$$

where $\mathbf{X}$ is a Banach space, A is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$ of linear operators of type ω, i.e.

$$||S(t)x - S(t)y|| \leq e^{\omega t}||x - y||, \ \forall \ t \geq 0, \ x, y \in \mathbf{X} \ ,$$

and B is an everywhere defined continuous operator from $\mathbf{X}$ to $\mathbf{X}$. Hereafter, by a mild solution $x(t), t \in [s, \tau]$ of equation (1.14) we mean a continuous solution of the integral equation

$$x(t) = S(t-s)x + \int_s^t S(t-\xi)Bx(\xi)d\xi, \ \forall s \leq t \leq \tau. \tag{1.15}$$

Before proceeding we recall some notions and results which will be frequently used later on. We define the bracket $[\cdot, \cdot]$ in a Banach space $\mathbf{Y}$ as follows (see e.g. [142] for more information)

$$[x, y] = \lim_{h \to +0} \frac{||x + hy|| - ||y||}{h} = \inf_{h > 0} \frac{||x + hy|| - ||y||}{h}$$

Definition 1.5 Suppose that F is a given operator on a Banach space $\mathbf{Y}$. Then $(F + \gamma I)$ is said to be *accretive* if and only if for every $\lambda > 0$ one of the following equivalent conditions is satisfied

i) $(1 - \lambda\gamma)||x - y|| \leq ||x - y + \lambda(Fx - Fy)||, \ \forall x, y \in D(F)$,

ii) $[x - y, Fx - Fy] \geq -\gamma||x - y||, \ \forall x, y \in D(F)$.

In particular, if $\gamma = 0$, then F is said to be accretive.

Remark 1.1 From this definition we may conclude that $(F + \gamma I)$ is accretive if and only if

$$||x - y|| \leq ||x - y + \lambda(Fx - Fy)|| + \lambda\gamma||x - y|| \tag{1.16}$$

for all $x, y \in D(F), \lambda > 0, 1 \geq \lambda\gamma$.

Theorem 1.11 *Let the above conditions hold true. Then for every fixed $s \in \mathbf{R}$ and $x \in \mathbf{X}$ there exists a unique mild solution $x(\cdot)$ of Eq.(1.14) defined on $[s, +\infty)$. Moreover, the mild solutions of Eq.(1.14) give rise to a semigroup of nonlinear operators $T(t), t \geq 0$ having the following properties:*

$$i) \quad T(t)x = S(t)x + \int_0^t S(t-\xi)BT(\xi)xd\xi, \ \forall t \geq 0, x \in \mathbf{X}, \tag{1.17}$$

$$ii) \quad ||T(t)x - T(t)y|| \leq e^{(\omega+\gamma)t}||x - y||, \ \forall t \geq 0, x, y \in \mathbf{X}. \tag{1.18}$$

More detailed information on this subject can be found in [142].