Advances in Applied Mathematics

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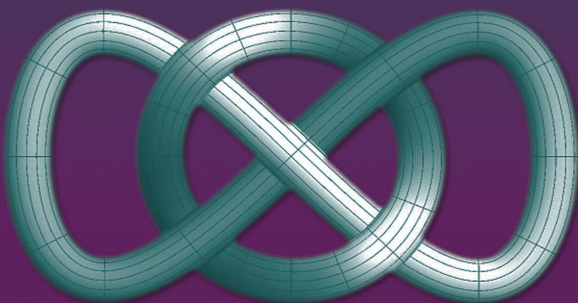
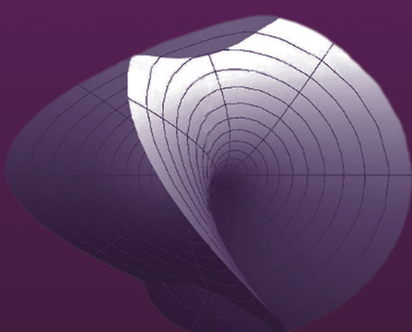
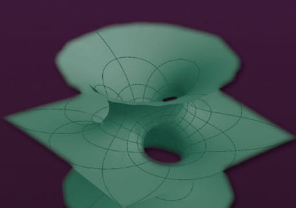
Third Edition





David H. von Seggern



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CRC Standard Curves and Surfaces with Mathematica®

Third Edition

Advances in Applied Mathematics

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CRC Standard Curves and Surfaces with Mathematica®

Third Edition

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Preface to the Third Edition

This third edition of this mathematical reference book (*CRC Standard Curves and Surfaces, with Mathematica*®) comes nine years after the second edition in 2007. (In fact, there was an earlier volume entitled *CRC Handbook of Mathematical Curves and Surfaces*, published in 1990; so the current volume may be considered as really a fourth edition.) The motivations for the current edition were several: 1) the *Mathematica* program has matured considerably since 2007, thus allowing more complex curves and surfaces to be presented; 2) the computing power of desktop computers has again increased several fold, thus allowing many 3-D graphical plots to be computed in a reasonable time; and 3) the *Mathematica* typesetting functionality has become sufficiently robust that the final copy for this edition of the book could be transformed directly from *Mathematica* notebooks to LaTex input, albeit with some editing afterward.

New curves and surfaces have been introduced in almost every chapter; several chapters have been reorganized; and better graphical representations have been produced for many curves and surfaces throughout. A new chapter on Laplace transforms has been added.

The overall format of the book is largely unchanged from the previous edition, with function definitions on the left-hand pages and corresponding function plots on the right-hand pages, thus maintaining the easy reference-like character of the volume. One significant change is that, instead of presenting a range of realizations for most functions, this edition presents only one curve associated with each function. The graphic output of the Manipulate function is shown exactly as rendered in *Mathematica*, with the exact parameters of the curve's equation shown as part of the graphic display. This enables the reader to gauge what a reasonable range of parameters might be while seeing the result of one particular choice of parameters.

In preparing the latest edition, the author has benefited from people, too numerous to mention here, who have communicated by letter or email concerning improvements, corrections, and possible additions; and the author here wants to extend his appreciation to these individuals. The author wishes to thank the Wolfram, Inc. developers for enabling this third edition with the many new and useful features of the *Mathematica* program and for providing stimulus in conferences, in newsletters, and in a rich, helpful, and extensive website. Wolfram, Inc. staff have also helped to solve some technical problems related to producing copy-ready text for this book and have responded quickly to special problems arising when employing the *Mathematica* program in this endeavor. The author is indebted to Robert Ross, the mathematics editor of CRC Press, for encouraging and facilitating this latest edition of the work.

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Author

David H. von Seggern, PhD, worked for Teledyne Geotech from 1967 to 1982 in Alexandria, Virginia, almost exclusively on analysis of seismic data related to underground nuclear explosions. This effort was supported by the Air Force Office of Scientific Research (AFOSR) and by the Defense Advanced Research Projects Agency (DARPA). His research there addressed detection and discrimination of explosions, physics of the explosive source, explosive yield estimation, wave propagation, and application of statistical methods. Dr. von Seggern earned his PhD at Pennsylvania State University in 1982. He followed that with a 10-year position in geophysics research at Phillips Petroleum Company, where he became involved with leading-edge implementation of seismic imaging of oil and gas prospects and with seismic-wave modeling. In 1992, Dr. von Seggern assumed the role of seismic network manager at the University of Nevada for the Yucca Mountain Project seismic studies. In this capacity, Dr. von Seggern continued to investigate detection and location of seismic events, elastic wave propagation, and seismic source properties. Dr. von Seggern retired from full-time work in September 2005 and now pursues various seismological studies as emeritus faculty at the University of Nevada.

1

Introduction

1.1 Concept of a Curve

Let E^n be the Euclidean space of dimension n. (According to this definition, E^1 is a line, E^2 is a plane, and E^3 is a volume.) A curve in n-space is defined as the set of points which result when a mapping from E^1 to E^n is performed. In this reference work, only curves in E^2 and E^3 will be considered. Let t represent the independent variable in E^1. An E^2 curve is then given by

$$x = f(t), y = g(t)$$

and an E^3 curve by

$$x = f(t), y = g(t), z = h(t)$$

where f, g, and h mean "function of." The domain of t is usually $(0,\ 2\pi)$, $(-\infty,\ \infty)$, or $(0,\ \infty)$. These are the parametric representations of a curve. However, in E^2 curves are commonly expressed as

$$y = f(x)$$

or as

$$f(x, y) = 0$$

which are the explicit and implicit forms, respectively. The explicit form is readily reducible from the parametric form when $x = f(t) = t$ in E^2 and when $x = f(t) = t$ and $y = g(t) = t$ in E^3. The implicit form of a curve will often comprise more points than a corresponding explicit form. For example, $y^2 - x = 0$ has two ranges in y, one positive and one negative, while the explicit form derived from solving the above equation gives $y = \sqrt{x}$ for which the range of y is positive only.

Generally, the definition of a curve imposes a *smoothness criterion*,[1] meaning that the trace of the curve has no abrupt changes of direction (continuous first derivative). However, for purposes of this reference work, a broader definition of curve is proposed. Here, a curve may be composed of smooth branches, each satisfying the above definition, provided that the intervals over which the curve branches are distinctly defined and are contiguous. This definition will encompass forms such as polygons or sawtooth functions.

1.2 Concept of a Surface

This reference work defines surfaces as existing only in E^3. Therefore a surface is defined as the mapping from E^2 to E^3 according to

$$x = f(s,\ t),$$
$$y = g(s,\ t),$$
$$z = h(s,\ t).$$

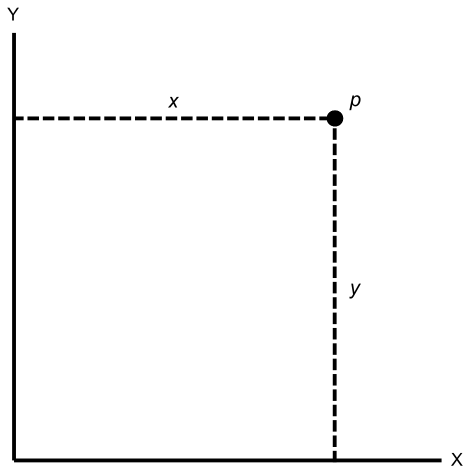

FIGURE 1.1
The Cartesian coordinate system for two dimensions.

As for curves, the conversion from this parametric form to more common forms

$$z = f(x, y)$$

or

$$f(x, y, z) = 0$$

may not be possible in some cases. Again, a *smoothness criterion*[1] is desirable; but the generalized definition of surface requires that this smoothness criterion only be satisfied piecewise for all distinct mappings of the (s,t) plane over which the surface is defined. These generalized surfaces are termed manifolds. Cubes are examples of surfaces which can be defined in this deterministic manner.

1.3 Coordinate Systems

The number of available coordinate systems for representing curves is large and even larger for surfaces. However, to maintain uniformity of presentation throughout this volume, only the following will be used:

2 – D	3 – D
Cartesian, polar	Cartesian, cylindrical, spherical

The term parametric is often used as though it were a coordinate system, but it is really a representation of coordinates in terms of an additional independent parameter which is not itself a coordinate of the E^3 space in which the curve or surface exists.

Cartesian Coordinates

The Cartesian coordinates system is illustrated in Figure 1.1 for two dimensions. This is the most natural, but not always the most convenient, system of coordinates for curves in two

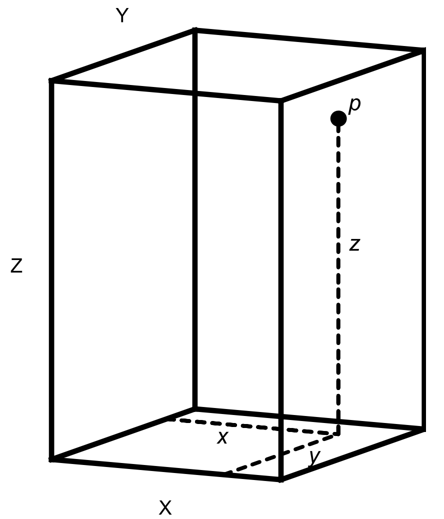

FIGURE 1.2
The Cartesian coordinate system for three dimensions.

dimensions. Coordinates of a point p are measured linearly along two axes which intersect with a right angle at the origin (0, 0). The Cartesian system is also called the rectangular coordinate system. For three dimensions, an additional axis, orthogonal to the other two, is placed as shown in Figure 1.2.

Polar Coordinates

Polar coordinates (r, θ) are defined for two dimensions and are a desirable alternative to Cartesian ones when the curve is point symmetric and exists only over a limited domain and range of the variables x and y. As illustrated in Figure 1.3, the coordinate r is the

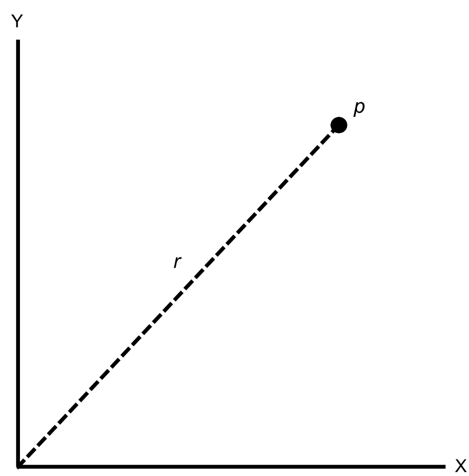

FIGURE 1.3
The polar coordinate system for two dimensions.

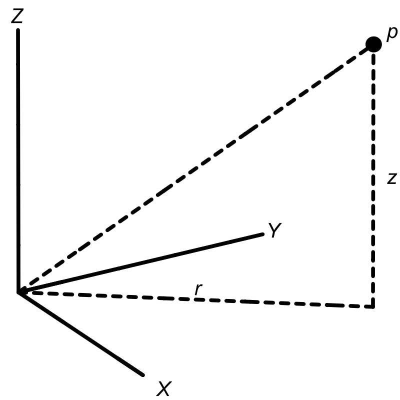

FIGURE 1.4
The cylindrical coordinate system for three dimensions.

distance of the point p from the origin and the coordinate θ is the counterclockwise angle which the line from the origin to p makes with the horizontal line through the origin to the right. Counterclockwise rotations are measured in positive θ, while clockwise rotations are measured in negative θ, relative to this line. Transformations from polar to Cartesian, and vice versa, are made according to:

$$x = r\ cos(\theta),$$
$$y = r\ sin(\theta)$$

and

$$r = \left(x^2 + y^2\right)^{1/2},$$
$$\theta = arctan(y/x).$$

Cylindrical Coordinates

Cylindrical coordinates are used in E^3. They combine the $(r,\ \theta)$ polar coordinates of two dimensions with the third coordinate z measured perpendicularly from the x-y plane at $(r,\ \theta)$ to the point p at $(r,\ \theta,\ z)$ as in Figure 1.4. The normal convention is for z to be positive upward. Transformation from cylindrical to Cartesian coordinates involves only the polar-to-Cartesian transformations given above because the z coordinate is unchanged. Cylindrical coordinates are often appropriate when surfaces are axially symmetric about the z axis; for example, in representing the form $r^2 = z$.

Spherical Coordinates

As illustrated in Figure 1.5, let a point in E^3 lie at a radial distance r along a vector from the origin. Project this vector to the x-y plane and let the angle between the vector and its projection be φ. Now measure the angle θ of the projected line in the x-y plane as for polar coordinates. Then $(r,\ \theta,\ \varphi)$ are the spherical coordinates of p. The transformations

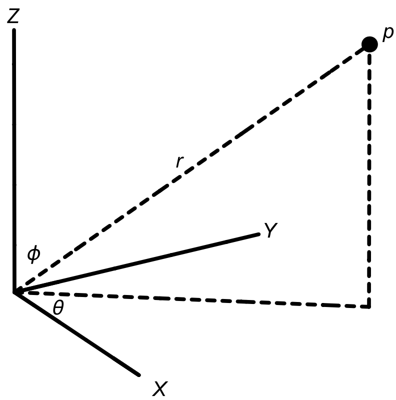

FIGURE 1.5
The spherical coordinate system for three dimensions.

from spherical to Cartesian coordinates, and vice versa, are given by

$$x = r\,cos\theta sin\varphi,$$
$$y = r\,sin\theta sin\varphi,$$
$$z = r\,cos\varphi$$

and

$$r = \left(x^2 + y^2\right)^{1/2},$$
$$\theta = arctan(y/x),$$
$$\varphi = arctan[\left(x^2 + y^2\right)^{1/2}/z].$$

Spherical coordinates are often appropriate for surfaces having point symmetry about the origin. The usual coordinates of geography, which refer to points on the earth by latitude and longitude, are a spherical system.

1.4 Qualitative Properties of Curves and Surfaces

Curves and surfaces exhibit a wide variety of forms. Particular attributes of form are derivable from the equations themselves, and many texts treat these in rigorous detail. The purpose here is not to duplicate such explicit and analytical treatment but rather to present the properties of curves and surfaces in a qualitative manner to which their visible forms are naturally and easily related. Understanding these properties enables one to choose the appropriate curve for a given purpose (for example, data fitting) or to modify, when necessary, an equation given in this volume into one more suitable for a given purpose.

Derivative

A fundamental quantity associated with a curve, or function, is the derivative. The derivative exists at all continuous points of the curve (except singular points as described in Section

1.4.7). Although the definition of derivative can be made with analytical rigor,[1] in graphical terms the derivative at any point is the slope of the tangent line at that point and is written as dy/dx for two-dimensional curves. For three-dimensional curves, the tangent line is along the trajectory of the curve, and three such derivatives are possible using the three pairs of $(x,\ y,\ z)$ coordinates. Closely associated with the derivative is a curve's normal which is the line perpendicular to the tangent. In two dimensions the normal is a single line, but in three dimensions the normal sweeps out a plane perpendicular to the tangent of the curve. As for curves, the derivative of a surface is a fundamental quantity. The derivative at any continuous point of a surface relates to the tangent plane of the surface at that point. For this plane, three partial derivatives exist, written as dy/dz, dz/dx, and dx/dy (or their inverses), which are the slopes of the lines formed at the intersection of the tangent plane with the y-z, z-x, and x-y planes, respectively. The normal n_p to the surface at a point is the vector orthogonal to the surface there. It is defined at all points for which the surface is smooth by the partial derivatives

$$(n_p) = \left[\begin{pmatrix} \frac{\delta y}{\delta s} & \frac{\delta y}{\delta t} \\ \frac{\delta z}{\delta s} & \frac{\delta z}{\delta t} \end{pmatrix}, \begin{pmatrix} \frac{\delta z}{\delta s} & \frac{\delta z}{\delta t} \\ \frac{\delta x}{\delta s} & \frac{\delta x}{\delta t} \end{pmatrix}, \begin{pmatrix} \frac{\delta x}{\delta s} & \frac{\delta x}{\delta t} \\ \frac{\delta y}{\delta s} & \frac{\delta y}{\delta t} \end{pmatrix}\right]_p$$

using the parametric representation equations. If the surface can be expressed in the implicit form $f(x,\ y,\ z) = 0$, then simply

$$(n_p) = \left[\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}\right]_p .$$

The above definitions give the $(x,\ y,\ z)$ components of the normal vector, and it is customary to normalize them to (x', y', z') by dividing them by $\left(x^2 + y^2 + z^2\right)^{1/2}$ so that $x'^2 + y'^2 + z'^2 = 1$.

Symmetry

For curves in two dimensions, if

$$y = f(x) = f(-x)$$

holds, then the curve is symmetric about the y axis. The curve is antisymmetric about the y axis when

$$y = f(x) = -f(-x).$$

A simple example is powers of x given by $y = x^n$. If n is even, the curve is symmetric; if n is odd, it is antisymmetric. Antisymmetry is also referred to as "symmetric with respect to the origin" or point symmetry about $(x,\ y) = (0,\ 0)$. For surfaces, three kinds of symmetry exist: point, axial, and plane. A surface has point symmetry when

$$z = f(x, y) = -f(-x, -y).$$

Simple examples of point symmetry are spheres or ellipsoids. A surface has axial symmetry when

$$z = f(x, y) = f(-x, -y).$$

An example of axial symmetry is a paraboloid. Finally, a surface has plane symmetry about the $(y,\ z)$ plane when

$$z = f(x, y) = f(-x, y).$$

Similarly, symmetry about the $(x,\ z)$ plane implies

$$z = f(x, y) = f(x, -y).$$

Finally, symmetry about the $(x,\ y)$ plane is represented by

$$z = f(x, y) = -f(x, y).$$

Examples of plane symmetry include $z = xy^2$ and $z = e^x \cos(y)$.

Extent

The extent of a curve is defined by the range (y variation) and domain (x variation) of the curve. The extent is unbounded if both x and y values can extend to infinity (for example, $y = x^2$). The extent is semibounded if either y or x has a bound less than infinity. The transcendental equation $y = \sin(x)$ is such a curve because the range is limited between negative and positive unity. A curve is fully bounded if both x and y bounds are less than infinity. A circle is a simple example of this type of extent.

For surfaces, the concept of extent can be applied in three dimensions where domain applies to x and y while range applies to z. Surfaces formed by revolution of a curve in the $(y,\ z)$ or $(x,\ z)$ plane about the z axis will possess the same extent property that the two-dimensional curve had. For example, an ellipse in the $(x,\ z)$ plane gives an ellipsoid as the surface of revolution—both have the fully bounded property. Similarly, any surface formed by continuous translation of a two-dimensional curve (for example, a parabolic sheet) will have the same extent property as the original curve.

Asymptotes

The y asymptotes of a curve are defined by

$$y_a = \lim_{x \to \pm\infty} f(x).$$

Although this definition includes asymptotes at infinity, only those with $|y_a| < \infty$ are of interest. Asymptotic values are often crucial in choosing and applying functions. Physically, an equation may or may not properly describe real phenomena, depending on its asymptotic behavior. Note that, even though a curve may be semi-bounded, its asymptote may not be determinable. An example of a semi-bounded function with a y asymptote is $y = e^{-x}$ while one without an asymptote is $y = \sin(x)$.

The x asymptotes of a curve may be defined in a similar manner with

$$(x_a) = \lim_{y \to \pm\infty} f(y)$$

when the function is inverted to give $x = f(y)$. An example of a curve with a finite x asymptote is $y = \left(c^2 - x^2\right)^{1/2}$ whose asymptote lies at $x = +c$ or $x = -c$.

In addition, curves may have asymptotes that are any arbitrary lines in the plane, not simply horizontal or vertical lines; and the limiting requirements are similar to the forms given above for horizontal or vertical asymptotes. For instance, the equation $y = x + 1/x$ has $y = x$ as its asymptote.

Periodicity

A curve is defined as periodic on x with period X if

$$y = f(x + nX)$$

is constant for all integers n. The transcendental function $y = \sin(ax)$ is an example of a periodic curve. A polar coordinate curve can also be defined as periodic with period α in terms of angle θ if

$$r = f(\theta + n\alpha)$$

is constant for all integers n. An example of such a periodic curve is $r = \cos(4\theta)$, which exhibits 8 "petals" evenly spaced around the origin. Surfaces are periodic on x and y with periods X and Y, respectively, if

$$z = f(x + nX, y + mY)$$

is constant for all integers n and m. A surface also may be periodic in only x or only y. A cylindrical-coordinate surface may be periodic with period a in terms of the angle θ if

$$z = f(r, \theta + n\alpha)$$

is constant for all integers n. Another type of periodicity expressible in cylindrical coordinates is in the radial direction with period R, when

$$z = f(r + nR, \theta)$$

is constant for all integers n. An example of such periodicity is given by $z = \cos(2\pi r)\cos(\theta)$, which has a period of $R = 1$.

Continuity

A curve is continuous at a point x_0, provided it is defined at x_0, when

$$y^+ = \lim_{x \to x_0^+} f(x)$$

and

$$y^- = \lim_{x \to x_0^-} f(x)$$

are finite and equal. Here "+" and "−" refer to approaching x_0 from the right and left, respectively. Discontinuities may be finite or infinite: the former implies $y^+ \neq y^-$ even though they are both finite while the latter implies one or both limits are infinite. For surfaces, the paths to a point $p_0 = (x_0,\ y_0)$ are infinite in number; and continuity exists only if the surface is defined at p_0 and

$$z = \lim_{p \to p_0} f(p)$$

is constant for all possible paths. When the curve or surface is undefined at x_0 or p_0 and the above relations hold, it is said to be discontinuous, but with a removable discontinuity. For any points at which the above relations do not hold, the curve or surface is discontinuous, with an essential discontinuity at such points. The curve $y = \sin(x)/x$ has a removable discontinuity and is therefore continuous in appearance while $y = 1/x$ has an essential discontinuity at $x = 0$ and is therefore discontinuous in appearance. Curves and surfaces are differentiable (meaning the derivative exists) everywhere that they are either continuous or have removable discontinuities.

Singular Points

Curves and surfaces may contain singular points. Writing the function for a two-dimensional curve as

$$f(x, y) = 0,$$

the derivative $\delta y/\delta x$ can be written as

$$\frac{\delta y}{\delta x} = \frac{g(x, y)}{h(x, y)}$$

where g and h are functions of x and y. If, for a given point $p(x,\ y)$, the functions g and h both vanish, the derivative becomes the indeterminate form 0/0, and $p(x,\ y)$ is then a singular point of the curve. Singular points imply that two or more branches of the curve meet or cross. If two branches are involved, it is a double point; if three are involved, it is a triple point; etc. Singularities at triple or higher points are not as commonly encountered as those at double points. Double-point singularities for two-dimensional curves are classified as follows:

1) Isolated points (also known as acnodes or conjugate points) are where a single point is disjoint from the remainder of the curve. In this case, the derivative is imaginary.

2) Node points (also known as crunodes) are where the two derivatives are real and unequal, such that the curve crosses itself.

3) Cusp points (also known as spinodes) are where the derivatives of two arcs on either side are unequal while the curve joins at this point. A cusp of the first kind involves second derivatives of opposite sign, and a cusp of the second kind involves second derivatives of the same sign.

4) Double cusp points (also known as tacnodes or osculation points) are where the derivatives of two arcs become equal while the two arcs of the curve are continuous along both directions away from such points. Double cusps may also be of the first or second kind, as for single cusps.

Curves having one or more nodes will exhibit loops that enclose areas. Curves having osculations may also exhibit loops, on one or both sides of the osculation point.

The concept of singular points is extendable to surfaces. Many surfaces are the result of the revolution of a two-dimensional curve about some line; such surfaces retain the singular points of the curve, except that each such point on the curve, unless on the axis of revolution, becomes a circular ring of singular points centered on the axis of revolution. Singular points appear on more complicated surfaces also, but an analysis of the possibilities is beyond the scope of this volume.

Critical Points

Points of a curve $y = f(x)$ at which the derivative $dy/dx = 0$ are termed critical points, of which there are three types:

1) Maximum points are where the curve is concave downward and thus the second derivative $d^2y/dx^2 > 0$.

2) Minimum points are where the curve is concave upward and thus the second derivative $d^2y/dx^2 < 0$.

3) Inflection points are where $d^2y/dx^2 = 0$ and the curve changes its direction of concavity.

For surfaces $z = f(x, y)$, the critical points lie at $dz/dx = dz/dy = 0$. Maximum and minimum points of surfaces are defined similar to those of curves, except both second derivatives must together be greater than zero or less than zero. In the case that they are of opposite sign, the critical point is termed a saddle. Such critical points are nondegenerate[2] and are isolated from other critical points. More complicated types of degenerate critical points occur for surfaces. Points can be classified as degenerate or nondegenerate, depending on whether the determinant of

$$\begin{pmatrix} \frac{\delta^2 z}{\delta x^2} & \frac{\delta^2 z}{\delta x \delta y} \\ \frac{\delta^2 z}{\delta x \delta y} & \frac{\delta^2 z}{\delta y^2} \end{pmatrix}$$

vanishes or not, respectively. For instance, the surface $z = x^2 + y^2$ has a single nondegenerate critical point while $z = x^2y^2$ has two continuous lines of degenerate critical points, intersecting at (0, 0).

Zeroes

The zeroes of a two-dimensional function $f(x)$ occur where $y = f(x) = 0$ and are isolated points on the x axis. (For polynomial functions, the zeroes are often referred to as the roots.) Similarly, the zeroes of a three-dimensional function $f(x, y)$ occur where $z = f(x, y) = 0$; but the loci of these points form one or more distinct, continuous curves in the x-y plane. The zeroes of certain functions are important in characterizing their oscillatory behavior; for example, the function $\sin(x)$. The zeroes of other functions may be unique points of interest in physical applications. Not all functions, as defined, have zeroes; for example, the function $f(x) = 2 - \cos(x)$ has unity as its lower bound. However, such a function can be translated in one or the other y directions to produce a function having zeroes in addition to all the qualitative properties of the original function. The definition of the exact zeroes of a function is often difficult and often must be accomplished by numerical methods on a computer. Zeroes of many functions are tabulated in standard references such as Abramowitz.[3]

Integrability

The function $y = f(x)$ defined over the interval $[a, b]$ has the integral

$$I = \int_a^b f(x)\,dx.$$

The integral exists if I converges to a single, bounded value for a given interval; and the function is said to be integrable. Note that the integral I may not exist under two abnormal circumstances:

1) Either a or b, or both, extend to infinity.

2) The function y has an infinite discontinuity at one or both endpoints or at one or more points interior to $[a, b]$.

Under either of these circumstances, the integral is an improper integral. Proving the existence of the integral of a given function is not always straightforward, and a discussion is beyond the scope of this volume. Transient functions always have an integral on the interval $[0, \infty]$ and are often given as solutions to physical problems in which the response of a medium to a given input or disturbance is sought. Such responses must possess an integral if the input was finite and measurable. Examples of such functions are $y = e^{-ax}\sin(bx)$ or $y = 1/(1 + x^2)$. Surfaces given by $z = f(x, y)$ are integrable when

$$I = \int_a^b \int_c^d f(x, y)\, dx\ dy$$

exists. Improper integrals of surfaces are defined in the same manner as those of two-dimensional curves. Transient responses exist for three dimensions and are integrable also. A curve property that has an important consequence for integration is that of even and odd functions. Even functions have $f(x) = f(\text{-}x)$, and for such curves

$$I = 2\int_0^a f(x)\, dx$$

if the one-sided I exists over $[0, a]$. For odd functions $f(x) = f(-x)$, and $I = 0$ over any interval$[-a, a]$. This concept can be easily extended to surfaces.

Multiple Values

A curve is multivalued if, for a given (x, y), it has two or more distinct values. A simple example is $y^2 = x$. Multivalued functions are not integrable in the normal sense, although one or more particular branches of the curve may have well-defined integrals. While a curve may be multivalued in its Cartesian-form equation, the polar form of the equation may be single-valued, in the sense that only one value of r exists for each value of angle θ. Compare, for example,

$$(x^2 + y^2)^3 = (x^2 - y^2)^2,$$

which is the equation of a quadrifolium, with its polar equation

$$r = cos(2\theta).$$

Integrability is affected by the choice of coordinate system; this example shows that, when an integral is not defined due to a function being multivalued, it may be well defined when the transformation to polar coordinates is made and the integral evaluated along the polar angle θ. Similarly, surfaces may be single-valued or multivalued depending upon whether z takes on one or more values for a given (x, y) point.

Curvature

Given that a unit of length along the curve path is δs and that the tangent line changes its direction over δs by an angle $\delta\theta$ where θ is the angle of the tangent with the x axis, then the principal curvature is given by

$$c = \frac{\delta\theta}{\delta s}.$$

The radius of curvature is simply the inverse of the curvature, or $\rho = 1/c$. At a point of inflection of a curve, $c = 0$ and $\rho = \infty$. Conversely, at a cusp of a curve, $c = \infty$ and $\rho = 0$.

The curvature can be expressed in terms of the derivatives of the curve also. If the curve is expressed implicitly as $f(x,\ y) = 0$ and if f_x and f_y are the first partial derivatives and f_{xx}, f_{yy}, and f_{xy} are the second partial derivatives, then

$$c = \frac{f_{xx}{f_y}^2 - 2f_{xy}f_xf_y + f_{yy}{f_x}^2}{\left(\ {f_x}^2 + \ \ {f_y}^2\ \right)^{3/2}}.$$

For curves defined parametrically as $\{x(\theta),\ y(\theta)\}$, letting $x' = d[x(\theta)]/d\theta$, $y' = d[y(\theta)]/d\theta$, $x" = d^2[x(\theta)]/d\theta^2$, and $y" = d^2[y(\theta)]/d\theta^2$, one obtains the curvature as

$$c = \frac{x'y" - y'x"}{\left(\ x'^2 + \ \ y'^2\ \right)^{3/2}}.$$

When the curve is expressed in polar coordinates $r = f(\theta)$ and the derivatives $dr/d\theta$ and $d^2r/d\theta^2$ are given by r' and $r"$, respectively, then the curvature is

$$c = \frac{r^2 + 2r'^2 - rr"}{\left(\ r^2 + \ \ r'^2\ \right)^{3/2}}.$$

The radius of curvature at lobes of polar curves is of interest in order to define the "tightness" of the lobes. At the peak of the lobe, $r' = 0$ and $\rho = r^2/(r - r")$. This reduces to $\rho = r$ in the case of a circle, for which $r" = 0$.

For surfaces, the geometry underlying the concept of curvature is more complex. Curvature of a surface at a point p is normally given as the Gaussian curvature

$$K = \kappa_1\kappa_2$$

where the κ's are the principal curvatures, with κ_1 being the minimum curvature at p and κ_2 being the maximum curvature at p. These curvatures are determined by the two-dimensional curvature of the intersections of the surface with all possible planes containing p. If κ_1 and κ_2 are both of the same sign, the point p is an elliptic point and the surface is dome-like at p. If κ_1 and κ_2 have opposite signs, the point p is a hyperbolic point and the surface is saddle-like at p. If either κ_1 or κ_2 is zero, the point p is a parabolic point. A line separating positive and negative K regions is a parabolic line.

If a surface is defined explicitly as $z = f(x,\ y)$, then the Gaussian curvature can be calculated as

$$K = \frac{z_{xx}z_{yy} - z_{xy}^2}{1 + z_x^2 z_y^2}$$

where

$$\begin{aligned} z_x &= \delta z/\delta x, \\ z_y &= \delta z/\delta y, \\ z_{xx} &= \delta^2 z/\delta x^2, \\ z_{yy} &= \delta^2 z/\delta y^2, \\ z_{xy} &= \delta^2 z/\delta x \delta y. \end{aligned}$$

1.5 Classification of Curves and Surfaces

The family of two-dimensional and three-dimensional curves can be illustrated as in Figure 1.6. This particular schematic reflects the organization of this reference work, and every

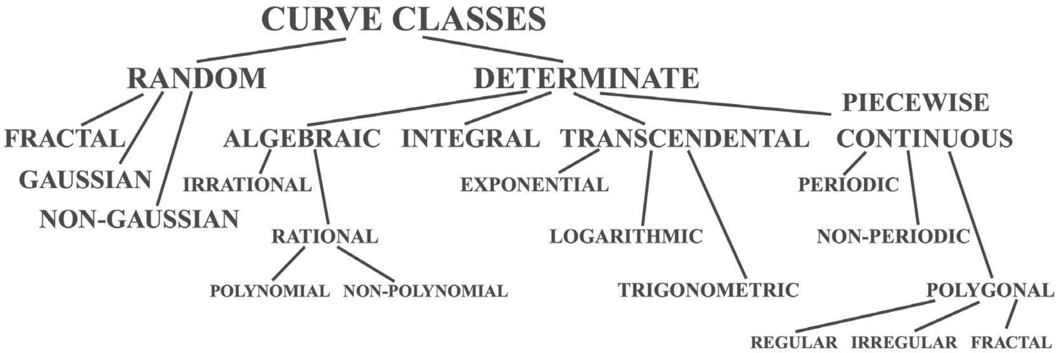

FIGURE 1.6
Classification of curves and surfaces.

curve which can be traced by a given mathematical equation or given set of mathematical rules can be placed in one of the categories shown. There is a top-level dichotomy between determinate and random curves. A determinate curve is one for which the functional relationship between x and y is known everywhere from the equation or set of rules. No realization is required to produce the curve, for it is contained wholly within its defining equations or rules. On the other hand, a random curve will have a random factor or term in its mathematical definition such that an actual realization is required to produce the curve, which will differ from any other realization. For example, $y = \sin(x) + w(x)$ where $w(x)$ is a random variable on x, defines a random curve. At the second level in Figure 1.6, the distinction is made between algebraic, transcendental, integral, and non-differentiable curves as described below.

Algebraic Curves

A polynomial is defined as a summation of terms composed of integer powers of x and y. An algebraic curve is one whose implicit function

$$f(x, y) = 0$$

is a polynomial in x and y (after rationalization as described below, if necessary). Because a curve is often defined in the explicit form

$$y = f(x),$$

there is a need to distinguish rational and irrational functions of x. A rational function of x is a quotient of two polynomials in x, both having only integer powers. An irrational function of x is a quotient of two polynomials, one or both of which has a term (or terms) with power p/q, where p and q are integers. Irrational functions can be rationalized, but the curves will not be identical before and after rationalization. In general, the rationalized form has more branches; for example, consider $y = \sqrt{x}$, which is rationalized to $y^2 = x$. The former curve has only one branch (for positive y) if a strict definition of the radical is used, whereas the latter has two branches, for $y < 0$ and $y > 0$. In this reference work, the rationalized curve will be presented graphically in all cases, even though the equation is printed in its irrational form for simplicity.

Besides simple polynomials, rational functions are often grouped into sets convenient for certain mathematical applications. Examples of such polynomial sets are Chebyshev polynomials, Laguerre polynomials, and Bernoulli polynomials. Most polynomial sets have the property of orthogonality, meaning that for any two functions f_1 and f_2 of the set,

$$\int w(x) f_1(x) f_2(x)\, dx = 0$$

over the defined domain of x for the particular set, where $w(x)$ is a weighting function. This property ensures that the different curves within the set make distinct contributions to the set.

Transcendental Curves

The transcendental curves cannot be expressed as finite polynomials in x and y. These are curves containing one or more of the following forms: exponential (e^x), logarithmic (log x), or trigonometric (sin x, cos x). The hyperbolic functions are often mentioned as part of this group, but they are not really distinct because they are forms composed of exponential functions. Any curve expressed as a mixture of transcendentals and polynomials is considered to be transcendental. All of the primary transcendental functions can, in fact, be expressed as infinite polynomial series:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} (-\infty < x < \infty), \\ cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} (-\infty < x < \infty), \\ sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} (-\infty < x < \infty), \\ log x &= 2 \sum_{n=0}^{\infty} \frac{1}{2n-1} \left(\frac{x-1}{x+1}\right)^{2n+1} \qquad (x > 0). \end{aligned}$$

Integral Curves

Certain continuous curves not expressible in algebraic or transcendental forms are familiar mathematical tools. These curves are equal to the integral of algebraic or transcendental curves by definition; examples include Bessel functions, Airy integrals, Fresnel integrals, and the error function. The integral curve is given by

$$y[a, b] = \int_a^b f(x)\, dx$$

where the lower limit of integration a is usually a fixed point such as $-\infty$ or 0. Like transcendental curves, these integral curves also have expansions in terms of power series or polynomial series, often making evaluation rather straightforward on computers.

Piecewise Continuous Functions

Members of the previous classes of curves (algebraic, transcendental, and integral) all have the property that (except at a few points, called singular points) the curve is smooth and differentiable. In the spirit of a broad definition of curve, a class of non-differentiable curves appears in Figure 1.6. These curves have discontinuity of the first derivative as a basic attribute and are quite often composed of straight-line segments. Such curves include the simple polygonal forms as well as the intricate "regular fractal" curves of Mandelbrot.[4]

Classification of Surfaces

In general, surfaces may follow the same classification scheme as curves (Figure 1.6). Many commonly used surfaces are either rotations of two-dimensional curves about an axis, thus giving axial, or possibly point, symmetry. In this case the independent variable x of the two-dimensional curve's equation can be replaced with the radial variable $r = \left(x^2 + y^2\right)^{1/2}$ to form the equation of the surface. Other commonly used surfaces are merely a continuous translation of a given two-dimensional curve along a straight line. Such surfaces will actually have only one independent variable if a coordinate system having one axis coincident with the straight line is chosen.

If the two independent variables of the explicit equation of the surface, $z = f(x, y)$, are separable in the sense that

$$z = f(x)f(y),$$

then the surface is orthogonal. In such a case, the x dependence may fall in one of the classes of Figure 1.6 while the y dependence falls in another. Orthogonal surfaces require fewer operations to evaluate over a grid of the domain of x and y because the defining equation only needs to be evaluated once along the x direction and once along the y direction, with all other points evaluated by simple multiplication of the x and y factors appropriate to each point on the (x, y) plane.

1.6 Basic Curve and Surface Operations

There are many simple operations that can be applied to curves and surfaces in order to change them. Knowledge of these operations enables one to adapt a given curve or surface to a particular need and to thus extend the curves and surfaces given in this reference work to a larger set of mathematical forms. Only a few of the most common operations are presented here. Of these, two (translation and rotation) are homomorphic operations, which means that the form of the curve is preserved, with merely its position or orientation in space being changed.

Translation

If one or more of the coordinates (x, y, z) of a point is changed according to

$$x' = x + a,$$
$$y' = y + b,$$
$$z' = z + c,$$

the curve or surface undergoes a translation of amount (a, b, c) along the (x, y, z) axes, respectively.

1.6.2 Rotation

In polar coordinates, if the angle θ is changed by a positive amount α thus

$$\theta' = \theta + \alpha,$$

the curve undergoes a counter-clockwise rotation of α degrees. This is convenient for polar coordinates, but the rotation can also be expressed in Cartesian coordinates as

$$x' = x\,cos(\alpha) + y\,sin(\alpha),$$
$$y' = -x\,sin(\alpha) + y\,cos(\alpha).$$

In three dimensions, a surface can be rotated about any of the three axes by using these equations on the coordinate pairs (x, y), (y, z), or (x, z) depending on whether the rotation is about the z, x, or y axis, respectively.

Linear Scaling

The relations for linear scaling are

$$x' = ax,$$
$$y' = by,$$
$$z' = cz.$$

These stretch the curve or surface by the factors a, b, and c along the respective axes. When using polar, cylindrical, or spherical coordinates, a similar relation

$$r' = dr$$

stretches or compresses the curve or surface along the radial coordinate by the factor d.

Reflection

A two-dimensional curve has a reflection about the x axis caused by letting

$$y' = -y$$

or about the y axis by letting

$$x' = -x$$

or through the origin by applying both these equations. In three dimensions, a curve or surface is reflected across the (y, z), (x, z), or (x, y) planes when

$$x' = -x,$$
$$y' = -y,$$
$$z' = -z,$$

respectively. It can be reflected through the origin when one sets

$$r' = -r$$

in spherical coordinates and mirrored through the z axis when the same operation is made on r for cylindrical coordinates. The application to two-dimensional polar coordinates follows from the cylindrical case.

Rotational Scaling

For two dimensions, let

$$\theta' = c\theta$$

for the polar angle; the polar curve is then stretched or compressed along the angular direction by a factor c in a rotational scaling. The same operation can be applied to θ for cylindrical coordinates in three dimensions or to both θ and φ for spherical coordinates in three dimensions.

Radial Translation

In two dimensions with polar coordinates, if the radial coordinate is translated according to

$$r' = r + a,$$

then the entire curve moves outward by the amount a from the origin. Note that this operation is not homomorphic like Cartesian translation because the curve is stretched in the angular direction while undergoing the radial translation. This operation can be performed on the radial coordinate of either cylindrical or spherical coordinate systems in three dimensions.

Weighting

In a two-dimensional Cartesian system, let

$$y' = |x|^a\, y.$$

This operation performs a weighting on the curve by the factor $|x|^a$, a symmetric operator. If $a > 0$, the curve is stretched in the y direction by a factor that increases with x; but if $a < 0$, the curve is compressed by a factor that decreases with x. Similar treatments can be performed on surfaces in three dimensions.

Nonlinear Scaling

If in two dimensions the nonlinear scaling

$$y' = y^a$$

is performed, the curve is progressively scaled upward or downward in absolute value, according to whether $a > 1$ or $a < 1$, respectively. Note that, if $y < 0$ and $a = 2, 4, 6, ...$, then the scaled curve will flip to the opposite side of the x axis. Similar scalings can be made in three dimensions using any of the appropriate coordinate systems.

Shear

A curve undergoes simple shear when either all its x coordinates or all its y coordinates remain constant while the other set is increased in proportion to x or y, respectively. The general transformations for simple shearing of a two-dimensional curve are

$$x' = x + ay,$$
$$y' = bx + y.$$

The transformations for simple x shear are

$$x' = x + ay,$$
$$y' = y.$$

and for simple y shear are

$$x' = x,$$
$$y' = bx + y.$$

Surfaces may be simply sheared along one or two axes with similar transformations. Another special case of shear is termed pure shear, and the transformations for a two-dimensional curve are given by

$$x' = kx,$$
$$y' = k^{-1}y.$$

For surfaces, pure shear will only apply to two of the three coordinate directions, with the remaining one having no change. Pure shear is a special case of linear scaling under this circumstance.

Matrix Method for Transformation

The foregoing transformations can all be expressed in matrix form, which is often convenient for computer algorithms. This is especially true when several transformations are concatenated together, for the matrices can then be simply multiplied together to obtain a single transformation matrix. Given a pair of coordinates (x, y), a matrix transformation to obtain the new coordinates (x', y') is written as

$$(x' y') = (x\ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

or explicitly

$$x' = ax + cy,$$
$$y' = bx + dy.$$

According to this definition, Table 1.1 lists several of the two-dimensional x-y transformations discussed previously with their corresponding matrix.

Translations cannot be treated with the above matrix definition. An extension is required to produce what is commonly referred to as the homogeneous coordinate representation in computer graphics programming. In its simplest form, an additional coordinate of unity is appended to the (x, y) pair to give $(x, y, 1)$. A translation by u and v in the x and y directions is then written using a 3-by-3 matrix

$$(x' y' 1) = (x\ y\ 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix}$$

where explicitly,

$$x' = x + u,$$
$$y' = y + v,$$
$$1 = 1.$$

With this representation, a radial translation by s units of a curve given in (r, θ) coordinates is effected by

$$(r' \theta' 1) = (r\ \theta\ 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s & 0 & 1 \end{pmatrix}$$

such that $r' = r + s$ and θ is unchanged. In three dimensions similar transformations exist, as shown in Table 1.2, mostly being simple extensions of those given in Table 1.1.

TABLE 1.1
2-D Transformations

Operation	Matrix
Rotation	$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$
Linear scaling	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
Reflection	$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$
Weighting	$\begin{pmatrix} 1 & 0 \\ 0 & x^a \end{pmatrix}$
Nonlinear scaling	$\begin{pmatrix} 1 & 0 \\ 0 & y^a \end{pmatrix}$
Simple shear	$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$
Rotational scaling	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$

Notes:
Rotation: α is the counterclockwise angle in the x-y plane.
Reflection: Use $+$ or $-$ according to the desired reflection.
Simple shear: Either a or b is zero, for x or y shear, respectively.
Rotational scaling: Use with (r, θ) coordinates.

1.7 Method of Presentation

This reference work is basically intended to be illustrative; therefore all functions, whether curves or surfaces, presented in this work will have an accompanying plot showing the form of the function. Curves and surfaces and their plots are numbered for easy reference and grouped according to type. Wherever popular names exist for certain curves or surfaces, they are placed with the equations themselves. Only basic explanatory information is provided with each curve, as needed. The interested reader can consult textbooks, or world-wide web resources for further information on specific functions.

Equations

The equation of each algebraic or transcendental curve will be given in the explicit form $y = f(x)$ or $r = f(\theta)$ wherever possible; similarly, surfaces will be given as $z = f(x, y)$ or $r = f(\theta, z)$ or $r = f(\theta, \varphi)$. Whenever polar, cylindrical, or spherical coordinate forms are used, the equation is also written in Cartesian coordinates, if possible. Because some curves and surfaces are not amenable to explicit forms, the parametric equations will be used as the alternative. In either case, whether explicit or parametric, the implicit functional form will also be given, if derivable. The explicit or parametric form is usually the most direct means to evaluate the curve or surface on a computer while the implicit form enables one to determine the degree of the equation (if algebraic) and also easily determine the derivatives in some cases. Notes pertinent to evaluation are given whenever they may help to understand the figures better. For integral curves and surfaces, the equation will be given

TABLE 1.2
3-D Transformations

Operation	Matrix
Rotation	$\begin{pmatrix} c\beta\cdot c\gamma & s\alpha\cdot s\beta\cdot c\gamma + c\alpha\cdot s\gamma & -c\alpha\cdot s\beta\cdot c\gamma + s\alpha\cdot s\gamma \\ -c\beta\cdot s\gamma & c\alpha\cdot c\gamma - s\alpha\cdot s\beta\cdot s\gamma & s\alpha\cdot c\gamma + c\alpha\cdot s\beta\cdot s\gamma \\ s\beta & -s\alpha\cdot c\beta & c\alpha\cdot c\beta \end{pmatrix}$
Linear scaling	$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$
Reflection	$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$
Weighting	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^a y^a \end{pmatrix}$
Nonlinear scaling	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^a \end{pmatrix}$
Simple shear	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Rotational scaling	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$

Notes:
cα, cβ , cγ = cos α , cos β , cos γ
sα, sβ, sγ = sin α, sin β , sin γ
Rotation: α, β , γ are the counterclockwise rotations about each positive axis.
Reflection: Use + or − according to the desired reflection.
Simple shear: Gives simple x shear depending on whether done along y or z direction. Similar expressions hold for simple y or z shear.
Rotational scaling: Use with (r, θ, ϕ) coordinates.

as the integral $y = \int f(x)$ or $z = \int f(x, y)$. Most of the integral forms have commonly used names (for example, "Bessel functions"). Other curves or surfaces in this reference work are expressed not by single equations, but rather by some set of mathematical rules. The method of presentation will vary in these cases, always with the objective of providing the reader with a means of easily constructing the curve or surface by machine computation.

Plots

Readers of previous editions will notice that, in this edition, only one realization of an equation is given. The plots of the curves and surfaces were enabled with the *Mathematica®* Manipulate function. All of the variable parameters were allowed to be manipulated, to within reasonable limits, and a representative choice for printing was made using one set of parameters. All graphs, unless there are no parameters to manipulate, include the readout of the parameters so that the reader can see the exact realization of the function. Many functions have a wide range of possible realizations, sometimes differing radically in appearance; and thus the single example shown may not adequately show the behavior of the function. Some curves and surfaces have no variable parameters, and so a simple static plotting function was used. Plots of two-dimensional curves are done on the $(x,\ y)$ plane, with the x and y axes being horizontal and vertical, respectively. Three-dimensional curves and surfaces have the additional z axis and are plotted in a projection that satisfactorily illustrates the form of each function. The implicit form of a curve often comprises more points than a corresponding explicit form. For example $y^2 - x = 0$ has two ranges in y, one positive and one negative, while the explicit form derived from solving the above equation gives $y = \sqrt{x}$ for which the range of y is positive only; in such cases both the positive and negative range of y are plotted.

References

[1] Buck, R.C., *Advanced Calculus*, McGraw-Hill, New York, 1965, chap. 5.

[2] Poston, T., and I. Stewart, *Catastrophe Theory and Its Applications, Pitman*, New York, 1978.

[3] Abramowitz, M., Ed., *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, Dover, 1974.

[4] Mandelbrot, B.B., *The Fractal Geometry of Nature*, W.H. Freeman, San Francisco, 1983.

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2

Algebraic Functions

The curves of this chapter are mostly familiar equations found in elementary algebra texts or in tables of integrals. Many have acquired traditional or accepted names in the mathematical literature, and these names are included wherever appropriate. The last two sections deal with curves more readily expressed in polar coordinates or parametrically; this allows much easier computation of the curves than with the form $y = f(x)$, especially when curves are multiple-valued in this form.

2.0 Plotting Information for This Chapter

The functions were plotted here utilizing the *Mathematica*® plotting functions Plot, ParametricPlot, and ContourPlot within the Manipulate function. The x axes run from -1 to $+1$; but, in order to show the true nature of the curves, it is often necessary to scale in y. Thus one sees the curve expressed as $y = c\ f(x)$ where c scales the y coordinate. Many of the curves have discontinuities at one or more discrete x values. For curves involving radicals, both the positive and negative branches are plotted to show the symmetry.

2.1 Functions with $x^{n/m}$

2.1.1 $y = cx^n$

Note that cases with n even are symmetrical about the y axis while cases with n odd are anti-symmetrical about the y axis. The curve corresponding to each power of n has a specific name:
n = 1 → linear
n = 2 → quadratic or parabola
n = 3 → cubic
n = 4 → quartic
n = 5 → quintic
n = 6 → sextic
n = 7 → septic
n = 8 → octic

n = 9 → nonic
n = 10 → decic

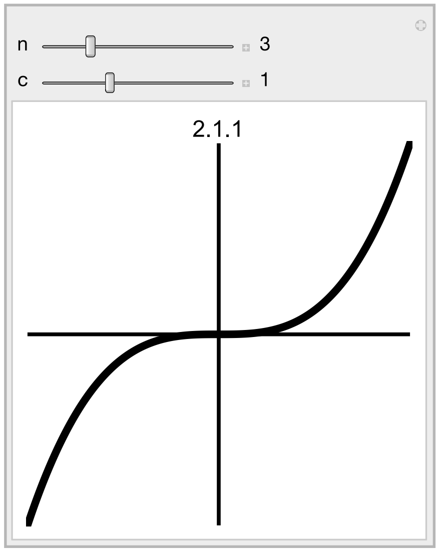

FIGURE 2.1.1

2.1.2 $y = c/x^n$

Note that cases with n even are symmetrical about the y axis while cases with n odd are anti-symmetrical about the y axis. The case $n = 1$ gives a hyperbola.

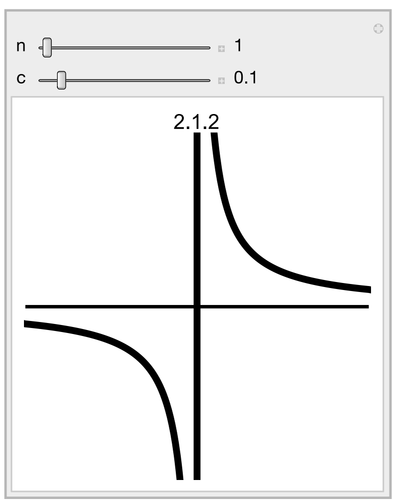

FIGURE 2.1.2

2.1.3 $y = c|x|^{n/m}$

$n = 3$; $m = 2 \rightarrow$ semicubical parabola
$n = 2$; $m = 3 \rightarrow$ cusp catastrophe

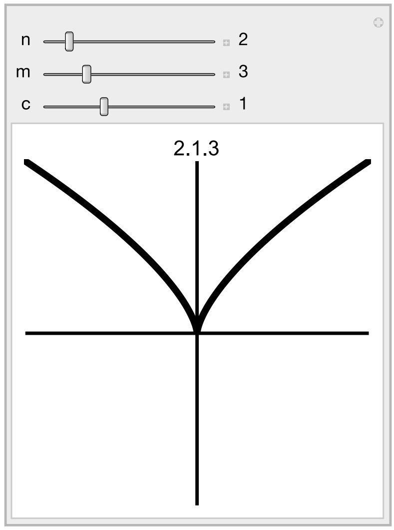

FIGURE 2.1.3

2.1.4 $y = c/|x|^{n/m}$

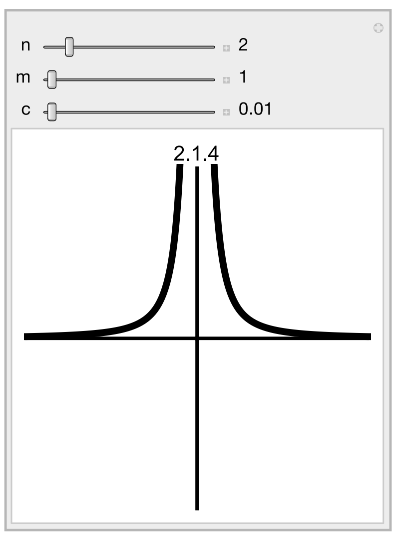

FIGURE 2.1.4

2.2 Functions with x^n and $(a+bx)^m$

2.2.1 $y = c(a + bx)$

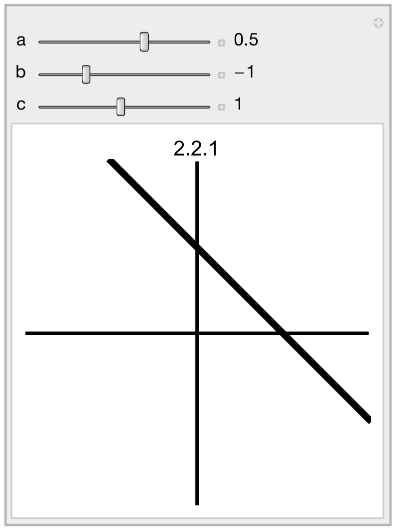

FIGURE 2.2.1

2.2.2 $y = c(a + bx)^2$

Parabola

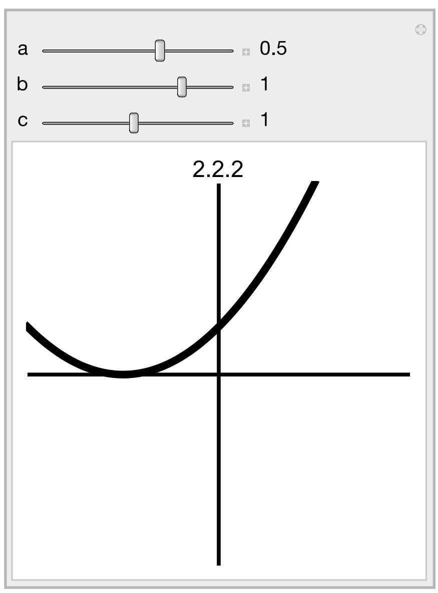

FIGURE 2.2.2

2.2.3 $y = c(a + bx)^3$

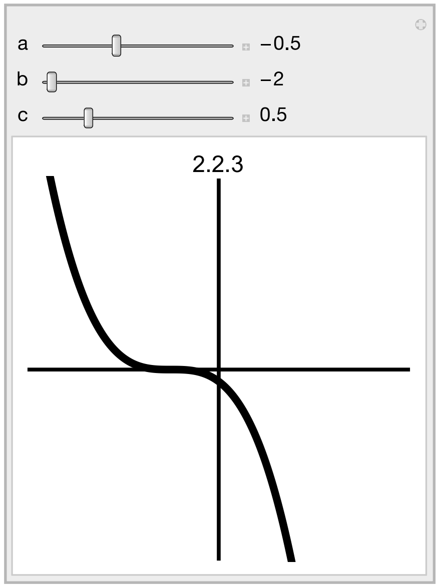

FIGURE 2.2.3

2.2.4 $y = cx(a + bx)$

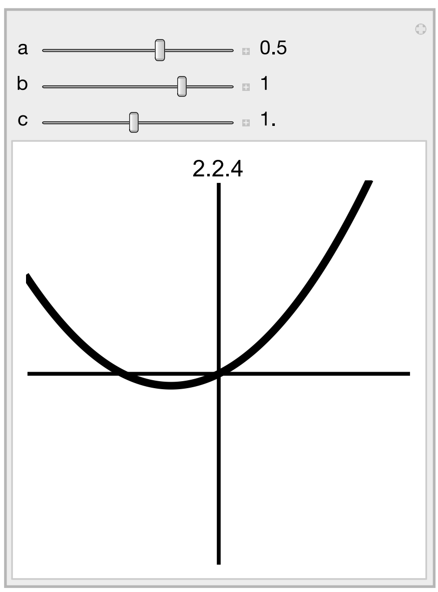

FIGURE 2.2.4

2.2.5 $y = cx(a + bx)^2$

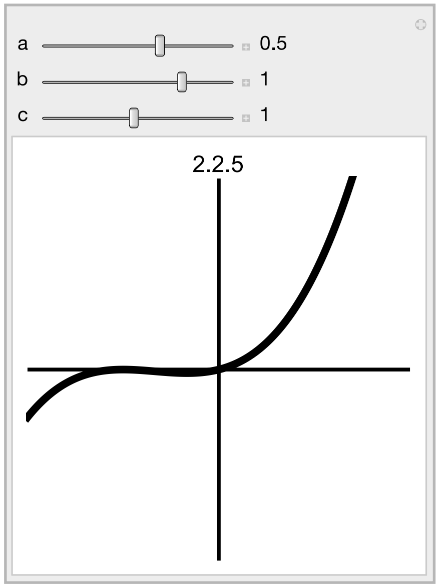

FIGURE 2.2.5

2.2.6 $y = cx(a + bx)^3$

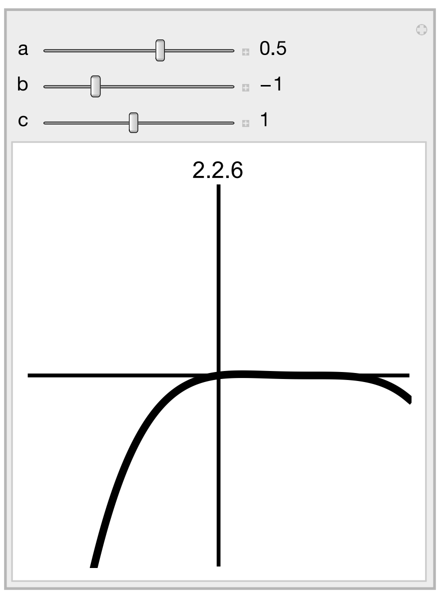

FIGURE 2.2.6

2.2.7 $y = cx^2(a + bx)$

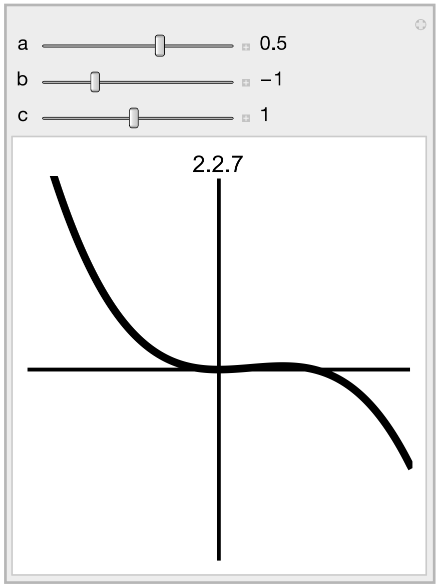

FIGURE 2.2.7

2.2.8 $y = cx^2(a + bx)^2$

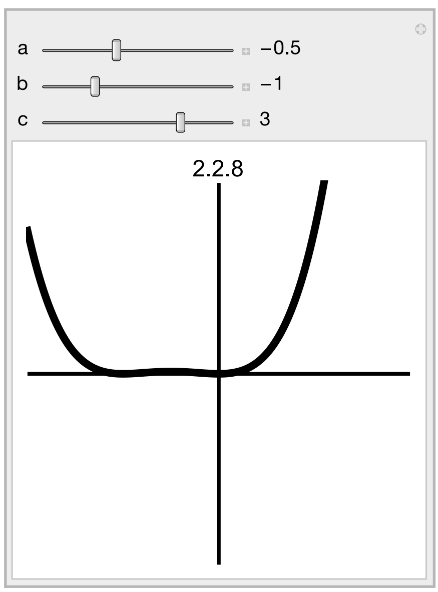

FIGURE 2.2.8

2.2.9 $y = cx^2(a + bx)^3$

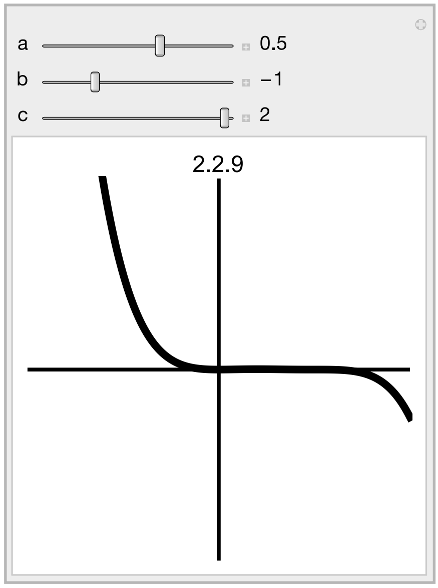

FIGURE 2.2.9

2.2.10 $y = cx^3(a + bx)$

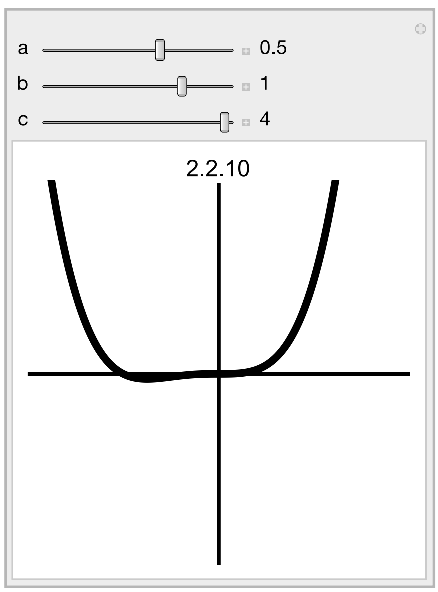

FIGURE 2.2.10

2.2.11 $y = cx^3(a + bx)^2$

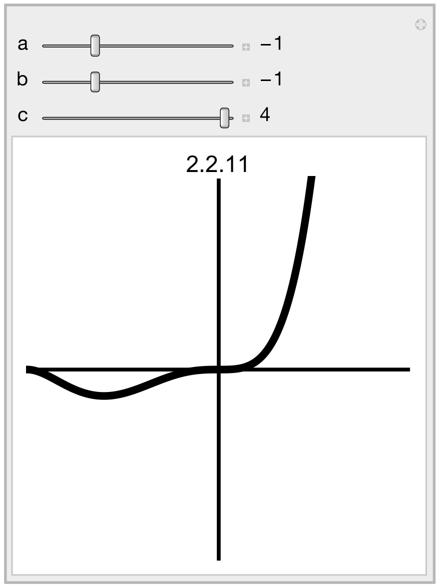

FIGURE 2.2.11

2.2.12 $y = cx^3(a + bx)^3$

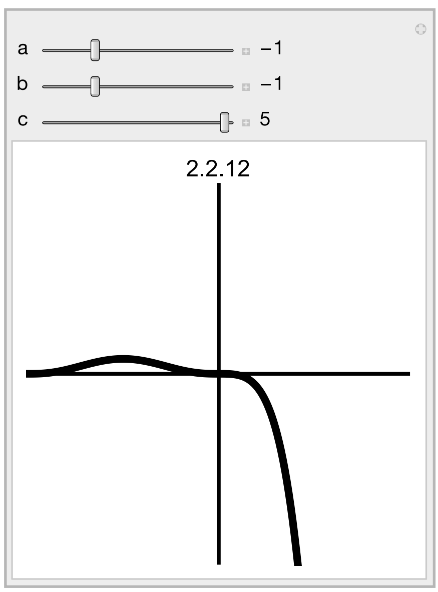

FIGURE 2.2.12