MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Actions and Invariants of Algebraic Groups

Second Edition

Walter Ricardo Ferrer Santos
Alvaro Rittatore

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Actions and Invariants of Algebraic Groups

Second Edition

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Preface to the Second Edition

More than ten years ago we wrote the first edition of this introduction to geometric invariant theory, with the intent to draw a bridge between the basic theory of affine algebraic groups and the more sophisticated theory of the geometric invariant theory à la Mumford. When the editors of the series "Current Monographs and Research Notes in Mathematics" by Chapman and Hall/CRC Press suggested the possibility of a new edition, we gladly took the opportunity in order to introduce some relevant subjects of recent appearance. Moreover, since the first edition saw the light, some colleagues and we, ourselves, thought that a few omitted subjects could profitably be introduced; these additions also were implemented. In both cases we tried to guarantee the stability of the general stylistic conception of the book, in particular its aspiration of self-containment. Some typos and mistakes have also been fixed.

The main changes implemented in this second edition are described in the introductions to the different chapters, especially chapters 7 and 14 — present in the first edition — and chapters 2, 15 and 16 — that are new chapters.

Below we describe the more relevant changes.

In order to introduce the new topics presented in this second edition, we added some additional results on the geometry of algebraic varieties, namely the construction of the *Proj variety* associated to a graded algebra and the concept of scheme — mainly affine schemes — and of rational points of an affine scheme — i.e., the categorical perspective of a scheme. In order to keep the balance of the chapter sizes, we split Chapter 1 into two: the title of the the first chapter *Algebraic geometry: basic definitions and results* is self-explanatory — almost all its results are presented with proofs; in the added chapter *Algebraic varieties*, we deal with the finer geometric aspects of this theory, and often we do not write detailed proofs — although the reader is given precise references for them. It is in this chapter that we introduce some elements of the theory of schemes, for their use mainly in the new Chapter 16, *Quotient varieties: an introduction to geometric invariant theory*. Also, we added an early introduction in Chapter 7 to the concept of affinized quotient that will be used to treat the theory of observability in more depth, and that might illustrate the situation of other more refined quotients. Chapter 14 has also seen many changes and intended improvements, mainly related to a preparation for the proofs of the results on observability in Chapter 15, *Observable actions of affine algebraic groups* — in the next paragraph we describe this new chapter, which is another addition to this second edition. But not only: the geometric counterpart of the concept of induced representation — that of *homogeneous fiber bundle* — seemed to be a necessary addition to this chapter in order to wrap up the important examples of quotients that can be dealt within the scope of this book.

In this second edition, there is the new Chapter 15, dealing in full detail with the subject of the *observability of a general action.* This concept of *observable action* was introduced recently, and it is a natural generalization of the notion of an observable subgroup. See [147], [148] where the basic definitions and initial results on observable actions appeared. In the case that we have an affine algebraic group and a closed subgroup acting by translations, the action is observable if and only if the subgroup is observable. The interested reader can look at the introduction of the corresponding chapter for a more detailed description of the concepts appearing therein. Here we only mention that this new idea of observability is closely related to the phenomenon of unipotency, and that almost all the results on observable subgroups as they appear in Chapter 11 can be generalized to this new context (see, for example, theorems 15.4.2 and 15.5.5).

One of the cornerstones of geometric invariant theory (and its application to moduli problems, see, for example, [120], [125] and [134]) is the construction of open subsets where the restriction of a given action has a quotient. An introduction to the subject is presented in the new Chapter 16. In there we treat the Hilbert–Mumford criterion, and its numerical counterpart, that permits the identification of the set of *semistable* and *stable points* of an action of a reductive group on a projective variety, in terms of the induced action of the multiplicative group *via* the one parameter subgroups of the reductive group. Whereas the notion of stable point is absolute, the notion of semistable point is relative to an equivariant closed immersion of the projective variety into a projective space — this immersion is given by a linearized ample line bundle. Since the full extent of this criterion falls out of the scope of this book, we present a simplified version in theorems 16.3.54, 16.4.15 and 16.4.21, which omits the construction of the closed immersion. However, the reader should be aware that even in this simplified context we have to use the so-called Iwahori decomposition of a reductive group as well as some results on morphisms from Chapter 2 that are presented with precise statements but without explicit proofs.

Acknowledgments

We would like to thank many careful readers of the first edition who pointed out an embarrassing number of typos and mistakes.

During the elaboration of this edition, both authors were partially financed by CSIC Research group “Geometría algebraica y teoría de invariantes” (Universidad de la República, Uruguay), ANII (Agencia nacional de investigación e innovación, Uruguay) and MathAmSud project “Gradings groups and Hopf algebras.” The second author would like to thank CIMAT (Guanajuato, México) for the hospitality he enjoyed during a period of the time spent working on this second edition.

Montevideo, February 2016

Walter Ferrer Santos Alvaro Rittatore

Preface to the First Edition

A tree that can fill the space
of a man's arm
Grows from a downy tip;
A terrace nine storeys high
Rises from hodfuls of earth;
A journey of a thousand miles
Starts from beneath one's feet.

Lao Tzu, Tao Te Ching
Tr. C.C. Lau, Penguin Classics

This book is an introduction to geometric invariant theory understood *à la* Mumford — as presented in his seminal book, *Geometric Invariant Theory* [121]. In this sense, we intend to draw a bridge between the basic theory of affine algebraic groups (that is inseparable from considerations related to the geometry of actions) and the more sophisticated theory mentioned above.

Many problems of invariants of abstract groups become naturally problems of invariants of *affine algebraic groups*. In fact, the view of an abstract group as a group of linear transformations of a vector space, or more generally of transformations of a certain set with additional structure, has been fundamental since the origins of group theory in the pioneering works of E. Galois and C. Jordan in the nineteenth century. In this situation, it becomes handy to consider the associated action of the Zariski closure of the group.

Once we are dealing with affine algebraic groups, the use of the geometric structure adds many useful tools to our workbench. For example, one can linearize the problem by considering the tangent space at the identity, and view it as a problem in the category of finite dimensional Lie algebras.

If we are considering actions, it is natural to search for invariants, i.e., for functions from the original space into a certain set that are constant along the orbits, and if we are working with affine groups, we ask these functions to be regular. In principle, once we find a large enough number — but finite following Hilbert's expectations — of invariant functions, one can use them to decide whether or not two points are in the same orbit. Thereafter, one is led to search for natural, e.g., algebraic geometric, structures in the set of orbits. To deal with this problem, i.e., to study the concept of quotient variety, is one of the main objectives of this book. In particular, we have paid special attention in chapters 8, 11 and 12 to the relationship between the geometric structure of quotients of the form G/H, i.e., of *homogeneous spaces*, and the interplay between the representations of H and of G.

As we mentioned before, this text was written with the intention of being a reasonably self-contained introduction to the specialized texts and papers in geometric invariant theory. This intent of self-containment is specially laborious as, in this theory, techniques from many different areas of mathematics come into play: commutative algebra and field theory, Hopf algebra theory, representation theory of groups and algebras, algebraic geometry, Lie algebra theory.

Being an introductory text, we added at the end of each chapter a list of exercises that hopefully will help the reader to acquire a certain expertise in working with the fundamental concepts. Frequently, examples and parts of the proofs are left as exercises.

Our serious labors start with the theory of affine algebraic groups in Chapter 4, but we have included in the text two initial chapters. The first of these chapters contains most of the needed prerequisites in commutative algebra and algebraic geometry. Its results and definitions are presented sometimes with proofs or sketches of proofs, but always with precise references. The other chapter deals with the necessary prerequisites in the theory of semisimple Lie algebras over fields of characteristic zero.

Every chapter has an introductory section with a summary of its contents. We will not attempt to iterate here that non easy summarizing task. The interested reader may — if he possesses a certain degree of tenacity — read all these as a global introduction to the contents of this book.

At the end of the book, in order to minimize notational confusion, we have added an appendix with some basic definitions from category theory, algebra and topology. Moreover, in order to help the reader to keep track of the notations and important concepts, we collected most of them in an exhaustive glossary and a comprehensive subject index.

Concerning other texts dealing with the topics we treat, the reader may consult the references at the end of the book. Our bibliography is far from being exhaustive; the industrious reader can find an excellent bibliographic job done in some of the books we cite (see, for example, [144]).

Here and there along the book we have made some amateurish historical comments with the intention to give the reader a hint of the genesis of some of the subjects; the author index may help the reader to find these remarks in the text. We dare to expect that these comments will induce the reader to look at some of the serious books that have recently appeared dealing with the history of these topics, e.g., [13] and [68].

Our debts to the many contributors to the theory are impossible to record in this preface, but should be clear to the attentive reader. Many comments about our sources appear along the text.

We have chosen to avoid, mainly for reasons of space and emphasis, the consideration of non algebraically closed fields. Concerning this point, the reader should be aware that not a few of the results we treat are valid, sometimes with small modifications, for general fields. Furthermore, with the exception of some considerations about proper morphisms, we deal only with algebraic varieties, avoiding the language of schemes. Even in that case we restrict mainly to the situation of affine schemes. For a scheme theoretical vision of the theory the reader can consult, for example, [34] and [121], or the more recent [92].

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We would like to thank all the colleagues that helped us in many different ways during the preparation of this manuscript, some of them finding missing references, others reading parts of the manuscript, others making suggestions and correcting embarrassing mistakes, etc. We specially mention: N. Andruskiewitsch and the members of his research group; the participants of our seminar, specially V. Ferrer; the participants of the course "Acciones e invariantes de grupos algebraicos" (Guanajuato, México, 2004), specially P. Del Ángel; and also M. Aguiar, F. Cukiermann, G. González–Sprinberg, A. Treibich, and A. Solotar.

We hope that the remaining blemishes of the manuscript — that, of course, are the sole responsibility of the authors — will not set an insurmountable barrier to the interested readers.

The first author would like to thank G. Hochschild, whose influence, as the alert reader can easily check, is conspicuous throughout the book. This is only natural as he learned from him, directly or through his papers, most of what he knows about these subjects. He would also like to thank the persons, institutions and organizations that via different kinds of means were indirectly instrumental for the existence of this monograph: Cimat–México; CSIC–Universidad de la República; Conicyt-Uruguay; Universidad de Almería; Université d' Artois; Universidad Nacional Autónoma de México, Instituto de Matemática.

The second author would like to thank M. Brion, P.-L. Montagard and N. Ressayre for many useful conversations regarding the topics covered in this book. During its preparation, he was partially supported by the following institutions and organizations: CSIC-Universidad de la República, Uruguay; project PICS 1514, France; Université de Grenoble, France; Université de Montpellier II, France.

Walter Ferrer Santos Alvaro Rittatore

Enumeration of items and cross-references

The chapters are enumerated with arabic numerals; the Appendix is not enumerated. Within each chapter, each section is enumerated with an arabic number.

Within a given section of a given chapter, theorems, lemmas, corollaries, observations, examples, definitions and notations are enumerated with the same series of numerals. Each of these items appears labeled with two arabic numbers, the first corresponding to the section, the second to the specific item.

The few figures and numbered equations that appear are numbered globally for all the book also with an arabic number. Within each chapter the exercises are enumerated with only one arabic numeral.

For example, in Chapter 3, one can find Example 2.5 preceded by Definition 2.4 and followed by Example 2.6, all in Section 2. In Chapter 6 we can find in Section 2, a picture labeled Figure 1.

When we wish to refer to a theorem, etc., we use the above system of two arabic numerals provided that the item appears in the same chapter as the reference; otherwise we use a system of three numerals, adding a first arabic numeral with the indication of the chapter where the item appears. A similar system, without reference to the section, is used for exercises.

For example, the first exercise of Chapter 2 would be cited in Chapter 3 as Exercise 2.1 and in Chapter 2 as Exercise 1. The first definition in the second section of Chapter 2 will be cited in Chapter 3 as Definition 2.2.1 and in Chapter 2 as Definition 2.1.

Some sections are divided into subsections (for example Section 4 of Chapter 1 is divided into four subsections). Subsections are enumerated within the section to which they belong, and referred to within the same chapter with two numerals, the first corresponding to the section and the second to the subsection. When referring to a subsection that is in another chapter we use a system of three numerals, adding in the first place the numeral of the chapter where the section and subsection are located.

The enumeration of theorems, etc., does not take into account the subsections.

For the results and the sections of the Appendix we proceed in a slightly different way — that is self explanatory.

The bibliography is presented in lexicographical order, enumerated with arabic numbers.

Most of the notations used throughout the book are listed — in lexicographical order — in the Glossary of notations; there we refer to the number of the page where the notation is introduced. In order to help the reader in an eventual search we have displayed multiple entries for the same notation. For example, the notation u_β for the Casimir element can be found listed under the words starting with the letter C or the letter U.

Most of the concepts introduced in the text are referred to in the Index: the reader is sent to the page where the concept is introduced and to some other parts where we thought it might be useful for the reader to look. In order to help the reader, we introduce multiple entries for the same concept.

Chapter 1

Algebraic geometry: basic definitions and results

1 Introduction

This is the first of two chapters where we deal with most of the background in algebraic geometry which is needed in the rest of the book.[1] Here, we describe the basic foundational definitions and results of the theory of algebraic varieties. Local algebraic geometry can be viewed as commutative algebra, and for that reason a few basic aspects of the theory of commutative rings and fields will also be treated in this chapter — as well as in the Appendix.

The reader should not expect to find a systematic development neither of the necessary commutative algebra prerequisites, nor of the more global algebro-geometric concepts.

For reasons of space and emphasis, in this book we have chosen to keep the treatment of the basic algebraic geometry that lies under the theory of algebraic groups at a minimum; hence, our presentation will be (most of the time) brief and sketchy. In spite of that, we have tried to state with precision all the concepts and theorems and to give adequate references for the proofs we do not present.

At some points we are not consistently brief and some results and/or definitions are treated with a certain degree of detail. The reasons for this change of pace are manifold: the lack of an adequate reference for the exact statement we need; our opinion about the importance of the subject; and many times merely the taste of the authors.

For a thorough treatment of these topics the reader can consult any of the following textbooks: [3], [17], [42] or [187] and [188] (commutative algebra); [43], [65], [66], [89], [124], [138], and many others (algebraic geometry).

We proceed to the description of the contents of each section.

In Section 2, we collect foundational results in commutative algebra that are needed for the development of the theory of algebraic varieties, e.g., E. Noether normalization theorem, Artin–Tate's lemma, different versions of Hilbert's Nullstellensatz, etc. Special subsections are dedicated to the algebraic version of the crucial concepts of separability, flatness and regularity. Only a few of the proofs are presented and most of the ones we omitted can be found in the standard references on the subject.

In Section 3 we introduce the Zariski topology of the affine space $\mathbb{A}^n = \Bbbk^n$; this topology has as closed sets the algebraic subsets, i.e., the set of zeroes of a family of polynomials in n variables. We also define the morphisms of algebraic sets completing the definition of the category where local algebraic geometry is developed.

[1] In the first edition, these subjects were covered in a single chapter which has been now split into two halves.

In Section 4 we introduce the first notions of the theory of algebraic varieties. First we define — in order to equip our objects with the algebras of functions that characterize the structure — the notion of a sheaf on a topological space, centering our attention on sheaves of functions. The spectrum and maximal spectrum of a ring are introduced in order to view abstractly the affine algebraic subsets. Afterwards, algebraic prevarieties are defined by pasting together these abstract affine pieces. The concept of prevariety is then strengthened in order to introduce the main geometrical object of study, algebraic varieties. We first observe that products exist in the category of prevarieties, and then define varieties as prevarieties that satisfy the so-called "Hausdorff axiom," i.e., prevarieties X with the additional property that the diagonal Δ is closed in the product $X \times X$. We present also the basic notions of dimension and tangent space.

Unless the contrary is explicitly said, the field $\Bbbk$ will be algebraically closed of arbitrary characteristic, and all the rings and $\Bbbk$-algebras we consider are unital and commutative.

---

2 Commutative algebra

2.1 Ring and field extensions

Let $\Bbbk \subset K$ be a field extension. The elements $a_1, \dots, a_n \in K$ are *algebraically independent* over $\Bbbk$ if $\operatorname{Ker}\big(\varepsilon_{(a_1,\dots,a_n)}\big) = \{0\}$, where $\varepsilon_{(a_1,\dots,a_n)} : \Bbbk[X_1, \dots, X_n] \to \Bbbk$ is the evaluation at $(a_1, \dots, a_n)$. In other words, the only polynomial in $\Bbbk[X_1, \dots, X_n]$ which is annihilated by $(a_1, \dots, a_n)$ is the zero polynomial. A maximal algebraically independent subset of K is called a *transcendence basis.* All transcendence basis have the same number of elements; this number is called the *transcendence degree* of the extension $\Bbbk \subset K$ and it is denoted as $\operatorname{tr.deg}_{\Bbbk} K$. In the case that the field K is finitely generated over $\Bbbk$, the transcendence degree is finite.

If R is a finitely generated integral domain $\Bbbk$-algebra, then R has finite Krull dimension $\kappa(R)$, and $\kappa(R) = \operatorname{tr.deg}_{\Bbbk}[R]$, where $[R]$ is as usual the field of fractions of R (see Observation 2.7 below).

Definition 2.1. Let $R \subset S$ be an extension of commutative rings. An element $s \in S$ is said to be *integral* over R if there exists a monic polynomial $f \in R[X]$ such that $f(s) = 0$. The extension is *integral* if for all $s \in S$, s is integral over R. The *integral closure* of R in S is the set of all elements of S integral over R; it is a subring of S containing R. If R is an integral domain we say that R is *integrally closed* if it equals its integral closure in $[R]$.

Theorem 2.2. *Let $R \subset S$ be an extension of commutative rings. If S is finitely generated as an R-module, then S is integral over R.*

Proof. See, for example, [3, Prop. 5.1]. □

The converse of the above theorem is false in general, but we have the following partial results.

Theorem 2.3. *If $R \subset S$ is a ring extension with S integral and finitely generated as an R-algebra, then S is finitely generated as an R-module.*

Proof. See, for example, [3, Cor. 5.2]. □

Theorem 2.4 (Artin–Tate's theorem). *Let $T \subset R \subset S$ be a tower of commutative rings* *and assume that: (1) T is Noetherian; (2) S is finitely generated as a T-algebra; (3) S is finitely generated as an R-module. Then R is finitely generated as a T-algebra.*

Proof. Using (2) and (3) we write $S = Rs_1 + \cdots + Rs_n$, $s_1 = 1$, and $S = T[s'_1, \ldots, s'_m]$. Express $s'_i = \sum_j r_{ij} s_j$ for $i = 1, \ldots, m$, $r_{ij} \in R$ and $s_k s_l = \sum r'_{klu} s_u$ for $k, l = 1, \ldots, n$, $r'_{klu} \in R$. The original tower extends to $T \subset R_0 \subset R \subset S$, where R_0 is the T-subalgebra of R generated by $\{r_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{r'_{klu} : 1 \leq k, l, u \leq n\}$. As T is Noetherian and R_0 is finitely generated as a T-algebra, using Hilbert's basis theorem we conclude that R_0 is Noetherian. As $R_0 s_1 + \cdots + R_0 s_n$ is a subalgebra of S that contains all the s'_i and also contains T, it follows that $R_0 s_1 + \cdots + R_0 s_n = S$. Then S is a finitely generated R_0-module and thus R is a finitely generated R_0-module. Write $R = R_0 p_1 + \cdots + R_0 p_v$ for certain $p_1, \ldots, p_v \in R$. It follows immediately that R is generated by $p_1, \ldots, p_v$ and $\{r_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{r'_{klu} : 1 \leq k, l, u \leq n\}$ as a T-algebra. □

In particular, we deduce the following corollary.

Corollary 2.5. *Let $\Bbbk \subset R \subset S$ be an extension of commutative rings where $\Bbbk$ is a field. Assume that S is finitely generated as a $\Bbbk$-algebra and integral over R. Then R is a finitely generated $\Bbbk$-algebra.*

Proof. As S is a finitely generated R-module (see Theorem 2.3), we are in the hypothesis of the Theorem 2.4 and the conclusion follows immediately. □

The following theorem is an algebraic tool of central importance for the manipulation of algebraic varieties.

Theorem 2.6 (E. Noether's normalization theorem). *Let R be an integral domain that is finitely generated as a $\Bbbk$-algebra, with* $\operatorname{tr.deg}_{\Bbbk}[R] = d$. *Then there exist $\Bbbk$-algebraically independent elements $r_1, \ldots, r_d \in R$, such that in the tower $\Bbbk \subset \Bbbk[r_1, \ldots, r_d] \subset R$ the top part $\Bbbk[r_1, \ldots, r_d] \subset R$ is integral.*

Proof. In [3, p. 69] a proof is sketched and in [82, Thm. X.1.2] a detailed proof is presented. In [42] the reader can find a proof for a different (but essentially equivalent) formulation of this result. □

Observation 2.7. Notice that in accordance with the considerations previous to Definition 2.1, the number d of algebraically independent elements $\{r_1, \ldots, r_d\}$ coincides with the Krull dimension of R.

Informally speaking, Noether's theorem guarantees that a finitely generated integral domain $\Bbbk$-algebra can be viewed as an integral extension of a polynomial algebra over $\Bbbk$ in $\kappa(R)$ variables.

There is a version of Noether normalization theorem that generalizes it to extensions of integral domains.

Corollary 2.8. *Let $S \subset R$ be an extension of integral domains with R a finitely generated S-algebra. Then there exist elements $r_1, \ldots, r_d \in R$ that are algebraically independent over $[S]$, and a nonzero element $s \in S$ with the property that in the tower of extensions $S_s \subset S_s[r_1, \ldots, r_d] \subset R_s$, the top part is integral.*

Proof. Consider the field extension $[S] \subset [R]$ and apply Theorem 2.6 to R' the $[S]$-subalgebra of $[R]$ generated by R. The details are left as an exercise for the reader (see Exercise 1). □

Observation 2.9. The number d of algebraically independent elements constructed in Corollary 2.8 equals $\kappa\big([S] \otimes_S R\big)$.

LEMMA 2.10. *Let $S \subset R$ be a finitely generated integral ring extension of commutative integral domains. Then, there exists an element $0 \neq s \in S$ with the property that $S_s \subset R_s$ is free.*

PROOF. From Theorem 2.3 we deduce that R is a finitely generated S-module. Hence, we can find an S-epimorphism of a finite direct sum of copies of S onto R, $\phi : \bigoplus_1^r S \to R$. This implies in particular that R admits a finite S-composition series. The following assertion that will be proved by induction on the length guarantees our result. Let S be a commutative integral domain and assume that M is a S-module of finite length. Then there exists an element $0 \neq s \in S$, M_s is a free S_s-module. Consider N a maximal S-submodule of M and the exact sequence $0 \to N \to M \to M/N \to 0$. The S-module M/N is simple and then isomorphic to a module of the form S/P for some maximal ideal P in S. If $P = \{0\}$ then $M/N \cong S$ and then $M \cong N \oplus S$ and the proof follows by induction on the length. If $P \neq 0$ and we consider $0 \neq s_P \in P$, it is clear that $(S/P)_{s_P} = S_{s_P}/PS_{s_P} = \{0\}$. Then, going back to the original exact sequence we deduce that $N_{s_P} \cong M_{s_P}$. By induction we deduce the existence of $s_0 \in S$ with the property that N_{s_0} is free as a S_{s_0}-module. Hence, $M_{s_0 s_P}$ is free as a $S_{s_0 s_P}$-module. $\square$

The next theorem, which is a consequence of Noether normalization theorem, will be used in the characterization of affine homogeneous spaces in terms of exactness (see Corollary 12.6.6 and Theorem 12.6.7).

THEOREM 2.11. *Let $S \subset R$ be an extension of commutative integral domains, and assume that R is a finitely generated S-algebra. Then there exists an element $s \in S$ such that R_s is free as a S_s-module.*

PROOF. First use Corollary 2.8 in order to find $r_1, \dots, r_d \in R$ that are algebraically independent over $[S]$ and $0 \neq s \in S$ such that in the tower of extensions $S_s \subset S_s[r_1, \dots, r_d] \subset R_s$ the top part is integral, with $d = \kappa\big([S] \otimes_S R\big)$. Next proceed by induction on d. If $d = 0$ then the extension $S_s \subset R_s$ is integral and the result follows from Lemma 2.10.

Without loss of generality and eventually changing notations we may assume that the result is valid for all extensions of dimension smaller than d and that $s = 1$. In other words, we suppose that $S \subset S' = S[r_1, \dots, r_d] \subset R$, being the top extension integral and R finitely generated as an S-algebra (observe that S' is a free S-module).

It follows that R is a S'-module of finite length. The result will be deduced once we prove the following assertion: let M be a $S' = S[r_1, \dots, r_n]$-module of finite length. Then there exists an element $s \in S$ such that M_s is free as a S_s-module.

We proceed by induction on the length of M. Consider N a maximal S'-submodule of M and consider the exact sequence: $0 \to N \to M \to M/N \to 0$. Since M/N is cyclic, there exists an ideal $P \subset S'$ such that $S'/P \cong M/N$. We will consider now three possibilities for the ideal P. If $P = \{0\}$, then $M/N \cong S'$ that is a free S-module, and in this case $M \cong N \oplus S'$; hence the proof follows by induction on the length. If $P \neq \{0\}$ and $P \cap S \neq \{0\}$, choose $0 \neq p \in P \cap S$. Then $M_p = N_p$, and the proof follows by induction on the length. The last alternative for P is that $P \neq \{0\}$ and $P \cap S = \{0\}$. Consider the injection $[S] \otimes_S P \to [S] \otimes_S S'$. The image of this map is a prime ideal in $[S] \otimes_S S'$ with $\kappa\big([S] \otimes_S S'/P\big) < d$. By induction we deduce that there exists an element $s \in S$ such that $(M/N)_s \cong (S'/P)_s$ is free as a S_s-module. If we localize with respect to s the sequence $0 \to N \to M \to M/N \to 0$, we deduce that $0 \to N_s \to M_s \to (M/N)_s \to 0$. Then, $M_s \cong N_s \oplus (M/N)_s$. As the length of N is smaller than the length of M our proof is finished. $\square$

The theorem that follows is a variation of the usual results of extension of ideals for integral extension of rings.

THEOREM 2.12. *Let $R \subset S$ be an integral extension of $\mathbb{k}$-algebras, where $\mathbb{k}$ is an algebraically closed field. A $\mathbb{k}$-algebra homomorphism from R into $\mathbb{k}$ extends to a $\mathbb{k}$-algebra homomorphism from S into $\mathbb{k}$.*

PROOF. See [17, Chap. V, 2.1, Cor. 4]. □

The next lemma will be useful when dealing with the problem of the finite generation of the rings of invariants in Chapter 13, more particularly in Lemma 13.3.4. Here we only present a brief sketch of the proof; for the missing details see [17, Chap. V, 3.2].

LEMMA 2.13. *Let $R \subset S$ be an extension of $\mathbb{k}$-algebras that are also integral domains. Assume that (1) R is a finitely generated $\mathbb{k}$-algebra; (2) the field extension $[R] \subset [S]$ is finite algebraic; (3) S is integral over R. Then S is a finitely generated R-module and also a finitely generated $\mathbb{k}$-algebra. In particular, if S is the integral closure in $[R]$ of R and R is a finitely generated $\mathbb{k}$-algebra, then S is also a finitely generated $\mathbb{k}$-algebra.*

PROOF. First, one proves that it can be assumed that S is integrally closed. Then, using Theorem 2.6 one can assume that R is a polynomial ring over $\mathbb{k}$ and that $[R]$ is the field of rational functions in n-variables. Moreover, the extension $[R] \subset [S]$ can be considered as a composition of a purely inseparable extension with a Galois extension. Each of these cases can be treated using standard methods in the theory of field extensions. □

The following classical theorem will be presented without proof.

THEOREM 2.14 (Krull's principal ideal theorem). *Suppose that R is a finitely generated integral domain $\mathbb{k}$-algebra. Let $r \in R$ be a fixed element and P a minimal prime ideal containing r, i.e., an isolated prime ideal of rR. Then* $\operatorname{tr.deg}_{\mathbb{k}}[R/P] = \operatorname{tr.deg}_{\mathbb{k}}[R] - 1$.

PROOF. See, for example, [188]. □

2.2 Hilbert's Nullstellensatz

Hilbert's Nullstellensatz is one of the basic building blocks of the theory of algebraic varieties, and should be considered as a deep generalization of the so-called fundamental theorem of algebra. In our presentation the theorem appears initially as a result concerning extensions of $\mathbb{k}$-algebra homomorphisms with values in algebraically closed fields. We start by defining the category of algebras that are the algebras of functions for affine algebraic varieties.

DEFINITION 2.15. A commutative $\mathbb{k}$-algebra A is said to be an *affine $\mathbb{k}$-algebra* if it is finitely generated and has no nilpotent elements.

THEOREM 2.16. *Let $\mathbb{k}$ be an algebraically closed field and assume that R is a commutative finitely generated $\mathbb{k}$-algebra. If $R \neq \{0\}$, there exists a $\mathbb{k}$-algebra homomorphism from R into $\mathbb{k}$.*

PROOF. In accordance to Theorem 2.6, there exist elements $r_1, \dots, r_d \in R$ such that in the tower of extensions $\mathbb{k} \subset \mathbb{k}[r_1, \dots, r_d] \subset R$, the lower part is isomorphic to a polynomial ring and the top part is an integral extension. The existence of a $\mathbb{k}$-algebra morphism from $\mathbb{k}[r_1, \dots, r_d]$ into $\mathbb{k}$ is evident. The extension from $\mathbb{k}[r_1, \dots, r_d]$ to R of the morphism previously constructed can be deduced from Theorem 2.12. □

We are ready to prove an abstract version of the Nullstellensatz.

THEOREM 2.17. *Assume that $\mathbb{k}$ is an algebraically closed field and R a commutative finitely generated $\mathbb{k}$-algebra with no nonzero nilpotent. If $r \neq s \in R$, then there exists a $\mathbb{k}$-algebra homomorphism $\phi : R \to \mathbb{k}$ such that $\phi(r) \neq \phi(s)$.*

Proof. We may assume that $s = 0$. In this case we consider a prime ideal $P \in R$ such that $r \notin P$ — to guarantee the existence of such an ideal, one uses a standard fact in commutative ring theory that asserts that in this situation the set of nilpotent elements coincides with the intersection of all prime ideals of the ring (see Appendix, Section 3). In the ring R/P, the element $\overline{r} = r + P \neq 0$ is not nilpotent, and the $\Bbbk$-algebra $(R/P)_{\overline{r}}$ is finitely generated and nonzero. Using Theorem 2.16 we deduce the existence of a morphism $\gamma : (R/P)_{\overline{r}} \to \Bbbk$ and as $\overline{r}$ is invertible in the localization, it follows that $\gamma(\overline{r}) \neq 0$. The map $\phi : R \to \Bbbk$ defined by the commutativity of the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi"'] & R/P \arrow[d] \\ \Bbbk & (R/P)_{\overline{r}} \arrow[l, "\gamma"] \end{tikzcd}$$

is a $\Bbbk$-algebra homomorphism that sends r into a nonzero element. $\square$

Next it follows a more classical version of the Nullstellensatz that is known as the weak Nullstellensatz.

Theorem 2.18 (Weak Nullstellensatz). *Let $\Bbbk$ be an algebraically closed field.*

(1) If $R = \Bbbk[r_1, \dots, r_n]$ is a finitely generated ring extension of $\Bbbk$ that is also a field, then $R = \Bbbk$.

(2) An ideal $M \subset \Bbbk[X_1, \dots, X_n]$ is maximal if and only if $M = \langle X_1 - a_1, \dots, X_n - a_n\rangle$, with $a_1, \dots, a_n \in \Bbbk$.

Proof. (1) Assume that one of the r_i's is not zero, say r_1, and consider the morphism $\phi : \Bbbk[r_1, \dots, r_n] \to \Bbbk$ that sends r_1 into a nonzero element (see Theorem 2.17). As $\Bbbk[r_1, \dots, r_n]$ is a field, it follows that ϕ is injective, so that if we compute $\phi\bigl(r_1 - \phi(r_1)1\bigr) = 0$ we deduce that $r_1 \in \Bbbk$ and then by an evident iteration that $R = \Bbbk$.

(2) Let M be a maximal ideal in $\Bbbk[X_1, \dots, X_n]$. Then $\Bbbk[X_1, \dots, X_n]/M$ is a field and by what we just proved it has to coincide with $\Bbbk$. If we fix i, $1 \leq i \leq n$, then there exists $a_i \in \Bbbk$ with the property that $X_i - a_i1 \in M$. It follows that the ideal $\langle X_1 - a_1, \dots, X_n - a_n\rangle \subset M$. Moreover, all the ideals of the form $\langle X_1 - a_1, \dots, X_n - a_n\rangle$ are maximal as these ideals are of the form $\operatorname{Ker}\bigl(\varepsilon_{(a_1,\dots,a_n)}\bigr)$ where $\varepsilon_{(a_1,\dots,a_n)} : \Bbbk[X_1, \dots, X_n] \to \Bbbk$ is the evaluation at $(a_1, \dots, a_n)$. Hence, $\langle X_1 - a_1, \dots, X_n - a_n\rangle = M$. $\square$

Theorem 2.19 (Hilbert's Nullstellensatz). *Let $I \subsetneq \Bbbk[X_1, \dots, X_n]$ be a proper ideal, where $\Bbbk$ is an algebraically closed field. Then, there exists a point $(a_1, \dots, a_n) \in \Bbbk^n$ such that $f(a_1, \dots, a_n) = 0$ for all $f \in I$.*

Proof. Let M be a maximal ideal of $\Bbbk[X_1, \dots, X_n]$ that contains I and write $M = \langle X_1 - a_1, \dots, X_n - a_n\rangle$. If $f \in I$, then there exist $g_i \in \Bbbk[X_1, \dots, X_n]$, $i = 1, \dots, n$, such that $f = g_1(X_1 - a_1) + \cdots + g_n(X_n - a_n)$. It follows that $f(a_1, \dots, a_n) = 0$. $\square$

Observation 2.20. It is clear that the Nullstellensatz (Theorem 2.19) implies the weak Nullstellensatz (Theorem 2.18).

Theorem 2.21. *Assume that $\Bbbk$ is an algebraically closed field and let $I \subsetneq \Bbbk[X_1, \dots, X_n]$ be a proper ideal. Then*

$$\sqrt{I} = \bigcap\bigl\{M \subset \Bbbk[X_1, \dots, X_n] : I \subset M\,,\ M \text{ maximal ideal }\bigr\}.$$

PROOF. Clearly if M is maximal and $I \subset M$, then $\sqrt{I} \subset \sqrt{M} = M$, so that

$$\sqrt{I} \subset \bigcap \{ M \subset \Bbbk[X_1, \dots, X_n] : I \subset M\,,\ M \text{ maximal ideal } \}.$$

Conversely, suppose that $f \in M$ for all maximal ideals M that contain I, and let $J = \big\langle I \cup \{1 - X_{n+1} f(X_1, \dots, X_n)\} \big\rangle \subset \Bbbk[X_1, \dots, X_n, X_{n+1}]$ be the ideal generated by I and the polynomial $1 - X_{n+1} f(X_1, \dots, X_n)$. Consider a common zero $(a_1, \dots, a_{n+1}) \in \Bbbk^{n+1}$ of the polynomials in J. Then $h(a_1, \dots, a_n) = 0$ for all $h \in I$ and this means that $I \subset \langle X_1 - a_1, \dots, X_n - a_n \rangle$ (see Exercise 3). As f is inside all maximal ideals that contain I, it follows that $f \in \langle X_1 - a_1, \dots, X_n - a_n \rangle$ and then that $f(a_1, \dots, a_n) = 0$. As $(a_1, \dots, a_{n+1})$ is a zero of the polynomial $1 - X_{n+1} f(X_1, \dots, X_n)$, we obtain a contradiction.

Therefore, the ideal J has no common zeroes and from Theorem 2.19 we deduce that $J = \Bbbk[X_1, \dots, X_{n+1}]$. Hence, we can find $g_1, \dots, g_s, g \in \Bbbk[X_1, \dots, X_{n+1}]$ and $f_1, \dots, f_s \in I \subset \Bbbk[X_1, \dots, X_n]$ such that $1 = g_1 f_1 + \cdots + g_s f_s + g(1 - X_{n+1} f)$. Writing $X_{n+1} = 1/f(X_1, \dots, X_n)$ we obtain the following equality in $\Bbbk(X_1, \dots, X_n)$:

$$\begin{aligned} 1 = g_1\big(X_1, \dots, X_n, 1/f(X_1, \dots, X_n)\big) f_1(X_1, \dots, X_n) + \cdots \\ \cdots + g_s\big(X_1, \dots, X_n, 1/f(X_1, \dots, X_n)\big) f_s(X_1, \dots, X_n). \end{aligned}$$

Eliminating denominators the above equality is transformed in: $f^m = h_1 f_1 + \cdots + h_s f_s$, with $h_i \in \Bbbk[X_1, \dots, X_n]$ and m a conveniently chosen exponent. Then $f^m \in I$ and thus $f \in \sqrt{I}$. □

2.3 Separability

In this paragraph the fields we consider are not necessarily algebraically closed.

DEFINITION 2.22. Let $\Bbbk \subset K$ be an algebraic field extension. An element $a \in K$ is *separable over* $\Bbbk$ if there exists a polynomial $f \in \Bbbk[X]$ with simple roots and such that $f(a) = 0$. The extension is *separable* if all the elements of K are separable over $\Bbbk$.

An element of $a \in K$ is *purely inseparable over* $\Bbbk$ if the only separable elements in $\Bbbk \subset \Bbbk(a)$ are those belonging to $\Bbbk$. The extension is *purely inseparable* if all the elements of K are purely inseparable over $\Bbbk$.

Concerning non algebraic extensions the notion of separability is defined in a different manner. The next result is the basis for this definition.

THEOREM 2.23. *If $\Bbbk \subset K$ is a fixed field extension, then the following conditions are equivalent.*

(1) If V is a K-vector space and $D : \Bbbk \to V$ is a derivation, then there exists a derivation $D' : K \to V$ that extends D, i.e., $D'\big|_{\Bbbk} = D$ (see Appendix, Definition 3.17).

(2) For an arbitrary field K' that extends $\Bbbk$, the tensor product $K \otimes_{\Bbbk} K'$ has no nonzero nilpotent.

In the case that the fields are of characteristic p, the above conditions are equivalent to:

(3) If $X \subset K$ is a $\Bbbk$-linearly independent set, then $X^p = \{x^p : x \in X\}$ is also a $\Bbbk$-linearly independent set.

PROOF. See, for example, [82, Chap. III]. □

DEFINITION 2.24. A field extension $\Bbbk \subset K$ is *separable* if the equivalent conditions (1),(2) or (3) (this last in the case of positive characteristic) of Theorem 2.23 are satisfied.

Observation 2.25. (1) It is not hard to prove that in characteristic zero all extensions are separable.

(2) A purely transcendental extension is separable.

(3) In the case of algebraic extensions both definitions of separability coincide. Indeed, assume that $a \in K$ is algebraic over $\Bbbk$ and separable in the sense of Definition 2.22. Let V be a K-space and endow it with a $\Bbbk[X]$-module structure as follows, for $g \in \Bbbk[X]$ and $v \in V$, then $g \cdot v = g(a)v$.

Extend an arbitrary derivation $D : \Bbbk \to V$ to $D' : \Bbbk[X] \to V$ by the rule: $D'(X) = -\big(\sum D(a_i)a^i/f'(a)\big)$, where $f = \sum a_i X^i$ is the minimal polynomial of a with coefficients in the base field $\Bbbk$. It is easy to prove that $D'(f) = 0$ and, hence, that D' factors to a derivation $D'' : \Bbbk(a) \to V$.

Conversely, if we call $f = \operatorname{Irr}(a, \Bbbk) \in \Bbbk[X]$, we want to prove that $f'(a) \neq 0$. If $f'(a) = 0$ we deduce that f divides f', and this may only happen if $f' = 0$, i.e., if for some polynomial $g \in \Bbbk[X]$, $f(X) = g(X^p)$. This means that the elements $1, a^p, \dots, a^{p(d-1)}$ are linearly dependent over $\Bbbk$, where $d = [\Bbbk(a) : \Bbbk]$. But this contradicts the fact that $1, a, \dots, a^{d-1}$ are linearly independent and Definition 2.24.

In the case of a separable extension, one can find a transcendence basis with special properties. The proof of this classical result will be omitted.

Theorem 2.26. *Assume that the extension $\Bbbk \subset K$ is separable and finitely generated. Then there exists a finite transcendence basis $\mathcal{B}$ such that the tower of extensions $\Bbbk \subset \Bbbk(\mathcal{B}) \subset K$ has the lower part purely transcendental and the top part separable algebraic.*

Proof. See [187, Chap. II, Thm. 30]. □

The next theorem relates the transcendence degree of a separable finitely generated extension with the dimension of the space of derivations $\mathcal{D}_\Bbbk(K)$.

Theorem 2.27. *Assume that the extension $\Bbbk \subset K$ is separable and finitely generated. Then* $\operatorname{tr.deg}_\Bbbk K = \dim_K \mathcal{D}_\Bbbk(K)$.

Proof. See, for example, [82, Chap. III]. □

The next lemma will be presented without proof.

Lemma 2.28. *Let $\Bbbk$ be an algebraically closed field and $S \subset R$ be an extension of integral domain $\Bbbk$-algebras and assume that R is finitely generated as an S-algebra. If $0 \neq r \in R$, there exists an element $0 \neq t \in S$ with the property that every homomorphism of $\Bbbk$-algebras $\alpha : S \to \Bbbk$ such that $\alpha(t) \neq 0$ extends to a homomorphism of $\Bbbk$-algebras from R into $\Bbbk$, such that $\alpha(r) \neq 0$.*

Proof. See, for example, [82, Thm. II.3.3]. □

The result that follows will be used when dealing with the structure of homogeneous spaces in Chapter 8.

Lemma 2.29. *Let $S \subset R$ be an extension of $\Bbbk$-algebras that are also integral domains and assume that R is finitely generated over $\Bbbk$. Assume that an element $r \in R$ has the following property: if $\alpha, \beta : R \to \Bbbk$ is a pair of $\Bbbk$-algebra homomorphisms that coincide over S, then $\alpha(r) = \beta(r)$. Then $r \in R$ is algebraic and purely inseparable over $[S]$.*

Proof. We prove first that r is algebraic over $[S]$. Assume that this is not the case, and consider $S[r] \subset R$. Using Lemma 2.28 we deduce that there exists an element $0 \neq t \in S[r]$ with the property that every $\Bbbk$-algebra homomorphism $\gamma : S[r] \to \Bbbk$ such that $\gamma(t) \neq 0$

extends to R, with $\gamma(r) \neq 0$. Write $t = s_0 + s_1 r + \cdots + s_n r^n$ with $s_i \in S$ and $s_n \neq 0$. Using the Nullstellensatz 2.17 we deduce the existence of a homomorphism of $\Bbbk$-algebras $\widehat{\gamma} : R \to \Bbbk$ such that $\widehat{\gamma}(s_n) \neq 0$, and by restriction to S we obtain a homomorphism of $\Bbbk$-algebras $\gamma_0 : S \to \Bbbk$ with the same property. It is clear that in order to extend γ_0 to $S[r]$ all we have to do is to assign a value to r. Assume that γ_1 is an extension of γ_0 and such that $\gamma_1(t) = 0$. Then $0 = \gamma_0(s_0) + \gamma_0(s_1)\gamma_1(r) + \cdots + \gamma_0(s_n)\gamma_1(r)^n$. Hence, if we assign a value to $\gamma_1(r)$ that is not a root of the above polynomial, we obtain an extension of the original morphism not vanishing at t. There are then infinite extensions of γ_0 to R and this contradicts the hypothesis about r.

The proof that r is purely inseparable is similar. Call p the characteristic exponent of the base field, and assume that r is not purely inseparable over $[S]$. Then for some exponent $m > 0$ the element r^{p^m} is separable, algebraic over $[S]$ and does not belong to $[S]$. After eliminating denominators we can find $0 \neq s \in S$ such that if we call $t = sr^{p^m}$, then $f = \operatorname{Irr}\big(t, [S]\big) \in S[X]$, with $\deg(f) = n > 1$.

Proceeding as before we can find $u = s_0 + s_1 t + \cdots + s_l t^l$, where $s_l \neq 0$, and $l < n$, with the property that all $\Bbbk$-algebra homomorphisms $\gamma : S[t] \to \Bbbk$ that do not annihilate u can be extended to R. Call $g = s_0 + s_1 X + \cdots + s_l X^l \in S[X]$. As f, g, as well as f, f', are relatively prime over$[S]$, there exist polynomials $h, k, q, w \in S[X]$ and nonzero elements $e, e' \in S$ such that $hf + kg = e$, $qf + wf' = e'$. We use the Nullstellensatz to construct $\beta : S \to \Bbbk$, such that $\beta(ee') \neq 0$. Given an arbitrary polynomial in $z \in S[X]$ we call $z_1 \in \Bbbk[X]$ the polynomial obtained by applying β to the coefficients of z. It is clear in the above construction that $(z_1)' = (z')_1$ and that if z is monic the degree of z_1 coincides with the degree of z. Then, $h_1 f_1 + k_1 g_1 = \beta(e)$, $q_1 f_1 + w_1 f_1' = \beta(e')$. Hence, the polynomials f_1 and g_1 are relatively prime and the same happens with f_1 and f_1'.

Then, f_1 has n roots in $\Bbbk$ and none of these roots is a root of g_1, and in this way we can obtain n different extensions of β to algebra homomorphisms from $S[t]$ into $\Bbbk$ and none of them annihilates u. Hence, all these extensions extend further to R. This is a contradiction: if β' is such an extension, then $\beta'(t) = \beta(s)\beta'(r)^{p^m}$ and all the values of $\beta'(r)$ should be equal by hypothesis. □

THEOREM 2.30. *If K is a field and G is a group of field automorphisms of K, then the extension ${}^G K \subset K$ is separable.*

PROOF. See, for example, [82, Thm. III.2.3] or [12, Prop. AG.2.4]. □

2.4 Faithfully flat ring extensions

DEFINITION 2.31. A commutative ring extension $S \subset R$ is said to be *faithfully flat* if for all sequences of S-modules: $\mathcal{E} : 0 \to M \to N \to T \to 0$, $\mathcal{E}$ is exact if and only if $\mathcal{E} \otimes_S R : 0 \to M \otimes_S R \to N \otimes_S R \to T \otimes_S R \to 0$ is exact.

Note that if the extension $S \subset R$ is free, i.e., if R is free as an S-module, then it is faithfully flat.

OBSERVATION 2.32. In the situation of Definition 2.31, $S \subset R$ is a faithfully flat ring extension if and only if:

(1) for all injective morphisms $\alpha : M \to N$ of S-modules, the morphism of R-modules, $\mathrm{id} \otimes \alpha : R \otimes_S M \to R \otimes_S N$ is injective;

(2) if M is an S-module such that $R \otimes_S M = \{0\}$, then $M = \{0\}$.

See Exercise 4.

LEMMA 2.33. *Let $S \subset R$ be a finitely generated commutative ring extension of integral domains. Suppose we can find $s_1, \dots, s_n \in S$ such that: (1) the elements $s_1, \dots, s_n$ generate the unit ideal of S; (2) R_{s_i} is faithfully flat as an S_{s_i}-module. Then R is faithfully flat as an S-module.*

PROOF. We use here Observation 2.32. First suppose that M is a S-module such that $M \otimes_S R = 0$. Then $M \otimes_S R \otimes_R R_{s_i} = 0$ or equivalently $M \otimes_S R_{s_i} = 0$. Therefore, $M \otimes_S S_{s_i} \otimes_{S_{s_i}} R_{s_i} = 0$ and from the hypothesis we conclude that $M \otimes_S S_{s_i} = 0$. Hence, for an arbitrary $m \in M$ there exists an exponent q such that for all $1 \leq i \leq n$, $s_i^q m = 0$. As the ideal generated by $\{s_1^q, \dots, s_n^q\}$ is also the unit ideal, we conclude that $m = 0$. Hence, $M = 0$.

Assume that $\alpha : M \to N$ is an injective morphism of S-modules. Then $\mathrm{id} \otimes \alpha : S_{s_i} \otimes_S M \to S_{s_i} \otimes_S N$ is injective and so is

$$\mathrm{id} \otimes \mathrm{id} \otimes \alpha : R_{s_i} \otimes_{S_{s_i}} S_{s_i} \otimes_S M \to R_{s_i} \otimes_{S_{s_i}} S_{s_i} \otimes_S N.$$

Hence, the morphism $\mathrm{id} \otimes \alpha : R_{s_i} \otimes_S M \to R_{s_i} \otimes_S N$ is injective. Looking at the diagram

$$\begin{tikzcd} R \otimes_S M \arrow[r, "\mathrm{id} \otimes \alpha"] \arrow[d] & R \otimes_S N \arrow[d] \\ R_{s_i} \otimes_S M \arrow[r, "\mathrm{id} \otimes \alpha"'] & R_{s_i} \otimes_S N \end{tikzcd}$$

we deduce that if an element $\sum r_k \otimes m_k \in R \otimes_S M$ satisfies that $0 = \sum r_k \otimes \alpha(m_k) \in R \otimes_S N$, then $0 = \sum r_k \otimes m_k \in R_{s_i} \otimes_S M$ for all $i = 1, \dots, n$. From Exercise 4 (d), we deduce that $0 = \sum r_k \otimes m_k \in R \otimes_S M$. $\square$

2.5 Regular local rings

In this section we deal with the algebraic version of the concept of non singular point (see Definition 2.4.1 below). The relevant idea is the concept of regular local ring.

Let R be a commutative integral Noetherian local ring and M its maximal ideal. It follows from general results in dimension theory of commutative rings (see, for example, [3, p. 119]) that the cardinality of an arbitrary set of generators of M as an R-module is larger than or equal to the Krull dimension of R.

DEFINITION 2.34. In the above situation, we say that the ring R is regular if M has a set of R-module generators of cardinality $\kappa(R)$, the Krull dimension of R.

The following basic result will be interpreted in geometric terms in Theorem 2.4.8.

THEOREM 2.35. *Let R be a Noetherian regular local ring. Then R is an integral domain that is also integrally closed in its field of fractions.*

PROOF. See, for example, [3, Lemma 11.23] or [82, Cor. XI.4.2]. $\square$

In the case of rings of Krull dimension 1, i.e., in the case of curves, there is an easy criterion for regularity.

THEOREM 2.36. *Assume that R is a Noetherian local integral domain of dimension 1. Then the following conditions are equivalent:*

(1) R is a discrete valuation ring;

(2) R is integrally closed;
(3) R is a regular local ring;
(4) the maximal ideal of R is principal.

PROOF. See [3, Chap. I. Prop. 9.2.]. $\square$

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3 Algebraic subsets of the affine space

From now on we assume that $\Bbbk$ is an algebraically closed field.

3.1 Basic definitions

DEFINITION 3.1. Consider the map $\mathcal{V}$ from the family of subsets of $\Bbbk[X_1, \dots, X_n]$ to the family of subsets of $\Bbbk^n$,

$$\mathcal{V}(S) = \bigl\{(a_1, \dots, a_n) \in \Bbbk^n : f(a_1, \dots, a_n) = 0, \forall\, f \in S\bigr\},$$

where $S \subset \Bbbk[X_1, \dots, X_n]$. The image of the map $\mathcal{V}$ is the family of closed sets of a topology of $\Bbbk^n$, called the *Zariski topology*. The set $\Bbbk^n$ when endowed with the Zariski topology will be denoted as $\mathbb{A}^n$ and called the *affine space*. An *algebraic set* is a Zariski closed subset of $\mathbb{A}^n$, for some $n \geq 0$. If $S \subset \mathbb{A}^n$ is a subset, the *Zariski topology* of S is the topology induced by the Zariski topology of $\mathbb{A}^n$.

The above is the basic construction for developing the local theory of algebraic varieties over a field $\Bbbk$.

OBSERVATION 3.2. In the situation above we have that:

(1) The map $\mathcal{V}$ is determined by the values it takes on the ideals of the algebra $\Bbbk[X_1, \dots, X_n]$. Indeed, if S is an arbitrary subset of the polynomial ring and $\langle S \rangle$ is the ideal generated by S then $\mathcal{V}(S) = \mathcal{V}\bigl(\langle S \rangle\bigr) = \mathcal{V}\bigl(\sqrt{\langle S \rangle}\bigr)$.

(2) If I and J are ideals in the polynomial ring, and $\sqrt{I} = \sqrt{J}$, then $\mathcal{V}(I) = \mathcal{V}(J) = \mathcal{V}\bigl(\sqrt{I}\bigr)$.

(3) An arbitrary algebraic subset of $\Bbbk^n$ is always the set of zeroes of a finite number of polynomials. Indeed, if $X \subset \Bbbk^n$ is algebraic, then $X = \mathcal{V}(I)$ for some ideal I in the corresponding polynomial ring. As $I = \langle f_1, \dots, f_m \rangle$ for a finite set of polynomials (see Appendix, Theorem 3.10), we have that $X = \mathcal{V}(f_1, \dots, f_m)$.

Next we reverse the above construction and associate to an arbitrary subset of $\mathbb{A}^n$ an ideal in the polynomial ring $\Bbbk[X_1, \dots, X_n]$.

DEFINITION 3.3. Let $X \subset \mathbb{A}^n$ be an arbitrary subset. Call

$$\mathcal{I}(X) = \bigl\{f \in \Bbbk[X_1, \dots, X_n] : f\big|_X = 0\bigr\} \subset \Bbbk[X_1, \dots, X_n].$$

Notice that $\mathcal{I}(X)$ is an ideal of $\Bbbk[X_1, \dots, X_n]$.

Below we list — and leave as an exercise for the reader to prove — the basic properties of the maps $\mathcal{I}$ and $\mathcal{V}$. See Exercise 6.

LEMMA 3.4. *Consider an algebraically closed field* $\Bbbk$ *and the maps* $\mathcal{V}$ *and* $\mathcal{I}$ *defined above.*

(1) If $S \subset T \subset \Bbbk[X_1,\dots,X_n]$*, then* $\mathcal{V}(T) \subset \mathcal{V}(S)$*. Also,* $\mathcal{V}(\{0\}) = \mathbb{A}^n$ *and* $\mathcal{V}(\Bbbk[X_1,\dots,X_n]) = \emptyset$*.*

(2) If $\{S_\alpha\}_\alpha$ *is a family of subsets of* $\Bbbk[X_1,\dots,X_n]$*, then* $\mathcal{V}\big(\bigcup_\alpha S_\alpha\big) = \bigcap_\alpha \mathcal{V}(S_\alpha)$*.*

(3) If $I, J \subset \Bbbk[X_1,\dots,X_n]$ *are ideals, then* $\mathcal{V}(IJ) = \mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$*.*

(4) If $X \subset Y \subset \mathbb{A}^n$*, then* $\mathcal{I}(Y) \subset \mathcal{I}(X)$*.* $\mathcal{I}(\emptyset) = \Bbbk[X_1,\dots,X_n]$ *and* $\mathcal{I}(\mathbb{A}^n) = \{0\}$*.*

(5) If $X, Y \subset \mathbb{A}^n$*, then* $\mathcal{I}(X \cup Y) = \mathcal{I}(X) \cap \mathcal{I}(Y)$*.*

(6) If $\{X_\alpha\}_\alpha$ *are closed subsets of* $\mathbb{A}^n$*, then* $\mathcal{I}\big(\bigcap_\alpha X_\alpha\big) = \sum_\alpha \mathcal{I}(X_\alpha)$*.*

(7) If $X \subset \mathbb{A}^n$*, then* $X \subset \mathcal{V}\big(\mathcal{I}(X)\big)$*.*

(8) If I *is an ideal in* $\Bbbk[X_1,\dots,X_n]$*, then* $I \subset \sqrt{I} \subset \mathcal{I}\big(\mathcal{V}(I)\big)$*.*

(9) The image of $\mathcal{I}$ *consists of radical ideals.*

OBSERVATION 3.5. In accordance with Lemma 3.4 parts (7) and (8), if $X \subset \mathbb{A}^n$, then $X \subset \mathcal{V}\big(\mathcal{I}(X)\big)$ and if $I \subset \Bbbk[x_1,\dots,x_n]$ is an ideal, then $I \subset \mathcal{I}\big(\mathcal{V}(I)\big)$. These inclusions are not necessarily equalities: take, for example, $X = \Bbbk \setminus \{0\} \subset \Bbbk$, and $I = \langle x^2 \rangle \subset \Bbbk[X]$ and perform the explicit computations.

LEMMA 3.6. *If* $X \subset \mathbb{A}^n$ *is an arbitrary subset of the affine space and* $\overline{X}$ *denotes its closure, then* $\overline{X} = \mathcal{V}\big(\mathcal{I}(X)\big)$*.*

PROOF. The proof of this lemma is left as an exercise (see Exercise 7). □

LEMMA 3.7. *Let* $X \subset \mathbb{A}^n$ *be an arbitrary subset. Then*

$$\mathcal{I}(X) = \bigcap_{(a_1,\dots,a_n)\in X} \langle X_1 - a_1, \dots, X_n - a_n \rangle.$$

PROOF. If $f \in \mathcal{I}(X)$, then $f(a_1,\dots,a_n) = 0$ for all $(a_1,\dots,a_n) \in X$, and thus $f \in \langle X_1 - a_1, \dots, X_n - a_n \rangle$ for all $(a_1,\dots,a_n) \in X$ (see Exercise 3). Conversely, if $f \in \langle X_1 - a_1, \dots, X_n - a_n \rangle$, it is clear that $f(a_1,\dots,a_n) = 0$. □

Another version of Hilbert's Nullstellensatz guarantees that the equality $\sqrt{I} = \mathcal{I}\big(\mathcal{V}(I)\big)$ holds. This result is due to D. Hilbert (see [70], [71]).

THEOREM 3.8 (Hilbert's Nullstellensatz). *Let* I *be an ideal in the polynomial ring* $I \subset \Bbbk[X_1,\dots,X_n]$*. Then* $\sqrt{I} = \mathcal{I}\big(\mathcal{V}(I)\big)$*.*

PROOF. Recall that (see Theorem 2.21)

$$\sqrt{I} = \bigcap \big\{ M \subset \Bbbk[X_1,\dots,X_n] : I \subset M, M \text{ maximal ideal} \big\}.$$

If M is maximal, then $M = \langle X_1 - a_1, \dots, X_n - a_n \rangle$ for some $a_1,\dots,a_n \in \Bbbk$ (see Theorem 2.18). Clearly, $I \subset \langle X_1 - a_1, \dots, X_n - a_n \rangle$ if and only if $f(a_1,\dots,a_n) = 0$ for all $f \in I$, i.e., if and only if $(a_1,\dots,a_n) \in \mathcal{V}(I)$. Thus, we conclude that

$$\sqrt{I} = \bigcap \big\{ \langle X_1 - a_1, \dots, X_n - a_n \rangle \subset \Bbbk[X_1,\dots,X_n] : (a_1,\dots,a_n) \in \mathcal{V}(I) \big\}.$$

By Lemma 3.7, $\mathcal{I}\big(\mathcal{V}(I)\big) = \bigcap_{(a_1,\dots,a_n)\in\mathcal{V}(I)} \langle X_1 - a_1, \dots, X_n - a_n \rangle$. It is then evident that $\sqrt{I} = \mathcal{I}\big(\mathcal{V}(I)\big)$. □

COROLLARY 3.9. *If we fix n and restrict the domain of the map $\mathcal{I}$ to the family of algebraic subsets of $\mathbb{A}^n$ and the domain of $\mathcal{V}$ to the family of radical ideals of $\Bbbk[X_1,\dots,X_n]$, the maps $\mathcal{V}$ and $\mathcal{I}$ are inclusion reversing inverse isomorphisms. Moreover, this correspondence takes points of $\mathbb{A}^n$ into maximal ideals of $\Bbbk[X_1,\dots,X_n]$.*

PROOF. The proof of this result follows easily from Hilbert's Nullstellensatz (Theorem 3.8) and lemmas 3.6 and 3.7. □

EXAMPLE 3.10. (1) Let $\Bbbk$ be an algebraically closed field. Then the algebraic subsets of $\mathbb{A}^1=\Bbbk$ are $\emptyset$, $\mathbb{A}^1$ and finite subsets of $\Bbbk$.

(2) The reader should be aware that many of the above conditions fail drastically for non algebraically closed fields. For example, the ideal generated by $X^2+1\in\mathbb{R}[X]$ is maximal, but its zero set in $\mathbb{R}^2$ is empty.

3.2 The Zariski topology

DEFINITION 3.11. Let $f\in\Bbbk[X_1,\dots,X_n]$ and consider the open subset of $\mathbb{A}^n$,

$$\mathbb{A}^n_f=\mathbb{A}^n\setminus f^{-1}(0)=\big\{(a_1,\dots,a_n)\in\mathbb{A}^n : f(a_1,\dots,a_n)\neq 0\big\}.$$

If X is an arbitrary algebraic subset of $\mathbb{A}^n$, and $f\in\Bbbk[X_1,\dots,X_n]$ then $X_f=X\setminus f^{-1}(0)=X\cap\mathbb{A}^n_f$ is open in X. The open subsets X_f will be called the *basic open subsets* of X.

LEMMA 3.12. *In the situation of Definition 3.11, the family of open sets $\big\{\mathbb{A}^n_f : f\in\Bbbk[X_1,\dots,X_n]\big\}$ forms a basis for the Zariski topology of $\mathbb{A}^n$. Similarly, the family of the open subsets $\big\{X_f : f\in\Bbbk[X_1,\dots,X_n]\big\}$ forms a basis for the Zariski topology of X.*

PROOF. The proof of this result is left as an exercise (see Exercise 8). □

As the reader can easily see in example 3.10, the Zariski topology in general is not Hausdorff. In fact, an algebraic set is Hausdorff if and only if it is a finite collection of points (see Exercise 9).

We leave as an exercise the proof that algebraic sets are quasi-compact (see Exercise 10).

LEMMA 3.13. *The Zariski topology when restricted to an arbitrary algebraic set of an affine space is Noetherian.*

PROOF. Clearly it is enough to prove this result for $\mathbb{A}^n$. The family of all closed, i.e., algebraic, subsets of $\mathbb{A}^n$ is in bijection with the family of radical ideals of $\Bbbk[X_1,\dots,X_n]$. But, since the polynomial algebra is Noetherian (see Appendix, Theorem 3.10), the ascending chains of ideals stabilize and hence the same happens with the descending chains of algebraic subsets of $\mathbb{A}^n$. □

In an informal sense, the Noetherian property tells us that in the Zariski topology the open subsets are large (see, for example, Theorem 3.15) and this accounts for the rigidity of the theory.

DEFINITION 3.14. A topological space X is *reducible* if it is the union of two proper closed subsets. It is *irreducible* if this is not the case. An *irreducible component* of X is a maximal irreducible subset of X.

THEOREM 3.15. *(1) A topological space X is irreducible if and only if any two nonempty open subsets intersect, i.e., $U\cap V\neq\emptyset$ for all $U,V\subset X$ nonempty open subsets.*

(2) The closure of an irreducible set is irreducible.

(3) The irreducible components of a topological space are closed.

PROOF. This is an easy exercise in general topology. □

OBSERVATION 3.16. The reader must be careful not to confuse irreducibility with connectedness. Clearly an irreducible topological space is connected. Since for a Hausdorff topological space given two different points we can find two disjoint nonempty open subsets, an irreducible Hausdorff topological space is necessarily a point.

OBSERVATION 3.17. If S is an arbitrary irreducible subset of X, then there exists an irreducible component Z of X that contains S.

Indeed, consider the family $\mathcal{F}_S$ consisting of all irreducible closed subsets of X that contain S with the order given by the inclusion. If $\{Z_i\}_{i\in I}$ is a chain in $\mathcal{F}_S$, then $Z = \overline{\bigcup_{i\in I} Z_i}$ is an irreducible closed subset of X that contains S, i.e., $Z \in \mathcal{F}_S$. To prove this assertion assume that $\emptyset = (U \cap Z) \cap (V \cap Z) = U \cap V \cap Z$, U, V open in X, with $U \cap Z \neq \emptyset$. Then $U \cap Z_i \neq \emptyset$ for some $i \in I$. Thus $U \cap Z_j \neq \emptyset$ and $U \cap V \cap Z_j = \emptyset$ for any $Z_j \supset Z_i$. As Z_j is irreducible, it follows that $V \cap Z_j = \emptyset$ for every $Z_j \supset Z_i$ and hence for every $j \in I$, $V \cap Z_j = \emptyset$.

Then, $V \cap Z = \emptyset$, and Z is irreducible. Using Zorn's lemma we conclude that every irreducible subset of X is contained in a maximal irreducible, i.e., in an irreducible component.

LEMMA 3.18. *Let X be a Noetherian topological space. Then in X there are at most a finite number of irreducible components. Moreover, $X = \bigcup_{i=1}^{n} X_i$, where $\{X_1, \dots, X_n\}$ are the irreducible components of X.*

PROOF. Let X_j, $j \in J$, be the family of irreducible components of X — as we observed before this family is nonempty. Since points are irreducible, it follows that $X = \bigcup_{j\in J} X_j$.

We prove now that an arbitrary nonempty closed subset of X can be written as a finite union of irreducible subsets. If not, call $\mathcal{F}$ the family of the closed subsets of X that cannot be written as above and take $X_{-\infty}$ a minimal set in this family. If $X_{-\infty}$ is irreducible we have a contradiction. Contrariwise write $X_{-\infty} = X_0 \cup X_1$, with $X_0, X_1 \subsetneq X_{-\infty}$ closed in X. Since $X_0, X_1 \notin \mathcal{F}$, we have a contradiction.

Assume now that $X = \bigcup_{i=1}^{n} X_i$, X_i irreducible, and eliminate all redundancies, i.e., assume that there are no inclusion relations between the X_i. If Z is an irreducible component of X we have that $Z = \bigcup_{i=1}^{n} (X_i \cap Z)$; then, using the irreducibility of Z, we conclude that for some $1 \leq i \leq n$, $Z = Z \cap X_i$. Then, $Z \subset X_i$ and, hence, $Z = X_i$. □

EXAMPLE 3.19. The algebraic subset $\mathcal{V}(XY) \subset \Bbbk^2$ (the union of the two coordinate axes) is reducible, with irreducible components

$$\mathcal{V}(XY) = \big\{(0, b) : b \in \Bbbk\big\} \cup \big\{(a, 0) : a \in \Bbbk\big\}.$$

It is very easy to see that the lines $\big\{(0, b) : b \in \Bbbk\big\}$ and $\big\{(a, 0) : a \in \Bbbk\big\}$ are irreducible.

The irreducibility of an algebraic set can be completely characterized in terms of the corresponding ideal.

THEOREM 3.20. *An algebraic set $X \subset \mathbb{A}^n$ is irreducible if and only if $\mathcal{I}(X)$ is a prime ideal. In particular, $\mathbb{A}^n$ is irreducible.*

PROOF. Let X be an irreducible algebraic subset and suppose that $f, g \in \Bbbk[X_1, \dots, X_n]$ are such that $fg \in \mathcal{I}(X)$. Consider the union $\mathcal{V}(f) \cup \mathcal{V}(g) = \mathcal{V}(fg)$. Since $fg \in \mathcal{I}(X)$, it follows that $X \subset \mathcal{V}(fg)$. Thus, either $X \subset \mathcal{V}(f)$ or $X \subset \mathcal{V}(g)$. We suppose without loss of generality that $X \subset \mathcal{V}(f)$. Then $\sqrt{(f)} \subset \mathcal{I}(X)$, and thus $f \in \mathcal{I}(X)$.

Suppose now that $\mathcal{I}(X)$ is a prime ideal. Let $X = Y \cup Z$, with $Y = \mathcal{V}(I)$, $Z = \mathcal{V}(J)$ two closed subsets. Then $X = \mathcal{V}(IJ)$, and thus $\mathcal{I}(X) = \sqrt{IJ} \supset IJ$. Suppose there exists a polynomial $f \in I \setminus \mathcal{I}(X)$. Since $fg \in IJ \subset \mathcal{I}(X)$ for any $g \in J$, and $\mathcal{I}(X)$ is prime, it follows that $J \subset \mathcal{I}(X)$, and thus $X \subset Z$. This concludes the proof. $\square$

Observation 3.21. Let $f \in \Bbbk[X_1, \dots, X_n]$ and consider the corresponding function $f : \mathbb{A}^n \to \Bbbk$. Then the function f is continuous in the Zariski topology. Indeed, $f^{-1}(a) = \mathcal{V}(f - a)$.

3.3 Polynomial maps. Morphisms

Observation 3.22. Let $X \subset \mathbb{A}^n$ be an algebraic set and call $\Bbbk^X$ the algebra of all functions from X into $\Bbbk$. Consider the map $R : \Bbbk[X_1, \dots, X_n] \to \Bbbk^X$, defined by the restriction of functions, i.e., $R(f) = f\big|_X$. If $I = \mathcal{I}(X)$ is the ideal of X, it is clear that the image of R is isomorphic to $\Bbbk[X_1, \dots, X_n]/I$. Observe also that for $f \in \Bbbk[X_1, \dots, X_n]$ the function $R(f) : X \to \Bbbk$, being the restriction of a continuous function, is also continuous.

Definition 3.23. Let $X \subset \mathbb{A}^n$ be an algebraic subset. We say that a function of $\Bbbk^X$ is a *regular function* or that it is a *polynomial* on X if it is the restriction to X of a polynomial in $\mathbb{A}^n$, i.e., if it belongs to $R\big(\Bbbk[X_1, \dots, X_n]\big)$. We denote the set of polynomial functions as $\Bbbk[X]$.

Observation 3.24. As $\Bbbk[X] \subset \Bbbk^X$ is $R\big(\Bbbk[X_1, \dots, X_n]\big)$, it follows that the algebra $\Bbbk[X]$ is isomorphic to $\Bbbk[X_1, \dots, X_n]/\mathcal{I}(X)$ (see Observation 3.22).

Observation 3.25. If we call $C_{\mathrm{Zar}}(X)$ the subalgebra of $\Bbbk^X$ consisting of the functions on X continuous with respect to the Zariski topology, it is clear that $\Bbbk[X] \subset C_{\mathrm{Zar}}(X)$. Notice that there exist continuous functions that are not regular. See Exercise 17.

Observation 3.26. Since the ideals of $\Bbbk[X_1, \dots, X_n]/I$ correspond to the ideals of $\Bbbk[X_1, \dots, X_n]$ that contain I, the closed subsets of X in the Zariski topology correspond to the ideals in $\Bbbk[X]$. In particular, the points in X correspond to the maximal ideals of $\Bbbk[X]$. It is also clear that the basis for the Zariski topology of an algebraic set X considered in Definition 3.11 is $\big\{X_f : f \in \Bbbk[X]\big\}$.

Definition 3.27. In the case that X and Y are abstract sets and $F : X \to Y$ is a function, define a $\Bbbk$-algebra homomorphism $F^\sharp : \Bbbk^Y \to \Bbbk^X$ as $F^\sharp(f) = f \circ F$.

The following definition of morphism between algebraic sets generalizes and is motivated by the construction of $\Bbbk[X]$.

Definition 3.28. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be algebraic sets. A *morphism of algebraic sets* $F : X \to Y$ is a set theoretical function from X into Y with the property that $F^\sharp\big(\Bbbk[Y]\big) \subset \Bbbk[X]$. Morphisms of algebraic sets are also called regular maps or polynomial maps.

Observation 3.29. (1) If $F : X \to Y$ is a morphism of algebraic sets, we denote the restriction $F^\sharp\big|_{\Bbbk[Y]}$ also as $F^\sharp : \Bbbk[Y] \to \Bbbk[X]$.

(2) The reader is asked to prove as an exercise (see Exercise 14) that, in the situation of the above Definition 3.28, if $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, a function $F : X \to Y$ is a morphism of algebraic sets if and only if there exists polynomials $f_1, \dots, f_m \in \Bbbk[X_1, \dots, X_n]$ such that if we call $G = (f_1, \dots, f_m) : \mathbb{A}^n \to \mathbb{A}^m$, then $G\big|_X = F$ (see also the proof of Theorem 3.32). In other words, the morphisms of algebraic sets are the restrictions of m-uples of polynomials viewed as maps in the ambient space. In particular the morphisms from $\mathbb{A}^n$ to $\mathbb{A}^m$ are the m-uples of polynomials in n variables.

Lemma 3.30. *Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be algebraic sets and assume that $F : X \to Y$ is a morphism of algebraic sets. Then, the map $F^\sharp : \Bbbk[Y] \to \Bbbk[X]$ is an algebra homomorphism.*

Proof. The proof follows immediately from Definition 3.28. □

Observation 3.31. The reader should be aware that the notation $F^\sharp$ for the map $\Bbbk[Y] \to \Bbbk[X]$, $f \mapsto f \circ F$, is not uniform in the literature; see, for example, [12], [66], [144].

The next theorem shows that the study of the geometry of the algebraic sets can be considered as a part of commutative algebra.

Theorem 3.32. *The contravariant functor*

$$X \mapsto \Bbbk[X]\,,\ (F : X \to Y) \mapsto \big(F^\sharp : \Bbbk[Y] \to \Bbbk[X]\big)$$

is an isomorphism between the category of algebraic sets and morphisms of algebraic sets and the category of affine $\Bbbk$-algebras and morphisms of $\Bbbk$-algebras.

Proof. Let A be an affine $\Bbbk$-algebra; it can be written as $A = \Bbbk[X_1, \dots, X_n]/I$, where I is a radical ideal. Call $X = \mathcal{V}(I)$ the algebraic subset of $\mathbb{A}^n$ consisting of the zeroes of I. Clearly $\Bbbk[X] \cong A$. Assume now that X and Y are algebraic subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$, respectively, and that $\alpha : \Bbbk[Y] \to \Bbbk[X]$ is a morphism of algebras. Write $\Bbbk[Y] = \Bbbk[Y_1, \dots, Y_m]/\mathcal{I}(Y)$ and $\Bbbk[X] = \Bbbk[X_1, \dots, X_n]/\mathcal{I}(X)$. Define polynomials $f_i \in \Bbbk[X_1, \dots, X_n], i = 1, \dots, m$ by the formulæ $\alpha\big(Y_i + \mathcal{I}(Y)\big) = f_i + \mathcal{I}(X)$, and consider the map $\widehat{\alpha} : \Bbbk[Y_1, \dots, Y_m] \to \Bbbk[X_1, \dots, X_n]$ given by extending multiplicatively the map that sends $\widehat{\alpha}(Y_i) = f_i$, for $i = 1, \dots, m$. Then, the diagram below commutes

$$\begin{tikzcd}
\Bbbk[Y_1, \dots, Y_m] \arrow[r, "\widehat{\alpha}"] \arrow[d] & \Bbbk[X_1, \dots, X_n] \arrow[d] \\
\Bbbk[Y_1, \dots, Y_m]/\mathcal{I}(Y) \arrow[r, "\alpha"'] & \Bbbk[X_1, \dots, X_n]/\mathcal{I}(X)
\end{tikzcd}$$

Consider the map $F = (f_1, \dots, f_m) : \mathbb{A}^n \to \mathbb{A}^m$. We want to prove that $F(X) \subset Y$ and that $F^\sharp = \alpha$ (see Lemma 3.30). If $f \in \Bbbk[Y_1, \dots, Y_m]$ then $f \circ F = f(f_1, \dots, f_m) = f\big(\widehat{\alpha}(Y_1), \dots, \widehat{\alpha}(Y_m)\big) = \widehat{\alpha}(f)$, i.e., $F^\sharp = \widehat{\alpha}$.

Also, if $f \in \mathcal{I}(Y)$, then $f \circ F \in \mathcal{I}(X)$ and hence map F sends X into Y. □

Lemma 3.33. *If $X \subset \mathbb{A}^n$ is an algebraic set, then the elements of $\Bbbk[X]$ separate the points of X. In other words, given $x \neq y \in X$, there exists $f \in \Bbbk[X]$ such that $f(x) = 0$, $f(y) \neq 0$. More generally, if $Y \subset X$ is a closed subset and $x \notin Y$, then there exists $f \in \mathcal{I}(Y)$ such that $f(x) \neq 0$*

Proof. Since $x \notin Y$, the maximal ideal M_x does not contain $\mathcal{I}(Y)$. Hence, there exists $f \in \mathcal{I}(Y) \setminus M_x$. □

Definition 3.34. (1) Let X be an algebraic set, $x \in X$ and U_x be an open subset of X containing x. We say that a function $h : U_x \to \Bbbk$ is *regular* at x, if there exist an open subset $x \in V \subset U_x$ and functions $f, g \in \Bbbk[X]$, such that $g(y) \neq 0$ for all $y \in V$ and $h\big|_V = (f/g)\big|_V$.

(2) We call $\mathcal{O}_{X,x}$, the *local ring of X at x*, the ring of functions that are regular at x.

(3) If U is an open subset of X we define the *ring of regular functions on an open subset U* as the ring of the functions $f : U \to \Bbbk$ that are regular at every point of U. We denote this ring as $\mathcal{O}_X(U)$.

OBSERVATION 3.35. Observe that in the above definition there is no loss of generality if we ask V to be a basic open subset of X.

LEMMA 3.36. *(1) Let X be an algebraic set and $x \in X$. Then $\mathcal{O}_{X,x} \cong \Bbbk[X]_{M_x}$ where M_x is the maximal ideal in the ring $\Bbbk[X]$ corresponding to x.*

(2) If X is an irreducible algebraic subset, and $0 \neq f \in \Bbbk[X]$, then $\Bbbk[X]_f \cong \mathcal{O}_X(X_f)$, and in particular $\mathcal{O}_X(X) = \Bbbk[X]$.

PROOF. (1) There exists an injective map $\Bbbk[X]_{M_x} \to \mathcal{O}_{X,x}$. Indeed, if we consider f/g with $f, g \in \Bbbk[X]$ and $g(x) \neq 0$ and take X_g, it is clear that g does not vanish in X_g and then the quotient f/g represents an element in $\mathcal{O}_{X,x}$. If $h \in \mathcal{O}_{X,x}$ is an arbitrary element, one can represent h as the quotient f/g of two polynomials $f, g \in \Bbbk[X]$, with $g(x) \neq 0$, in a conveniently chosen neighborhood of x. It follows that the above morphism is surjective.

(2) It is clear that $\Bbbk[X]_f$ injects into $\mathcal{O}_X(X_f)$. Consider an element $g \in \mathcal{O}_X(X_f)$; then $g \in \mathcal{O}_{X,x}$ for all $x \in X_f$ or equivalently, $g \in \Bbbk[X]_M$ for all the ideals M corresponding to points of X_f. Now, $x \in X_f$ if and only if $f(x) \neq 0$, if and only if $f \notin M$, where M is the maximal ideal corresponding to the point x. In other words, $g \in \mathcal{O}_X(X_f)$ if and only if $g \in \Bbbk[X]_M$ for all maximal ideals $M \subset \Bbbk[X]$ such that $f \notin M$, i.e., $\mathcal{O}_X(X_f) = \bigcap \big\{\Bbbk[X]_M : f \notin M,\ M \subset \Bbbk[X] \text{ is maximal}\big\}$. But, the localization map establishes a bijective correspondence between the set of maximal ideals of $\Bbbk[X]$ that do not contain f and the set of maximal ideals of $\Bbbk[X]_f$. Moreover, as $\Bbbk[X]_M = (\Bbbk[X]_f)_{M_f}$ we conclude that $\mathcal{O}_X(X_f) = \bigcap \big\{(\Bbbk[X]_f)_{\widetilde{M}} : \widetilde{M} \subset \Bbbk[X]_f, \widetilde{M} \text{ is maximal}\big\} = \Bbbk[X]_f$. For this last equality see Appendix, Observation 3.15. □

OBSERVATION 3.37. If $U \subset V \subset X$ are open subsets, the restriction of functions from V to U induces a morphism of $\Bbbk$-algebras $\rho_{VU} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Given two open subsets $U, V \subset X$, $f \in \mathcal{O}_X(U)$ and $g \in \mathcal{O}_X(V)$, such that $f\big|_{U\cap V} = g\big|_{U\cap V}$, the function $h : U \cup V \to \Bbbk$ defined as $h(x) = f(x)$ if $x \in U$, $h(x) = g(x)$ if $x \in V$, belongs to $\mathcal{O}_X(U \cup V)$.

Then the assignment $U \mapsto \mathcal{O}_X(U)$ together with the restriction maps form a sheaf of rings in the topological space X (see Section 4.1, and in particular Example 4.6).

COROLLARY 3.38. *(1) Let X be an irreducible algebraic subset and $U \subset X$ an open subset. Then every function $f \in \mathcal{O}_X(U)$ is continuous.*

(2) If X and Y are affine algebraic sets and $f : X \to Y$ is a morphism of affine algebraic sets, then for any V open subset of Y the map given by composition with f sends $\mathcal{O}_Y(V)$ into $\mathcal{O}_X\big(f^{-1}(V)\big)$. □

The last assertion of the above Corollary is better interpreted in terms of morphisms of sheaves (see, for example, Observation 4.40).

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4 Algebraic varieties

In this section we continue with the development of algebraic geometry by defining the category of algebraic varieties.

4.1 Sheaves on topological spaces

DEFINITION 4.1. A *presheaf of rings* $\mathcal{F}$ on a topological space X associates to each open subset $U \subset X$ a ring $\mathcal{F}(U)$ and to each pair of open subsets $U \subset V \subset X$ a morphism of rings $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$ such that:

(a) $\mathcal{F}(\emptyset) = \{0\}$;

(b) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for all open subsets $U \subset X$;

(c) if $U \subset V \subset W \subset X$ are three open subsets, then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$;

We say that $\mathcal{F}$ is a *sheaf of rings*, or simply a *sheaf*, if it also satisfies:

(d) for every open subset $U \subset X$, for every cover $\{V_i\}_{i \in I}$ of U by open subsets, and for every family $s_i \in \mathcal{F}(V_i)$ such that $\rho_{V_i\, V_i \cap V_j}(s_i) = \rho_{V_j\, V_i \cap V_j}(s_j)$ for all $i, j \in I$, there exists $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = s_i$ for all $i \in I$;

(e) if U and $\{V_i\}_{i \in I}$ are as in (d) and $s \in \mathcal{F}(U)$ is such that $\rho_{UV_i}(s) = 0$ for all $i \in I$, then $s = 0$.

For $U \subset X$ open, the ring $\mathcal{F}(U)$ is called the ring of *sections of* $\mathcal{F}$ *on* U and the maps ρ_{VU} are called the restriction maps. The elements of $\mathcal{F}(U)$ are called the sections of the sheaf on U.

If $\mathcal{F}$ is a sheaf on X, a *subsheaf* $\mathcal{G} \subset \mathcal{F}$ is a sheaf such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ is a subring, for all open subsets $U \subset X$.

OBSERVATION 4.2. (1) Most of the sheaves used in this book are sheaves of $\Bbbk$-algebras — i.e., the rings $\mathcal{F}(U)$ are $\Bbbk$-algebras, and the restriction maps are morphisms of $\Bbbk$-algebras. In this context, by a subsheaf we mean a subsheaf such that $\mathcal{G}(U)$ is a subalgebra of $\mathcal{F}(U)$ for all U open subset of X.

(2) Usually — and the motivation for this abuse of notation will become clear in what follows — if $U \subset V$ and $s \in \mathcal{F}(V)$, we write $s\big|_U = \rho_{VU}(s)$.

(3) If X is a topological space a more formal definition of a presheaf on X would be the following. Consider the topology $\mathcal{T}$ as a category — viewing it as an ordered set. A presheaf on X is a contravariant functor from the topology into the category of rings. In this interpretation sheaves are functors satisfying certain equalization properties.

EXAMPLE 4.3. Let X and Z be topological spaces. To each open subset $U \subset X$ we associate the set of continuous functions from U to Z, and if $V \subset U$ are open, we consider the restriction of functions from U to V. Since continuity is a local property, in this manner we obtain a sheaf. If $Z = \mathbb{R}$, this is a sheaf of $\mathbb{R}$-algebras.

DEFINITION 4.4. Let $\mathcal{F}$ be a presheaf of rings on X, and $x \in X$. We define the *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at x as the direct limit of the directed family of rings

$$\big\{\mathcal{F}(U) : x \in U\,,\ \rho_{VU}, U \subset V \text{ open in } X\big\}.$$

OBSERVATION 4.5. (1) Explicitly, $\mathcal{F}_x$ is the quotient of the set of pairs $\big\{(U,s) : s \in \mathcal{F}(U)\,,\ x \in U \text{ open in } X\big\}$ with respect to the equivalence relation: $(U,s) \sim (V,t)$ if and only if there exists an open set $x \in W \subset V \cap U$ such that $s\big|_W = t\big|_W$.

(2) Notice that for all $x \in X$ the fiber $\mathcal{F}_x$ is a commutative ring, and a $\Bbbk$-algebra if the $\mathcal{F}$ is a presheaf of $\Bbbk$-algebras.

(3) If $U \subset X$ is an open subset, then the canonical map associated to the direct limit is a ring homomorphism for all $x \in U$ — recall that this canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ sends $s \in \mathcal{F}(U)$ into the equivalence class of the pair (U,s).

The image of s in the stalk $\mathcal{F}_x$ can be thought as the value of s at x. Thus, the stalk

$\mathcal{F}_x$ represents the germs of the sections of $\mathcal{F}$ at x, and a section $s \in \mathcal{F}(U)$ can be thought as a function $s : U \to \bigsqcup_{x\in U} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ — the symbol $\bigsqcup$ represents the disjoint union. Notice that not all functions as above produce elements of $\mathcal{F}(U)$, as the elements of $\mathcal{F}(U)$ satisfy additional coherence properties.

The following example is central in the development of the theory of algebraic varieties.

EXAMPLE 4.6 (The sheaf of regular functions). Let $X \subset \mathbb{A}^n$ be an algebraic set. In accordance to Definition 3.34 we associate to each open subset $U \subset X$ the algebra of regular functions $\mathcal{O}_X(U)$. This, together with the restriction maps, produces a sheaf of $\Bbbk$-algebras on X, called the *structure sheaf* of X and denoted as $\mathcal{O}_X$.

It is more or less obvious that $\mathcal{O}_X$ satisfies properties (a), (b), (c) and (e) of Definition 4.1. Condition (d) follows from the local character of the definition of regular function.

We leave as an exercise the proof that the stalk of the sheaf $\mathcal{O}_X$ is what we called $\mathcal{O}_{X,x}$ in Definition 3.34 (see Exercise 23).

DEFINITION 4.7. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves of rings on a topological space X. A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a family of ring homomorphisms $\{\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U), U \subset X, U \text{ open}\}$ such that whenever there is an inclusion $U \subset V \subset X$ of open subsets, the following diagram is commutative:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\rho^{\mathcal{F}}_{VU}"'] & \mathcal{G}(V) \arrow[d, "\rho^{\mathcal{G}}_{VU}"] \\ \mathcal{F}(U) \arrow[r, "\varphi(U)"'] & \mathcal{G}(U) \end{tikzcd}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, a morphism of sheaves from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of presheaves. The morphisms $\varphi(U)$ will frequently be denoted as $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$.

OBSERVATION 4.8. (1) A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ induces, for all $x \in X$, a ring homomorphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$.

(2) We say that φ is *injective* (resp. *surjective*) if φ_x is injective (resp. surjective) for all $x \in X$.

(3) Considering presheaves as functors (see Observation 4.2), the morphisms between presheaves can be interpreted as natural transformations between the functors.

(4) If the presheaves have additional structure, for example if they are presheaves of $\Bbbk$-algebras, we additionally require in the definition of morphism that for all open sets U of the base space X, the maps $\varphi(U)$ are morphisms of $\Bbbk$-algebras.

DEFINITION 4.9. Let X, Y be topological spaces, $\mathcal{F}$ a sheaf of rings on X, and $f : X \to Y$ a continuous function.

We define the *direct image* sheaf $f_*\mathcal{F}$ as the sheaf on Y given as follows: $f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$, $V \subset Y$ open, with restriction morphisms

$$\rho^{f_*\mathcal{F}}_{VW} = \rho^{\mathcal{F}}_{f^{-1}(V)f^{-1}(W)} : \mathcal{F}\big(f^{-1}(V)\big) \to \mathcal{F}\big(f^{-1}(W)\big).$$

OBSERVATION 4.10. Assume that X and Y are topological spaces and call $\mathcal{C}_X$ and $\mathcal{C}_Y$ the sheaves of $\Bbbk$-valued continuous functions on X and Y respectively — we endow $\Bbbk$ with the Zariski topology. Given a continuous function $f : X \to Y$ we define a morphism of sheaves $f^\sharp : \mathcal{C}_Y \to f_*\mathcal{C}_X$ as follows: if $V \subset Y$ is open, then $f^\sharp_V : \mathcal{C}_Y(V) \to f_*\mathcal{C}_X(V) = \mathcal{C}_X\big(f^{-1}(V)\big)$ is given by composition with f.

More generally, if $f : X \to Y$ is a continuous function, a pair of sheaves of continuous $\Bbbk$-valued functions $\mathcal{F}_X$ and $\mathcal{F}_Y$ defined on X and Y, respectively, i.e., subsheaves of $\mathcal{C}_X$ and $\mathcal{C}_Y$ respectively, are said to be *f-compatible* if for all $V \subset Y$ open in Y, $f^\sharp_V\big(\mathcal{F}_Y(V)\big) \subset f_*\mathcal{F}_X(V) = \mathcal{F}_X\big(f^{-1}(V)\big)$. For f-compatible sheaves, the diagram that follows is commutative

$$\begin{tikzcd} \mathcal{C}_Y \arrow[r, "f^\sharp"] & f_*(\mathcal{C}_X) \\ \mathcal{F}_Y \arrow[u, hook] \arrow[r, "f^\sharp"'] & f_*(\mathcal{F}_X) \arrow[u, hook] \end{tikzcd}$$

In explicit terms, the f-compatibility means that if $V \subset Y$ is an arbitrary open subset of Y and $\alpha : V \to \Bbbk$ is a function on $\mathcal{F}_Y(V)$, then the function $\alpha \circ f : f^{-1}(V) \to \Bbbk$ belongs to $\mathcal{F}_X\big(f^{-1}(V)\big)$.

4.2 The maximal spectrum

We need to introduce a few elements of the abstract theory of spectra of commutative rings.

Definition 4.11. Let A be a commutative ring. The *prime spectrum* of A — denoted as $\mathrm{Sp}(A)$ — is the set

$$\mathrm{Sp}(A) = \big\{P \subset A : P \text{ is a prime ideal of } A\big\}.$$

The subset $\mathrm{Spm}(A) = \big\{M \subset A : M \text{ is a maximal ideal of } A\big\}$ is called the *maximal spectrum* of A.

Definition 4.12. Let A be a commutative ring and call $X = \mathrm{Sp}(A)$. If $f \in A$ we define

$$X_f = \big\{P \in \mathrm{Sp}(A) : f \notin P\big\}.$$

If $Y = \mathrm{Spm}(A)$, we define

$$Y_f = X_f \cap Y = X_f = \big\{M \in \mathrm{Spm}(A) : f \notin M\big\}.$$

The proof of the theorem that follows is an easy exercise in commutative algebra.

Theorem 4.13. *Let A be a commutative ring and $X = \mathrm{Sp}(A)$ or $X = \mathrm{Spm}(A)$. Then the family of sets $\big\{X_f : f \in A\big\}$ considered in Definition 4.12 is the basis of a topology of X that is called the Zariski topology. A subset $Y \subset X$ is closed in this topology if and only if $Y = \big\{Q \in X : Q \supset I\big\}$, where I is an ideal of A.* □

Observation 4.14. (1) The assignment $A \mapsto \mathrm{Sp}(A)$ can be extended to a contravariant functor from the category of commutative rings to the category of topological spaces. If $\alpha : A \to B$ is a morphism of commutative rings, we define $\alpha^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ as $\alpha^*(Q) = \alpha^{-1}(Q)$ for a prime ideal $Q \subset B$.

(2) If we consider the inclusion $\mathbb{Z} \subset \mathbb{Q}$, then the maximal ideal $\{0\} \subset \mathbb{Q}$ when intersected with $\mathbb{Z}$ is not maximal. Hence, one does not have a natural way to view Spm as a functor in all the category of commutative rings.

Lemma 4.15. *Let A and B be commutative finitely generated $\Bbbk$-algebras, $\alpha : A \to B$ a morphism of $\Bbbk$-algebras and $M \in \mathrm{Spm}(B)$. Then $\alpha^{-1}(M) \in \mathrm{Spm}(A)$. In other words, $\alpha^*\big(\mathrm{Spm}(B)\big) \subset \mathrm{Spm}(A)$.*

PROOF. Let M be a maximal ideal in B, consider $M' = \alpha^{-1}(M)$ and the map $\overline{\alpha} : A/M' \to B/M$. As B is a quotient of a polynomial algebra the Nullstellensatz guarantees that B/M coincides with the base field $\Bbbk$. Then, as $\overline{\alpha}$ is $\Bbbk$-linear and injective, we conclude that A/M' is also the field $\Bbbk$ and hence that M' is a maximal ideal. $\square$

The theorem that follows can be viewed as a more formal presentation of Observation 3.26.

THEOREM 4.16. *Assume that X is an algebraic subset of $\mathbb{A}^n$ and consider $\operatorname{Spm}(\Bbbk[X])$ as defined before. Then the map $\iota_X : X \to \operatorname{Spm}(\Bbbk[X])$ defined as*

$$\iota_X(a_1, \dots, a_n) = \langle X_1 - a_1, \dots, X_n - a_n \rangle + \mathcal{I}(X) \subset \Bbbk[X_1, \dots, X_n]/\mathcal{I}(X)$$

is a natural homeomorphism when we endow the domain and codomain with the corresponding Zariski topologies.

PROOF. The proof is a direct consequence of the theory developed so far. We only verify the assertions concerning the topology. Consider $f \in \Bbbk[X]$; then

$$\begin{aligned}\iota_X(X_f) = \big\{ \langle X_1 - a_1, \dots, X_n - a_n \rangle + \mathcal{I}(X) : f(a_1, \dots, a_n) \neq 0 \big\} = \\ \big\{ M \subset \Bbbk[X] : f \notin M \text{ maximal} \big\}.\end{aligned}$$

$\square$

The triple $(X, \Bbbk[X], \iota_X)$ is an example of the concept of "abstract" affine algebraic variety (see Definition 4.17).

4.3 Affine algebraic varieties

In order to eliminate the dependency of an algebraic set on the affine ambient space, we present the following intrinsic definition of affine algebraic variety.

DEFINITION 4.17. Let $\Bbbk$ be an algebraically closed field. An *affine variety over* $\Bbbk$ consists of a triple (X, A, φ), where X is a topological space — the *underlying topological space* of the affine variety — A is an affine $\Bbbk$-algebra — the *algebra of regular functions* of the affine variety — and $\varphi : X \to \operatorname{Spm}(A)$ is a homeomorphism. If there is no danger of confusion A is denoted as $\Bbbk[X]$, or $\mathcal{O}_X(X)$, and the affine variety (X, A, φ) is written as $(X, \Bbbk[X])$ or even as X.

A *morphism* of affine algebraic varieties with domain (X, A, φ) and codomain (Y, B, ψ) is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map and $f^\sharp : B \to A$ is a morphism of $\Bbbk$-algebras such that ${f^\sharp}^* : \operatorname{Spm}(A) \to \operatorname{Spm}(B)$ makes the diagram below commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \operatorname{Spm}(A) \arrow[r, "{f^\sharp}^*"'] & \operatorname{Spm}(B) \end{tikzcd}$$

In accordance with the standard notations, we denote $\varphi(x) = M_x$.

EXAMPLE 4.18. Assume that (X, A, φ) is an affine algebraic variety and Y a closed subset of X. In this case Y also becomes naturally an affine algebraic variety as follows. The homeomorphism $\varphi : X \to \operatorname{Spm}(A)$ sends Y onto $\varphi(Y)$, that is a closed subset, and then

$$\varphi(Y) = \big\{ M \subset A : I \subset M \text{ maximal ideal of } A \big\}$$

for some ideal $I \subset A$ (see Theorem 4.13).

Consider $\big(Y, A/I, \varphi\big|_Y\big)$; as

$$\operatorname{Spm}(A/I) \cong \big\{M \subset A : I \subset M \text{ maximal ideal of } A\big\},$$

it is clear that $\big(Y, A/I, \varphi\big|_Y\big)$ is an affine algebraic variety. Moreover, the pair (ι, π) is a morphism of affine algebraic varieties where $\iota : Y \subset X$ is the inclusion and $\pi : A \to A/I$ is the canonical projection.

Observation 4.19. (1) Let $X \subset \mathbb{A}^n$ be an algebraic subset. In accordance with Theorem 4.16 the triple $\big(X, \Bbbk[X], \iota_X\big)$ is an affine algebraic variety.

(2) In the Definition 4.17, if $x \in X$, then the $\Bbbk$-algebra A/M_x is canonically isomorphic to $\Bbbk$ (see Theorem 2.18 and Lemma 4.15).

(3) The elements of A can be interpreted as functions on X as follows. Consider the morphism of $\Bbbk$-algebras $\iota_X : A \to \Bbbk^X$ defined as $\iota_X(a)(x) = a + M_x \in A/M_x = \Bbbk$. The map ι_X is injective because if $\iota_X(a) = 0$, then $a \in M_x$ for all $x \in X$, and it follows from Exercise 5 that $a = 0$. Hence, A can be identified with a subalgebra of $\Bbbk^X$, i.e., A is an algebra of functions on X with values on the base field $\Bbbk$. Observe that if a is fixed, then

$$\big\{x \in X : \iota_X(a)(x) \neq 0\big\} = \ \big\{x \in X : a \notin M_x\big\} = \big\{x \in X : a \notin \varphi(x)\big\} = \varphi^{-1}\big(\big(\operatorname{Spm}(A)\big)_a\big),$$

that is open in X. Hence, the functions of the form $\iota_X(a)$ are continuous. We call $\iota_X(A) = \Bbbk[X]$.

(4) Viewing the $\Bbbk$-algebra A as a subalgebra of $\Bbbk^X$ as before, the map $f^\sharp$ can be visualized as the composition by f or, in other words, the diagram below is commutative.

$$\begin{tikzcd} B \arrow[r, "f^\sharp"] \arrow[d, "\iota_Y"'] & A \arrow[d, "\iota_X"] \\ \Bbbk^Y \arrow[r, "-\circ f"'] & \Bbbk^X \end{tikzcd}$$

This will be shown in Lemma 4.22.

Hence, in this situation the map $f^\sharp$ is determined by f.

(5) It follows from the previous definitions above that an affine algebraic variety is isomorphic to the affine algebraic variety associated to an algebraic subset of some $\mathbb{A}^n$.

Indeed, if we have a triple (X, A, φ) the affine $\Bbbk$-algebra A is isomorphic to a quotient $\Bbbk[X_1, \dots, X_n]/I$, where I is a radical ideal. If we call X_A the corresponding algebraic subset of $\mathbb{A}^n$, and consider $\big(X_A, \Bbbk[X_A], \iota_{X_A}\big)$, it is easy to show (and left as an exercise for the reader, see Exercise 24) that the two affine algebraic varieties (X, A, φ) and $\big(X_A, \Bbbk[X_A], \iota_{X_A}\big)$ are isomorphic.

(6) Let (X, A, φ) be an affine variety and $\big(X_1, \Bbbk[X_1], \iota_{X_1}\big)$, $\big(X_2, k[X_2], \iota_{X_2}\big)$ be affine varieties associated to the affine algebraic sets X_1 and X_2, that are also isomorphic to (X, A, φ). Then the algebraic sets X_1 and X_2 are isomorphic (see Theorem 3.32 and Definition 3.28).

Observation 4.19 justifies the definition that follows.

Definition 4.20. Let X be an affine variety. Consider an algebraic subset Y isomorphic with X, and call $\psi : X \to Y$ an isomorphism. We define the *structure sheaf* of X as $\mathcal{O}_X(U) = \mathcal{O}_Y\big(\psi(U)\big)$, where $\mathcal{O}_Y$ is as usual the structure sheaf of Y. The restriction morphism is defined in the same manner.

Observation 4.21. (1) The construction of the structure sheaf above is independent of the chosen isomorphism ψ; see Observation 4.19, (6).

(2) Referring to the situation of Example 4.18, if we consider the corresponding associated structure sheaves on X and Y, then the morphism $\iota^\sharp : \mathcal{O}_X \to \iota_*(\mathcal{O}_Y)$, given by composition with the inclusion, is surjective. Equivalently, if I is the ideal of $\Bbbk[X]$ associated to Y, then for an arbitrary point $y \in Y$ the morphism $\Bbbk[X]_{M_y} \to \big(\Bbbk[X]/I\big)_{M_y/I}$ is surjective. This follows immediately from the fact that the projection $\Bbbk[X] \to \Bbbk[X]/I$ is surjective.

Given two topological spaces X, Y underlying to affine algebraic varieties, the following is a criterion to decide if a given continuous map between X and Y is the first component of a morphism.

Lemma 4.22. *Let (X, A, φ) and (Y, B, ψ) be affine algebraic varieties and assume that $f : X \to Y$ is a continuous map. Then, f is the first component of a morphism of affine algebraic varieties if and only if $\alpha \circ f \in \Bbbk[X] \subset \Bbbk^X$ for all $\alpha \in \Bbbk[Y] \subset \Bbbk^Y$. Moreover, $f^\sharp$ is uniquely determined by f, as asserted in Observation 4.19.*

Proof. Assume that f is the first component of the morphism $(f, f^\sharp)$. Then the diagram below is commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \operatorname{Spm}(A) \arrow[r, "f^{\sharp *}"'] & \operatorname{Spm}(B) \end{tikzcd}$$

Given the morphism $f^\sharp : B \to A$, if M is a maximal ideal of A there is an isomorphism $B/(f^\sharp)^{-1}(M) \cong A/M$ and then, via the identification of both sides with $\Bbbk$, we see that $b + (f^\sharp)^{-1}(M) = f^\sharp(b) + M$. It follows that the diagram below commutes (here we are using the notations of Observation 4.19).

$$\begin{tikzcd} B \arrow[r, "f^\sharp"] \arrow[d, "\iota_Y"'] & A \arrow[d, "\iota_X"] \\ \Bbbk^Y \arrow[r, "-\circ f"'] & \Bbbk^X \end{tikzcd}$$

Indeed, we have that $\iota_X\big(f^\sharp(b)\big)(x) = f^\sharp(b) + M_x = b + (f^\sharp)^{-1}(M_x)$ and $\iota_Y(b)\big(f(x)\big) = b + \psi\big(f(x)\big) = b + (f^\sharp)^*(M_x) = b + (f^\sharp)^{-1}(M_x)$. As $\Bbbk[Y] = \iota_Y(B)$ and $\Bbbk[X] = \iota_X(A)$, the conclusion follows.

The converse is proved similarly. First observe that if we call $\mathrm{E}_X : X \to \operatorname{Spm}\big(\Bbbk[X]\big)$ the map defined as $\mathrm{E}_X(x) = \operatorname{Ker}(\varepsilon_x)$, where $\varepsilon_x : \Bbbk[X] \to \Bbbk$ is as usual the evaluation at x, the triangle that follows is commutative

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "\mathrm{E}_X"] & \\ \operatorname{Spm}(A) & & \operatorname{Spm}\big(\Bbbk[X]\big) \arrow[ll, "{\iota_X}^*"] \end{tikzcd}$$

This commutativity follows by explicit computations:

$$\begin{gathered} \iota_X^*\big(\mathrm{E}_X(x)\big) = \iota_X^*\big(\operatorname{Ker}(\varepsilon_x)\big) = \iota_X^{-1}\big(\operatorname{Ker}(\varepsilon_x)\big) = \big\{a \in A : \iota_X(a) \in \operatorname{Ker}(\varepsilon_x)\big\} = \\ \big\{a \in A : \iota_X(a)(x) = a + \varphi(x) = 0\big\} = \varphi(x). \end{gathered}$$

We define $f^\sharp$, i.e., the second component of the morphism of affine varieties, by the commutativity of the diagram

$$\begin{tikzcd} B \arrow[r, "f^\sharp"] \arrow[d, "\iota_Y"'] & A \arrow[d, "\iota_X"] \\ \Bbbk[Y] \arrow[r, "-\circ f"'] & \Bbbk[X] \end{tikzcd}$$

Considering the corresponding diagram at the level of the spectra, we obtain another commutative diagram

$$\begin{tikzcd} \operatorname{Spm}\big(\Bbbk[X]\big) \arrow[r, "(-\circ f)^*"] \arrow[d, "\iota_X^*"'] & \operatorname{Spm}\big(\Bbbk[Y]\big) \arrow[d, "\iota_Y^*"] \\ \operatorname{Spm}(A) \arrow[r, "{f^\sharp}^*"'] & \operatorname{Spm}(B) \end{tikzcd}$$

Next consider the diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "\varphi"] \arrow[dl, "\mathrm{E}_X"'] & Y \arrow[d, "\psi"'] \arrow[dr, "\mathrm{E}_Y"] & \\ & \operatorname{Spm}(A) \arrow[r, "{f^\sharp}^*"] & \operatorname{Spm}(B) & \\ \operatorname{Spm}\big(\Bbbk[X]\big) \arrow[ur, "i_X^*"'] \arrow[rrr, "(-\circ f)^*"'] & & & \operatorname{Spm}\big(\Bbbk[Y]\big) \arrow[ul, "i_Y^*"] \end{tikzcd}$$

This diagram is formed by two triangular and two quadrangular blocks, and the two triangles as well as the lower quadrangular block are commutative. Hence, the commutativity of the central square (that is our thesis) will follow from the commutativity of the diagram that follows, which is the outer diagram of the above.

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "\mathrm{E}_X"'] & Y \arrow[dr, "\mathrm{E}_Y"] & \\ \operatorname{Spm}\big(\Bbbk[X]\big) \arrow[rrr, "(-\circ f)^*"'] & & & \operatorname{Spm}\big(\Bbbk[Y]\big) \end{tikzcd}$$

The commutativity of this diagram is a direct computation. □

Observation 4.23. Let X be an affine variety and $f \in \Bbbk[X]$. Then the basic open subset $X_f \subset X$ can be viewed as an affine variety. In this sense we interpret X_f as the triple $\big(X_f, \Bbbk[X]_f, \iota_f\big)$, where $\iota_f : X_f \to \operatorname{Spm}\big(\Bbbk[X]_f\big)$ is the map defined by the commutativity of the diagram

$$\begin{tikzcd} X_f \arrow[r, "\iota_f"] \arrow[d, hook] & \operatorname{Spm}\big(\Bbbk[X]_f\big) \arrow[d, hook] \\ X \arrow[r, "\iota_X"'] & \operatorname{Spm}\big(\Bbbk[X]\big) \end{tikzcd}$$

In other words, the map ι_f is the restriction of the homeomorphism ι_X considered in Theorem 4.16. The reader should verify that if $x \in X$ and M is its associated maximal ideal, then $f(x) \neq 0$ if and only if $f \notin M$. This means that the restriction of ι_X has the codomain we need.

We show now how to give in an explicit way an isomorphism between X_f and a closed subset in an affine space.

Assume that $X \subset \mathbb{A}^n$ is irreducible and consider $\varphi : X_f \to X \times \mathbb{A}^1$, $\varphi(x) = \big(x, \frac{1}{f(x)}\big)$. The image of φ is the algebraic subset $Y \subset X \times \mathbb{A}^1 \subset \mathbb{A}^n \times \mathbb{A}^1$, $Y = \big\{(x,z) : x \in X, z \in \mathbb{A}^1, f(x)z - 1 = 0\big\}$. It is clear that Y is an algebraic subset of $\mathbb{A}^{n+1}$. In Exercise 25 we ask the reader to prove that $\Bbbk[Y] \cong \Bbbk[X]_f$ and that the diagram below is commutative.

$$\begin{tikzcd} X_f \arrow[r, "\varphi"] \arrow[d, "\iota_f"'] & Y \arrow[d, "\iota_Y"] \\ \operatorname{Spm}\big(\Bbbk[X]_f\big) \arrow[r, equal] & \operatorname{Spm}\big(\Bbbk[Y]\big) \end{tikzcd}$$

It is clear that the map $\varphi : X_f \to Y$ is bijective and its inverse is the restriction to Y of the projection $p_1 : X \times \mathbb{A}^1 \to X$. To prove that φ is an homeomorphism we only have to prove that it is continuous, as its inverse is the projection that is clearly continuous. Take $g \in \Bbbk[Y]$; we want to prove that $\varphi^{-1}(Y_g)$ is open in X_f. Now, $x \in \varphi^{-1}(Y_g)$ if and only if $g\big(x, \frac{1}{f(x)}\big) \neq 0$. If we multiply by a large enough power f^r, then $h(x) = f^r(x) g\big(x, \frac{1}{f(x)}\big)$ is a polynomial in X, and then $\varphi^{-1}(Y_g) = X_f \cap X_h$.

Observe that we have constructed X_f as the graph of the function $\frac{1}{f}$. One can show in general that if $g : X \to Y$ is a morphism of affine algebraic varieties, then the graph of g is an affine variety (see Exercise 15).

The next lemma shows how to produce dense affine basic open subsets in the case when the affine algebraic variety X is not irreducible.

Lemma 4.24. *Let X be an affine algebraic variety and $X = \bigcup_{i=1}^n X_i$ its decomposition in irreducible components. Then X_i is affine, and there exists $f \in \Bbbk[X]$ such that for all $i, j = 1 \ldots, n$, then $X_f \cap X_i \neq \emptyset$ and $X_f \cap X_i \cap X_j = \emptyset$. In other words, X_f is a dense open subset of X, and the irreducible components of $X_f = \bigcup_{i=1}^n X_f \cap X_i$ are its connected components.*

Proof. Since $X_i \subset X$ is a closed subset, it follows from Example 4.18 the X_i is an affine algebraic variety. We proceed by induction and assume that $n = 2$. If $X_1 \cap X_2 = \emptyset$, then we can take $f = 1$. If $X_1 \cap X_2 \neq set$, observe that $X_1 \cap X_2 \subsetneq X_1$, since otherwise $X_1 \subset X_2$ and therefore $X_1 = X_2$; anagolously, $X_1 \cap X_2 \subsetneq X_2$.

Let $0 \neq f_1 \in \Bbbk[X_1]$ be such that $f_1(X_1 \cap X_2) = 0$, and consider $\widetilde{f_1} \in \Bbbk[X]$ such that $f\big|_{X_1} = f_1$ — the regular function f exists because $X_1 \subset X$ is a closed subset. If $\widetilde{f_1}\big|_{X_2} \neq 0$, then we take $f = \widetilde{f_1}$ and we are done. If $\widetilde{f_1}\big|_{X_2} = 0$, we consider $0 \neq f_2 \in \mathcal{I}(X_1) \subset \Bbbk[X]$, and let $f = \widetilde{f_1} + f_2$. Then $f\big|_{X_1} = f_1 \neq 0$, and $f\big|_{X_2} = f_2\big|_{X_2} \neq 0$ — otherwise $f_2 = 0$ and this is a contradiction. It is clear that $f \in \mathcal{I}(X_1 \cap X_2)$; therefore, f verifies the required property.

We left as an exercise to the reader to complete this proof; see Exercise 18. $\square$

Example 4.25. Assume that A and B are commutative $\Bbbk$-algebras. The maximal ideals of $A \otimes B$ are of the form $M \otimes B + A \otimes N$ for M and N maximal ideals of A and B, respectively. Hence, as (abstract) sets $\operatorname{Spm}(A \otimes B)$ and $\operatorname{Spm}(A) \times \operatorname{Spm}(B)$ are isomorphic. See Appendix, Section 3 and Exercise 19.

Let X and Y be affine varieties. Then $\big(X \times Y, \Bbbk[X] \otimes \Bbbk[Y]\big)$ is an affine variety, when we endow the set $X \times Y$ with the topology induced by the isomorphism $X \times Y = \operatorname{Spm}\big(\Bbbk[X] \otimes \Bbbk[Y]\big)$. This topology in general is *not* the product topology (see Exercise 19). Moreover, if X is an algebraic subset of $\mathbb{A}^n$ and Y of $\mathbb{A}^m$, we can consider in a natural way $X \times Y$ as a subset of $\mathbb{A}^{n+m}$ and as such it is also an affine algebraic set. In Exercise 19 we ask the reader to prove that in this case both structures of affine algebraic varieties coincide. In particular the element $\sum f_i \otimes g_i \in \Bbbk[X] \otimes \Bbbk[Y]$ can be viewed as the function on $X \times Y$ given by $\big(\sum f_i \otimes g_i\big)(x, y) = \sum f_i(x) g_i(y)$.

In this context, it is instructive to describe explicitly the topology on $X \times Y$. A basis for the topology of $X \times Y$ is given as follows: for arbitrary regular functions $f_1, \dots, f_n \in \Bbbk[X]$, $g_1, \dots, g_n \in \Bbbk[Y]$, define

$$U_{f_1,\dots,f_n,g_1,\dots,g_n} = \Big\{(x, y) \in X \times Y : \sum_{i=1}^{n} f_i(x) g_i(y) \neq 0\Big\}.$$

Then, the family of subsets $U_{f_1,\dots,f_n,g_1,\dots,g_n}$ is a basis for the topology of $X \times Y$. Indeed, if $\sum f_i \otimes g_i$ a generic element of $\Bbbk[X] \otimes \Bbbk[Y]$, it follows from Observation 4.23 that $(X \times Y)_{\sum f_i \otimes g_i}$ is isomorphic to the affine variety $\operatorname{Spm}\big(\Bbbk[X] \otimes \Bbbk[Y]\big)_{\sum f_i \otimes g_i}$. Moreover, it is clear that $(X \times Y)_{\sum f_i \otimes g_i} = U_{f_1,\dots,f_n,g_1,\dots,g_n}$.

Lemma 4.26. *Let X be an affine algebraic variety. Then the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$ is a morphism of affine varieties. Moreover, $\Delta(X)$ is closed in $X \times X$.*

Proof. The composition of a regular function $\alpha = \sum f_i \otimes g_i : X \times X \to \Bbbk$ with Δ yields the function $\alpha \circ \Delta = \sum f_i g_i : X \to \Bbbk$. Using Lemma 4.22 we conclude that Δ is a morphism of affine varieties. Moreover, the image of Δ can be described as $\Delta(X) = \mathcal{V}\big(\{f \otimes 1 - 1 \otimes f : f \in \Bbbk[X]\}\big)$. Indeed, the elements of $\Bbbk[X]$ separate the points of X (see Lemma 3.33); thus, given $(x, y) \in X \times X$ with $x \neq y$, there exists $f \in \Bbbk[X]$ such that $f(x) = 0$ and $f(y) \neq 0$. Then, $(f \otimes 1 - 1 \otimes f)(x, y) = f(x) - f(y) \neq 0$. ☐

4.4 Algebraic varieties

Definition 4.27. Assume that X is a topological space and that U and V are open subsets of X such that each of them supports a structure of affine algebraic $\Bbbk$-variety. We say that U and V are *compatible affine charts*, if for all $W \subset U \cap V$ open in X, then $\mathcal{O}_U(W) = \mathcal{O}_V(W) \subset \Bbbk^W$ (see Observation 4.19 and Definition 4.20).

Definition 4.28. Let X be a topological space. An *affine $\Bbbk$-atlas* for X — or simply an *affine atlas* — is a covering of X by open subsets U_i, $i \in I$, such that each U_i is equipped with a structure of affine algebraic $\Bbbk$-variety, in such a way that U_i and U_j are compatible for every $i, j \in I$. Two atlases are said to be *equivalent* if their union is also an atlas. A *finite atlas* is an atlas with a finite number of affine charts.

Lemma 4.29. *Let X be a topological space that admits an affine $\Bbbk$-atlas $\{U_i\}_{i \in I}$. There exists a unique sheaf of $\Bbbk$-algebras on X (denoted $\mathcal{O}_X$) such that $\mathcal{O}_X(U_i) = \mathcal{O}_{U_i}(U_i)$ for all $i \in I$. Moreover, if $x \in X$, then the stalk $\mathcal{O}_{X,x}$ is a local ring.*

Proof. Given an open subset $U \subset X$ we define $\mathcal{O}_X(U)$ as the $\Bbbk$-algebra of all the functions $f : U \to \Bbbk$ such that for all $i \in I$, $f\big|_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$.

It is clear that $\mathcal{O}_X$ is a sheaf, and it follows from the very definition that $\mathcal{O}_X(U_i) = \mathcal{O}_{U_i}(U_i)$.

The uniqueness is also clear and the assertion about the stalks follows from the fact that locally we are dealing with affine varieties whose stalks are local rings. ☐

Observation 4.30. (1) If there is no danger of confusion we omit the reference to the base field $\Bbbk$, and refer to *affine atlas and algebraic varieties* instead of *affine $\Bbbk$-atlas and algebraic $\Bbbk$-varieties.*

(2) If there is no danger of confusion we omit the subscript X in the structure sheaf of the algebraic variety and in the notation for the stalk. Hence the variety will be denoted as $(X, \mathcal{O})$ and the stalk as $\mathcal{O}_x$.

(3) It is important to observe (see the proof of the above lemma) that the structure sheaf is a subsheaf of the sheaf of continuous functions on the topological space X with values in $\Bbbk$. The continuity follows immediately from the local definition of the sheaf.

(4) The stalk $\mathcal{O}_x$ is also an augmented $\Bbbk$-algebra. The augmentation map is called $\varepsilon_x : \mathcal{O}_x \to \Bbbk$ and is the evaluation at x. The kernel of this augmentation map is the maximal ideal of $\mathcal{O}_x$ that is denoted as $\mathcal{M}_x$.

Observation 4.31. If X is a topological space which admits an affine atlas U_i, $i \in I$, then the covering $\{U_i : i \in I\}$ induces a covering $U_i \times U_j$ of $X \times X$, and thus the open subsets of the affine variety $U_i \times U_j$ are a basis for a topology in $X \times X$. For this topology, $U_i \times U_j$ is an affine atlas (see Exercise 19).

First we define prevarieties that are obtained by pasting together affine algebraic varieties. Then we add a “Hausdorff” separability condition to obtain the general definition of algebraic variety.

Definition 4.32. A structure of *algebraic $\Bbbk$-prevariety* on a topological space X is a equivalence class of finite $\Bbbk$-atlases. If the (set theoretical) diagonal morphism $\Delta : X \to X \times X$ has closed image (for the topology on $X \times X$ considered in Observation 4.31) we say that the above is a structure of *algebraic $\Bbbk$-variety.*

An algebraic $\Bbbk$-prevariety is a pair $(X, \mathcal{O}_X)$, where X is as above and $\mathcal{O}_X$ is the corresponding structure sheaf — similarly for an algebraic $\Bbbk$-variety.

Observation 4.33. If U_i, $i \in I$, is an atlas for the topological space X, it is easy to show that the preceding closedness condition is equivalent to the condition that for all $i, j \in I$, $\Delta(U_i \cap U_j)$ is closed in $U_i \times U_j$.

Observation 4.34. In the more general context of schemes, the condition of the diagonal being closed in the product is called the *separability* condition. Exercise 29 gives some insight on the way this condition is used in algebraic geometry; see also Lemma 2.2.7.

The main example of a non affine algebraic variety is the projective space.

Example 4.35. Let $n \in \mathbb{N}$, and consider in $\mathbb{A}^{n+1} \setminus \{0\}$ the equivalence relation defined as $x \sim y$ if and only if for some $\lambda \in \Bbbk^*$, $x = \lambda y$ — in geometric terms $x \sim y$ if and only if x and y belong to the same straight line through the origin.

The *projective space* $\mathbb{P}^n(\Bbbk)$ (or $\mathbb{P}(\Bbbk^n)$, or even $\mathbb{P}^n$) is defined (set theoretically) as the quotient $\big(\mathbb{A}^n \setminus \{0\}\big)/\sim$. It is customary to denote the equivalence class of $(x_0, \dots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ as $[x_0 : \cdots : x_n]$. If $V \cong \Bbbk^n$ is a finite dimensional $\Bbbk$-space, then $\mathbb{P}(V)$ is identified with $\mathbb{P}(\Bbbk^n)$. We endow $\mathbb{P}^n$ with the quotient topology.

To describe explicitly the topology of $\mathbb{P}^n$ first observe that even though for an arbitrary polynomial $p \in \Bbbk[X_0, \dots, X_n]$ we cannot evaluate it at a point in $\mathbb{P}^n$, if p is homogeneous, then the expression $p\big([a_0 : \cdots : a_n]\big) = 0$ is meaningful. In a similar way than for subsets of $\mathbb{A}^n$, we can define the map $\mathcal{V}$ from homogeneous ideals to subsets:

$$\mathcal{V}(I) = \big\{[a_0 : \cdots : a_n] \in \mathbb{P}^n : p_i\big([a_0 : \cdots : a_n]\big) = 0\,,\ i = 1, \dots, m\big\},$$

where $\{p_i : i = 1, \dots, m\}$ is a set of homogeneous generators of I.