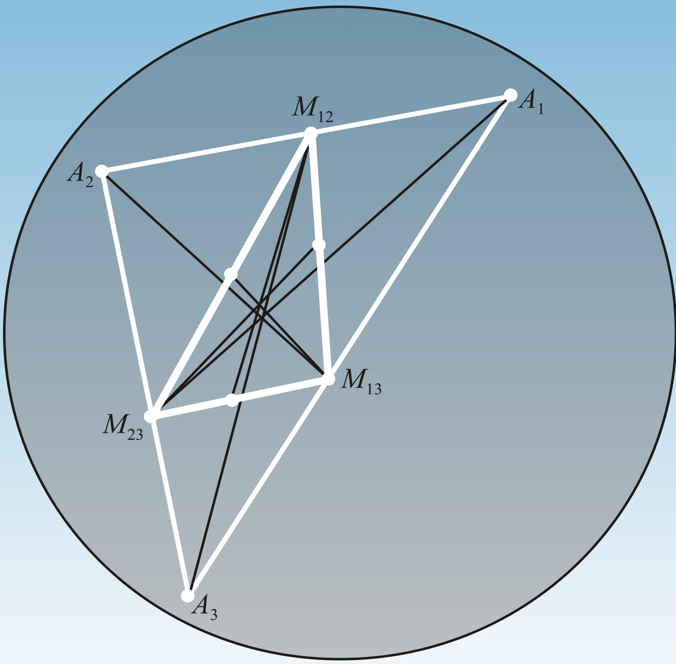
# Analytic Hyperbolic Geometry in *N* Dimensions

## AN INTRODUCTION

Abraham A. Ungar

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# Preface

Analytic hyperbolic geometry in $n$ dimensions is a new interdisciplinary subject between hyperbolic geometry of Lobachevsky and Bolyai and the special theory of relativity of Einstein. As Duncan MacLaren Young Sommerville (1879–1934) emphasized in his 1930 classic *An Introduction to the Geometry of N Dimensions*, when a geometry is extended to higher dimensions, one acquires both greater generality and greater succinctness in related expressions. In the book, the theory of Einstein's addition law of relativistically admissible velocities, extended to $n$ dimensions, is a rich playground for analytic hyperbolic geometry in $n$ dimensions. The book encourages researchers to cross traditional boundaries between hyperbolic geometry and special relativity theory.

It is natural to expect that important developments in science will come from interdisciplinary research. A merger of analytic hyperbolic geometry and special relativity theory stems from the author's two discoveries in the 1980s:

1. Einstein's addition law encodes rich structures that became known as a *gyrogroup* and a *gyrovector space*; and the resulting
2. Einstein gyrogroups and gyrovector spaces form the algebraic setting for the relativistic model (known as the Beltrami-Klein model) of $n$-dimensional hyperbolic geometry, just as groups and vector spaces form the algebraic setting for the standard model of $n$-dimensional Euclidean geometry.

The binary operation in Einstein gyrogroups and gyrovector spaces, which plays the role analogous to vector addition, is Einstein addition, which is neither commutative nor associative. Einstein addition, in turn, admits special automorphisms called *gyroautomorphisms* (or *gyrations,* in short), which come to the rescue. Indeed, gyroautomorphisms establish a formalism that remedies the breakdown of commutativity and associativity in gyrogroups and gyrovector spaces.

The book demonstrates that when special relativity theory and hyperbolic geometry meet, they cross-pollinate ideas from one area to the other. Techniques and tools from one area lead to advances in the other. Among outstanding examples found in the book are the topics listed in Items 1 and 2 below:

1. The hyperbolic counterparts of the following tools, commonly used in Euclidean geometry,

a) Cartesian coordinates;
b) barycentric coordinates;
c) trigonometry; and
d) vector algebra,

are adapted for use in hyperbolic geometry as well.

2. The hyperbolic counterparts of the following well-known theorems in Euclidean geometry:
   a) the Inscribed Angle Theorem;
   b) the Tangent–Secant Theorem;
   c) the Intersecting Secants Theorem; and
   d) the Intersecting Chords Theorem,

   are established in hyperbolic geometry as well.

Furthermore, (1) the relativistic effect known as *Thomas precession* and (2) the *relativistic mass* emerge in the book as relativistic concepts that possess a natural, crucially important hyperbolic geometric interpretation. Indeed,

1. *Thomas precession* is extended by abstraction to the *gyrator*, an operator that generates automorphisms called *gyrations*. Gyrations, in turn, capture remarkable analogies that Euclidean and hyperbolic geometry share. In fact, it is the incorporation of gyrations that turns Euclidean geometry into hyperbolic geometry, as demonstrated in the book.
2. *Relativistic mass* of particle systems suggests hyperbolic barycentric (gyrobarycentric) coordinates to be introduced as a tool into hyperbolic geometry, just as Newtonian mass of particle systems suggests barycentric coordinates to be introduced as a tool into Euclidean geometry. Moreover, the use of gyrobarycentric coordinates enables interesting results in hyperbolic geometry to be discovered, just as the use of barycentric coordinates enables interesting results in Euclidean geometry to be discovered.

Due to the novel analogies with vector addition that Einstein addition captures, the book provides a new look at Einstein's special relativity theory, an example of which is Einstein's addition law, which gives rise to a binary operation, $\oplus$, in the ball of all relativistically admissible velocities:

1. In the same way that vector addition is both commutative and associative, Einstein addition, $\oplus$, is both *gyrocommutative* and *gyroassociative*. Consequently,
2. in the same way that vector addition admits scalar multiplication that gives rise to vector spaces, Einstein addition admits scalar multiplication, $\otimes$, that gives rise to gyrovector spaces.

The resulting new looks at Einstein's special relativity theory are best illustrated by considering the following novel analogy between classical and relativistic kinetic energy that Einstein scalar multiplication captures:

1. Classically, the kinetic energy, $K_{cls}$, of a particle with mass $m$ that moves uniformly with velocity $\mathbf{v}$ relative to a rest frame $\Sigma_0$ is given by $K_{cls} = \frac{1}{2} m\mathbf{v}^2$, where $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v}$. It can be viewed as the inner product of the particle "*classical half-velocity*" $\frac{1}{2}\mathbf{v}$ and its classical momentum $m\mathbf{v}$, that is,

$$K_{cls} = \tfrac{1}{2}m\mathbf{v}^2 = (\tfrac{1}{2}\mathbf{v})\cdot(m\mathbf{v}). \qquad (*)$$

2. Relativistically, the kinetic energy, $K_{rel}$, of a particle with relativistic mass $m\gamma_{\mathbf{v}}$ that moves uniformly with relativistically admissible velocity $\mathbf{v}$ relative to a rest frame $\Sigma_0$ is given by the well-known equation $K_{rel} = c^2m(\gamma_{\mathbf{v}} - 1)$. Here, $c$ is the speed of light in empty space and $\gamma_{\mathbf{v}} = (1-\mathbf{v}^2/c^2)^{-1/2}$ is the Lorentz gamma factor of special relativity. Surprisingly, the relativistic kinetic energy $K_{rel}$ satisfies the identity (Sect. 3.2)

$$K_{rel} = c^2m(\gamma_{\mathbf{v}} - 1) = (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v}). \qquad (**)$$

   Identity (**) presents a new look at the relativistic kinetic energy. It enables the relativistic kinetic energy $K_{rel}$ of a particle to be viewed as the inner product of the particle "*relativistic half-velocity*" $\frac{1}{2}\otimes\mathbf{v}$ and its relativistic momentum $m\gamma_{\mathbf{v}}\mathbf{v}$, in full analogy with its classical counterpart in (*).

The analogies between Identities (*) and (**) that the relativistic scalar multiplication, $\otimes$, captures illustrate the new looks at Einstein's special relativity that the study of analytic hyperbolic geometry provides in the book.

Cayley-Menger matrices and determinants of order $(N + 1) \times (N + 1)$ are classically assigned to $(N - 1)$-simplices in higher dimensional Euclidean geometry. Hence, of particular interest are analogies with Cayley-Menger matrices that $N \times N$ gamma matrices and determinants, assigned to $(N - 1)$-hyperbolic-simplices in higher dimensional hyperbolic geometry are captured in the book. Remarkably, entries of a gamma matrix are gamma factors of special relativity.

The book demonstrates the power and elegance that emerge when Einstein's special theory of relativity, now a part of classical mechanics, is treated integrally with its underlying hyperbolic geometry. As such, the book creates interdisciplinarity in the research and in the teaching of hyperbolic geometry and special relativity, along with an algebraic language, called *gyrolanguage*, in which both hyperbolic geometry and special relativity find an aesthetically pleasing formulation.

The first chapter of the book is an introductory chapter. By presenting selected topics from the book, the introductory chapter describes the way analytic hyperbolic geometry evolves in the book from Einstein's velocity addition law. Each of the other chapters of the book ends with a set of exercises, some of which require the use of a computer algebra system, like Mathematica or Maple. Computer algebra is an indispensable tool in the book, allowing complicated algebraic manipulations to yield novel results that capture analogies with familiar results while taking on unexpected grace, elegance and simplicity. Indeed, the unexpected grace, elegance and simplicity that the analogies between Identities (*) and (**) exhibit, are just the tip of the giant iceberg of analogies with classical results that the book uncovers. Thus, putting hyperbolic geometry and special relativity together, the book produces a unified, analytic theory of enriched content.

It is assumed familiarity with Euclidean geometry from the point of view of vectors and with basic elements of linear algebra. Readers of this book are not required to have a prior acquaintance with either hyperbolic geometry, special relativity or nonassociative algebra.

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October, 2014

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# Author's Biography

Abraham Ungar is professor in the Department of Mathematics at North Dakota State University. After gaining his M.sc. from the Hebrew University in Pure Mathematics (1967) and Ph.D. from Tel-Aviv University in Applied Mathematics (1973), he held a postdoctoral position at the University of Toronto. His favored research areas are related to hyperbolic geometry and its applications in relativity physics. He currently serves on the editorial boards of *Journal of Geometry and Symmetry in Physics* and *Communications in Applied Geometry*.

# CHAPTER 1

# Introduction



## 1.1 Gyrovector Spaces in the Service of Analytic Hyperbolic Geometry

This introductory chapter indicates the role of analogies that Einstein's addition law of relativistically admissible velocities captures. The story of the book, unfolded here, begins in Chapter 2 with the introduction of a new look at Einstein addition and the way it gives rise to the novel algebraic structures known as *gyrogroups* and *gyrovector spaces*. The aim of this introductory chapter is to briefly illustrate the use of gyrovector spaces in the service of analytic hyperbolic geometry [118], in full analogy with the common use of vector spaces in the service of analytic Euclidean geometry.

## 1.2 When Two Counterintuitive Theories Meet

Hyperbolic geometry was introduced by Lobachevsky in 1829 and by Bolyai in 1832 as a counterintuitive geometry that denies the Euclid's postulate according to which there exists in the plane only one line parallel to a given line through a given point not on the line. Several decades later, Einstein introduced his special theory of relativity in 1905 [29, 30]. This physical theory is counterintuitive as well since, for instance, it implies that velocity addition is, in general, neither commutative nor associative.

The counterintuitive hyperbolic geometry of Lobachevsky and Bolyai, and the counterintuitive special relativity theory of Einstein were discovered independently. However, they met each other in 1908 when Varičak discovered that special relativity has a natural interpretation in hyperbolic geometry [130, 139, 140, 141].

In fact, we will see in the book that when hyperbolic geometry and special relativity meet, they cross-pollinate ideas from one area to the other, thus producing a novel way to study these two disciplines under the same umbrella. Techniques and tools in one area lead to advances in the other. Indeed,

1. Einstein addition law of relativistically admissible velocities encodes the novel algebraic structures known as a *gyrogroup* and a *gyrovector space*.
2. The resulting Einstein gyrovector spaces form the algebraic setting for hyperbolic geometry, just as vector spaces form the algebraic setting for Euclidean geometry. As such, they enable Cartesian and barycentric coordinates to be introduced into hyperbolic geometry. The mathematical tools that Cartesian and barycentric coordinates provide, commonly used in the study of Euclidean geometry, can now be used in the study of hyperbolic geometry as well.
3. Being the geometry that underlies special relativity, hyperbolic geometry, now equipped with Cartesian and barycentric coordinates, improves the study of special relativity, demonstrating the cross-fertilization of special relativity and hyperbolic geometry at work. Special attention is paid to the relativistic mass and to the relativistic effect called Thomas precession, since they play an important role in analytic hyperbolic geometry.

The resulting study of analytic hyperbolic geometry in $n$ dimensions thus begins with a new look at the Einstein velocity addition law that Einstein introduced in 1905. We employ *gyroalgebra*, the algebra that Einstein's relativistic velocity addition law encodes, to enrich, enliven, and enhance the study of analytic hyperbolic geometry. The sparkling beauty of Einstein's special relativistic velocity addition law manifests itself when it is placed in the framework of hyperbolic geometry, giving rise to the story of the book. The hyperbolic space that we use is the $s$-ball $\mathbb{R}^n_s$,

$$\mathbb{R}^n_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}, \tag{1.1}$$

$n = 1, 2, 3, \ldots$, of the Euclidean $n$-space $\mathbb{R}^n$, where $s$ is an arbitrarily fixed positive constant. In physical applications $n = 3$, but in geometry $n \geq 1$ is any positive integer.

Einstein's special relativity stems from his addition law of relativistically admissible velocities that he introduced in his 1905 paper that founded the theory. The resulting Einstein addition, $\oplus$, is a binary operation in the $s$-ball $\mathbb{R}^n_s$ of relativistically admissible velocities, which takes the vectorial form

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}. \tag{1.2}$$

Here (i) $s > 0$ is a constant that, when $n = 3$, represents the speed of light, $s = c$, in empty space, (ii) the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$ are $n$-dimensional relativistically admissible velocities, (iii) $\gamma_{\mathbf{u}}$ is the gamma factor of special relativity,

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}} \geq 1\,, \tag{1.3}$$

and (iv) $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm that the $s$-ball $\mathbb{R}_s^n$ inherits from its space $\mathbb{R}^n$. Einstein addition in the $s$-ball $\mathbb{R}_s^n$ thus gives rise to pairs $(\mathbb{R}_s^n, \oplus)$ known as Einstein *gyrogroups*. In Einstein gyrogroups we define $\ominus\mathbf{v} = -\mathbf{v}$, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{v}\oplus(-\mathbf{v}) = \mathbf{0}$, $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$, $\ominus\mathbf{u}\oplus\mathbf{v} = (-\mathbf{u})\oplus\mathbf{v}$ and $\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v}$. The formal definitions of the abstract gyrogroup and related algebraic structures are presented in Sect. 2.8 on the road from Einstein addition to gyrogroups.

In the non-relativistic limit, when $s$ approaches infinity, Einstein addition, $\oplus$, in $\mathbb{R}_s^n$ and ordinary vector addition, $+$, in $\mathbb{R}^n$ coalesce.

Here we have to remember that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. In 1905 [29], Einstein calculated the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (1.2) of Einstein addition.

Einstein addition underlies the Lorentz transformation of special relativity theory. Being neither commutative nor associative, Einstein addition, $\oplus$, is seemingly structureless, as opposed to the Lorentz transformation of special relativity, which enjoys the algebraic structure known as a *group*. As a result, much to Albert Einstein's chagrin [120], the pristine clarity of Einstein addition is obscured behind the cloud of Lorentz transformation. Einstein's intuition was, therefore, left dormant for about 80 years until it was brought back into a new mathematical life in 1988 in the author's article: "*The Thomas rotation formalism underlying a nonassociative group structure for relativistic velocities*" [112] and in its predecessor [111].

The pair $(\mathbb{R}_s^n, \oplus)$ is a *groupoid* in the sense that it is a nonempty set, $\mathbb{R}_s^n$, with a binary operation, $\oplus$, and an *automorphism* of the groupoid $(\mathbb{R}_s^n, \oplus)$ is a bijective (one-to-one) map $f$ of $\mathbb{R}_s^n$, $f: \mathbb{R}_s^n \to \mathbb{R}_s^n$, which respects its binary operation $\oplus$, that is, $f(\mathbf{a}\oplus\mathbf{b}) = f(\mathbf{a})\oplus f(\mathbf{b})$. The set of all automorphisms of any groupoid $(G, \oplus)$ forms a group, denoted $\mathrm{Aut}(G, \oplus)$, with group operation given by automorphism composition.

Being nonassociative, Einstein addition gives rise to automorphisms of the Einstein groupoids $(\mathbb{R}_s^n, \oplus)$, called *gyrations*, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. For each pair $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}_s^n \times \mathbb{R}_s^n$ the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}_s^n \to \mathbb{R}_s^n, \tag{1.4}$$

is an automorphism of $(\mathbb{R}_s^n, \oplus)$, given by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\}, \tag{1.5}$$

for any $\mathbf{w} \in \mathbb{R}_s^n$. Being automorphisms, the gyrations of an Einstein gyrogroup $(\mathbb{R}_s^n, \oplus)$ form a subset of the automorphism group $\mathrm{Aut}(\mathbb{R}_s^n, \oplus)$.

The *gyrator* gyr,

$$\mathrm{gyr} : \mathbb{R}_s^n \times \mathbb{R}_s^n \to \mathrm{Aut}(\mathbb{R}_s^n, \oplus), \tag{1.6}$$

is thus an operator that generates the special automorphisms, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, that we call *gyrations*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is, in general, nontrivial since the binary operation $\oplus$ is nonassociative. Note that in the special case when the binary operation $\oplus$ is associative, the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ in (1.5) is trivial, that is, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}_s^n$. Accordingly, gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ measure the extent to which the binary operation $\oplus$ deviates from associativity.

Moreover, Einstein addition is noncommutative, satisfying

$$\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}), \tag{1.7}$$

so that gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ measure the extent to which $\oplus$ deviates from commutativity as well.

## 1.3 The Fascinating Rich Mathematical Life of Einstein's Velocity Addition Law

Being neither commutative nor associative, Einstein addition is seemingly void of mathematical life. However, Einstein addition turns out to be both *gyrocommutative* and *gyroassociative*, signifying rich mathematical life, as the identities in (1.8) below indicate.

The gyrations to which Einstein addition gives rise in (1.4)–(1.6) regulate Einstein addition in a powerful and elegant way, giving rise to the following laws and properties for all relativistically admissible velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}, \mathbf{b} \in \mathbb{R}_s^n$:

$$\begin{array}{ll}
\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}) & \text{Gyrocommutative Law} \\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) = (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} & \text{Left Gyroassociative Law} \\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) & \text{Right Gyroassociative Law} \\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Gyration Left Reduction Property} \\
\mathrm{gyr}[\mathbf{u}, \mathbf{v}\oplus\mathbf{u}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Gyration Right Reduction Property} \\
\mathrm{gyr}[\ominus\mathbf{u}, \ominus\mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] & \text{Gyration Even Property} \\
(\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] & \text{Gyration Inversion Law} \\
\mathbf{a}\cdot\mathbf{b} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} & \text{Inner Product Gyroinvariance.}
\end{array} \tag{1.8}$$

The reduction properties of gyrations in (1.8) trigger a remarkable reduction in complexity, as we will see in Chapter 2. Following the algebraic properties in (1.8), Einstein addition can be interpreted as a peculiar vector addition in the $s$-ball $\mathbb{R}_s^n$, whose departure from commutativity and associativity is controlled by gyrations which, in turn, possess their own rich structure.

We thus see that Einstein addition, $\oplus$, is the gem of special relativity theory that cries out to be admired and studied with gyroalgebra theoretic techniques, as we do in the book.

The coincidences involved in the gyrocommutative-gyroassociative laws of Einstein addition in (1.8) cry for explanation and application. Indeed, explanation in terms of group theoretic techniques is available in [40, 41], and application is provided by the resulting *gyroalgebra* that we use extensively in the book. These coincidences are amazing, compelling the reader to ask: why? How can it be that the same gyration, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, that remedies the breakdown of commutativity in Einstein addition, remedies the breakdown of associativity in Einstein addition as well? Seeing the gyrocommutative-gyroassociative laws for the first time is like watching a magician pull a rabbit out of a hat. After studying the resulting gyrogroups and gyrovector spaces since 1988 [111, 112], the author still has that reaction.

Indeed, the mere introduction of gyrations turns Euclidean geometry into hyperbolic geometry, where Einstein addition is regulated by gyrations, playing the role of vector addition. Accordingly, Einstein addition is the hyperbolic analog of vector addition. It is more complex than vector addition, but much richer in structure.

As the reader has noted, in gyroalgebra we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix “gyro” stems from “gyration”, which is the mathematical abstraction of the special relativistic effect known as “Thomas precession”, studied in Chapter 13. The resulting group-like structure to which Einstein addition gives rise is thus naturally called a *gyrocommutative gyrogroup*. Interestingly, Einstein addition can be complexified, giving rise to *nongyrocommutative gyrogroups* [100].

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication, giving rise to Einstein *gyrovector spaces*. The latter, in turn, form the algebraic setting for the relativistic velocity model of hyperbolic geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

In order to extract Einstein scalar multiplication, $\otimes$, from Einstein addition, $\oplus$, let $k\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v} \ldots \oplus\mathbf{v}$ ($k$ terms) be the Einstein addition of $k$ copies of $\mathbf{v} \in \mathbb{R}^n_s$, defined inductively as

$$(k + 1)\otimes\mathbf{v} = \mathbf{v}\oplus(k\otimes\mathbf{v}), \qquad 1\otimes\mathbf{v} = \mathbf{v}, \tag{1.9}$$

for any $\mathbf{v} \in \mathbb{R}^n_s$. Then,

$$k\otimes\mathbf{v} = s\frac{\left(1+\dfrac{\|\mathbf{v}\|}{s}\right)^k - \left(1-\dfrac{\|\mathbf{v}\|}{s}\right)^k}{\left(1+\dfrac{\|\mathbf{v}\|}{s}\right)^k + \left(1-\dfrac{\|\mathbf{v}\|}{s}\right)^k}\frac{\mathbf{v}}{\|\mathbf{v}\|}\,, \tag{1.10}$$

as one can readily check.

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing $k$ off the positive integers, obtaining from (1.10) the Einstein scalar multiplication, $\otimes$. It is given by the equation

$$r\otimes\mathbf{v} = s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} = s\ \tanh(r\ \tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|}\,, \tag{1.11}$$

where $r$ is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$. Thus, for instance, the *Einstein half* is given by (3.20), p. 78,

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\,, \tag{1.12}$$

enabling one to recast the relativistic kinetic energy into a novel form that captures remarkable analogies with its classical counterpart, as shown in (3.22)–(3.23), p. 78.

The gyrodistance $d_\oplus(A, B)$ between two points $A, B \in \mathbb{R}_s^n$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is the *gyrolength* of the gyrovector $\ominus A\oplus B$,

$$d_\oplus(A, B) = \|\ominus A\oplus B\|, \tag{1.13}$$

illustrated in Fig. 1.1, just as the distance $d_+(A, B)$ between two points $A, B \in \mathbb{R}^n$ in a Euclidean space $\mathbb{R}^n$ is the length of the vector $-A + B$,

$$d_+(A, B) = \|-A + B\|, \tag{1.14}$$

illustrated in Fig. 1.2.

Interestingly, the gyrodistance function obeys the gyrotriangle inequality

$$d_\oplus(A, C) \le d_\oplus(A, B)\oplus d_\oplus(B, C) \tag{1.15}$$

for any $A, B, C \in \mathbb{R}_s^n$, just as the distance function obeys the triangle inequality

$$d_+(A, C) \le d_+(A, B) + d_+(B, C) \tag{1.16}$$

for any $A, B, C \in \mathbb{R}^n$.

Having Einstein addition and scalar multiplication in hand, we explore graphically in Fig. 1.1 the minimizing gyrolength curve

$$A\oplus(\ominus A\oplus B)\otimes t, \tag{1.17}$$

$t \in \mathbb{R}$, where $A$ and $B$ are two distinct points in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. The graph of the function of $t$ in (1.17) is a hyperbolic geodesic line, called a *gyroline*, shown in Fig. 1.1 for $0 \le t \le 1$. Figures. 1.1 and 1.2 indicate that the

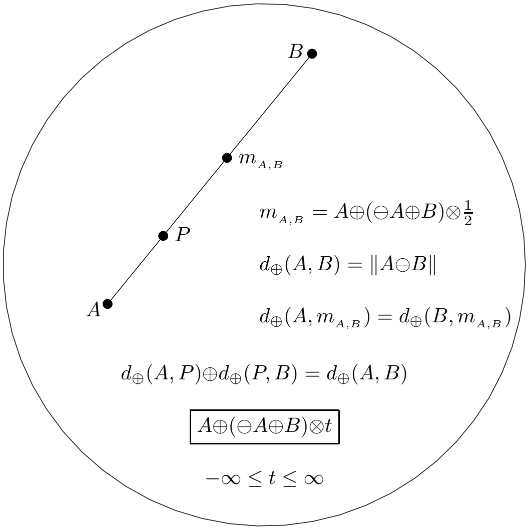


**Figure 1.1.** Gyroline, the hyperbolic line. The gyroline $L_{AB} = A \oplus (\ominus A \oplus B) \otimes t$, $t \in \mathbb{R}$, that passes through the points $A$ and $B$ in an Einstein gyrovector plane is a geodesic line in the Beltrami-Klein disk model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in a Euclidean plane. The points $A$ and $B$ correspond to $t = 0$ and $t = 1$, respectively. The point $P$ is a generic point on the gyroline through the points $A$ and $B$ lying between these points. The gyrosum, $\oplus$, of the gyrodistance from $A$ to $P$ and from $P$ to $B$ equals the gyrodistance from $A$ to $B$. The point $m_{A,B}$ is the gyromidpoint of the points $A$ and $B$, corresponding to $t = 1/2$. The analogies between lines and gyrolines, as illustrated in Figs. 1.2 and 1.1, are obvious.

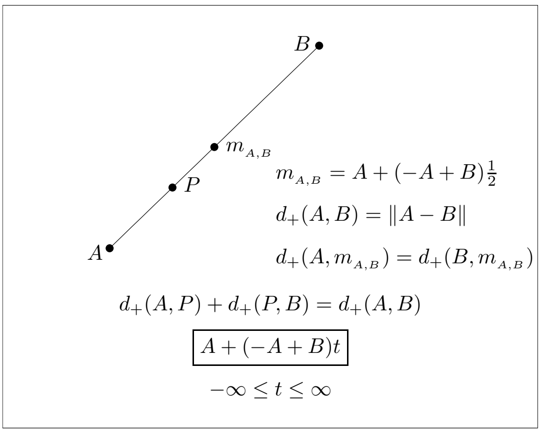


**Figure 1.2.** The Euclidean line. This figure depicts the vector space approach to the Euclidean line, and is presented as the Euclidean counterpart of Fig. 1.1. The line $A + (-A + B)t$, $t \in \mathbb{R}$, that passes through the points $A$ and $B$ in a Euclidean vector plane is shown. The points $A$ and $B$ correspond to $t = 0$ and $t = 1$, respectively. The point $P$ is a generic point on the line through the points $A$ and $B$ lying between these points. The sum, +, of the distance from $A$ to $P$ and from $P$ to $B$ equals the distance from $A$ to $B$. The point $m_{A,B}$ is the midpoint of the points $A$ and $B$, corresponding to $t = 1/2$.

gyroline (1.17) is fully analogous to its Euclidean counterpart, the minimizing length curve, which is the Euclidean straight line

$$A + (-A + B)t, \tag{1.18}$$

shown graphically in Fig. 1.2 for $0 \le t \le 1$.

The hyperbolic line (1.17) and its Euclidean counterpart (1.18) are presented graphically in Figs. 1.1 and 1.2 with respect to unseen Cartesian coordinates. The use of Cartesian coordinates in Euclidean geometry is common. Here we see that Einstein addition and scalar multiplication allow us to use Cartesian coordinates in hyperbolic geometry as well.

The analogies between lines and gyrolines, described symbolically in (1.18) and (1.17), and illustrated graphically in Figs. 1.2 and 1.1, are extended in the book to many other analogies including, in particular, analogies

1. between parameters of triangles and parameters of gyrotriangles, illustrated in Figs. 1.3 and 1.4;
2. between trigonometry and gyrotrigonometry, illustrated in Figs. 1.3 and 1.4, and studied in Chapter 7;

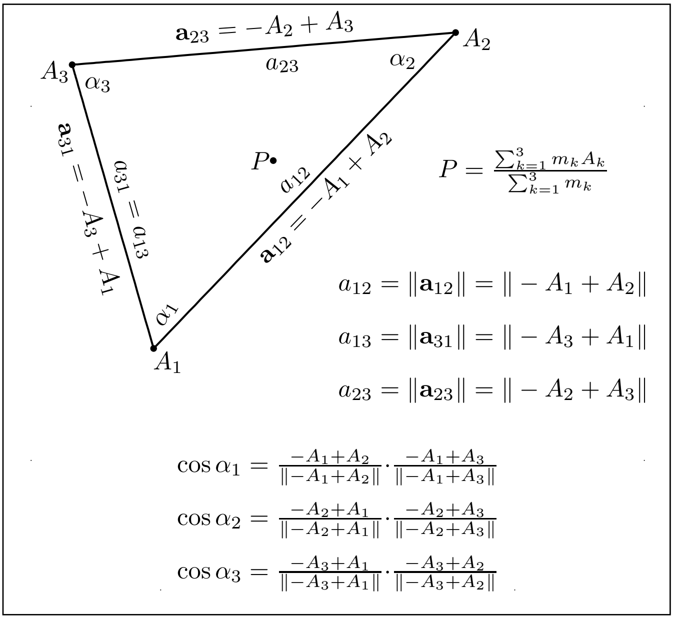


**Figure 1.3.** The index notation for triangle parameters. The barycentric coordinate representation of a generic point $P$ with respect to the reference triangle $A_1A_2A_3$ is shown, the barycentric coordinates of $P$ being $m_1$, $m_2$ and $m_3$. Trigonometry is the discipline that studies relationships between a triangle angles $\alpha_i$ and its side-lengths $a_{ij}$, $i, j = 1, 2, 3$, $i < j$. This figure sets the stage for its hyperbolic counterpart in Fig. 1.4.

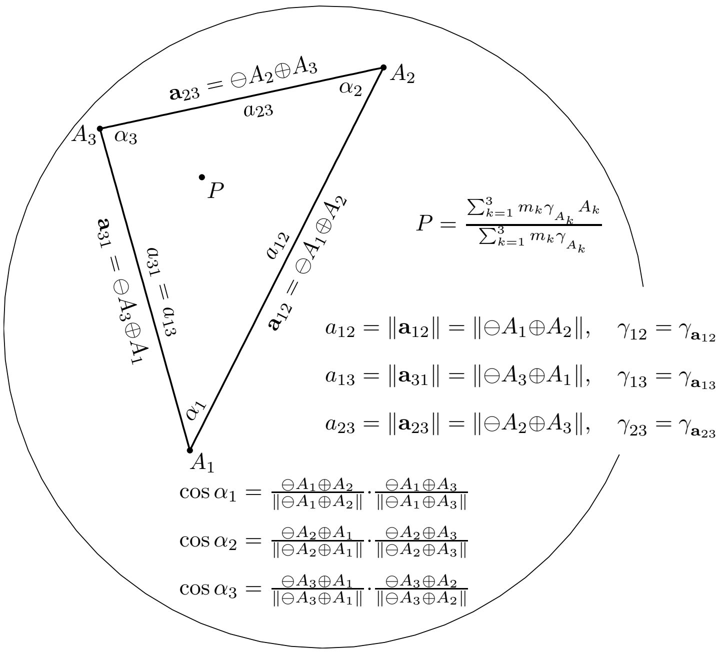


**Figure 1.4.** The index notation for gyrotriangle parameters. The gyrobarycentric coordinate representation of a generic point $P$ with respect to the reference gyrotriangle $A_1A_2A_3$ is shown, the gyrobarycentric coordinates of $P$ being $m_1$, $m_2$ and $m_3$. Gyrotrigonometry, introduced in Chapter 6, is the discipline that studies relationships between a gyrotriangle gyroangles $\alpha_i$ and its gyrosides $a_{ij}$, $i, j = 1, 2, 3$, $i < j$. Gamma factors $\gamma_{ij}$ of gyrosides play an important role.

3. between the parallelogram law of vector addition and the gyroparallelogram law of gyrovector addition, illustrated in Figs. 1.5 and 1.6; and
4. between barycentric coordinates and gyrobarycentric coordinates, studied in Chapter 5, and employed in Chapters 5–12.

The formal link between Einstein addition and the differential geometry that underlies the Beltrami-Klein model of the hyperbolic geometry of Lobachevsky and Bolyai is presented in Sect. 3.4.

Einstein addition, $\oplus$, in $\mathbb{R}^n_s$ comes with a dual binary operation, $\boxplus$ in $\mathbb{R}^n_s$, called *Einstein coaddition*, given by the equation

$$\mathbf{u} \boxplus \mathbf{v} = \mathbf{u} \oplus \mathrm{gyr}[\mathbf{u}, \ominus\mathbf{v}]\mathbf{v}. \tag{1.19}$$

Surprisingly, while Einstein addition is gyrocommutative, Einstein coaddition is commutative (and weakly associative in some general sense studied in Sect. 6.4). Additionally, while Einstein addition obeys the gyrotriangle inequality (1.15), Einstein coaddition obeys a cogyrotriangle inequality that involves a gyration, as shown in [129, Eq. (6.19), p. 158].

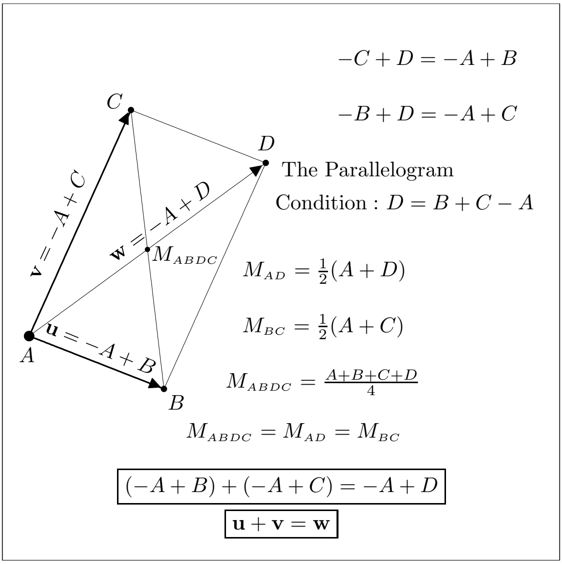


**Figure 1.5.** The Euclidean parallelogram and its addition law in a Euclidean vector plane $(\mathbb{R}^2, +, \cdot)$. The diagonals $AD$ and $BC$ of parallelogram $ABDC$ intersect each other at their midpoints. The midpoints of the diagonals $AD$ and $BC$ are, respectively, $M_{AD}$ and $M_{BC}$, each of which coincides with the parallelogram center $M_{ABDC}$. This figure shares obvious analogies with its hyperbolic counterpart in Fig. 1.6. As such, this figure sets the stage for Fig. 1.6.

The presence of Einstein coaddition in Einstein gyrovector spaces, along with the presence of Einstein addition, enables us to capture important analogies with classical results. Thus, for instance, Einstein addition obeys the following cancellation laws, two of which involve Einstein coaddition and cosubtraction:

$$\begin{aligned}
&\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} && \text{Left Cancellation Law} \\
&(\mathbf{u} \boxplus \mathbf{v})\ominus\mathbf{v} = \mathbf{u} && \text{First Right Cancellation Law} \\
&(\mathbf{u}\oplus\mathbf{v}) \boxminus \mathbf{v} = \mathbf{u} && \text{Second Right Cancellation Law.}
\end{aligned} \tag{1.20}$$

Remarkably, in particular, Einstein coaddition allows us to capture analogies between the common parallelogram law in Euclidean geometry and its hyperbolic counterpart, the gyroparallelogram law, illustrated in Figs. 1.5 and 1.6.

Gyroparallelograms are hyperbolic parallelograms. At first glance, the term *hyperbolic parallelogram* sounds as a contradiction in terms, since parallelism is denied in hyperbolic geometry. However, there is no need to employ parallelism in the definition of hyperbolic parallelograms. A hyperbolic parallelogram, called a gyroparallelogram, is a gyroquadrangle the two gyrodiagonals of which intersect at their gyromidpoints, just as a Euclidean parallelogram is a quadrangle the two diagonals of which intersect at their midpoints.

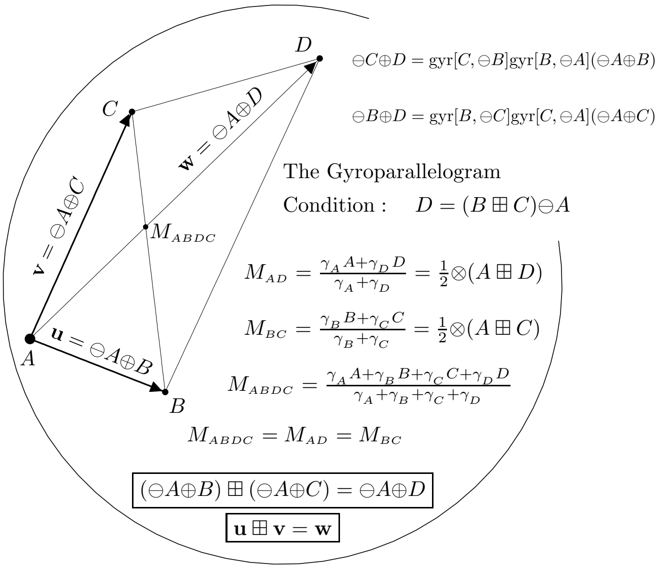


**Figure 1.6.** The Einstein gyroparallelogram law of gyrovector addition. Let $A, B, C \in \mathbb{R}^n_s$ be any three points of an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, giving rise to the two gyrovectors $\mathbf{u} = \ominus A\oplus B$ and $\mathbf{v} = \ominus A\oplus C$. Furthermore, let $D$ be a point of the gyrovector space such that $ABDC$ is a gyroparallelogram, that is, $D = (B \boxplus C)\ominus A$ by Def. 6.2, p. 174, of the gyroparallelogram. Then, Einstein coaddition of gyrovectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \boxplus \mathbf{v} = \mathbf{w}$, expresses the gyroparallelogram law, where $\mathbf{w} = \ominus A\oplus D$. Einstein coaddition, $\boxplus$, thus gives rise to the gyroparallelogram addition law of Einsteinian velocities, which is commutative and fully analogous to the parallelogram addition law of Newtonian velocities. Einsteinian velocities are, thus, gyrovectors that add according to the gyroparallelogram law just as Newtonian velocities are vectors that add according to the parallelogram law. Like vectors, a gyrovector $\ominus A\oplus B$ in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $n = 2, 3$, is described graphically as a straight arrow from the tail $A$ to the head $B$ with gyrolength $\|\ominus A \oplus B\|$.

For the sake of comparison with its hyperbolic counterpart in Fig. 1.6, Fig. 1.5 depicts the well-known *parallelogram law* of vector addition,

$$(-A + B) + (-A + C) = (-A + D). \tag{1.21}$$

In Fig. 1.5 we see arbitrarily selected three noncollinear points $A, B, C \in \mathbb{R}^2$, together with a fourth point $D \in \mathbb{R}^2$, which satisfies the *parallelogram condition*, $D = B + C - A$. The parallelogram condition insures that quadrangle $ABDC$ is a parallelogram (that is, the two diagonals of $ABDC$ intersect at their midpoints). In parallelogram $ABDC$ three vectors emanate from vertex $A$. These are the two side vectors $\mathbf{u} = -A + B$ and $\mathbf{v} = -A + C$ and the diagonal vector $\mathbf{w} = -A + D$. The diagonal vector turns out to be the resultant of the two side vectors, given by the parallelogram law (1.21).

Fig. 1.6 is fully analogous to Fig. 1.5. It depicts the *gyroparallelogram law* of gyrovector addition,

$$(\ominus A \oplus B) \boxplus (\ominus A \oplus C) = (\ominus A \oplus D). \tag{1.22}$$

In Fig. 1.6 we see arbitrarily selected three nongyrocollinear points $A, B, C \in \mathbb{R}_s^2$ (that is, the points $A, B, C$ do not lie on the same gyroline), together with a fourth point $D \in \mathbb{R}_s^2$, which satisfies the *gyroparallelogram condition*, $D = (B \boxplus C) \ominus A$. The gyroparallelogram condition insures that gyroquadrangle $ABDC$ is a gyroparallelogram (that is, the two gyrodiagonals of $ABDC$ intersect at their gyromidpoints). In gyroparallelogram $ABDC$ three gyrovectors emanate from vertex $A$. These are the two side gyrovectors $\mathbf{u} = \ominus A \oplus B$ and $\mathbf{v} = \ominus A \oplus C$ and the gyrodiagonal gyrovector $\mathbf{w} = \ominus A \oplus D$. The gyrodiagonal gyrovector $\ominus A \oplus D$ in the gyroparallelogram turns out to be the gyroresultant of the two side gyrovectors, given by the gyroparallelogram law (1.22).

The parallelogram law for the composition of (Newtonian) velocities was known to the ancients, traditionally ascribed to Aristotle [26, p. 21],[67]. Einstein addition captures the notion of the Einsteinian velocity vector, $\mathbf{u} = \ominus A \oplus B$, called a velocity gyrovector. Interestingly, it is Einstein coaddition that captures the hyperbolic parallelogram (gyroparallelogram) law, $\mathbf{u} \boxplus \mathbf{v}$, for the composition of Einsteinian velocity gyrovectors. Experimental evidence that supports the physical significance of Einstein gyroparallelogram law of velocity addition is provided by the relativistic interpretation of the cosmological stellar aberration phenomenon, as explained in Sect. 6.3 and, in detail, in [129, Chapter 13].

In Euclidean geometry, the extension of the parallelogram law of addition of two vectors to a parallelotope law of addition of more than two vectors is obvious. In hyperbolic geometry, however, the extension of the gyroparallelogram law of addition of two gyrovectors to a gyroparallelotope law of addition of more than two gyrovectors, presented in Chapter 6, is challenging and interesting, demonstrating the power and elegance of gyroalgebra. Thus, after over more than two decades of development, since 1988 [112], gyroalgebra has been proved to be an important tool in the study of analytic hyperbolic geometry.

The invention of Cartesian coordinates in the 17th century by René Descartes (Latinized name: Cartesius) (1596–1650) and Pierre de Fermat (1601 or 1607/8–1665) revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra. Using the resulting standard Cartesian model of Euclidean geometry, geometric shapes are described by algebraic equations involving the Cartesian coordinates of the points lying on the shape. The standard Cartesian model of Euclidean geometry is the foundation of analytic Euclidean geometry, where Cartesian coordinates play the role of a tool, allowing geometric content expressed through them to be studied algebraically [13]. Studying Euclidean geometry by its Cartesian model has the advantage of having the whole mathematical machinery of algebra and calculus to hand. The task of reviving interest in hyperbolic geometry by the adaptation of Cartesian coordinates for use in that geometry, resulting in *analytic hyperbolic geometry*, has thus begun with the appearance of the author's books since 2001.

According to Klein's 1871 paper (an English translation of which is available in [106, pp. 69–111]), non-Euclidean geometry was encountered by Gauss, who coined this term, by Lobachevsky (1829) and by Bolyai (1832). The term *hyperbolic geometry* for non-Euclidean geometry was coined by Klein in his 1871 paper. About 75 years later, in 1905, Einstein discovered the special theory of relativity [29, 30]. Soon later, the link between Einstein's special theory of relativity and hyperbolic geometry was discovered and developed during the period 1908–1912 by Varičak, Robb, Wilson and Lewis, and Borel [143]. The subsequent major development that followed 1912 appeared about 80 years later, in 2001 [119].

Following the emergence of gyroalgebra since 1988 [111, 112, 113], the author has crafted gyrolanguage, the algebraic language that sheds natural light on hyperbolic geometry and special relativity, in several books [119, 122, 129, 131, 133, 134], [144, 89]. Several authors have successfully employed gyroalgebra in their explorations, for instance, [2, 3, 4, 5, 87, 99], [24, 25], [32], [33, 34, 35, 36], [86], [104], [66,80,147], noting in [16, p. 523] that the computation language that Einstein addition encodes plays a universal computational role, which extends far beyond the domain of special relativity.

Euclidean geometry is very different from hyperbolic geometry, so that it was not clear before 1988 that lessons from Euclidean geometry would routinely translate into hyperbolic geometry. About a quarter century later, the gamble has paid off owing to the gyrovector space structure that Einstein addition encodes. It is now clear that the Einstein gyrovector space approach to relativistic hyperbolic geometry is fully analogous to the standard vector space approach to Euclidean geometry. The resulting analogies allow, in particular, the adaptation of tools that are commonly used in Euclidean geometry for use in hyperbolic geometry as well.

According to Leo Corry [19], Einstein considered Minkowski's reformulation of his special relativity theory in terms of four-dimensional spacetime to be no more than "*superfluous erudition*". Einstein could have made a better case for his program to adopt his three-dimensional relativistic velocity addition law as the primitive notion of special relativity (rather than the Lorentz transformation group), had he but known of the fascinating rich mathematical life that his velocity addition law possesses.

## 1.4 Matrices Assigned to Simplices and to Gyrosimplices

The index notation for triangles and gyrotriangles in Figs. 1.3 and 1.4 is naturally extended to higher dimensional simplices and gyrosimplices. In the study of higher dimensional simplices it proves useful to assign to each $(N-1)$-simplex $A_1 \dots A_N$ the so called $(N+1)\times(N+1)$ Cayley–Menger matrix $M_N$, [38, Sect. 1.4], [10, Sect. 9.7.3], (10.462), p. 462,

$$M_N = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_{12}^2 & \dots & a_{1N}^2 \\ 1 & a_{12}^2 & 0 & \dots & a_{2N}^2 \\ \vdots & & & \ddots & \\ 1 & a_{1N}^2 & a_{2N}^2 & \dots & 0 \end{pmatrix}, \tag{1.23}$$

along with its Cayley–Menger determinant, Det $M_N$, where $a_{ij}^2 = \|-A_i + A_j\|^2$. Here we use the notation illustrated in Fig. 1.3.

Analogously, in the study of higher dimensional gyrosimplices it proves useful to assign to each $(N-1)$-gyrosimplex $A_1 \dots A_N$ the so called $N \times N$ gamma matrix $\Gamma_N$, (10.40), p. 378,

$$\Gamma_N = \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1N} \\ \gamma_{12} & 1 & \gamma_{23} & \cdots & \gamma_{2N} \\ \vdots & & & \ddots & \\ \gamma_{1N} & \gamma_{2N} & \gamma_{3N} & \cdots & 1 \end{pmatrix}, \tag{1.24}$$

along with its gamma determinant, Det $\Gamma_N$, where $\gamma_{ij} = \gamma_{a_{ij}} = \gamma_{\|\ominus A_i \oplus A_j\|}$. Here we use the notation illustrated in Fig. 1.4.

On first glance it seems that the two determinants, Det $M_N$ and Det $\Gamma_N$, share no analogies between Euclidean and hyperbolic geometry that justify viewing each of them as the counterpart of the other one. Surprisingly, however, by Theorem 10.50, p. 463, it turns out that the Cayley–Menger determinant Det $M_N$, commonly used in the study of higher dimensional Euclidean geometry, is in some sense the Euclidean limit of the gamma determinant Det $\Gamma_N$, which we use in the study of higher dimensional hyperbolic geometry. Indeed, by (10.468), p. 463,

$$\lim_{s\to\infty} s^{2(N-1)} \mathrm{Det}\,\Gamma_N = -\frac{1}{2^{N-1}} \mathrm{Det}\, M_N. \tag{1.25}$$

Accordingly, the gamma determinant, Det $\Gamma_N$, that we use in the study of higher dimensional hyperbolic geometry is the hyperbolic counterpart of the well-known Cayley–Menger determinant, Det $M_N$. Yet, undoubtedly, our gamma matrix $\Gamma_N$ appears to be more elegant than its Euclidean counterpart, the Cayley–Menger matrix $M_N$. By discovering the hyperbolic counterpart of Cayley–Menger determinant, we pave the road to the study of analytic hyperbolic geometry in $n$ dimensions, guided by analogies with the common study of analytic Euclidean geometry in $n$ dimensions.

Owing to the advantage of the use of $\Gamma_N$ in hyperbolic geometry over the use of $M_N$ in Euclidean geometry it is sometimes easy to solve a difficult problem in

Euclidean geometry by solving the analogous problem in hyperbolic geometry. A point in case is the problem of determining the barycentric coordinate representation of the tetrahedron circumcenter. The gyrobarycentric coordinate representation of the circumgyrocenter of any $(N-1)$-gyrosimplex, $N \geq 3$, is determined in Theorem 10.18, p. 396. The special case when $N = 4$ (gyrotetrahedron) is presented earlier, in Sect. 10.1. The barycentric coordinate representation of the tetrahedron circumcenter is not determined directly. Rather, it is extracted from the gyrobarycentric coordinate representation of the gyrotetrahedron circumgyrocenter.

## 1.5 Parts of the Book

The book is self-contained. The required background in the theory of gyrogroups and gyrovector spaces and in gyrotrigonometry is presented in Parts I and II. More about these topics is found in [122, 129, 133, 134].

The book is divided into six parts:

1. **Part I: Einstein Gyrogroups and Gyrovector Spaces.** The first part of the book reveals the emergence of mathematical beauty and regularity that results from decoding the algebraic structures that the Einstein relativistic velocity addition law encodes. Part I of the book, Chapters 2–4, presents the Einstein velocity addition law of special relativity theory, revealing the novel algebra, called *gyroalgebra*, that it encodes. The resulting gyroalgebra stems from the notions of
   a) the *gyrogroup*, which is a natural generalization of the group concept in algebra; and
   b) the *gyrovector space*, which is a natural generalization of the vector space concept in algebra.

   It is demonstrated that gyroalgebra regulates Einstein addition and, hence, sheds a natural light on the special relativity theory of Einstein and on its underlying hyperbolic geometry of Lobachevsky and Bolyai. As such, gyroalgebra is used extensively in the book in the study of analytic hyperbolic geometry in $n$ dimensions.
2. **Part II: Mathematical Tools for Hyperbolic Geometry.** Part II of the book, Chapters 5–7, presents the adaptation of classical tools that are commonly used in Euclidean geometry for use in hyperbolic geometry. Specifically, the classical tools are:
   a) Cartesian coordinates (in Euclidean geometry);
   b) Barycentric coordinates;
   c) trigonometry; and
   d) vector algebra,

and their respective hyperbolic counterparts are:

a) Cartesian coordinates (in hyperbolic geometry);
b) gyrobarycentric coordinates;
c) gyrotrigonometry; and
d) gyrovector gyroalgebra.

3. **Part III: Hyperbolic Triangles and Circles.** Part III of the book, Chapters 8–9, employs the tools developed in Part II for the discovery of properties of hyperbolic triangles (gyrotriangles) and hyperbolic circles (gyrocircles). Several important, well-known results in Euclidean geometry are translated into corresponding results in hyperbolic geometry. Thus, for instance,

   a) the Inscribed Angle Theorem;
   b) the Tangent–Secant Theorem, p. 319;
   c) the Intersecting Secants Theorem, p. 320; and
   d) the Intersecting Chords Theorem, p. 359,

   are translated into their counterparts in hyperbolic geometry. The resulting counter-part theorems in hyperbolic geometry, respectively, are:

   a) the Inscribed Gyroangle Theorem, p. 304, 305;
   b) the Gyrotangent–Gyrosecant Theorem, p. 313, 318;
   c) the Intersecting Gyrosecants Theorem, p. 319; and
   d) the Intersecting Gyrochords Theorem, p. 358.

4. **Part IV: Hyperbolic Simplices, Hyperplanes and Hyperspheres in *n* Dimensions.** In Part IV of the book, Chapters 10–11, the gyrosimplex (hyperbolic simplex) is the extension of the gyrotriangle and the gyrotetrahedron to higher dimensions. Based on experience about gyrotriangles and gyrotetrahedra studied in previous parts of the book, this part presents the study of the gyrosimplex circumgyrohypersphere, along with its circumgyrocenter and circumgyroradius in higher dimensions, $n \geq 2$. Special attention is paid to the gyrotetrahedron in Chapter 11.
5. **Part V: Hyperbolic Ellipses and Parabolas.** Part V of the book, Chapter 12, employs the tools developed in Part II for the discovery of properties of hyperbolic ellipses (gyroellipses) and hyperbolic parabolas (gyroparabolas).
6. **Part VI: Thomas Precession.** Gyrations play an important role, enabling analogies that hyperbolic and Euclidean geometry share to be captured. The gyration, in turn, is a mathematical abstraction of the special relativistic effect known as *Thomas precession*. Therefore, Part VI of the book, Chapter 13, is devoted to the study of Thomas precession and its frequency in the framework of special relativity theory and its underlying hyperbolic geometry. Accordingly, this part of the book illustrates the physical background of gyrations in hyperbolic geometry, and the usefulness of the study of special relativity theory and hyperbolic geometry under the same umbrella.

The study of special relativity theory and hyperbolic geometry under the same umbrella is rewarding. It reveals, for instance, that the Einstein relativistic, velocity dependent mass conforms with the Minkowskian formalism of special relativity theory, as explained in Chapter 4. The relativistic, velocity dependent mass plays an important role since it enables the adaptation of barycentric coordinates, commonly used in Euclidean geometry, for use in hyperbolic geometry.

The study of analytic hyperbolic geometry in $n$ dimensions, guided by analogies with classical results, thus begins with the study of Einstein gyrogroups and gyrovector spaces in Part I.

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# PART I

# Einstein Gyrogroups and Gyrovector Spaces

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# CHAPTER 2

# Einstein Gyrogroups



## 2.1 Introduction

Einstein's addition law of three-dimensional relativistically admissible velocities is the corner stone [125] of Einstein's three-vector formalism of the special theory of relativity that he founded in 1905 [29, 71]. The resulting binary operation, $\oplus$, called Einstein addition, is employed along with the nonassociative algebraic structures that it encodes. These algebraic structures are the gyrocommutative gyrogroup structure, studied in this chapter, and the gyrovector space structure, studied in Chapter 3. It will turn out that Einstein gyrovector spaces form the algebraic setting for the $n$-dimensional Cartesian-Beltrami-Klein ball model of analytic hyperbolic geometry, just as vector spaces form the algebraic setting for the standard $n$-dimensional Cartesian model of analytic Euclidean geometry.

Connections between the theory of relativity of Einstein and the hyperbolic geometry of Lobachevsky and Bolyai were encountered even before the introduction of the theory of relativity by Einstein in 1905. Owing mainly to the work of Tibor Toró, cited in [62], it is now known that János Bolyai was the forerunner of geometrizing physics. According to Kiss [62], Lajos Dávid drew attention in a 1924 series of articles in Italian journals to the precursory role which János Bolyai played in the constructions of Einstein's relativity theory.

According to A.I. Miller [79, p. 266], one of the first demonstrations that non-Euclidean geometry could be used to present concisely results of relativity theory was obtained by Sommerfeld in 1909 [101] when he was led to the result that relativistically admissible velocities add according to a spherical geometry. Sommerfeld's 1909 work is described by Rosenfeld in his book [94, pp. 270–273]:

> Although Sommerfeld established the connections between the formula for the addition of velocities in the theory of relativity and the trigonometric formulas for hyperbolic functions he was not aware [in 1909; but, see our next quotation] that these formulas are formulas of Lobačevskian geometry. This was shown by the Yugoslav geometer Vladimir Varičak (1865–1942) . . .

From Varičak's acknowledgment of Sommerfeld's 1909 paper [101] it appears that there was a causal link between the latter paper and Varičak's 1910 discovery in [140] of the role that hyperbolic geometry plays in special relativity theory. Thus, it was Sommerfeld's 1909 paper that sparked Varičak's non-Euclidean program for special relativity; see [94, p. 270]. Ironically, however, not only did Sommerfeld employ an imaginary temporal coordinate, following the space-time formalism of Minkowski, he deplored the non-Euclidean style in print, as Walter noted in [143, p. 114]:

> ... just after Varičak's first exposé of the non-Euclidean style ([140], 1910), Sommerfeld completed his signal work on the four-dimensional vector calculus for the *Annalen der Physik*. In a footnote to his work, Sommerfeld remarked that the geometrical relations he presented in terms of three real and one imaginary coordinate could be reinterpreted in terms of non-Euclidean geometry. The latter approach, Sommerfeld cautioned in [102, p. 752], could "hardly be recommended".

Furthermore, Walter notes in [143] that following the competition between the two geometrical approaches to relativity physics:

> Minkowski neither mentioned the [Einstein] law of velocity addition, nor expressed it in formal terms.

Instead, however [143],

> Minkowski retained the geometric interpretation of the Lorentz transformations that had accompanied the *now-banished* non-Euclidean interpretation of velocity vectors. [italics added].

The trend initiated by Minkowski continues today, with the full Einstein addition and its associated Thomas precession receiving scant attention, and modern texts on relativity physics reflect this with the only single, outstanding exception being the book of Sexl and Urbantke [96], along with the forerunners [119, 122, 129, 131, 133, 134], of the present book.

Being neither commutative nor associative, Einstein addition is seemingly structureless, as opposed to Lorentz transformations, which form a transformation group. The resulting almost forgotten attempt of the famous mathematician É mile Borel to "repair" the seemingly "defective" Einstein's velocity addition law in the years following 1912 is described by Walter in [143, p. 117]: "Borel could construct a tetrahedron in kinematic space, and determined thereby both the direction and magnitude of relative [composite] velocity in a symmetric manner." Borel has, thus, "repaired" the breakdown of commutativity in Einstein addition, but did not pay attention to the breakdown of associativity in Einstein addition. Accordingly, it seemed appropriate to consider the Lorentz transformation, rather than Einstein addition, as a primitive notion in special relativity.

However, in 1988 it was discovered in [111, 114, 115] that Einstein addition encodes rich noncommutative and nonassociative algebraic structures. Following the 1988 discovery, it is now rewarding to consider Einstein addition, rather than Lorentz transformation, as a primitive notion in special relativity, from which the Lorentz transformation is derived.

Soon after its introduction by Einstein in 1905 [29] special relativity theory, as named by Einstein ten years later, became overshadowed by the appearance of general relativity. Subsequently, the study of special relativity followed the lines laid down by Minkowski, in which the role of Einstein velocity addition is ignored. Following Minkowski, therefore, the general Einstein velocity addition law of relativistically admissible velocities that need not be parallel is unheard of in most texts on special and general relativity theory. Rather, it is only the special case of Einstein addition, corresponding to parallel velocities, which is presented. Among outstanding exceptions we note the relativity physics books by Fock [39] and by Sexl and Urbantke [96].

## 2.2 Einstein Velocity Addition

Let $s > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean $n$-space, $n = 1, 2, 3, \ldots$, equipped with the common vector addition, $+$, and inner product, $\cdot$. The home of all $n$-dimensional Einsteinian velocities is the $s$-ball

$$\mathbb{R}_s^n = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}. \tag{2.1}$$

The $s$-ball $\mathbb{R}_s^n$ is the open ball of radius $s$, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than $s$.

Einstein velocity addition is a binary operation, $\oplus$, in the $s$-ball $\mathbb{R}_s^n$ given by the equation [119], [96, Eq. 2.9.2], [83, p. 55], [39],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}, \tag{2.2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, where $\gamma_{\mathbf{u}}$ is the Lorentz gamma factor,

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}} \geq 1, \tag{2.3}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}_s^n$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. A nonempty set with a binary operation is called a *groupoid* so that the pair $(\mathbb{R}_s^n, \oplus)$ is an *Einstein groupoid*.

In analytic hyperbolic geometry the parameter $s > 0$ plays the role of the vacuum speed of light, $c$, in special relativity theory. In the Euclidean-Newtonian limit of large $s$, $s \to \infty$, the ball $\mathbb{R}_s^n$ expands to the whole of its space $\mathbb{R}^n$, as we see from (2.1), and Einstein addition $\oplus$ in $\mathbb{R}_s^n$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (2.2) and (2.3).

When the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{R}_s^n$ of $\mathbb{R}^n$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, Einstein addition (2.2) reduces to the Einstein addition of parallel velocities,

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{1}{s^2}\mathbf{u}\cdot\mathbf{v}}, \qquad \mathbf{u}\|\mathbf{v}, \tag{2.4}$$

which was partially confirmed experimentally by the Fizeau's 1851 experiment [79]. Following (2.4) we have, for instance,

$$\|\mathbf{u}\|\oplus\|\mathbf{v}\| = \frac{\|\mathbf{u}\|+\|\mathbf{v}\|}{1+\frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|} \tag{2.5}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$.

The *restricted Einstein addition* in (2.4) and (2.5) is both commutative and associative. Accordingly, the restricted Einstein addition is a group operation, as Einstein noted in [29]; see [30, p. 142]. In contrast, Einstein made no remark about group properties of his addition (2.2) of velocities that need not be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure discovered more than 80 years later, in 1988 [111, 112, 115], which we will study in Sect. 2.8.

Einstein addition (2.2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [29] [30, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2.2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [29] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2.2) of Einstein addition. Einstein was aware of the nonassociativity of his velocity addition law of relativistically admissible velocities that need not be collinear. He therefore emphasized in his 1905 paper that his velocity addition law of relativistically admissible collinear velocities forms a group operation [29, p. 907].

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$ and

$$\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}. \tag{2.6}$$

Einstein addition and subtraction satisfy the equations

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{2.7}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{2.8}$$

for all $\mathbf{u}$, $\mathbf{v}$ in the ball $\mathbb{R}_s^n$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (2.7) is called the *gyroautomorphic inverse property* of Einstein addition, and Identity (2.8) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (2.8) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u}. \tag{2.9}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired in (2.112), p. 49.

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right), \tag{2.10}$$

which can be written, equivalently, as

$$\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right) \tag{2.11}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. Here, (2.11) is obtained from (2.10) by replacing $\mathbf{u}$ by $\ominus\mathbf{u} = -\mathbf{u}$ in (2.10).

A frequently used identity that follows immediately from (2.3) is

$$\frac{\mathbf{v}^2}{s^2} = \frac{\|\mathbf{v}\|^2}{s^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \tag{2.12}$$

and useful identities that follow immediately from (2.10)–(2.11) are

$$\frac{\mathbf{u}\cdot\mathbf{v}}{s^2} = -1 + \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = 1 - \frac{\gamma_{\ominus\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}. \tag{2.13}$$

It is the gamma identity (2.10) that signaled the emergence of the link between hyperbolic geometry and special relativity. It was first studied by Sommerfeld [101] and Varičak [139, 140] in terms of *rapidities*, a term coined by Robb [93]. Indeed, if we replace the velocity parameter $\mathbf{v}/c$ by the parameter $\phi_{\mathbf{v}}$, called *rapidity*,

$$\phi_{\mathbf{v}} = \tanh^{-1} \frac{\|\mathbf{v}\|}{s}, \tag{2.14}$$

then the gamma factor $\gamma_{\mathbf{v}}$ of $\mathbf{v} \in \mathbb{R}_s^n$ is related to the rapidity $\phi_{\mathbf{v}}$ of $\mathbf{v}$ by

$$\begin{aligned} \cosh \phi_{\mathbf{v}} &= \gamma_{\mathbf{v}} \\ \sinh \phi_{\mathbf{v}} &= \gamma_{\mathbf{v}} \frac{\|\mathbf{v}\|}{s}. \end{aligned} \tag{2.15}$$

The gamma identity plays in hyperbolic geometry a role analogous to the role that the law of cosines plays in Euclidean geometry, as we will see in Sect. 7.3, p. 218. Historically, the gamma identity (2.10) formed the first link between special relativity and the hyperbolic geometry of Lobachevsky and Bolyai.

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|, \tag{2.16}$$

in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}, \tag{2.17}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. Moreover, Einstein addition is also nonassociative since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}), \tag{2.18}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_s^n$.

As an application of the gamma identity (2.10), we prove the Einstein gyrotriangle inequality.

**Theorem 2.1 (Gyrotriangle Inequality, I).**

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{2.19}$$

*for all* $\mathbf{u}, \mathbf{v}$ *in an Einstein groupoid* $(\mathbb{R}_s^n, \oplus)$.

***Proof.*** By the gamma identity (2.10) and by the Cauchy-Schwarz inequality [76], we have

$$\begin{aligned}\gamma_{\|\mathbf{u}\|\oplus\|\mathbf{v}\|} &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\|\mathbf{u}\|\,\|\mathbf{v}\|}{s^2}\right)\\ &\geq \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right)\\ &= \gamma_{\mathbf{u}\oplus\mathbf{v}}\\ &= \gamma_{\|\mathbf{u}\oplus\mathbf{v}\|}\end{aligned} \tag{2.20}$$

for all $\mathbf{u}$, $\mathbf{v}$ in an Einstein groupoid $(\mathbb{R}^n_s, \oplus)$. But $\gamma_{\mathbf{x}} = \gamma_{\|\mathbf{x}\|}$ is a monotonically increasing function of $\|\mathbf{x}\|$, $0 \leq \|\mathbf{x}\| < s$. Hence (2.20) implies

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{2.21}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$. □

**Remark 2.2 (Einstein Addition Domain Extension).** *Einstein addition* $\mathbf{u}\oplus\mathbf{v}$ *in* (2.2) *involves the gamma factor* $\gamma_{\mathbf{u}}$ *of* $\mathbf{u}$*, while it is free of the gamma factor* $\gamma_{\mathbf{v}}$ of $\mathbf{v}$. *Hence, unlike* $\mathbf{u}$*, which must be restricted to the ball* $\mathbb{R}^n_s$ *in order to insure the reality of a gamma factor,* $\mathbf{v}$ *need not be restricted to the ball. Hence, the domain of* $\mathbf{v}$ *can be extended from the ball* $\mathbb{R}^n_s$ *to the whole of the space* $\mathbb{R}^n$*. Moreover, also the gamma identity* (2.10) *remains valid for all* $\mathbf{u} \in \mathbb{R}^n_s$ *and* $\mathbf{v} \in \mathbb{R}^n$ *under appropriate choice of the square root of negative numbers. If* $1 + \mathbf{u}\cdot\mathbf{v}/s = 0$, *then* $\mathbf{u}\oplus\mathbf{v}$ *is undefined, and, by* (2.10), $\gamma_{\mathbf{u}\oplus\mathbf{v}} = 0$, *so that* $\|\mathbf{u}\oplus\mathbf{v}\| = \infty$.

## 2.3 Einstein Addition for Computer Algebra

Various identities that involve Einstein addition play important role, but the detailed proof of some of these identities is left to the interested reader. In general, the proof of these identities is lengthy, but straightforward, so that the use of a computer software that facilitates symbolic mathematics, like Mathematica [145] or Maple, is required. For the use of computer algebra in proving algebraic identities that involve Einstein addition, it is convenient to rewrite Einstein addition as a linear combination of two vectors. Indeed, following (2.2), Einstein addition in $\mathbb{R}^n_s$ can be written as

$$\mathbf{u}\oplus\mathbf{v} = A_{\mathbf{u},\mathbf{v}}\mathbf{u} + B_{\mathbf{u},\mathbf{v}}\mathbf{v}, \tag{2.22a}$$

where

$$A_{\mathbf{u},\mathbf{v}} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left(1+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}\mathbf{u}\cdot\mathbf{v}\right) \tag{2.22b}$$

and

$$B_{\mathbf{u},\mathbf{v}} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\frac{1}{\gamma_{\mathbf{u}}} . \tag{2.22c}$$

The form (2.22) of Einstein addition is convenient for use in computer algebra. Readers who wish to obtain their own proof, by computer algebra, of many identities that appear in the book, particularly in Problem Sections, are likely to employ Einstein addition in the form (2.22).

As an illustrative example for the use of (2.22) in computer algebra for proving identities that involve Einstein addition, we present and prove the following interesting theorem.

**Theorem 2.3 (Cocycle Equation).** *The cocycle form*

$$S(\mathbf{u},\mathbf{v}) = \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = 1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}, \tag{2.23}$$

*which appears as a factor in Einstein's velocity addition law, satisfies the functional equation and the normalization conditions*

$$\begin{aligned} F(\mathbf{u}, \mathbf{v}\oplus\mathbf{w})F(\mathbf{v}, \mathbf{w}) &= F(\mathbf{v}\oplus\mathbf{u}, \mathbf{w})F(\mathbf{u}, \mathbf{v}) \\ F(\mathbf{u}, \mathbf{0}) = F(\mathbf{0}, \mathbf{v}) &= 1 \end{aligned} \tag{2.24}$$

*in* $\mathbb{R}^n_s$.

***Proof.*** By means of (2.22), we have

$$\begin{aligned} \mathbf{u}\cdot(\mathbf{v}\oplus\mathbf{w}) &= A_{\mathbf{v},\mathbf{w}}\mathbf{u}\cdot\mathbf{v} + B_{\mathbf{v},\mathbf{w}}\mathbf{u}\cdot\mathbf{w} \\ &= \frac{1}{1+\frac{\mathbf{v}\cdot\mathbf{w}}{s^2}}\left\{(1+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\cdot\mathbf{w})\mathbf{u}\cdot\mathbf{v} + \frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}\right\} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \mathbf{w}\cdot(\mathbf{v}\oplus\mathbf{u}) &= A_{\mathbf{v},\mathbf{u}}\mathbf{v}\cdot\mathbf{w} + B_{\mathbf{v},\mathbf{u}}\mathbf{u}\cdot\mathbf{w} \\ &= \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{(1+\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{w} + \frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}\right\}. \end{aligned} \tag{2.26}$$

With the definition of the cocycle form $S(\mathbf{u}, \mathbf{v})$ in (2.23) we have from (2.25) and (2.26) (taking $s = 1$ without loss of generality)

$$\begin{aligned}
&S(\mathbf{u},\mathbf{v}\oplus\mathbf{w})S(\mathbf{v},\mathbf{w})\\
&\qquad=\{1+\mathbf{u}\cdot(\mathbf{v}\oplus\mathbf{w})\}\,(1+\mathbf{v}\cdot\mathbf{w})\\
&\qquad=\left(1+\frac{1}{1+\mathbf{v}\cdot\mathbf{w}}\left\{(1+\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\cdot\mathbf{w})\mathbf{u}\cdot\mathbf{v}+\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}\right\}\right)(1+\mathbf{v}\cdot\mathbf{w}) \qquad (2.27)\\
&\qquad=1+\mathbf{v}\cdot\mathbf{w}+\left(1+\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\cdot\mathbf{w}\right)\mathbf{u}\cdot\mathbf{v}+\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}
\end{aligned}$$

and

$$\begin{aligned}
&S(\mathbf{v}\oplus\mathbf{u},\mathbf{w})S(\mathbf{u},\mathbf{v})\\
&\qquad=\{1+\mathbf{w}\cdot(\mathbf{v}\oplus\mathbf{u})\}\,(1+\mathbf{u}\cdot\mathbf{v})\\
&\qquad=\left(1+\frac{1}{1+\mathbf{u}\cdot\mathbf{v}}\left\{(1+\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{w}+\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}\right\}\right)(1+\mathbf{u}\cdot\mathbf{v}) \qquad (2.28)\\
&\qquad=1+\mathbf{u}\cdot\mathbf{v}+\left(1+\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{w}+\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}\cdot\mathbf{w}
\end{aligned}$$

implying

$$S(\mathbf{u},\mathbf{v}\oplus\mathbf{w})S(\mathbf{v},\mathbf{w})=S(\mathbf{v}\oplus\mathbf{u},\mathbf{w})S(\mathbf{u},\mathbf{v}) \qquad (2.29)$$

so that $S(\mathbf{u}, \mathbf{v})$ in (2.23) satisfies the functional equation (2.24) as desired. □

Applications of the Einstein cocycle equation (2.29) and the Einstein cocycle form (2.23) to the Lorentz transformation of special relativity theory are studied in [119].

## 2.4 Thomas Precession Angle

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s \subset \mathbb{R}^n$ be two relativistically admissible velocities such that $\mathbf{u} \neq -\mathbf{v}$, so that $\mathbf{u}\oplus\mathbf{v} \neq \mathbf{0}$, and let $\theta$, $0 \le \theta < 2\pi$, be the angle between $\mathbf{u}$ and $\mathbf{v}$. Furthermore, let $\varepsilon$ be the angle between the two Einstein sums $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$. Then,

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}{\sqrt{\gamma_{\mathbf{u}}^2-1}\sqrt{\gamma_{\mathbf{v}}^2-1}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}, \qquad (2.30)$$

by (2.12), and

$$\cos\varepsilon=\frac{(\mathbf{u}\oplus\mathbf{v})\cdot(\mathbf{v}\oplus\mathbf{u})}{\|\mathbf{u}\oplus\mathbf{v}\|^2}, \qquad (2.31)$$

noting that $\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|$, as explained in (2.16). The angle $\varepsilon$, $0 \le \varepsilon < \pi$, is the rotation angle of Thomas precession, called the *Thomas precession angle* generated by $\mathbf{u}$ and $\mathbf{v}$. Suggestively, in the context of Thomas precession, we call $\theta$ the generating angle that generates the Thomas precession angle $\varepsilon$. These two angles are depicted in Fig. 13.2, p. 567, in the study of Thomas precession in Chapter 11.

In this section we employ Einstein addition in the form (2.22) to express Thomas precession angle $\varepsilon$ in terms of its generating angle $\theta$. For this sake we define the velocities parameter $\rho > 1$ by the equation

$$\rho = \sqrt{\frac{\gamma_{\mathbf{u}}+1}{\gamma_{\mathbf{u}}-1}\frac{\gamma_{\mathbf{v}}+1}{\gamma_{\mathbf{v}}-1}}, \tag{2.32}$$

noting the useful identity

$$\rho\sqrt{\gamma_{\mathbf{u}}^2-1}\sqrt{\gamma_{\mathbf{v}}^2-1} = (\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1). \tag{2.33}$$

Employing Einstein addition in the form (2.22), which is suitable for computer algebra, one can readily check by computer algebra that, following (2.30)–(2.33), the Thomas precession angle $\varepsilon$ is related to its generating angle $\theta$ by the first equation in (2.34) below,

$$\begin{aligned} \cos\varepsilon &= \frac{(\rho+\cos\theta)^2-\sin^2\theta}{(\rho+\cos\theta)^2+\sin^2\theta} \\ \sin\varepsilon &= \frac{-2(\rho+\cos\theta)\,\sin\theta}{(\rho+\cos\theta)^2+\sin^2\theta}. \end{aligned} \tag{2.34}$$

The second equation in (2.34) is determined from the first by the trigonometric identity $\sin\varepsilon = \pm\sqrt{1-\cos^2\varepsilon}$. The ambiguous sign for $\sin\varepsilon$ is selected in (2.34) such that $\varepsilon$ and $\theta$ have opposite signs. The selection of the physically correct sign for $\sin\varepsilon$ is important, as explained in Chapter 11. The graphs of $\cos\varepsilon$ and $-\sin\varepsilon$ as functions of $\theta$, $0 \le \theta \le 2\pi$, for several values of the velocities parameter $\rho$, are presented in Figs. 13.3–13.4, pp. 569–570.

Thomas precession angle $\varepsilon$ possesses the *exclusion property*: $\varepsilon \ne \pi$. Indeed, by (2.34), the equation $\cos\varepsilon = -1$ implies $\rho + \cos\theta = 0$, which results in a contradiction since $\rho > 1$ while $|\cos\theta| \le 1$.

## 2.5 Einstein Addition with Respect to Cartesian Coordinates

Like any physical law, Einstein velocity addition law (2.2) is coordinate independent. Indeed, it is presented in (2.2) in terms of vectors, noting that one of the great advantages of vectors is their ability to express results independent of any coordinate system.

However, in order to generate numerical and graphical demonstrations of laws in physics and results in geometry, we need coordinates. Accordingly, we introduce

Cartesian coordinates into the Euclidean $n$-space $\mathbb{R}^n$ and its ball $\mathbb{R}^n_s$, with respect to which we generate the graphical presentations. Introducing the Cartesian coordinate system $\Sigma$ into $\mathbb{R}^n$ and $\mathbb{R}^n_s$, each point $P \in \mathbb{R}^n$ is given by an $n$-tuple

$$P = (x_1, x_2, \ldots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < \infty, \tag{2.35}$$

of real numbers, which are the coordinates, or components, of $P$ with respect to $\Sigma$. Similarly, each point $P \in \mathbb{R}^n_s$ is given by an $n$-tuple

$$P = (x_1, x_2, \ldots, x_n), \qquad x_1^2 + x_2^2 + \cdots + x_n^2 < s^2, \tag{2.36}$$

of real numbers, which are the coordinates, or components of $P$ with respect to $\Sigma$.

Equipped with a Cartesian coordinate system $\Sigma$ and its standard vector addition given by component addition, along with its resulting scalar multiplication, $\mathbb{R}^n$ forms the standard Cartesian model of $n$-dimensional Euclidean geometry. In full analogy, equipped with a Cartesian coordinate system $\Sigma$ and its Einstein addition, along with its resulting scalar multiplication (to be studied in Sect. 3.3, p. 79), the ball $\mathbb{R}^n_s$ forms the Cartesian-Beltrami-Klein ball model of $n$-dimensional hyperbolic geometry (as we will see in Chapter 3, particularly, in (3.39)–(3.40), pp. 83–84).

As an illustrative example, we present below the Einstein velocity addition law (2.2) in $\mathbb{R}^3_s$ with respect to a Cartesian coordinate system.

Let $\mathbb{R}^3_s$ be the $s$-ball of the Euclidean 3-space, equipped with a Cartesian coordinate system $\Sigma$,

$$\mathbb{R}^3_s = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + x_3^2} < s \right\}. \tag{2.37}$$

Accordingly, each point of the ball is represented by its coordinates $(x_1, x_2, x_3)^t$ (exponent $t$ denotes transposition) with respect to $\Sigma$, satisfying the condition $x_1^2 + x_2^2 + x_3^2 < s^2$.

Furthermore, let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3_s$ be three points in $\mathbb{R}^3_s \subset \mathbb{R}^3$ given by their coordinates with respect to $\Sigma$,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{2.38}$$

where

$$\mathbf{w} = \mathbf{u} \oplus \mathbf{v}. \tag{2.39}$$

The dot (inner) product of $\mathbf{u}$ and $\mathbf{v}$ is given in $\Sigma$ by the equation

$$\mathbf{u}\cdot\mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3, \tag{2.40}$$

and the squared norm $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$ of $\mathbf{v}$ is given by the equation

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2. \tag{2.41}$$

Hence, it follows from the coordinate free, vector representation (2.2) of Einstein addition that the coordinate Einstein addition (2.39) with respect to the Cartesian coordinate system $\Sigma$ takes the form

$$\begin{aligned}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{1}{1 + \dfrac{u_1v_1 + u_2v_2 + u_3v_3}{s^2}} \\ &\times \left\{ [1 + \frac{1}{s^2} \frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(u_1v_1 + u_2v_2 + u_3v_3)] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \frac{1}{\gamma_{\mathbf{u}}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\},\end{aligned} \tag{2.42}$$

where

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \dfrac{u_1^2 + u_2^2 + u_3^2}{s^2}}}. \tag{2.43}$$

Note that (i) $\gamma_{\mathbf{u}}$ is real if and only if $\|\mathbf{u}\| < s$, (ii) $\gamma_{\mathbf{u}} = \infty$ if and only if $\|\mathbf{u}\| = s$, and (iii) $\gamma_{\mathbf{u}}$ is purely imaginary if and only if $\|\mathbf{u}\| > s$.

The three components of Einstein addition (2.39) are $w_1$, $w_2$ and $w_3$ in (2.42). For a two-dimensional illustration of Einstein addition (2.42) one may impose the condition $u_3 = v_3 = 0$, implying $w_3 = 0$. An illustrative example in two dimensions is presented in Example 2.4 below.

In the Newtonian-Euclidean limit, $s \to \infty$, the ball $\mathbb{R}_s^3$ expands to the Euclidean 3-space $\mathbb{R}^3$, and Einstein addition (2.42) reduces to the common vector addition in $\mathbb{R}^3$,

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{2.44}$$

We will find that Einstein addition plays in the Cartesian model of the Beltrami-Klein ball model of hyperbolic geometry the same role that vector addition plays in the Cartesian model of Euclidean geometry. Suggestively, the Cartesian-Beltrami-Klein ball model of hyperbolic geometry is also known as the *relativistic velocity model* [2, 5].

Vector equations and identities are represented by coordinate free expressions, like Einstein addition in (2.2). For numerical and graphical presentations, however, these must be converted into a coordinate dependent form relative to a Cartesian coordinate system that must be introduced. The latter, in turn, can be presented relative to Cartesian coordinates numerically and graphically, as we do in the generation of figures. In general, Cartesian coordinates are not shown in figures. For the sake of demonstration, however, they are shown in Figs. 3.3 and 3.4, p. 87.

**Example 2.4** *As an illustrative example of a 2-dimensional Einstein addition with respect to a Cartesian coordinate system, we employ* (2.42) *to calculate the elegant result of the Einstein sum* $\ominus(0, b)^t \oplus (x, b)^t$ *in the relativistic velocity plane*

$$\mathbb{R}_s^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{x^2 + y^2} < s \right\} \tag{2.45}$$

*of 2-dimensional relativistically admissible velocities, equipped with the Cartesian coordinate system* $\Sigma = (x, y)$. *Following* (2.42) *we have*

$$\begin{aligned}
\ominus \begin{pmatrix} 0 \\ b \end{pmatrix} \oplus \begin{pmatrix} x \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ -b \end{pmatrix} \oplus \begin{pmatrix} x \\ b \end{pmatrix} \\
&= \frac{1}{1 - \frac{b^2}{s^2}} \left\{ (1 - \frac{\gamma_b}{1 + \gamma_b} \frac{b^2}{c^2}) \begin{pmatrix} 0 \\ -b \end{pmatrix} + \frac{1}{\gamma_b} \begin{pmatrix} x \\ b \end{pmatrix} \right\} \\
&= \gamma_b^2 \left\{ (1 - \frac{\gamma_b}{1 + \gamma_b} \frac{\gamma_b^2 - 1}{\gamma_b^2}) \begin{pmatrix} 0 \\ -b \end{pmatrix} + \frac{1}{\gamma_b} \begin{pmatrix} x \\ b \end{pmatrix} \right\} \\
&= \gamma_b^2 \left\{ \frac{1}{\gamma_b} \begin{pmatrix} 0 \\ -b \end{pmatrix} + \frac{1}{\gamma_b} \begin{pmatrix} x \\ b \end{pmatrix} \right\} \\
&= \gamma_b \begin{pmatrix} x \\ 0 \end{pmatrix} ,
\end{aligned} \tag{2.46}$$

*so that*

$$\left\| \ominus \begin{pmatrix} 0 \\ b \end{pmatrix} \oplus \begin{pmatrix} x \\ b \end{pmatrix} \right\| = \gamma_b \; |x|. \tag{2.47}$$

## 2.6 Einstein Addition vs. Vector Addition

Vector addition, +, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} && \text{Commutative Law} \\
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} && \text{Associative Law}
\end{aligned} \tag{2.48}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}_s^n$ is neither commutative nor associative.

Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}_s^3, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^3$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u} \oplus \mathbf{v}) \oplus \{\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})\} \tag{2.49}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_s^3$. Equation (2.49) presents the application to $\mathbf{w}$ of the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ generated by $\mathbf{u}$ and $\mathbf{v}$. Gyrations turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}_s^3, \oplus)$.

An automorphism of a groupoid $(S, \oplus)$ is a bijective map $f$ of $S$ onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, denoted by $\mathrm{Aut}(S, \oplus)$, where the group operation is given by automorphism composition. To emphasize that the gyrations of an Einstein groupoid $(\mathbb{R}_s^3, \oplus)$ are automorphisms of the groupoid, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^3$, is *trivial* if $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}_s^3$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}[\mathbf{v}, \mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}]$ are trivial for all $\mathbf{v} \in \mathbb{R}_s^3$, as we see from (2.49). More generally, gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ are trivial when $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n \subset \mathbb{R}^n$ are parallel in $\mathbb{R}^n$.

Possessing their own rich structure, gyrations measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the following list of identities [119, 122, 129]:

$$
\begin{aligned}
&\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law}\\
&\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) = (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law}\\
&(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}\\
&\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Left Reduction Property}\\
&\mathrm{gyr}[\mathbf{u}, \mathbf{v}\oplus\mathbf{u}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Right Reduction Property}\\
&\mathrm{gyr}[\ominus\mathbf{u}, \ominus\mathbf{v}] = \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Even Property}\\
&(\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1} = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] && \text{Gyration Inversion Law}
\end{aligned}
\tag{2.50}
$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_s^n$.

It is clear from (2.50) that the departure of Einstein addition, $\oplus$, from commutativity and associativity is strictly controlled by gyrations. The reduction properties in (2.50) present important gyration identities. One of them, the left reduction property, will soon demonstrate its power and elegance in solving the gyrogroup equation $x\oplus a = b$ in (2.105) and (2.107), p. 47.

Einstein addition plays in hyperbolic geometry the role that vector addition plays in the vector space approach to Euclidean geometry. Einstein addition is more complex than vector addition, but richer in structure. Hence, a computer algebra system, like Mathematica or Maple, is an indispensable tool. Indeed, the identities in (2.50) can be verified by lengthy, but straightforward algebra that can be handled easily by employing the computer algebra system Mathematica [145], as explained in Sect. 2.3 and illustrated in Sect. 2.4. Related details are found in Prob. 2.5, p. 71.

In the wide area of nonassociativity in physics and mathematics [65] the gyroassociative law of Einstein addition in (2.50) is the most marvelous law of both

1. the special theory of relativity of Einstein and
2. the hyperbolic geometry of Lobachevsky and Bolyai.

This marvelous law, along with its associated gyrocommutative law, enables the Einstein addition law of relativistically admissible velocities to be employed as the hyperbolic vector (gyrovector) addition in Cartesian models of hyperbolic geometry. Thus, in particular, it is the 1988 discovery [111, 112, 113] of the most marvelous law, the gyroassociative law of Einstein addition, that enables vector algebra and Cartesian coordinates to be adapted for use in hyperbolic geometry.

The concept of the gyration is a mathematical abstraction of the relativistic effect known as *Thomas precession* [129, Sect. 10.3], [136], which we will study in Chapter 13. An excellent description of the 3-space rotation which, since 1926, is named after Thomas [108] can be found in Silberstein's 1914 book [97]. In 1914 the Thomas precession did not have a name, and Silberstein calls it in his 1914 book a "certain space-rotation" [97, p. 169]. An early study of Thomas rotation, made by the famous mathematician É mile Borel in 1913, is described in his 1914 book [12] and, more recently, in [105]. According to Belloni and Reina [8] and Malykin [74], Sommerfeld's route to the Thomas precession dates back to 1909.

Following the gyrocommutative-gyroassociative laws, Einstein addition and the gyrations to which it gives rise are inextricably linked.

## 2.7 Gyrations

Owing to its nonassociativity, Einstein addition gives rise in (2.49) to gyrations,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}^n_s \to \mathbb{R}^n_s, \tag{2.51}$$

of an Einstein groupoid $(\mathbb{R}^n_s, \oplus)$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s$. Gyrations, in turn, regulate Einstein addition, $\oplus$, endowing it with the rich structure of a gyrocommutative gyrogroup, as we will see in Sect. 2.8, and a gyrovector space, as we will see in Sect. 3.3.

In the formal approach to gyrogroups in Def. 2.14, p. 39, the left reduction property is elevated to the *Reduction Axiom*. The gyration left reduction axiom is also known as the *left loop property*. The more revealing term, reduction axiom, was coined by F. Chatelin in [15] since it triggers remarkable reduction in complexity as, for instance, in (2.107), p. 47.

Gyrations are defined in (2.49) in terms of Einstein addition. An explicit presentation of the gyrations of Einstein groupoids $(\mathbb{R}^n_s, \oplus)$ in terms of vector addition rather than Einstein addition is given by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w} + \frac{A\mathbf{u} + B\mathbf{v}}{D}, \tag{2.52}$$

where

$$
\begin{aligned}
A &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}^2}{(\gamma_{\mathbf{u}}+1)}(\gamma_{\mathbf{v}}-1)(\mathbf{u}\cdot\mathbf{w}) + \frac{1}{s^2}\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w}) \\
&\quad + \frac{2}{s^4}\frac{\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2}{(\gamma_{\mathbf{u}}+1)(\gamma_{\mathbf{v}}+1)}(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{w}) \\
B &= -\frac{1}{s^2}\frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}\{\gamma_{\mathbf{u}}(\gamma_{\mathbf{v}}+1)(\mathbf{u}\cdot\mathbf{w}) + (\gamma_{\mathbf{u}}-1)\gamma_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{w})\} \\
D &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) + 1 = \gamma_{\mathbf{u}\oplus\mathbf{v}} + 1 \geq 2
\end{aligned}
\tag{2.53}
$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_s^n$.

**Remark 2.5 (Gyration Domain Extension).** *The domain of* $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n \subset \mathbb{R}^n$ *in* (2.52)–(2.53) *is restricted to* $\mathbb{R}_s^n$ *in order to insure the reality of the gamma factors of* $\mathbf{u}$ *and* $\mathbf{v}$ *in* (2.53). *However, while the expressions in* (2.52)–(2.53) *involve gamma factors of* $\mathbf{u}$ *and* $\mathbf{v}$, *they involve no gamma factors of* $\mathbf{w}$. *Hence, the domain of* $\mathbf{w}$ *in* (2.52)–(2.53) *can be extended from* $\mathbb{R}_s^n$ to $\mathbb{R}^n$. *Indeed, extending in* (2.52)–(2.53) *the domain of* $\mathbf{w}$ *from* $\mathbb{R}_s^n$ *to* $\mathbb{R}^n$, *gyrations* gyr[$\mathbf{u}, \mathbf{v}$] *are expanded from maps of* $\mathbb{R}_s^n$ *to linear maps of* $\mathbb{R}^n$ *for any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, gyr[$\mathbf{u}, \mathbf{v}$] : $\mathbb{R}^n \to \mathbb{R}^n$.

In each of the three special cases when (i) $\mathbf{u} = \mathbf{0}$, or (ii) $\mathbf{v} = \mathbf{0}$, or (iii) $\mathbf{u}$ and $\mathbf{v}$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, we have $A\mathbf{u} + B\mathbf{v} = \mathbf{0}$ so that gyr[$\mathbf{u}, \mathbf{v}$] is trivial. Thus, we have

$$
\begin{aligned}
&\mathrm{gyr}[0, \mathbf{v}]\mathbf{w} = \mathbf{w} \\
&\mathrm{gyr}[\mathbf{u}, 0]\mathbf{w} = \mathbf{w} \\
&\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}, \qquad\qquad \mathbf{u}\|\mathbf{v},
\end{aligned}
\tag{2.54}
$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{u}\|\mathbf{v}$ in the third equation, and all $\mathbf{w} \in \mathbb{R}^n$.

It follows from (2.52) by straightforward algebra that

$$\mathrm{gyr}[\mathbf{v}, \mathbf{u}](\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}) = \mathbf{w} \tag{2.55}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, $\mathbf{w} \in \mathbb{R}^n$, or equivalently,

$$\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathrm{gyr}[\mathbf{u}, \mathbf{v}] = I \tag{2.56}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, where $I$ denotes the trivial map, also called the *identity map*.

Hence, gyrations are invertible linear maps of $\mathbb{R}^n$, the inverse, $\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}]$, of gyr[$\mathbf{u}, \mathbf{v}$] being gyr[$\mathbf{v}, \mathbf{u}$]. We thus have the gyration inversion property

$$\mathrm{gyr}^{-1}[\mathbf{u}, \mathbf{v}] = \mathrm{gyr}[\mathbf{v}, \mathbf{u}] \tag{2.57}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$.

Gyrations keep the inner product of elements of the ball $\mathbb{R}_s^n$ invariant, that is,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{2.58}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. Hence, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an *isometry* of $\mathbb{R}_s^n$, keeping the norm of elements of the ball $\mathbb{R}_s^n$ invariant,

$$\|\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}\| = \|\mathbf{w}\|. \tag{2.59}$$

Accordingly, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball $\mathbb{R}_s^n$ about its origin for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$.

The invertible map $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ of $\mathbb{R}_s^n$ respects Einstein addition in $\mathbb{R}_s^n$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} \tag{2.60}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, so that $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of the Einstein groupoid $(\mathbb{R}_s^n, \oplus)$.

**Example 2.6** *As an example that illustrates the use of the invariance of the norm under gyrations, we note that*

$$\|\ominus\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{u}\ominus\mathbf{v}\| = \|\ominus\mathbf{v}\oplus\mathbf{u}\|. \tag{2.61}$$

*Indeed, we have the following chain of equations, which are numbered for subsequent derivation,*

$$\begin{aligned}
\|\ominus\mathbf{u}\oplus\mathbf{v}\| &\overbrace{=\!=}^{(1)} \|\ominus(\ominus\mathbf{u}\oplus\mathbf{v})\| \\
&\overbrace{=\!=}^{(2)} \|\mathbf{u}\ominus\mathbf{v}\| \\
&\overbrace{=\!=}^{(3)} \|\mathrm{gyr}[\mathbf{u}, \ominus\mathbf{v}](\ominus\mathbf{v}\oplus\mathbf{u})\| \\
&\overbrace{=\!=}^{(4)} \|\ominus\mathbf{v}\oplus\mathbf{u}\|
\end{aligned} \tag{2.62}$$

*for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$. *Derivation of the numbered equalities in* (2.62) *follows:*

1) *Follows from the result that* $\ominus\mathbf{w} = -\mathbf{w}$, *so that* $\|\ominus\mathbf{w}\| = \|-\mathbf{w}\| = \|\mathbf{w}\|$ *for all* $\mathbf{w} \in \mathbb{R}_s^n$.
2) *Follows from the automorphic inverse property* (2.7)*, p. 25, of Einstein addition.*
3) *Follows from the gyrocommutative law of Einstein addition.*
4) *Follows from the result that, by* (2.59)*, gyrations keep the norm invariant.*

Let $\mathbf{z} \in \mathbb{R}^n$ be a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}_s^n \subset \mathbb{R}^n$, $n \geq 3$, that is, $\mathbf{u}\cdot\mathbf{z} = \mathbf{v}\cdot\mathbf{z} = 0$. Then, by (2.52)–(2.53) with $\mathbf{w} = \mathbf{z}$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{z} = \mathbf{z}. \tag{2.63}$$

Motivated by the special case when $n = 3$, following (2.63) we say that the gyration axis in $\mathbb{R}^n$ of the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}^n \to \mathbb{R}^n$, generated by $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$, is parallel to the vector $\mathbf{z}$. The gyration angle $\varepsilon$ of the gyration gyr[$\mathbf{u}$, $\mathbf{v}$] of $\mathbb{R}^n$ is given by the equation

$$\cos\varepsilon = \frac{\mathbf{x}\cdot \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}}{\|\mathbf{x}\|^2} \tag{2.64}$$

for any $\mathbf{x} \in \mathbb{R}^n$ that lies on the plane spanned by $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_s \subset \mathbb{R}^n$, that is,

$$\mathbf{x} = c_u\mathbf{u} + c_v\mathbf{v}, \tag{2.65}$$

$\mathbf{x} \neq \mathbf{0}$, for any coefficients $c_u, c_v \in \mathbb{R}$, excluding $c_u = c_v = 0$.

As expected, the angle $\varepsilon$ in (2.64) is independent of the choice of $\mathbf{x}$ in (2.65). Moreover, we have the following result.

**Theorem 2.7 (Gyration–Thomas Precession Angle).** *Let* $\mathbf{u}, \mathbf{v}, \mathbf{x} \in \mathbb{R}^n_s$ *be relativistically admissible velocities such that* $\mathbf{u} \neq -\mathbf{v}$ (*so that* $\mathbf{u}\oplus\mathbf{v} \neq \mathbf{0}$). *Then,*

$$\cos\varepsilon := \frac{\mathbf{x}\cdot \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x}}{\|\mathbf{x}\|^2} = \frac{(\mathbf{u}\oplus\mathbf{v})\cdot(\mathbf{v}\oplus\mathbf{u})}{\|\mathbf{u}\oplus\mathbf{v}\|^2} \tag{2.66}$$

***Proof.*** The proof of the identity in (2.66) is obtained straightforwardly by computer algebra, (i) where Einstein addition is expressed by (2.22), pp. 27–28, and (ii) where the application gyr[$\mathbf{u}$, $\mathbf{v}$]$\mathbf{x}$ of gyration gyr[$\mathbf{u}$, $\mathbf{v}$] to x is expressed by (2.52)–(2.53). □

Theorem 2.7 states that the gyration angle $\varepsilon$ in (2.64) and the Thomas precession angle $\varepsilon$ in (2.31), p. 29, coincide.

Special attention to three dimensional gyrations, which are of interest in physical applications, is paid in Chapter 13 in the study of Thomas precession.

## 2.8 From Einstein Velocity Addition to Gyrogroups

Guided by analogies with groups, the key features of Einstein groupoids $(\mathbb{R}^n_s, \oplus)$, $n = 1, 2, 3, \ldots$, suggest the formal gyrogroup definition in which gyrogroups form a most natural generalization of groups. Accordingly, definitions related to groups and gyrogroups follow.

**Definition 2.8 (Binary Operations).** *A binary operation + in a set S is a function* $+ : S \times S \to S$. *We use the notation* $a + b$ *to denote* $+(a, b)$ *for any* $a, b \in S$.

**Definition 2.9 (Groupoids, Automorphisms).** *A groupoid* $(S, +)$ *is a nonempty set, S, with a binary operation, +. An automorphism* $\phi$ *of a groupoid* $(S, +)$ *is a bijective self-map of S which respects its groupoid operation, that is,* $\phi(a + b) = \phi(a) + \phi(b)$ *for all* $a, b \in S$.

**Definition 2.10 (Groups).** *A groupoid* $(G, +)$ *is a group if its binary operation satisfies the following axioms. In G there is at least one element,* 0, *called a left identity, satisfying*

$(G1)$ $\qquad 0 + a = a$

*for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of $a$, satisfying*

$(G2)$ $\qquad -a + a = 0.$

*Moreover, the binary operation obeys the associative law*

$(G3)$ $\qquad (a + b) + c = a + (b + c)$

*for all $a, b, c \in G$.*

Groups are classified into commutative and noncommutative groups.

**Definition 2.11 (Commutative Groups).** *A group $(G, +)$ is commutative if its binary operation obeys the commutative law*

$(G6)$ $\qquad a + b = b + a$

*for all $a, b \in G$.*

**Definition 2.12 (Subgroups).** *A subset $H$ of a group $(G, +)$ is a subgroup of $G$ if it is nonempty, and $H$ is closed under group compositions and inverses in $G$, that is, $x, y \in H$ implies $x + y \in H$ and $-x \in H$.*

**Theorem 2.13 (The Subgroup Criterion).** *A subset $H$ of a group $(G, +)$ is a subgroup of $G$ if and only if (i) $H$ is nonempty, and (ii) $x, y \in H$ implies $x - y \in H$.*

For a proof of the Subgroup Criterion see any book on group theory.

**Definition 2.14 (Gyrogroups).** *A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In $G$ there is at least one element, $0$, called a left identity, satisfying*

$(G1)$ $\qquad 0 \oplus a = a$

*for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of $a$, satisfying*

$(G2)$ $\qquad \ominus a \oplus a = 0.$

*Moreover, for any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law*

$(G3)$ $\qquad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$

*The map $\mathrm{gyr}[a, b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,*

$(G4)$ $\qquad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus),$

*and the automorphism* gyr$[a, b]$ *of G is called the gyroautomorphism, or the gyration, of G generated by a,* $b \in G$*. The operator* gyr : $G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of G. Finally, the gyroautomorphism* gyr$[a, b]$ *generated by any* $a, b \in G$ *possesses the left reduction property*

$$(G5) \qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b],$$

*called the reduction axiom.*

The gyrogroup axioms (*G1*)–(*G5*) in Definition 2.14 are classified into three classes:

1. The first pair of axioms, (*G1*) and (*G2*), is a reminiscent of the group axioms.
2. The last pair of axioms, (*G4*) and (*G5*), presents the gyrator axioms.
3. The middle axiom, (*G3*), is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and nongyrocommutative gyrogroups.

**Definition 2.15 (Gyrocommutative Gyrogroups).** *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

$$(G6) \qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$$

*for all* $a, b \in G$.

The abstract gyrocommutative gyrogroup is an algebraic structure tailor made to suit Einstein velocity addition of relativistically admissible velocities. Indeed, the Einstein groupoid $(\mathbb{R}^n_s, \oplus)$ is a gyrocommutative gyrogroup. Gyrogroups, both gyrocommutative and nongyrocommutative, abound in group theory as shown in [40] and [41]. A finite, nongyrocommutative gyrogroup of order 16, $K_{16}$, is presented in [119, Figs. 2.1-2.2, p. 41].

Einstein addition in the real ball $\mathbb{R}^n_s$ can straightforwardly be extended to the complex ball $\mathbb{C}^n_s$, giving rise to the complex Einstein groupoid $(\mathbb{C}^n_s, \oplus)$. The latter turns out to be a nongyrocommutative gyrogroup, studied in [100, 121, 128, 36].

## 2.9 Gyrogroup Cooperation (Coaddition)

Our plan to capture analogies with groups dictates the introduction into the abstract gyrogroup $(G, \oplus)$ a second binary operation, $\boxplus$, called the gyrogroup *cooperation*, or *coaddition*.

**Definition 2.16 (Gyrogroup Cooperation (Coaddition)).** *Let* $(G, \oplus)$ *be a gyrogroup. The gyrogroup cooperation (or, coaddition),* $\boxplus$*, is a second binary operation in G related to the gyrogroup operation (or, addition),* $\oplus$*, by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{2.67}$$

*for all* $a, b \in G$.

Naturally, we use the notation $a \boxminus b = a \boxplus (\ominus b)$ where $\ominus b = -b$, so that

$$a \boxminus b = a \ominus \mathrm{gyr}[a, b]b. \tag{2.68}$$

The gyrogroup cooperation is commutative if and only if the gyrogroup operation is gyrocommutative, as we will see in Theorem 2.45, p. 63.

Hence, in particular, Einstein coaddition $\boxplus$ is commutative since Einstein addition $\oplus$ is gyrocommutative.

Indeed, let us calculate, as a concrete example of (2.67), the Einstein coaddition $\boxplus$. By substituting into (2.67) both

1. Einstein addition in (2.2), p. 24, and
2. Einstein gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$ in (2.49), p. 33,

lengthy, but straightforward, algebra (that can be handled easily by employing a computer algebra system like Mathematica) reveals the following important result:

Einstein coaddition $\boxplus$ is given explicitly by the equation

$$\begin{aligned} \mathbf{u} \boxplus \mathbf{v} &= \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) - 1}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \\ &= 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \end{aligned} \tag{2.69}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_s^n$ where, by definition, $2\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v}$. Einstein coaddition (2.69) of two summands will be extended to $k$ summands, $k \geq 2$ in (6.23), p. 180, and (6.84), p. 194.

## 2.10 First Gyrogroup Properties

While it is clear how to define a right identity and a right inverse in a gyrogroup, the existence of such elements is not presumed. Indeed, the existence of a unique identity and a unique inverse, both left and right, is a consequence of the gyrogroup axioms, as the following theorem shows, along with other immediate results about gyrogroups.