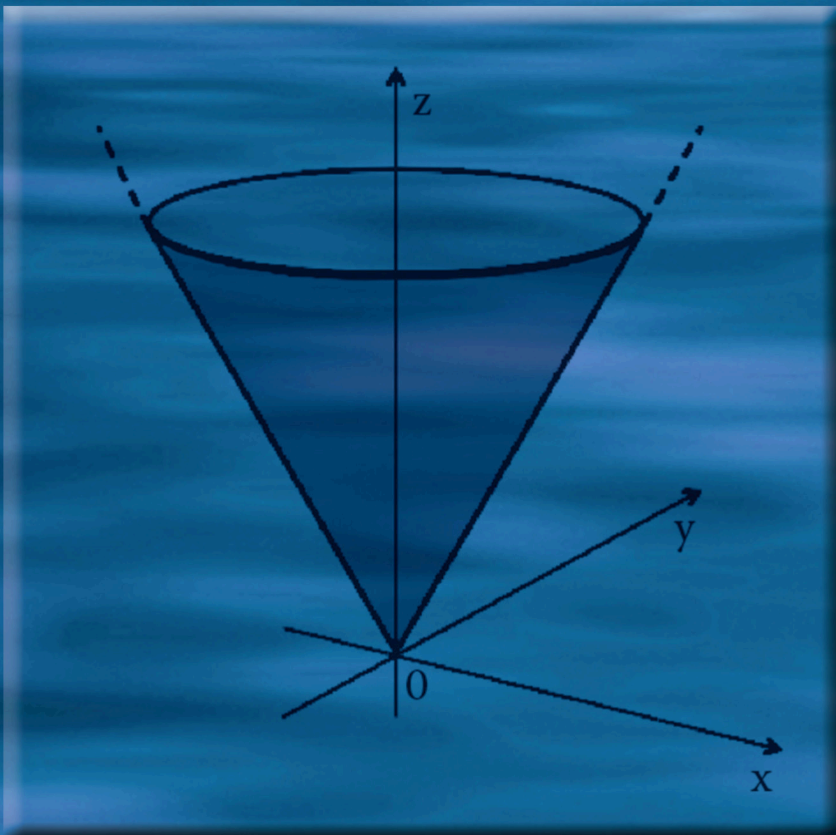
FIXED POINT THEORY, VARIATIONAL ANALYSIS, AND OPTIMIZATION



Edited by

Saleh A. R. Al-Mezel,
Falleh R. M. Al-Solamy, and
Qamrul H. Ansari

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Preface

Nonlinear analysis is one of the most interesting and fascinating branches of pure and applied mathematics. During the last five decades, several branches of nonlinear analysis have been developed and extensively studied. The main aim of this volume is to include those branches of nonlinear analysis which have different applications in different areas. We divide this volume into three parts: Fixed-Point Theory, Convex Analysis and Variational Analysis, and Vector Optimization.

The first part consists of the first three chapters.

Chapter 1 is devoted to the study of Mann-type iterations for nonlinear mappings on some classes of a metric space. This is achieved through the convex structure introduced by W. Takahashi. The common fixed-point results for asymptotically (quasi-) nonexpansive mappings through their explicit and implicit iterative schemes on nonlinear domains such as $CAT(0)$ spaces, hyperbolic spaces, and convex metric spaces are presented, which provide metric space version of the corresponding well-known results in Banach spaces.

Chapter 2 provides an outline of the recent results in fixed-point theory in modular function spaces. Modular function spaces are natural generalizations of both function and sequence variants of many important (from an applications perspective) spaces such as Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii, and many others. In the context of fixed-point theory, the foundations of the geometry of modular function spaces and other important techniques like extensions of the Opial property to modular spaces are discussed. A series of existence theorems of fixed points for nonlinear mappings, and of common fixed points for semigroups of mappings, is presented.

Chapter 3 discusses key results on the existence of continuous approximations and selections for set-valued maps with an emphasis on the non-convex case (non-convex domains, co-domains, and non-convex values) and in a general and generic framework allowing the passage, by approximation, from simple domains to more elaborate ones. Applications of the approximation and selection results to topological fixed-point and coincidence theory for set-valued maps are also presented.

The second part of the volume consists of Chapters 4, 5, and 6.

Chapter 4 contains the basic definitions, properties, and characterizations of convex, quasiconvex, and pseudoconvex functions, and of their strict counterparts. The aim of this chapter is to present the basic techniques that will help the reader in his/her further reading, so we include almost all proofs of the results presented. At the same time, the definitions of some classes of generalized monotone operators are recalled; it is shown how they are related to corresponding classes of generalized convex functions. Finally, some of the many applications of generalized convex functions and generalized monotone operators are given, which are related to optimization and microeconomics, and especially to consumer theory.

After the huge development of convex optimization during several decades, quasiconvex optimization, or optimization problems involving a quasiconvex objective function, can be considered as a new step to embrace a larger class of problem with powerful mathematical tools. The main aim of Chapter 5 is to show that, using some adapted tools, a sharp and powerful first-order analysis can be developed for quasiconvex optimization.

Chapter 6 gives an introduction to the theory of variational-like inequalities. Some relations between a nonconvex optimization problem and a variational-like inequality problem are provided. Some existence results for a solution of variational-like inequalities are presented under different kinds of assumptions. Two solution methods—auxiliary principle method and the proximal method—for finding the approximate solutions of variational-like inequalities are discussed.

The last part is devoted to vector optimization.

Chapter 7 presents basic concepts of vector optimization, starting with partial orders in a vector space with respect to which optimality is defined. Some criteria for existence of maximal elements of a set in a partially ordered space by using coverings of a set and monotone functions are discussed. For a vector optimization problem with equality and inequality constraints, one can express optimality conditions in terms of derivatives when the data of the problem are differentiable, or in terms of subdifferentials when the data are nonsmooth. Finally, three methods for solving nonconvex vector optimization problems are presented. Two of them are well-known and the other one is more recent, but both are interesting from mathematical and practical points of view.

The last chapter is devoted to multi-objective combinatorial optimization (MOCO) problems, which are integer programs with multiple objectives. The goal in solving a MOCO problem is to find efficient (or Pareto optimal) solutions and their counterparts in objective space, called non-dominated points. Various types of efficient solutions and non-dominated points as well as lexicographic optima are defined. It is shown that MOCO problems are usually NP-hard, #P-hard, and intractable, that is, they can have an exponential number of non-dominated points. The multi-objective shortest-path and spanning-tree problems are presented as examples of MOCO problems for which single objective algorithms can be extended. The two-phase method is an effective tool for problems that are polynomially solvable in the single objective case and for which efficient ranking algorithms to find r-best solutions exist. For problems for which the two-phase approach is not computationally effective, one must resort to general scalarization techniques or adapt general integer programming techniques, such as branch and bound, to deal with multiple objectives. Some popular scalarization methods in the context of a general formulation are presented. Bound sets for non-dominated points are natural multi-objective counterparts of lower and upper bounds that enable multi-objective branch and bound algorithms and several examples are cited. Exact algorithms based on integer programming techniques and scalarization can easily result in prohibitive computation times even for relatively small-sized problems. Metaheuristics may be applied in this case. The main concepts of metaheuristics that have been applied to MOCO problems are introduced and their evolution over time is illustrated.

We thank our friends and colleagues, whose encouragement and help influenced the development of this volume. We mainly are grateful to the Rector of the University of Tabuk Dr. Abdulaziz S. Al-Enazi, for his support in organizing an International Workshop on Nonlinear Analysis and Optimization, March 18–19, 2013 at University of Tabuk, Tabuk, Saudi Arabia. Most of the contributors attended this workshop and agreed to be a part of this volume.

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October 2013

Saleh A. R. Al-Mezel,
Falleh R. M. Al-Solamy,
and Qamrul H. Ansari

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Contributors

Qamrul Hasan Ansari
Department of Mathematics
Aligarh Muslim University
Aligarh, India
E-mail: qhansari@gmail.com

D. Aussel
Lab. PROMES CNRS
University of Perpignan
Perpignan, France
E-mail: aussel@univ-perp.fr

Hichem Ben-El-Mechaiekh
Department of Mathematics
Brock University
Saint Catharines, Ontario Canada
E-mail: hmechaie@brocku.ca

B. A. Bin Dehaish
Department of Mathematics
King Abdulaziz University
Jeddah, Saudi Arabia
E-mail: bbindehaish@yahoo.com

Matthias Ehrgott
Department of Management Science
Lancaster University
Lancaster, United Kingdom
E-mail: m.ehrgott@lancaster.ac.uk

Hafiz Fukhar-ud-din
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia and
Department of Mathematics
The Islamia University of Bahawalpur
Bahawalpur, Pakistan
E-mail: hfdin@kfupm.edu.sa

Xavier Gandibleux
Laboratoire d'Informatique de Nantes Atlantique
Université de Nantes
Nantes, France
E-mail: x.gandibleux@univ-nantes.fr

Nicolas Hadjisavvas
Department of Product and Systems Design
University of the Aegean
Hermoupolis, Syros, Greece
E-mail: nhad@aegean.gr

Mohamed Amine Khamsi
Department of Mathematical Sciences
University of Texas at El Paso
El Paso, Texas, USA
and
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia
E-mail: mohamed@utep.edu

Abdul Rahim Khan
Department of Mathematics and Statistics
King Fahd University of Petroleum and Minerals

Dhahran, Saudi Arabia
E-mail: arahim@kfupm.edu.sa

Dinh The Luc
Avignon University
Avignon, France
E-mail: dtluc@univ-avignon.fr

Augusta Raţiu
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
E-mail: augu2003@yahoo.com

Part I

Fixed Point Theory

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Chapter 1

Iterative Construction of Common Fixed Points in Convex Metric Spaces

Abdul Rahim Khan
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

Hafiz Fukhar-ud-din
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia; and Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan

1.1 Introduction

The Banach Contraction Principle (BCP) asserts that a contraction on a complete metric space has a unique fixed point and its proof hinges on "Picard iterations." This principle is applicable to a variety of subjects such as integral equations, partial differential equations and image processing. This principle breaks down for nonexpansive mappings on metric spaces. This led to the introduction of Mann iterations in a Banach space [33]. Our aim is to study Mann-type iterations for some classes of nonlinear mappings in a metric space. We achieve it through the convex structure introduced by Takahashi [45]. In this chapter, iterative construction of common fixed points of asymptotically (quasi-) nonexpansive mappings [11] by using their explicit and implicit schemes on nonlinear domains such as $CAT(0)$ spaces, hyperbolic spaces, and convex metric spaces[1, 7, 22, 24, 45] will be presented. The new results provide a metric space version of the corresponding known results in Banach spaces and $CAT(0)$ spaces (for example, [12, 25, 32, 26, 47]).

1.2 Preliminaries

Let C be a nonempty subset of a metric space (X, d) and T be a mapping on C. Denote the set of fixed points of T by $F = \{x \in C : Tx = x\}$. The mapping T is said to be:

- *contraction* if there exists a constant $k \in [0, 1)$ such that $d(T(x), T(y)) \leq k\ d(x, y)$, for all $x, y \in C$;
- *uniformly L-Lipschitzian* if $d(T^n(x), T^n(y)) \leq L\ d(x, y)$, for some $L > 0$, $x, y \in C$, $n \geq 1$;
- *asymptotically nonexpansive* [11] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that $d(T^n(x), T^n(y)) \leq k_n\ d(x, y)$, for all $x, y \in C$ and $n \geq 1$;
- *asymptotically quasi-nonexpansive* if $F \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that $d(T^n(x), p) \leq k_n\ d(x, p)$, for all $x \in C$, $p \in F$ and $n \geq 1$.

For $n = 1$, the uniform L-Lipschitzian mapping is known as *L-Lipschitzian.* For $k_n = 1$ for $n \geq 1$ in the definitions above, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping becomes *nonexpansive mapping* and *quasi-nonexpansive mapping*, respectively.

A nonlinear mapping may or may not have a fixed point.

Example 1.1. Define $T : [1, \infty) \to [1, \infty)$ by

(a) $T(x) = \frac{x}{2} + 3$. Then, T is a contraction with $k = \frac{1}{2}$ and $F = \{6\}$,

(b) $T(x) = \frac{25}{26}(x + \frac{1}{x})$. Then, T is a contraction with $k = \frac{25}{26}$ and $F = \{5\}$,

(c) $T(x) = x + \frac{1}{x}$. Then, T is not a contraction and so by the Banach contraction principle, it has no fixed point.

The concept of quasi-nonexpansiveness is more general than that of nonexpansiveness. A nonexpansive mapping with at least one fixed point is quasi-nonexpansive. There are quasi-nonexpansive mappings which are not nonexpansive.

Example 1.2. (a) [36] Let $X = \mathbb{R}$ (the set of real numbers). Define $T_1 : \mathbb{R} \to \mathbb{R}$ by $T_1(x) = \frac{x}{2} \sin \frac{1}{x}$ with $T_1(0) = 0$. The only fixed point of T_1 is 0 as follows: if $x \neq 0$ and $T_1(x) = x$, then $x = \frac{x}{2} \sin \frac{1}{x}$ or $2 = \sin \frac{1}{x}$, which is impossible. T_1 is quasi-nonexpansive because $|T_1(x) - 0| = \left|\frac{x}{2}\right| \left|\sin \frac{1}{x}\right| \leq \frac{|x|}{2} < |x - 0|$ for all $x \in X$. However, T_1 is not nonexpansive mapping. This can be verified by choosing $x = \frac{2}{\pi}$, $y = \frac{2}{3\pi}$; $|T(x) - T(y)| = \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{2}{3\pi} \sin \frac{3\pi}{2} = \frac{8}{3\pi}$ and $|x - y| = \frac{4}{3\pi}$.

(b) [11] Let B be the unit ball in the Hilbert space l^2 and $\{a_i\}$ be a sequence of numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Define $T : B \to B$ by $T(x_1, x_2, x_3, \ldots) = \big(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\big)$. Then,

$$\|T^n(x) - T^n(y)\| \leq 2 \prod_{i=2}^{n} a_i \|x - y\| \text{ for } n \geq 2 \quad \text{and}$$
$$\|T(x) - T(y)\| \leq 2 \|x - y\|$$

give that T is asymptotically nonexpansive but not nonexpansive.

Remark 1.1. (a) The linear quasi-nonexpansive mappings are nonexpansive, but it is easy to verify that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive; for example, the above T_1.

(b) It is obvious that, if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

If a fixed point of a certain mapping exists and its exact value is not known, then we use an iterative procedure to find it. Here is an example : Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = \left(\frac{\sin y}{4}, \frac{\sin z}{3} + 1, \frac{\sin x}{5} + 2\right)$. Then, T is a contraction with $k = \frac{1}{3}$. By (BCP), T has a unique fixed point p. Its exact value is not known. Using the method of proof of (BCP), we can find an approximate value of p with a required accuracy. An answer is $p = (x_6, y_6, z_6) = (0.2406, 1.2961, 2.0477)$ within the accuracy of 0.001 to the fixed point (by measuring it through Euclidean distance).

The reader interested in the iterative approximation of fixed points(common fixed points) for various classes of mappings in the context of Banach spaces and metric spaces is referred to Berinde [3].

Many problems in science and engineering are nonlinear. So translating a linear version of known problems (usually in Banach spaces) into an equivalent nonlinear version (metric spaces) has great importance. This basic problem is usually considered in a $CAT(0)$ space. We include a brief description of a $CAT(0)$ space.

Let (X, d) be a metric space. A *geodesic* from x to y in X is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic (or metric) segment* joining x and y. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the *segment* joining x to y.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Such a triangle always exists [5].

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom:

Let Δ be a geodesic triangle in X and let $\overline{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

A complete $CAT(0)$ space is often called a *Hadamard space* (see [28]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies:

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This inequality is the (CN)-inequality of Bruhat and Titz [6]. The above inequality has been extended by Khamsi and Kirk [18] as follows:

$$\begin{aligned} d(z, \lambda x \oplus (1-\lambda) y)^2 \leq \lambda d(z, x)^2 + (1-\lambda) d(z, y)^2 \\ - \lambda(1-\lambda) d(x, y)^2, \end{aligned} \tag{CN*}$$

for any $\lambda \in [0, 1] = I$ and $x, y, z \in X$.

In 1970, Takahashi [45] introduced a concept of convex structure in a metric space (X, d) as a mapping $W : X^2 \times I \to X$ satisfying

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in I$.

A metric space (X, d) together with a convex structure W is a *convex metric space* (X, d, W), which will be denoted by X for simplicity. A nonempty subset C of a convex metric space X is *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in I$.

If X is a $CAT(0)$ space and $x, y \in X$, then for any $\lambda \in I$, there exists a unique point $\lambda x \oplus (1-\lambda) y \in [x, y] = \{\lambda x \oplus (1-\lambda) y : \lambda \in I\}$ such that

$$d(z, \lambda x \oplus (1-\lambda) y) \leq \lambda d(z, x) + (1-\lambda) d(z, y)$$

for any $z \in X$ (see [9] for details).

In view of the above inequality, a $CAT(0)$ space has Takahashi's convex structure $W(x, y, \alpha) = \alpha x \oplus (1-\alpha) y$.

We now give examples of convex metric spaces which cannot be Banach spaces [45].

Example 1.3. (a) Let $X = \{[a_i, b_i] : 0 \le a_i \le b_i \le 1\}$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$, and $\lambda \in I$, we define

$$W(I_i, I_j, \lambda) = [\lambda a_i + (1-\lambda) a_j, \lambda b_i + (1-\lambda) b_j],$$

$$d(I_i, I_j) = \sup_{\alpha \in I} \left\{ \left| \inf_{b \in I_i} \{|a-b|\} - \inf_{c \in I_j} \{|a-c|\} \right| \right\}, \text{Hausdorff distance on } X.$$

(b) A linear space X with a metric d on it and satisfying the properties: (i) $d(x, y) = d(x - y, 0)$, (ii) $d(\lambda x + (1-\lambda)y, 0) \le \lambda d(x, 0) + (1-\lambda)d(y, 0)$ for $x, y \in X$ and $\lambda \in I$, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1-\lambda)y$.

Recently, Kohlenbach [27] defined the concept of a hyperbolic space by including the following additional conditions in the definition of a convex metric space X:

$$\begin{array}{ll} (1) & d(W(x,y,\lambda_1), W(x,y,\lambda_2)) = |\lambda_1 - \lambda_2|\, d(x,y), \\ (2) & W(x,y,\lambda) = W(y,x,1-\lambda), \\ (3) & d(W(x,z,\lambda), W(y,w,\lambda)) \le \lambda d(x,y) + (1-\lambda)\, d(z,w), \end{array}$$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in I$.

If $X = \mathbb{R}$, $W(x, y, \lambda) = \lambda x + (1-\lambda)y$, and $d(x, y) = \frac{|x-y|}{1+|x-y|}$, for all $x, y \in \mathbb{R}$, then X is a convex metric space but not a hyperbolic space (the above condition (1) does not hold). In fact, every normed space and its convex subsets are hyperbolic spaces, but the converse is not true, in general.

Some other notions of hyperbolic space have been introduced and studied by Goebel and Reich [13], Khamsi and Khan [17], and Reich and Shafrir [39].

Now we prove some elementary properties of a convex metric space.

Lemma 1.1. Let X be a convex metric space. Then, for all $x, y \in X$ and $\lambda \in I$, we have the following:

(a) $W(x, y, 1) = x$ and $W(x, y, 0) = y$;

(b) $W(x, x, \lambda) = x$;

(c) $d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y) = d(x, y)$;

(d) the open sphere $S_r(x) = \{y \in X : d(y, x) < r\}$ and the closed sphere $S_r[x] = \{y \in X : d(y, x) \le r\}$ are convex subsets of X; and

(e) the intersection of convex subsets of X is convex.

Proof. (a) and (b) follow easily from the definition of W.

(c) Since X is a convex metric space, we obtain

$$\begin{aligned} d(x,y) &\le d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y) \\ &\le \lambda d(x,x) + (1-\lambda) d(x,y) + \lambda d(x,y) + (1-\lambda) d(y,y) \\ &= (1-\lambda) d(x,y) + \lambda d(x,y) \\ &= d(x,y), \end{aligned}$$

for all $x, y \in X$ and $\lambda \in I$.

It follows by the above inequalities that

$$d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y).$$

(d) Since X is a convex metric space,

$$\begin{aligned} d(x, W(y,z,\lambda)) &\leq \lambda d(x,y) + (1-\lambda)d(x,z) \\ &< \lambda r + (1-\lambda)r = r, \end{aligned}$$

for any $y, z \in S_r(x)$ and $\lambda \in I$. Hence, $W(y,z,\lambda) \in S_r(x)$. This proves that $S_r(x)$ is a convex subset of X. Similarly, we can prove that $S_r[x]$ is a convex subset of X.

(e) Follows by routine calculations. □

A convex metric space X is said to satisfy *Property* (G) [46]: whenever $w \in X$ and there is $(x,y,\lambda) \in X^2 \times I$ for which

$$d(z,w) \leq \lambda d(z,x) + (1-\lambda)d(z,y), \text{ for every } z \in X,$$

then $w = W(x,y,\lambda)$.

The Property (G) holds in the Euclidean plane equipped with the norm $\|(x_1,x_2)\| = |x_1| + |x_2|$.

The proof of next lemma depends on the Property (G).

Lemma 1.2. Let X be a convex metric space satisfying the Property (G). Then, we have the following assertions:

(a) $W(W(x,y,\lambda_1), y, \lambda_2) = W(x,y,\lambda_1\lambda_2)$, for every $x, y \in X$ and $\lambda_1, \lambda_2 \in I$;

(b) The function $f(\lambda) = W(x,y,\lambda)$ is an embedding (one-to-one function) of I into X, for every pair $x, y \in X$ with $x \neq y$.

Proof. (a) Let $z \in X$. Then

$$\begin{aligned} d(z, W(W(x,y,\lambda_1), y, \lambda_2)) &\leq \lambda_2 d(z, W(x,y,\lambda_1)) + (1-\lambda_2) d(z,y) \\ &\leq \lambda_2 [\lambda_1 d(z,x) + (1-\lambda_1) d(z,y)] \\ &\quad + (1-\lambda_2) d(z,y) \\ &\leq \lambda_2\lambda_1 d(z,x) + (1-\lambda_2\lambda_1) d(z,y). \end{aligned}$$

Hence, by Property (G), $W(W(x,y,\lambda_1), y, \lambda_2) = W(x,y,\lambda_1\lambda_2)$.

(b) Let $\lambda_1, \lambda_2 \in I$ such that $\lambda_1 \neq \lambda_2$. Assume, without loss of generality, that $\lambda_1 < \lambda_2$. Then,

$$\begin{aligned} d\left(f\left(\lambda_1\right), f\left(\lambda_2\right)\right) &= d\left(W\left(x, y, \lambda_1\right), W\left(x, y, \lambda_2\right)\right) \\ &= d\left(W\left(x, y, \lambda_2\left(\frac{\lambda_1}{\lambda_2}\right)\right), W\left(x, y, \lambda_2\right)\right) \\ &= d\left(W\left(W\left(x, y, \lambda_2\right), y, \left(\frac{\lambda_1}{\lambda_2}\right)\right), W\left(x, y, \lambda_2\right)\right) \\ &= \left[1 - \left(\frac{\lambda_1}{\lambda_2}\right)\right] d\left(W\left(x, y, \lambda_2\right), y\right) \\ &= \left(\lambda_2 - \lambda_1\right) d\left(x, y\right) > 0. \end{aligned}$$

That is, $f\left(\lambda_1\right) \neq f\left(\lambda_2\right)$ for $\lambda_1 \neq \lambda_2$. This proves that the function f is an embedding of I into X for every pair $x, y \in X$ with $x \neq y$. □

The argument in the proof of Lemma 1.2 (b) shows that the mapping $W(x, y, \lambda) \mapsto \lambda d(x, y)$ is an isometry of the subspace $\{W(x, y, \lambda) : \lambda \in I\}$ of X onto the closed interval $[0, d(x, y)]$. In particular, $\{W(x, y, \lambda) : \lambda \in I\}$ is homeomorphic with I if $x \neq y$ and, is a singleton if $x = y$. It is not clear whether a convex structure W satisfying the Property (G) is necessarily a continuous function.

However, we have the following result:

Lemma 1.3. Let X be a convex metric space. Then, W is continuous at each point (x, x, λ) of $X^2 \times I$.

Proof. Let $\{(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ be a sequence in $X^2 \times I$ that converges to (x, x, λ). Because $W\left(x, x, \lambda\right) = x$, it suffices to show that $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ converges to x. This is immediate as both the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge to x, and hence the definition of W yields

$$d\left(x, W(x_n, y_n, \lambda_n)\right) \leq \lambda_n d\left(x, x_n\right) + \left(1 - \lambda_n\right) d\left(x, y_n\right), \quad \text{for each } n \geq 1.$$

Thus, $d\left(x, x_n\right) \to 0$, $d\left(x, y_n\right) \to 0$, and $\lambda_n \to \lambda$ imply the conclusion. □

The difficulty in obtaining continuity of W as a mapping from the product lies in the fact that there seems to be no way to guarantee that the sequence $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ will converge when $\{(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ converges to (x, y, λ) with $x \neq y$. When X is compact, manage this difficulty as follows:

Lemma 1.4. Let X be a compact convex metric space satisfying the Property (G). Then, W is a continuous function.

Proof. Let $\{(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ be a sequence in $X^2 \times I$ that converges to (x, y, λ), and let w be a limit point of the sequence $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$. Select a subsequence $\{W(x_{n_k}, y_{n_k}, \lambda_{n_k})\}_{k=1}^{\infty}$ that converges to w. Then, for any $z \in X$, we have $d\left(z, W(x_{n_k}, y_{n_k}, \lambda_{n_k})\right) \leq \lambda_{n_k} d\left(z, x_{n_k}\right) + \left(1 - \lambda_{n_k}\right) d\left(z, y_{n_k}\right)$ for $k \geq 1$. By continuity of d, we conclude that $d(z, w) \leq \lambda d(z, x) + (1 - \lambda) d(z, y)$. The

Property (G) now guarantees that $w = W(x, y, \lambda)$. Hence, it follows that $W(x, y, \lambda)$ is the only limit point of the sequence $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$. Since X is compact, $\{W(x_n, y_n, \lambda_n)\}_{n=1}^{\infty}$ must converge to $W(x, y, \lambda)$ and we are done. $\square$

Next we define two geometric structures in a convex metric space and present their basic properties.

A convex metric (hyperbolic) space X is strictly convex [45] if for any $x, y \in X$ and $\lambda \in I$, there exists a unique element $z \in X$ such that $d(z, x) = \lambda d(x, y)$ and $d(z, y) = (1 - \lambda)d(x, y)$, and uniformly convex [43] if for any $\varepsilon > 0$, there exists $\alpha > 0$ such that $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r\,(1 - \alpha) < r$ for all $r > 0$ and $x, y, z \in X$ with $d\,(z, x) \leq r, d\,(z, y) \leq r$ and $d\,(x, y) \geq r\varepsilon$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ that provides such an $\alpha = \eta(r, \varepsilon)$ for $u, x, y \in X$, $r > 0$, and $\varepsilon \in (0, 2]$, is called *modulus of uniform convexity* [24] of X. We call η monotone if it decreases with respect to r (for a fixed ε).

Example 1.4. Let H be a Hilbert space and $C = \{x \in H : \|x\| = 1\}$. If $x, y \in C$ and $a, b \in I$ with $a + b = 1$, then $\frac{ax+by}{\|ax+bk\|} \in C$ and $\delta(C) \leq \sqrt{2}/2$, where $\delta(C)$ denotes the diameter of C. Let $d(x, y) = \cos^{-1}\{\langle x, y\rangle\}$ for every $x, y \in C$, where $\langle ., .\rangle$ is the inner product of H. Then, C is uniformly convex under $W\,(x, y, \lambda) = \lambda x + (1 - \lambda)\,y$.

Now we present some basic properties of a uniformly convex metric space.

Lemma 1.5. Let X be a uniformly convex metric space. Then, we have the following assertions:

(a) X is strictly convex.

(b) If $d\,(x, z) + d\,(z, y) = d\,(x, y)$ for all $x, y, z \in X$, then $z \in \{W\,(x, y, \lambda) : \lambda \in I\}$.

(c) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|\, d(x, y)$, for all $x, y \in X$ and $\lambda_1, \lambda_2 \in I$.

(d) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$, for all $x, y \in X$ and $\lambda \in I$.

Proof. (a) Assume that X is not strictly convex. If $x, y \in X$ and $\lambda \in I$, then there exist z_1, z_2 in X such that $z_1 \neq z_2$ and

$$d(z_1, x) = \lambda d(x, y) = d(z_2, x), d(z_1, y) = (1 - \lambda)d(x, y) = d(z_2, y).$$

It follows by $z_1 \neq z_2$ and the above identities that $x \neq y$ and $\lambda \in (0, 1)$. Let $r_1 = \lambda d(x, y) > 0, r_2 = (1 - \lambda)d(x, y) > 0$. Obviously, $\varepsilon_1 = \frac{d(z_1,z_2)}{r_1} > 0$ and $\varepsilon_2 = \frac{d(z_1,z_2)}{r_2} > 0$. Since X is uniformly convex, we have

$$d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_1\,(1 - \alpha_1) < r_1$$

and

$$d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_2\left(1-\alpha_2\right) < r_2.$$

Consider

$$\begin{aligned} d(x,y) &\leq d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) + d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq r_1\left(1-\alpha_1\right) + r_2\left(1-\alpha_2\right) \\ &< r_1 + r_2 \\ &= \lambda d(x,y) + (1-\lambda) d(x,y) \\ &= d(x,y), \end{aligned}$$

a contradiction to the reflexive property of real numbers.

(b) Let $x, y, z \in X$ be such that

$$d(x,z) + d(z,y) = d(x,y). \tag{1.1}$$

Let $u \in \{W(x,y,\lambda) : \lambda \in I\}$ be such that $d(x,u) = d(x,z)$. Then, by Lemma 1.1 (c),

$$d(x,u) + d(u,y) = d(x,y). \tag{1.2}$$

Comparing (1.1) and (1.2), we have that $d(z,y) = d(u,y)$. Now, we show that $z = u$. Assume instead that $z \neq u$. Let $v = W\left(x, y, \frac{1}{2}\right)$ and $r = d(x,u) = d(x,z)$. Since $d(z,u) > 0$, choose $\varepsilon > 0$ so that $d(z,u) > r\varepsilon$. By the uniform convexity of X, there exists $\alpha > 0$ such that

$$d(x,v) \leq r(1-\alpha) < r = d(x,z).$$

Similarly, we can show that $d(y,v) < d(y,z)$.

Therefore,

$$d(x,y) \leq d(x,v) + d(y,v) < d(x,z) + d(y,z) = d(x,y).$$

This is a contradiction to the reflexive property of real numbers. Hence, $z = u \in \{W(x,y,\lambda) : \lambda \in I\}$.

(c) Note that the conclusion holds if $\lambda_1 = 0$ or $\lambda_2 = 0$. Let $x, y \in X, \lambda_1, \lambda_2 \in (0,1]$, $u = W(y,x,\lambda_1)$, and $z = W(y,x,\lambda_2)$. Without loss of generality, we may assume that $\lambda_1 < \lambda_2$. Let $v = W\left(z, x, \frac{\lambda_1}{\lambda_2}\right)$. Then,

$$d(x,v) = \frac{\lambda_1}{\lambda_2} d(x,z) = \lambda_1 d(x,y),$$

and

$$d(v,y) \leq \left(1 - \frac{\lambda_1}{\lambda_2}\right) d(x,y) + \frac{\lambda_1}{\lambda_2} d(z,y) = (1-\lambda_1) d(x,y).$$

If $u \neq v$, let $w = W\left(u, v, \frac{1}{2}\right)$. By the uniform convexity of X, we can prove that $d(x, w) < d(x, u)$ and $d(y, w) < d(y, u)$. Therefore,

$$d(x, y) < d(x, u) + d(u, y) = d(x, y).$$

This contradicts the reflexive property of real numbers. Hence, $u = v$.

Now, it follows that

$$d(z, u) = d(z, v) = \left(1 - \frac{\lambda_1}{\lambda_2}\right) d(x, z) = |\lambda_2 - \lambda_1| \, d(x, y).$$

(d) Let $x, y \in X$ and $\lambda \in I$. Obviously, the conclusion holds if $\lambda = 0$ or $\lambda = 1$. By the definition of W, we have

$$d(x, W(x, y, \lambda)) = (1 - \lambda) d(x, y), \quad d(y, W(x, y, \lambda)) = \lambda d(x, y),$$

and

$$d(x, W(y, x, 1 - \lambda)) = (1 - \lambda) d(x, y), \quad d(y, W(y, x, 1 - \lambda)) = \lambda d(x, y).$$

Suppose that $W(x, y, \lambda) = z_1 \neq z_2 = W(y, x, 1 - \lambda)$.

Let $r_1 = (1 - \lambda) d(x, y) > 0$, $r_2 = \lambda d(x, y) > 0$, $\varepsilon_1 = \frac{d(z_1, z_2)}{r_1}$, and $\varepsilon_2 = \frac{d(z_1, z_2)}{r_2}$. Obviously $\varepsilon_1, \varepsilon_2 > 0$.

By uniform convexity of X, we have

$$d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_1 (1 - \alpha_1) < r_1;$$

$$d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \leq r_2 (1 - \alpha_2) < r_2.$$

Since $\lambda \in (0, 1)$, we get $x \neq y$.

Finally,

$$\begin{aligned} d(x, y) &\leq d\left(x, W\left(z_1, z_2, \frac{1}{2}\right)\right) + d\left(y, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq r_1 (1 - \alpha_1) + r_2 (1 - \alpha_2) \\ &< r_1 + r_2 = d(x, y), \end{aligned}$$

which is against the reflexivity of reals. Therefore, $W(x, y, \lambda) = W(y, x, 1 - \lambda)$. $\square$

A convex metric space X is said to satisfy the Property (H) [10] if

$$d(W(x, y, \lambda), W(z, y, \lambda)) \leq \lambda d(x, z) \quad \text{for all } x, y, z \in X \text{ and } \lambda \in I.$$

Lemma 1.6. Let X be a uniformly convex metric space satisfying the Property (H). Then, X is a uniformly hyperbolic space.

Proof. In the light of Lemma 1.5 (c)–(d), it is sufficient to show that

$$d(W(x,z,\lambda),W(y,w,\lambda)) \le \lambda d(x,y) + (1-\lambda)\, d(z,w),$$

for all $x, y, z, w \in X$, $\lambda \in I$. Using the triangle inequality, Lemma 1.5 (d), and the Property (H), we have

$$\begin{aligned} d(W(x,z,\lambda),W(y,w,\lambda)) &\le d(W(x,z,\lambda),W(x,w,\lambda)) \\ &\quad + d(W(x,w,\lambda),W(y,w,\lambda)) \\ &= d(W(z,x,1-\lambda),W(w,x,1-\lambda)) \\ &\quad + d(W(x,w,\lambda),W(y,w,\lambda)) \\ &\le (1-\lambda)\, d(z,w) + \lambda d(x,y) \\ &= \lambda d(x,y) + (1-\lambda)\, d(z,w). \end{aligned}$$

□

Lemma 1.7. Let X be a uniformly convex metric space satisfying the Property (H). Then, the convex structure W is continuous.

Proof. It has been shown in Lemma 1.6 that

$$d(W(x,z,\lambda),W(y,w,\lambda)) \le \lambda d(x,y) + (1-\lambda)\, d(z,w),$$

for all $x, y, z, w \in X$ and $\lambda \in I$.

Let $\{(x_n, y_n, \lambda_n)\}$ be any sequence in $X^2 \times I$ such that $(x_n, y_n, \lambda_n) \to (x, y, \lambda)$ for all $x, y \in X$ and $\lambda \in I$. We show that $W(x_n, y_n, \lambda_n) \to W(x, y, \lambda)$.

An application of Lemma 1.5 (c) and Lemma 1.6 provide:

$$\begin{aligned} d(W(x_n,y_n,\lambda_n),W(x,y,\lambda)) &\le d(W(x_n,y_n,\lambda_n),W(x,y,\lambda_n)) \\ &\quad + d(W(x,y,\lambda_n),W(x,y,\lambda)) \\ &\le \lambda_n d(x_n,x) + (1-\lambda_n)\, d(y_n,y) \\ &\quad + |\lambda_n - \lambda|\, d(x,y). \end{aligned}$$

Since $d(x_n, x) \to 0$, $d(\lambda_n, \lambda) \to 0$ and $|\lambda_n - \lambda| \to 0$, therefore $W(x_n, y_n, \lambda_n) \to W(x, y, \lambda)$. □

Lemma 1.8. Let X be a uniformly convex metric space with modulus of uniform convexity α(decreases for a fixed ε). If $d(x,z) \le r, d(y,z) \le r$, and $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \ge h > 0$ for all $x, y, z \in X$, then $d(x,y) \le r\eta\left(\frac{r-h}{r}\right)$ where η is the inverse of α.

Proof. Let $d(x,z) \le r, d(y,z) \le r$ and $d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \ge h > 0$ for all $x, y, z \in X$. To show that $d(x,y) \le r\eta\left(\frac{r-h}{r}\right)$, we assume instead that $d(x,y) > r\eta\left(\frac{r-h}{r}\right)$. Take $\frac{r-h}{r} < \varepsilon_1$ such that $d(x,y) \ge r\,\eta\left(\frac{r-h}{r}\right)$. Now using the uniform convexity of X, we have

$$\begin{aligned} d\left(z, W\left(x, y, \frac{1}{2}\right)\right) &\leq (1-\alpha\left(\eta\left(\varepsilon_1\right)\right)) r \\ &= (1-\varepsilon_1) r \\ &< \left(1-\frac{r-h}{r}\right) r \\ &= h. \end{aligned}$$

That is,

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) < h,$$

a contradiction to a given inequality. □

Lemma 1.9. Let X be a uniformly convex metric space with modulus of uniform convexity α (decreases for a fixed ε) and satisfies the Property (H). Let $x_1, x_2, x_3 \in B_r[u] \subset X$ and satisfy $d(x_1, x_2) \geq d(x_2, x_3) \geq l > 0$. If

$$d(u, x_2) \geq \left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) r, \tag{1.3}$$

then

$$d(x_1, x_3) \leq \eta\left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) d(x_1, x_2),$$

where η is the inverse of α.

Proof. Denote $z_1 = W\left(x_1, x_2, \frac{1}{2}\right)$, $z_2 = W\left(x_3, x_2, \frac{1}{2}\right)$, and $z = W\left(z_1, z_2, \frac{1}{2}\right)$. By the uniform convexity of X, we have

$$\begin{aligned} d(u, z) &= d\left(u, W\left(z_1, z_2, \frac{1}{2}\right)\right) \\ &\leq \frac{1}{2} d(u, z_1) + \frac{1}{2} d(u, z_2) \\ &= \frac{1}{2} d\left(u, W\left(x_1, x_2, \frac{1}{2}\right)\right) + \frac{1}{2} d\left(u, W\left(x_3, x_2, \frac{1}{2}\right)\right) \\ &\leq \left(1 - \alpha\left(\frac{l}{r}\right)\right) r. \end{aligned} \tag{1.4}$$

Using (1.4) in (1.3), we get

$$\begin{aligned} d(u, x_2) &\geq \left(1 - \frac{1}{2}\alpha\left(\frac{l}{r}\right)\right) r \\ &= \left(1 - \alpha\left(\frac{l}{r}\right)\right) r + \frac{1}{2}\alpha\left(\frac{l}{r}\right) r \\ &\geq d(u, z) + \frac{1}{2}\alpha\left(\frac{l}{r}\right) r. \end{aligned}$$

That is,

$$\begin{aligned}\frac{1}{2}\alpha\left(\frac{l}{r}\right)r &\leq d(u,x_2)-d(u,z)\\ &\leq d(x_2,z). \qquad (1.5)\end{aligned}$$

Since $d(x_2,z_i) \leq \frac{1}{2}d(x_1,x_2)$ for $i=1,2$, and $d(z_1,z_2) \geq \frac{1}{2}d(x_1,x_2)$, therefore by uniform convexity of X (with $r=\frac{1}{2}d(x_1,x_2), \varepsilon=1$), Property (H), and (1.5), we have

$$\begin{aligned}\frac{1}{2}\alpha\left(\frac{l}{r}\right)r &\leq d(x_2,z)\\ &= d\left(x_2, W\left(z_1,z_2,\frac{1}{2}\right)\right)\\ &\leq (1-\alpha(1))\,r\\ &\leq \left(1-\alpha\left(\frac{d(z_1,z_2)}{\frac{1}{2}d\,(x_1,x_2)}\right)\right)r\\ &\leq \left(1-\alpha\left(\frac{\frac{1}{2}d(x_1,x_3)}{\frac{1}{2}d(x_1,x_2)}\right)\right)r.\end{aligned}$$

That is,

$$\frac{1}{2}\alpha\left(\frac{l}{r}\right) \leq 1-\alpha\left(\frac{d(x_1,x_3)}{d(x_1,x_2)}\right).$$

Therefore,

$$d(x_1,x_3) \leq \eta\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)d(x_1,x_2),$$

where η is the inverse of α. □

The condition (1.3) in the above lemma holds as indicated by the following example with $\alpha(n)=\frac{n}{2}$.

Example 1.5. Define $d(x,y)=|x-y|$ on $B_1[0]=[-1,1]\subset\mathbb{R}$. Let $u=0, x_1=0.1, x_2=0.99$, and $x_3=0.3$. Note that $d(x_1,x_2)\geq d(x_2,x_3)\geq 0.2=l>0$, $\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)r=0.95$, and $d(u,x_2)\leq\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)r$. All the conditions of Lemma 1.9 are satisfied. Moreover, $d(x_1,x_3)\leq\eta\left(1-\frac{1}{2}\alpha\left(\frac{l}{r}\right)\right)d(x_1,x_2)$, where η is the inverse of α.

1.3 Ishikawa Iterative Scheme

Mann [33] and Ishikawa [15] iterative schemes for nonexpansive and quasi-nonexpansive mappings have been extensively studied in a uniformly convex

Banach space. Senter and Dotson [41] established convergence of the Mann iterative scheme of quasi-nonexpansive mappings satisfying two special conditions in a uniformly convex Banach space. A mapping T on a nonempty set C is a *generalized nonexpansive* [4] if

$$d(T(x), T(y)) \leq a\, d(x, y) + b\{d(x, T(x)) + d(y, T(y))\} + c\{d(x, T(y)) + d(y, T(x))\}, \tag{1.6}$$

for all $x, y \in C$, where $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$.

In 1973, Goebel et al. [12] proved that a generalized nonexpansive mapping has a fixed point in a uniformly convex Banach space. Based on their work, Bose and Mukerjee [4] proved convergence theorems for the Mann iterative scheme of generalized nonexpansive mapping and got the result obtained by Kannan [16] under relaxed conditions. Maiti and Ghosh [32] generalized the results of Bose and Mukerjee [4] for the Ishikawa iterative scheme using a modified version of the conditions of Senter and Dotson [41].

Based on Lemma 1.8 and Lemma 1.9, Fukhar-ud-din et al. [10] have obtained the following fixed point theorem for a continuous mapping satisfying (1.6) in a uniformly convex metric space.

Theorem 1.1. Let C be a nonempty, closed, convex, and bounded subset of a complete and uniformly convex metric space X satisfying the Property (H). If T is a continuous mapping on C satisfying (1.6), then T has a fixed point in C.

In this section, we approximate the fixed point of this continuous mapping satisfying (1.6). We assume that C is a nonempty, closed, and convex subset of a convex metric space X, and T is a mapping on C. For an initial value $x_1 \in C$, we define the *Ishikawa iterative scheme* in C as follows:

$$\begin{aligned} x_1 &\in C, \\ x_{n+1} &= W(T(y_n), x_n, \alpha_n), \\ y_n &= W(T(x_n), x_n, \beta_n), \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $\alpha_n, \beta_n \in I$.

If we choose $\beta_n = 0$, then (1.7) reduces to the following *Mann iterative scheme*:

$$x_1 \in C, \quad x_{n+1} = W(T(x_n), x_n, \alpha_n), \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\} \in I$.

On a convex subset C of a linear space X, $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ is a convex structure on X; (1.7) and (1.8), respectively, become Ishikawa [15] and Mann [33] schemes:

$$\begin{aligned} &x_1 \in C,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), \\ &y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \quad n \geq 1, \end{aligned} \tag{1.9}$$

and

$$x_1 \in C,\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n \geq 1, \tag{1.10}$$