Thierry Roncalli

Introduction to Risk Parity and Budgeting

Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20130607

International Standard Book Number-13: 978-1-4822-0716-3 (eBook - PDF)

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Introduction

The death of Markowitz optimization?

For a long time, investment theory and practice has been summarized as follows. The capital asset pricing model stated that the market portfolio is optimal. During the 1990s, the development of passive management confirmed the work done by William Sharpe. At that same time, the number of institutional investors grew at an impressive pace. Many of these investors used passive management for their equity and bond exposures. For asset allocation, they used the optimization model developed by Harry Markowitz, even though they knew that such an approach was very sensitive to input parameters, and in particular, to expected returns (Merton, 1980). One reason is that there was no other alternative model. Another reason is that the Markowitz model is easy to use and simple to explain. For expected returns, these investors generally considered long-term historical figures, stating that past history can serve as a reliable guide for the future. Management boards of pension funds were won over by this scientific approach to asset allocation.

The first serious warning shot came with the dot-com crisis. Some institutional investors, in particular defined benefit pension plans, lost substantial amounts of money because of their high exposure to equities (Ryan and Fabozzi, 2002). In November 2001, the pension plan of The Boots Company, a UK pharmacy retailer, decided to invest 100% in bonds (Sutcliffe, 2005). Nevertheless, the performance of the equity market between 2003 and 2007 restored confidence that standard financial models would continue to work and that the dot-com crisis was a non-recurring exception. However, the 2008 financial crisis highlighted the risk inherent in many strategic asset allocations. Moreover, for institutional investors, the crisis was unprecedentedly severe. In 2000, the internet crisis was limited to large capitalization stocks and certain sectors. Small capitalizations and value stocks were not affected, while the performance of hedge funds was flat. In 2008, the subprime crisis led to a violent drop in credit strategies and asset-backed securities. Equities posted negative returns of about -50%. The performance of hedge funds and alternative assets was poor. There was also a paradox. Many institutional investors diversified their portfolios by considering several asset classes and different regions. Unfortunately, this diversification was not enough to protect them. In the end, the 2008 financial crisis was more damaging than the dot-com crisis. This was particularly true for institutional investors in continental Europe, who were relatively well protected against the collapse of the internet bubble because of their low exposure to equities. This is why the 2008 financial crisis was a deep trauma for world-wide institutional investors.

Most institutional portfolios were calibrated through portfolio optimization. In this context, Markowitz's modern portfolio theory was strongly criticized by professionals, and several journal articles announced the death of the Markowitz model[1]. These extreme reactions can be explained by the fact that diversification is traditionally associated with Markowitz optimization, and it failed during the financial crisis. However, the problem was not entirely due to the allocation method. Indeed, much of the failure was caused by the input parameters. With expected returns calibrated to past figures, the model induced an overweight in equities. It also promoted assets that were supposed to have a low correlation to equities. Nonetheless, correlations between asset classes increased significantly during the crisis. In the end, the promised diversification did not occur.

Today, it is hard to find investors who defend Markowitz optimization. However, the criticisms concern not so much the model itself but the way it is used. In the 1990s, researchers began to develop regularization techniques to limit the impact of estimation errors in input parameters and many improvements have been made in recent years. In addition, we now have a better understanding of how this model works. Moreover, we also have a theoretical framework to measure the impact of constraints (Jagannathan and Ma, 2003). More recently, robust optimization based on the lasso approach has improved optimized portfolios (DeMiguel *et al.*, 2009). So the Markowitz model is certainly not dead. Investors must understand that it is a fabulous tool for combining risks and expected returns. The goal of Markowitz optimization is to find arbitrage factors and build a portfolio that will play on them. By construction, this approach is an aggressive model of active management. In this case, it is normal that the model should be sensitive to input parameters (Green and Hollifield, 1992). Changing the parameter values modifies the implied bets. Accordingly, if input parameters are wrong, then arbitrage factors and bets are also wrong, and the resulting portfolio is not satisfied. If investors want a more defensive model, they have to define less aggressive parameter values. This is the main message behind portfolio regularization. In consequence, reports of the death of the Markowitz model have been greatly exaggerated, because it will continue to be used intensively in active management strategies. Moreover, there are no other serious and powerful models to take into account return forecasts.

[1]See for example the article "*Is Markowitz Dead? Goldman Thinks So*" published in December 2012 by AsianInvestor.

The rise of risk parity portfolios

There are different ways to obtain less aggressive active portfolios. The first one is to use less aggressive parameters. For instance, if we assume that expected returns are the same for all of the assets, we obtain the minimum variance (or MV) portfolio. The second way is to use heuristic methods of asset allocation. The term 'heuristic' refers to experience-based techniques and trial-and-error methods to find an acceptable solution, which does not correspond to the optimal solution of an optimization problem. The equally weighted (or EW) portfolio is an example of such non-optimized 'rule of thumb' portfolio. By allocating the same weight to all the assets of the investment universe, we considerably reduce the sensitivity to input parameters. In fact, there are no active bets any longer. Although these two allocation methods have been known for a long time, they only became popular after the collapse of the internet bubble.

Risk parity is another example of heuristic methods. The underlying idea is to build a balanced portfolio in such a way that the risk contribution is the same for different assets. It is then an equally weighted portfolio in terms of risk, not in terms of weights. Like the minimum variance and equally weighted portfolios, it is impossible to date the risk parity portfolio. The term risk parity was coined by Qian (2005). However, the risk parity approach was certainly used before 2005 by some CTA and equity market neutral funds. For instance, it was the core approach of the All Weather fund managed by Bridgewater for many years (Dalio, 2004). At this point, we note that the risk parity portfolio is used, because it makes sense from a practical point of view. However, it was not until the theoretical work of Maillard *et al.* (2010), first published in 2008, that the analytical properties were explored. In particular, they showed that this portfolio exists, is unique and is located between the minimum variance and equally weighted portfolios.

Since 2008, we have observed an increasing popularity of the risk parity portfolio. For example, Journal of Investing and Investment and Pensions Europe (IPE) ran special issues on risk parity in 2012. In the same year, The Financial Times and Wall Street Journal published several articles on this topic[2]. In fact today, the term risk parity covers different allocation methods. For instance, some professionals use the term risk parity when the asset weight is inversely proportional to the asset return volatility. Others consider that the risk parity portfolio corresponds to the equally weighted risk contribution (or ERC) portfolio. Sometimes, risk parity is equivalent to a risk budgeting (or RB) portfolio. In this case, the risk budgets are not necessarily the same for all of the assets that compose the portfolio. Initially, risk parity

[2] "*New Allocation Funds Redefine Idea of Balance*" (February 2012), "*Same Returns, Less Risk*" (June 2012), "*Risk Parity Strategy Has Its Critics as Well as Fans*" (June 2012), "*Investors Rush for Risk Parity Shield*" (September 2012), etc.

only concerned a portfolio of bonds and equities. Today, risk parity is applied to all investment universes. Nowadays, risk parity is a *marketing term* used by the asset management industry to design a portfolio based on risk budgeting techniques.

More interesting than this marketing operation is the way risk budgeting portfolios are defined. Whereas the objective of Markowitz portfolios is to reach an expected return or to target ex-ante volatility, the goal of risk parity is to assign a risk budget to each asset. Like for the other heuristic approaches, the performance dimension is then absent and the risk management dimension is highlighted. In addition, this last point is certainly truer for the risk parity approach than for the other approaches. We also note that contrary to minimum variance portfolios, which have only seduced equity investors, risk parity portfolios concern not only different traditional asset classes (equities and bonds), but also alternative asset classes (commodities and hedge funds) and multi-asset classes (stock/bond asset mix policy and diversified funds). By placing risk management at the heart of these different management processes, risk parity represents a substantial break with respect to the previous period of Markowitz optimization. Over the last decades, the main objective of institutional investors was to generate performance well beyond the risk-free rate (sometimes approaching double-digit returns). After the 2008 crisis, investors largely revised their expected return targets. Their risk aversion level increased and they do not want to experience another period of such losses. In this context, risk management has become more important than performance management.

Nevertheless, like for many other hot topics, there is some exaggeration about risk parity. Although there are people who think that it represents a definitive solution to asset allocation problems, one should remain prudent. Risk parity remains a financial model of investment and its performance also depends on the investor's choice regarding parameters. Choosing the right investment universe or having the right risk budgets is as important as using the right allocation method. As a consequence, risk parity may be useful when defining a reliable allocation, but it cannot free investors of their duty of making their own choices.

About this book

The subject of this book is risk parity approaches. As noted above, risk parity is now a generic term used by the asset management industry to designate risk-based management processes. In this book, the term risk parity is used as a synonym of risk budgeting. When risk budgets are identical, we prefer to use the term ERC portfolio, which is more explicit and less overused by the investment industry. When we speak of a risk parity fund, it corresponds to an equally weighted risk contribution portfolio of equities and bonds.

This book comprises two parts. The first part is more theoretical. Its first chapter is dedicated to modern portfolio theory whereas the second chapter is a comprehensive guide to risk budgeting. The second part contains four chapters, each of which presents an application of risk parity to a specific asset class. The third chapter concerns risk-based equity indexation, also called smart indexing. In the fourth chapter, we show how risk budgeting techniques can be applied to the management of bond portfolios. The fifth chapter deals with alternative investments, such as commodities and hedge funds. Finally, the sixth chapter applies risk parity techniques to multi-asset classes. The book also contains two appendices. The first appendix provides the reader with technical materials on optimization problems, copula functions and dynamic asset allocation. The second appendix contains 30 tutorial exercises. The relevant solutions are not included in this book, but can be accessed at the following web page[3]:

`http://www.thierry-roncalli.com/riskparitybook.html`

This book began with an invitation by Professor Diethelm Würtz to present a tutorial on risk parity at the 6^{th} R/Rmetrics Meielisalp Workshop & Summer School on Computational Finance and Financial Engineering. This seminar is organized every year at the end of June in Meielisalp, Lake Thune, Switzerland. The idea of tutorial sessions is to offer an overview on a specialized topic in statistics or finance. When preparing this tutorial, I realized that I had sufficient material to write a book on risk parity. First of all, I would like to thank Diethelm Würtz and the participants of the Meielisalp Summer School for their warm welcome and the different discussions we had about risk parity. I would also like to thank all of the people who have invited me to academic and professional conferences in order to speak about risk parity techniques and applications since 2008, in particular Yann Braouezec, Rama Cont, Nathalie Columelli, Felix Goltz, Marie Kratz, Jean-Luc Prigent, Fahd Rachidy and Peter Tankov. I would also like to thank Jérôme Glachant and my other colleagues of the Master of Science in Asset and Risk Management program at the Évry University where I teach the course on Risk Parity. I am also grateful to the CRC editorial staff, in particular Sunil Nair, for their support, encouragement and suggestions.

I would also like to thank my different co-authors on this subject, Benjamin Bruder, Pierre Hereil, Sébastien Maillard, Jérôme Teïletche and Guillaume Weisang, my colleagues at Lyxor Asset Management who work or have worked with me on risk parity strategies, in particular Cyrille Albert-Roulhac, Florence Barjou, Cédric Baron, Benjamin Bruder, Zélia Cazalet, Léo Culerier, Raphael Dieterlen, Nicolas Gaussel, Pierre Hereil, Julien Laplante, Guillaume

[3]This web page also provides readers and instructors other materials related to the book (errata, code, slides, etc.).

Lasserre, Sébastien Maillard, François Millet and Jean-Charles Richard. I am also grateful to Abdelkader Bousabaa, Jean-Charles Richard and Zhengwei Wu for their careful reading of the preliminary versions of this book. Special thanks to Zhengwei Wu who has been a helpful and efficient research assistant.

Last but not least, I express my deep gratitude to Théo, Eva, Sarah, Lucie and Nathalie for their support and encouragement during the writing of this book.

Paris, January 2013 Thierry Roncalli

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List of Symbols and Notations

Symbol Description

Symbol	Description
$\cdot$	Scalar multiplication
$\circ$	Hadamard product: $(x \circ y)_i = x_i y_i$
$\otimes$	Kronecker product $A \otimes B$
$\lvert\mathcal{E}\rvert$	Cardinality of the set $\mathcal{E}$
$\mathbf{1}$	Vector of ones
$\mathbb{1}\{\mathcal{A}\}$	The indicator function is equal to 1 if $\mathcal{A}$ is true, 0 otherwise
$\mathbb{1}_{\mathcal{A}}\{x\}$	The characteristic function is equal to 1 if $x \in \mathcal{A}$, 0 otherwise
$\mathbf{0}$	Vector of zeros
$(A_{i,j})$	Matrix A with entry $A_{i,j}$ in row i and column j
A^{-1}	Inverse of the matrix A
$A^{\top}$	Transpose of the matrix A
A^{+}	Moore-Penrose pseudo-inverse of the matrix A
b	Vector of weights $(b_1, \ldots, b_n)$ for the benchmark b
$B_t(T)$	Price of the zero-coupon bond at time t for the maturity T
β_i	Beta of asset i with respect to portfolio x
$\beta_i(x)$	Another notation for the symbol β_i
$\beta(x \mid b)$	Beta of portfolio x when the benchmark is b
C (or ρ)	Correlation matrix
$\mathbf{C}$	Copula function
$C(t_m)$	Coupon paid at time t_m
$\operatorname{cov}(X)$	Covariance of the random vector X
$C_n(\rho)$	Constant correlation matrix $(n \times n)$ with $\rho_{i,j} = \rho$
D	Covariance matrix of idiosyncratic risks
$\det(A)$	Determinant of the matrix A
$\mathcal{DR}(x)$	Diversification ratio of portfolio x
$\mathbf{e}_i$	The value of the vector is 1 for the row i and 0 elsewhere
$\mathbb{E}[X]$	Mathematical expectation of the random variable X
$\mathcal{E}(\lambda)$	Exponential probability distribution with parameter λ
$\mathrm{ES}_\alpha(x)$	Expected shortfall of portfolio x at the confidence level α
$f(x)$	Probability density function (pdf)
$\mathbf{F}(x)$	Cumulative distribution function (cdf)
$\mathcal{F}$	Vector of risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$
$\mathcal{F}_j$	Risk factor j
$F_t(T)$	Instantaneous forward rate at time t for the maturity T
$F_t(T, m)$	Forward interest rate at

	time t for the period $[T, T+m]$
$\mathcal{G}$	Gini coefficient
γ	Parameter $\gamma = \phi^{-1}$ of the Markowitz γ-problem
γ_1	Skewness
γ_2	Excess kurtosis
$\mathcal{H}$	Herfindahl index
i	Asset i
I_n	Identity matrix of dimension n
$\mathrm{IR}(x \mid b)$	Information ratio of portfolio x when the benchmark is b
$\ell(\theta)$	Log-likelihood function with θ the vector of parameters to estimate
ℓ_t	Log-likelihood function for the observation t
$L(x)$	Loss of portfolio x
$\mathcal{L}(x)$	Leverage measure of portfolio x
$\mathbb{L}(x)$	Lorenz function
λ	Parameter of exponential survival times
$\mathcal{MDD}$	Maximum drawdown
$\mathcal{MR}_i$	Marginal risk of asset i
μ	Vector of expected returns $(\mu_1, \ldots, \mu_n)$
μ_i	Expected return of asset i
$\hat{\mu}$	Empirical mean
$\hat{\mu}_{1\mathrm{Y}}$	Annualized return
$\mu(x)$	Expected return of portfolio x: $\mu(x) = x^\top \mu$
$\mu(x \mid b)$	Expected return of the tracking error of portfolio x when the benchmark is b
$\mathcal{N}\left(\mu, \sigma^2\right)$	Probability distribution of a Gaussian random variable with mean μ and standard deviation σ
$\mathcal{N}(\mu, \Sigma)$	Probability distribution of a Gaussian random vector with mean μ and covariance matrix Σ
Ω	Covariance matrix of risk factors
π	Vector of risk premia $(\pi_1, \ldots, \pi_n)$
$\tilde{\pi}$	Vector of implied risk premia $(\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$
π_i	Risk premium of asset i: $\pi_i = \mu_i - r$
$\tilde{\pi}_i$	Implied risk premium of asset i
$\pi(y \mid x)$	Risk premium of portfolio y if the tangency portfolio is x: $\pi(y \mid x) = \beta(y \mid x)(\mu(x) - r)$
Π	P&L of the portfolio
ϕ	Risk aversion parameter of the quadratic utility function
$\phi(x)$	Probability density function of the standardized normal distribution
$\Phi(x)$	Cumulative distribution function of the standardized normal distribution
$\Phi^{-1}(\alpha)$	Inverse of the cdf of the standardized normal distribution
r	Return of the risk-free asset
$r^\star$	Yield to maturity
R	Vector of asset returns $(R_1, \ldots, R_n)$
R_i	Return of asset i
$R_{i,t}$	Return of asset i at time t
$R(x)$	Return of portfolio x: $R(x) = x^\top R$
$\mathcal{R}(x)$	Risk measure of portfolio x
$R_t(T)$	Zero-coupon rate at time t for the maturity T
$\mathcal{RC}_i$	Risk contribution of asset i
$\mathcal{RC}_i^\star$	Relative risk contribution of asset i
$\Re$	Recovery rate
ρ (or C)	Correlation matrix of asset returns

$\rho_{i,j}$	Correlation between asset returns i and j
$\rho(x, y)$	Correlation between portfolios x and y
$\mathfrak{s}$	Credit spread
$\mathbf{S}_t(x)$	Survival function at time t
Σ	Covariance matrix
$\hat{\Sigma}$	Empirical covariance matrix
σ_i	Volatility of asset i
σ_m	Volatility of the market portfolio
$\tilde{\sigma}_i$	Idiosyncratic volatility of asset i
$\hat{\sigma}$	Empirical volatility
$\hat{\sigma}_{1\mathrm{Y}}$	Annualized volatility
$\sigma(x)$	Volatility of portfolio x: $\sigma(x) = \sqrt{x^\top \Sigma x}$
$\sigma(x \mid b)$	Standard deviation of the tracking error of portfolio x when the benchmark is b
$\sigma(x, y)$	Covariance between portfolios x and y
$\sigma(X)$	Standard deviation of the random variable X
SR_i	Sharpe ratio of asset i: $\mathrm{SR}_i = \mathrm{SR}(\mathbf{e}_i \mid r)$
$\mathrm{SR}(x \mid r)$	Sharpe ratio of portfolio x when the risk-free asset is r
$\mathbf{t}_v(x)$	Cumulative distribution function of the Student's t distribution with ν the number of degrees of freedom
$\mathbf{t}_v^{-1}(\alpha)$	Inverse of the cdf of the Student's t distribution with ν the number of degrees of freedom
$\mathbf{t}_{\rho,v}(x)$	Cumulative distribution function of the multivariate Student's t distribution with parameters ρ and ν
$\tau(x)$	Turnover of portfolio x
$\mathrm{tr}(A)$	Trace of the matrix A
$\mathrm{TR}(x \mid b)$	Treynor ratio of portfolio x when the benchmark is b
$\mathrm{VaR}_\alpha(x)$	Value-at-risk of portfolio x at the confidence level α
x	Vector of weights $(x_1, \ldots, x_n)$ for portfolio x
x_i	Weight of asset i in portfolio x
$x^\star$	Optimized portfolio

Portfolio Notation

ERC	Equally weighted risk contribution portfolio x_{erc}
EW	Equally weighted portfolio x_{ew}
MDP	Most diversified portfolio x_{mdp}
MSR	Max Sharpe ratio portfolio x_{msr}
MV	Minimum variance portfolio x_{mv}
MVO	Mean-variance optimized (or Markowitz) portfolio x_{mvo}
RB	Risk budgeting portfolio x_{rb}
RFP	Risk factor parity portfolio x_{rfp}
RP	Risk parity portfolio x_{rp}
WB	Weight budgeting portfolio x_{wb}

Part I

From Portfolio Optimization to Risk Parity

This part comprises two chapters. In the first chapter, we present the theoretical foundations of modern portfolio theory. We also show how this framework is implemented in practice and describe its limitations. The second chapter presents the risk budgeting approach. The main difference with the previous approach comes from the investor objective. Indeed, his objective is not to maximize a utility function or a risk-adjusted performance, but only to allocate the risk between assets. Consequently, the risk parity method does not need assumptions about expected returns and therefore constitutes a pure method of risk management.

Chapter 1

Modern Portfolio Theory

The concept of the market portfolio has a long history and dates back to the seminal work of Markowitz (1952). In his paper, Markowitz defined precisely what *portfolio selection* means: "*the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing*". Indeed, Markowitz showed that an efficient portfolio is the portfolio that maximizes the expected return for a given level of risk (corresponding to the variance of portfolio return). Markowitz concluded that there is not only one optimal portfolio, but a set of optimal portfolios which is called the efficient frontier.

By studying the liquidity preference, Tobin (1958) showed that the efficient frontier becomes a straight line in the presence of a risk-free asset. In this case, optimal portfolios correspond to a combination of the risk-free asset and one particular efficient portfolio named the tangency portfolio. Sharpe (1964) summarized the results of Markowitz and Tobin as follows: "*the process of investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets*[1]*; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless asset*". This two-step procedure is today known as the *Separation Theorem* (Lintner, 1965).

One difficulty when computing the tangency portfolio is to precisely define the vector of expected returns of the risky assets and the corresponding covariance matrix of asset returns. In 1964, Sharpe developed the CAPM theory and highlighted the relationship between the risk premium of the asset (the difference between the expected return and the risk-free rate) and its beta (the systematic risk with respect to the tangency portfolio). By assuming that the market is at equilibrium, he showed that the prices of assets are such that the tangency portfolio is the market portfolio, which is composed of all risky assets in proportion to their market capitalization. This implies that we do not need assumptions about the expected returns, volatilities and correlations of assets to characterize the tangency portfolio. This major contribution of Sharpe led to the emergence of index funds and to the increasing development of passive management.

In the active management domain, fund managers use the Markowitz framework to optimize portfolios in order to take into account their views

[1] It is precisely the tangency portfolio.

and to play their bets. However, the implementation of portfolio theory is not simple. It requires the estimation of the covariance matrix and the forecasting of asset returns. One problem is that optimized portfolios are very sensitive to these inputs. Some stability issues make the practice of portfolio optimization less attractive than the theory (Michaud, 1989). In this case, regularization techniques may be employed to attenuate these problems. This approach is largely supported by Ledoit and Wolf (2003), who propose to combine different covariance matrix estimators to stabilize the solution. Today, the most promising approach consists in interpreting optimized portfolios as the solution of a linear regression problem and to use lasso or ridge penalization.

However, regularization is not sufficient to obtain satisfactory solutions, which is why practitioners introduce some constraints in the optimization problem. These constraints may be interpreted as a shrinkage method (Jagannathan and Ma, 2003). By imposing weight constraints, the portfolio manager implicitly changes the covariance matrix. This approach is then equivalent to having some views and is therefore related to the model of Black and Litterman (1992).

1.1 From optimized portfolios to the market portfolio

In this section, we review the seminal framework of Markowitz and the CAPM theory of Sharpe.

1.1.1 The efficient frontier

Sixty years ago, Markowitz introduced the concept of the efficient frontier. It was the first mathematical formulation of optimized portfolios. For him, "*the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing*". By translating these principles into a problem of mean-variance optimization, Markowitz (1952) showed that there is no one optimal portfolio, but a set of optimized portfolios.

We consider a universe of n assets. Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio. We assume that the portfolio is fully invested meaning that $\sum_{i=1}^n x_i = \mathbf{1}^\top x = 1$. We denote $R = (R_1, \ldots, R_n)$ the vector of asset returns where R_i is the return of asset i. The return of the portfolio is then equal to $R(x) = \sum_{i=1}^n x_i R_i$. In a matrix form, we also obtain $R(x) = x^\top R$. Let $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right]$ be the vector of expected returns and the covariance matrix of asset returns. The expected return of the portfolio is:

$$\mu(x) = \mathbb{E}[R(x)] = \mathbb{E}\left[x^\top R\right] = x^\top \mathbb{E}[R] = x^\top \mu$$

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(x) &= \mathbb{E}\left[(R(x)-\mu(x))(R(x)-\mu(x))^\top\right] \\
&= \mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(x^\top R - x^\top \mu\right)^\top\right] \\
&= \mathbb{E}\left[x^\top (R-\mu)(R-\mu)^\top x\right] \\
&= x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] x \\
&= x^\top \Sigma x
\end{aligned}$$

We can then formulate the investor's financial problem as follows:

1. Maximizing the expected return of the portfolio under a volatility constraint (σ-problem):
$$\max \mu(x) \quad \text{u.c.} \quad \sigma(x) \le \sigma^\star \tag{1.1}$$

2. Or minimizing the volatility of the portfolio under a return constraint (μ-problem):
$$\min \sigma(x) \quad \text{u.c.} \quad \mu(x) \ge \mu^\star \tag{1.2}$$

Example 1 *We consider four assets. Their expected returns are equal to* 5%, 6%, 8% *and* 6% *while their volatilities are equal to* 15%, 20%, 25% *and* 30%. *The correlation matrix of asset returns is given by the following matrix:*

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

In Figure 1.1, we have simulated 1 000 portfolios and reported their expected return and their volatility (cross symbol). Let us consider the σ-problem with $\sigma^\star = 30\%$. Portfolio C could not be the solution even if it reached the volatility constraint, because it is dominated by portfolio B. However, this portfolio is not optimal, as we can find other portfolios with a higher expected return. Finally, the solution is portfolio A. In the same way, the optimal portfolio is D in the case of the μ-problem with $\mu^\star = 7\%$. The efficient frontier is then defined as the convex hull of the points $(\sigma(x), \mu(x))$ of all the possible portfolios. This convex hull may be computed numerically. In Figure 1.1, we have indicated the portfolios belonging to the convex hull by a solid circle symbol. In particular, the two optimal portfolios A and D are on the efficient frontier.

By considering all the portfolios belonging to the simplex set defined by $\left\{x \in [0,1]^n : \mathbf{1}^\top x = 1\right\}$, we can compute the expected return and volatility bounds of the portfolios: $\mu^- \le \mu(x) \le \mu^+$ and $\sigma^- \le \sigma(x) \le \sigma^+$. There is also a solution to the first problem if $\sigma^\star \ge \sigma^-$. The second problem has a solution if $\mu^\star \le \mu^+$. If these two conditions are verified, the inequality constraints becomes $\sigma(x) = \min(\sigma^\star, \sigma^+)$ and $\mu(x) = \max(\mu^-, \mu^\star)$.

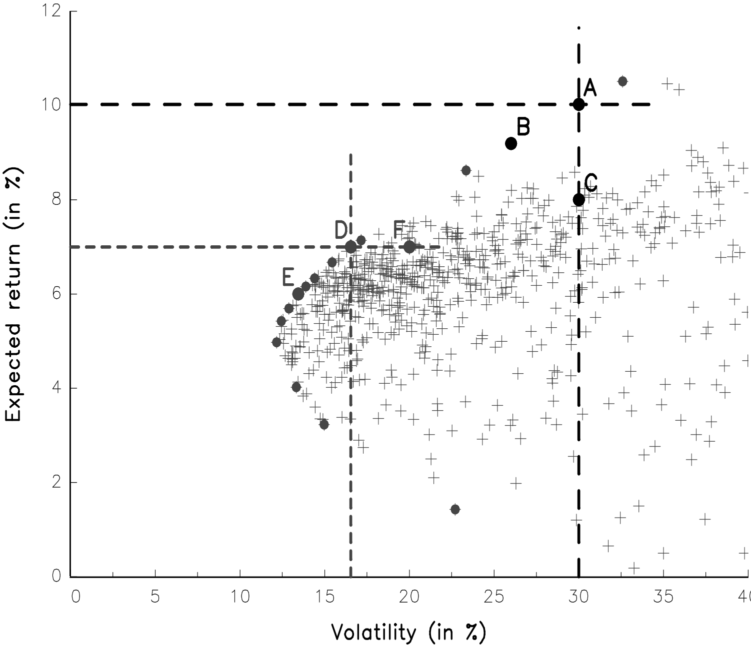

FIGURE 1.1: Optimized Markowitz portfolios

1.1.1.1 Introducing the quadratic utility function

The key idea of Markowitz (1956) was to transform the original non-linear optimization problem (1.1) into a quadratic optimization problem which is easier to solve numerically:

$$\begin{aligned} x^{\star}(\phi) &= \arg\max x^{\top}\mu - \frac{\phi}{2} x^{\top}\Sigma x \\ & \text{u.c.} \quad \mathbf{1}^{\top} x = 1 \end{aligned} \tag{1.3}$$

We can interpret ϕ as a risk-aversion parameter. If $\phi = 0$, the optimized portfolio is the one that maximizes the expected return and we have $\mu(x^{\star}(0)) = \mu^{+}$. If $\phi = \infty$, the optimization problem becomes:

$$\begin{aligned} x^{\star}(\infty) &= \arg\min \frac{1}{2} x^{\top}\Sigma x \\ & \text{u.c.} \quad \mathbf{1}^{\top} x = 1 \end{aligned}$$

The optimized portfolio is the one that minimizes the volatility and we have $\sigma(x^{\star}(\infty)) = \sigma^{-}$. It is called the minimum variance (or MV) portfolio.

Remark 1 *We note that the previous problem can also be written as follows:*

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ \text{u.c.} &\quad \mathbf{1}^\top x = 1 \end{aligned} \tag{1.4}$$

with $\gamma = \phi^{-1}$*. From a numerical point of view, this formulation has the advantage to be a standard quadratic programming (QP) problem (see Appendix A.1.1 on page 301). In this case, the minimum variance portfolio corresponds to* $\gamma = 0$*. Depending on the objective, we will use either the* ϕ*-problem or the* γ*-problem to calculate optimized portfolios.*

We consider Example 1. We have reported[2] in Table 1.1 the optimal portfolio for different values of ϕ. We verify that $\mu(x^\star(\phi))$ and $\sigma(x^\star(\phi))$ are two decreasing functions with respect to the parameter ϕ. It implies that the expected return $\mu(x^\star)$ is an increasing function of the volatility $\sigma(x^\star)$.

TABLE 1.1: Solving the ϕ-problem

ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(x^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(x^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

The formulation (1.3) allows to give a new characterization of the efficient frontier. It is the parametric function $(\sigma(x^\star(\phi)), \mu(x^\star(\phi)))$ with $\phi \in \mathbb{R}_+$. If we consider the previous example, we obtain the efficient frontier in Figure 1.2. We note that optimized portfolios substantially improve the risk/return profile with respect to the four assets, which are represented by a cross symbol.

Solving the μ-problem or the σ-problem is equivalent to finding the optimal value of ϕ such that $\mu(x^\star(\phi)) = \mu^\star$ or $\sigma(x^\star(\phi)) = \sigma^\star$. We know that the functions $\mu(x^\star(\phi))$ and $\sigma(x^\star(\phi))$ are decreasing with respect to ϕ and are bounded. The optimal value of ϕ can then be easily computed using the Newton-Raphson algorithm. We have reported some numerical solutions in Tables 1.2 and 1.3. For example, if $\mu^\star$ is set to 7%, we obtain a portfolio with a volatility $\sigma(x^\star)$ equal to 16.54%. It corresponds to portfolio D in Figure 1.1. If we target a volatility equal to 30%, the expected return of the optimized portfolio is 10.02% and the solution is portfolio A in Figure 1.1.

[2]In this book, the values of weights, expected returns and volatilities are expressed in % except if another unit is specified.

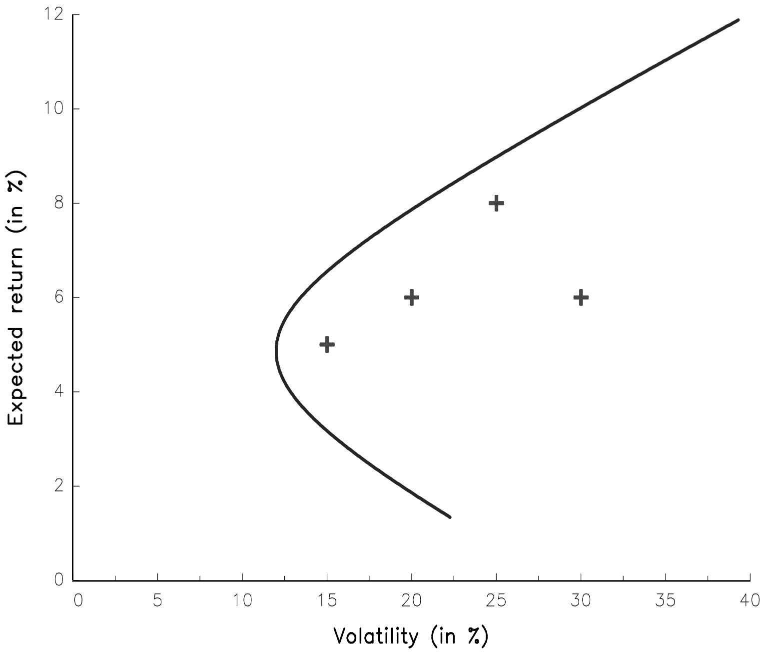

FIGURE 1.2: The efficient frontier of Markowitz

TABLE 1.2: Solving the unconstrained μ-problem

$\mu^\star$	5.00	6.00	7.00	8.00	9.00
$x_1^\star$	71.92	65.87	59.81	53.76	47.71
$x_2^\star$	46.73	26.67	6.62	−13.44	−33.50
$x_3^\star$	−14.04	32.93	79.91	126.88	173.86
$x_4^\star$	−4.60	−25.47	−46.34	−67.20	−88.07
$\sigma(x^\star)$	12.02	13.44	16.54	20.58	25.10
ϕ	25.79	3.10	1.65	1.12	0.85

TABLE 1.3: Solving the unconstrained σ-problem

$\sigma^\star$	15.00	20.00	25.00	30.00	35.00
$x_1^\star$	62.52	54.57	47.84	41.53	35.42
$x_2^\star$	15.58	−10.75	−33.07	−54.00	−74.25
$x_3^\star$	58.92	120.58	172.85	221.88	269.31
$x_4^\star$	−37.01	−64.41	−87.62	−109.40	−130.48
$\mu(x^\star)$	6.55	7.87	8.98	10.02	11.03
ϕ	2.08	1.17	0.86	0.68	0.57

1.1.1.2 Adding some constraints

The introduction of constraints consists in modifying the specification of the optimization problem (1.3):

$$\begin{aligned} x^{\star}(\phi) &= \arg\max x^{\top}\mu - \frac{\phi}{2}x^{\top}\Sigma x \\ \text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}^{\top}x = 1 \\ x \in \Omega \end{array} \right. \end{aligned} \tag{1.5}$$

where $x \in \Omega$ corresponds to the set of restrictions[3]. These restrictions may be linear or non-linear. In the latter case, the optimization problem cannot be solved by the standard quadratic programming algorithm, but by enhanced non-linear optimization algorithms. The imposition of constraints will impact the set of optimized portfolios by reducing opportunity arbitrages. It implies that the constrained efficient frontier is located at the right of the unconstrained efficient frontier in the mean-variance map.

The most frequent constraint is certainly the no short-selling restriction. In this case, $x_i \geq 0$ and $\Omega = [0,1]^n$. Let us define the leverage measure of the portfolio x as the sum of the absolute values of the weights:

$$\mathcal{L}(x) = \sum_{i=1}^{n} |x_i|$$

With the no short-selling restriction, the leverage measure is 100% because $\mathcal{L}(x) = \sum_{i=1}^{n} x_i = 1$ whereas it is larger than 100% without this constraint[4].

Let us introduce some constraints in Example 1. In Figure 1.3, we have reported two constrained efficient frontiers, the first one by imposing no short-selling and the second one by imposing that the weights are between 0% and 40%. We verify that we may substantially reduce opportunity arbitrages. Solutions of the σ-problem are given in Table 1.4. If we target a volatility equal of 15%, the expected return of the optimized portfolio is 6.55% for the unconstrained problem, 6.14% for the shortsale constrained problem and 6.11% if we impose an upper bound of 40%. So, by imposing no short positions, we have reduced the expected return by 41 bps. The impact of the upper bound is small. If the target volatility becomes 20%, the results become 7.87%, 7.15% and 6.74% and the impact is larger than in the previous case.

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[3]The restriction $\mathbf{1}^{\top}x = 1$ is already a constraint influencing the optimized portfolio (DeMiguel *et al.*, 2009).

[4]Let $x_i^- = -\min(0, x_i)$ and $x_i^+ = \max(0, x_i)$ be respectively the negative and positive parts of the weight x_i. We have $x_i = x_i^+ - x_i^-$. It follows that $\mathcal{L}(x) = \sum_{i=1}^{n} \left| x_i^+ - x_i^- \right| = \sum_{i=1}^{n} x_i^+ + \sum_{i=1}^{n} x_i^-$ with $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^+ - \sum_{i=1}^{n} x_i^- = 1$. It implies that $\mathcal{L}(x) = 1 + 2\sum_{i=1}^{n} x_i^-$ meaning that the leverage measure is larger than 1 because $\sum_{i=1}^{n} x_i^- \geq 0$.

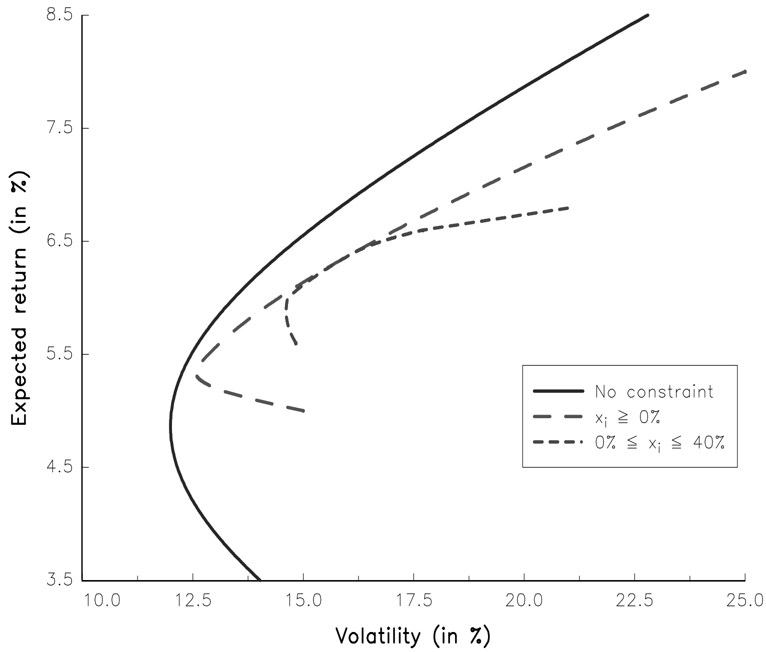

FIGURE 1.3: The efficient frontier with some weight constraints

TABLE 1.4: Solving the σ-problem with weight constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	−10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	−37.01	−64.41	0.00	0.00	0.00	13.87
$\mu(x^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

1.1.1.3 Analytical solution

The Lagrange function of the optimization problem (1.3) is:

$$\mathcal{L}(x;\lambda_0) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}^\top x - 1\right)$$

where λ_0 is the Lagrange coefficients associated with the constraint $\mathbf{1}^\top x = 1$. The solution $x^\star$ verifies the following first-order conditions:

$$\left\{ \begin{array}{l} \partial_x \mathcal{L}(x;\lambda_0) = \mu - \phi \Sigma x + \lambda_0 \mathbf{1} = \mathbf{0} \\ \partial_{\lambda_0} \mathcal{L}(x;\lambda_0) = \mathbf{1}^\top x - 1 = 0 \end{array} \right.$$

We obtain $x = \phi^{-1}\Sigma^{-1}(\mu + \lambda_0 \mathbf{1})$. Because $\mathbf{1}^\top x - 1 = 0$, we have $\mathbf{1}^\top \phi^{-1}\Sigma^{-1}\mu + \lambda_0 \left(\mathbf{1}^\top \phi^{-1}\Sigma^{-1}\mathbf{1}\right) = 1$. It follows that:

$$\lambda_0 = \frac{1 - \mathbf{1}^\top \phi^{-1}\Sigma^{-1}\mu}{\mathbf{1}^\top \phi^{-1}\Sigma^{-1}\mathbf{1}}$$

The solution is then[5]:

$$x^\star(\phi) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} + \frac{1}{\phi} \cdot \frac{\left(\mathbf{1}^\top \Sigma^{-1}\mathbf{1}\right)\Sigma^{-1}\mu - \left(\mathbf{1}^\top \Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \tag{1.6}$$

We deduce also that the global minimum variance portfolio has the following expression:

$$x_{\mathrm{mv}} = x^\star(\infty) = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}}$$

If we introduce other constraints, it is not possible to obtain a comprehensive analytical solution. Let us consider for example the no short-selling constraint. The Lagrange function becomes:

$$\mathcal{L}(x;\lambda_0,\lambda) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}^\top x - 1\right) + \lambda^\top x$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the vector of Lagrange coefficients associated with the constraints $x_i \geq 0$. The first-order condition is then $\mu - \phi \Sigma x + \lambda_0 \mathbf{1} + \lambda = \mathbf{0}$. It follows that $x = \phi^{-1}\Sigma^{-1}(\mu + \lambda_0 \mathbf{1} + \lambda)$. The Kuhn-Tucker conditions are $\min(\lambda_i, x_i) = 0$ for all $i = 1, \ldots, n$. It implies that if $x_i = 0$ then $\lambda_i > 0$ and if $x_i > 0$ then $\lambda_i = 0$. We find also a formula close to the previous one, but the universe is limited to assets which present positive weights. This formula is therefore endogenous.

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[5] If we do not impose the constraint $\mathbf{1}^\top x = 1$, the solution becomes:

$$x^\star(\phi) = \frac{1}{\phi}\Sigma^{-1}\mu$$

1.1.2 The tangency portfolio

We recall that in the view of Markowitz, there is a set of optimized portfolios. However, Tobin showed in 1958 that one optimized portfolio dominates all the others if there is a risk-free asset.

Let us consider a combination of the risk-free asset and a portfolio x. We denote r the return of the risk-free asset. We have[6]:

$$R(y) = (1-\alpha) r + \alpha R(x)$$

where $y = \begin{pmatrix} \alpha x \\ 1-\alpha \end{pmatrix}$ is a vector of dimension $(n+1)$ and $\alpha \geq 0$ is the proportion of the wealth invested in the risky portfolio. It follows that:

$$\mu(y) = (1-\alpha) r + \alpha \mu(x) = r + \alpha (\mu(x) - r)$$

and:

$$\sigma^2(y) = \alpha^2 \sigma^2(x)$$

We deduce that:

$$\mu(y) = r + \frac{(\mu(x) - r)}{\sigma(x)} \sigma(y)$$

It is the equation of a linear function between the volatility and the expected return of the combined portfolio y. In Figure 1.4, we reported the previous (unconstrained) efficient frontier. The dashed line corresponds to the combination between the risk-free asset (r is equal to 1.5%) and the optimized portfolio A. Nevertheless this combination is suboptimal, because it is dominated by other combinations. We note that a straight line dominates all the other straight lines and the efficient frontier. This line is tangent to the efficient frontier and is called the capital market line. It implies that one optimized risky portfolio dominates all the other risky portfolios, namely the tangency portfolio.

Let $\mathrm{SR}(x \mid r)$ be the Sharpe ratio of portfolio x:

$$\mathrm{SR}(x \mid r) = \frac{\mu(x) - r}{\sigma(x)}$$

We note that we can write the previous equation as follows:

$$\frac{\mu(y) - r}{\sigma(y)} = \frac{\mu(x) - r}{\sigma(x)} \Leftrightarrow \mathrm{SR}(y \mid r) = \mathrm{SR}(x \mid r)$$

We deduce that the tangency portfolio is the one that maximizes the angle θ or equivalently $\tan\theta$ which is equal to the Sharpe ratio. The tangency portfolio is also the risky portfolio corresponding to the maximum Sharpe ratio. We

[6] We have $n+1$ assets in the universe where the first n assets correspond to the previous risky assets and the last asset is the risk-free asset.