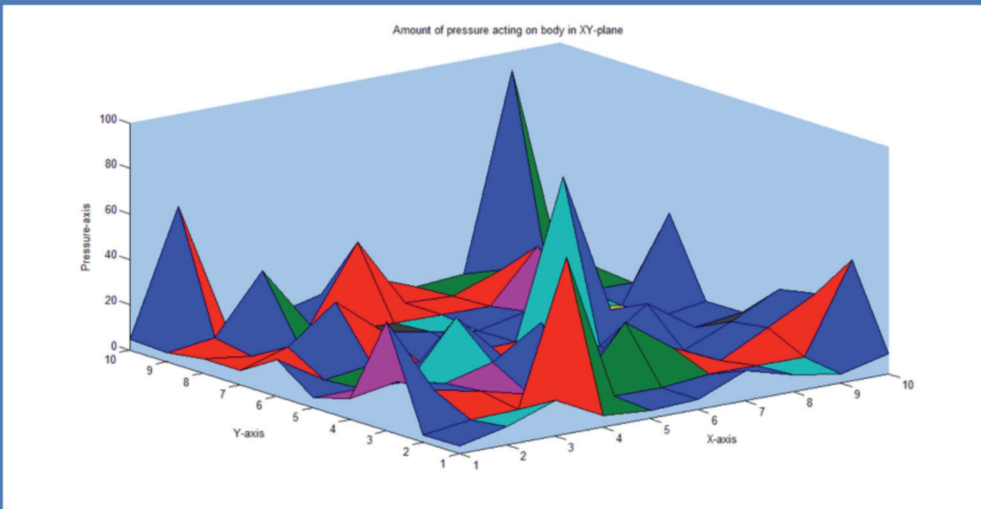
APPROXIMATE ANALYTICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS



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Preface

It is needless to say that differential equations play an important role in modeling many physical, engineering, technological, and biological processes. The differential equations in these diverse contexts may not be directly solvable by the usual elementary methods and hence in general do not have exact or closed-form solutions. In all such cases, researchers have tried to obtain either analytical approximate solutions or numerical approximate solutions. With the available high-speed computers and excellent computational algorithms, considerable advancement has been made in obtaining good numerical solutions. However, there have been trials to obtain approximate analytical solutions, and several approximate analytical methods have been developed to cater to the needs that have arisen and with a view to obtain "better-and-better" solutions. The methods range from the classical series solution method to the diverse perturbation methods and from the pioneering asymptotic methods to the recent ingenious homotopy methods.

This book aims to present some important approximate methods for solving ordinary differential equations and provides a number of illustrations. While teaching some related courses, we felt the need for a book of this type because there is no single book with all the available approximate methods for solving ordinary differential equations. At present, a student or a researcher interested in understanding the state of the art has to wade through several books and research articles to grasp the diverse methods. This book covers both the well-established techniques and the recently developed procedures along with detailed examples elucidating the applications of these methods to solve related real-world problems. It aims to give a complete description of the methods considered and discusses them through illustrative examples without going into several of the rigorous mathematical aspects.

Chapter 1 is introductory. We explain briefly the methods chosen for discussion in the present work.

Chapter 2 introduces the classical method of solving differential equations through the power series method. In fact, this method has been the basis for the introduction and the development of various special functions found in the literature. We explain and illustrate the method with a number of examples and proceed to describe the Taylor series method.

Chapter 3 deals with the asymptotic methods, which can be used to find asymptotic solutions to the differential equations that are valid for large values of the independent variable and in other cases as well.

The introduction of perturbation methods, which constitutes one of the top ten progresses of theoretical and applied mechanics of the 20th century, is the focus of Chapter 4. Attention is drawn to some research articles in which the perturbation methods are used successfully in understanding some physical phenomena whose mathematical formulation involves a so-called perturbation parameter.

Chapter 5 focuses on a special asymptotic technique called the multiple-scale technique for solving the problems whose solution cannot be completely described on a single timescale.

Chapter 6 describes an important asymptotic method called the WKB (for its developers, Wentzel, Kramers, and Brillown) method that helps construct solutions to problems that oscillate rapidly and problems for which there is a sudden change in the behavior of the solution function at a point in the interval of interest.

Chapter 7 deals with some nonperturbation methods, such as the Adomian decomposition method, delta expansion method, and others, that were developed during the last two decades and can provide solutions to a much wider class of problems.

Chapter 8 presents the most recent analytical methods developed, which are based on the concept of homotopy of topology and were initiated by Liao. The methods are the homotopy analysis method, homotopy perturbation method, and optimal homotopy asymptotic method.

Our principal aim is to present and explain the methods with emphasis on problem solving. Many illustrations are presented in each chapter.

The content of this book was drawn from diverse sources, which are cited in each chapter. Further, attention is drawn to some research articles that discuss use of the methods. We are grateful to the authors of all the works cited.

We believe this book will serve as a handbook not only for mathematicians and engineers but also for biologists, physicists, and economists. The book presupposes knowledge of advanced calculus and an elementary course on differential equations.

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1
Introduction

Many real-world problems involving change of variables with reference to spatial variables or time in science and engineering are modeled by researchers through differential equations with imposed boundary or initial conditions or both. Even if the governing equations are partial differential equations, solutions of the problems, in general, involve solutions of ordinary differential equations. As nonlinearity is the law of nature, the models involve solving nonlinear ordinary differential equations. Here again, the nonlinearity may come under weak nonlinearity or strong nonlinearity. Thanks to the advancement of high-speed computers and remarkable computer algorithms, obtaining reasonably good numerical solutions to linear problems is not now an uphill task. However, we come across some problems that are fairly simple to understand and visualize but involve solving highly nonlinear differential equations that cannot be solved exactly/analytically.

We come across problems of the following types in the literature:

1. Problems that involve differential equations exactly representing the problems realistically for which exact solutions can be obtained.
2. Problems that involve differential equations arising because of some simplifying assumptions and thus are approximate representations of the original problems but can be solved exactly.
3. Problems that involve differential equations exactly representing the problems under consideration but for which solutions cannot be obtained exactly so we may have to try to obtain an approximate solution.
4. Problems that involve differential equations arising because of some simplifying assumptions and thus are approximate representations of the original problems and also cannot be solved exactly.

Problems from various branches of science and engineering, unfortunately, do not always fall under category 1. Problems of category 2 are all right, but the solutions of the problems are not the exact solutions of the original problems.

When we formulate the real-world problems either exactly or approximately, suppose we are unable to obtain exact solutions, as in categories 3 and 4; then, we necessarily have to resort to obtaining approximate solutions of the problems formulated.

The approximate solutions may be

1. analytical approximate solutions
2. numerical approximate solutions

As mentioned, because of the availability of high-speed computers and techniques in programming, a good number of numerical techniques have been developed to solve problems in differential equations with considerable complexity and nonlinearity. In spite of this, the quest for analytical solutions, whether exact or approximate, has always existed, and a number of newer methods proposed by ingenious workers are making their presence felt.

Consider a differential equation of the form

$$L(u,x,\varepsilon)=0 \quad \text{or} \quad L(u,x)=0 \tag{1.1}$$

where x is an independent variable, u is the dependent variable, ε is a small parameter, and L is a differential operator.

Let

$$B(u,\varepsilon)=0 \quad \text{or} \quad B(u)=0 \tag{1.2}$$

be a condition to be satisfied by u. This may be taken as a boundary condition or an initial condition.

If we can, by some method, find $u(x)$, which satisfies Equations (1.1) and (1.2) exactly, that would be ideal. In this context, note that for an equation of the form of Equation (1.1) it is far more tractable to obtain an exact analytical solution if it is linear than if it is nonlinear. As mentioned, when we cannot obtain an exact analytical solution, the question is whether it is possible to obtain an analytical solution that is approximate. Over many decades, several analytical procedures, mostly approximate analytical methods, were developed by researchers to partially answer this question. Some of the popular methods in this direction are the following:

1. Power series solution method
2. Asymptotic series solution method
3. Perturbation methods
4. Nonperturbation methods such as Lyapunov's artificial parameter method, δ expansion method, and so on
5. Adomian decomposition method
6. Homotopy methods

The series solution method is classical in the sense that it led to a vast field and development in the form of special functions, such as Legendre polynomials, Bessel functions, Hermite polynomials, wave functions, and so on. The series solutions constructed using the power series method are all convergent in nature, although in some cases the region of convergence is small. But, the major drawback is that this method fails to provide solutions to the equations at so-called irregular singular points and the solutions that are valid for the point at infinity. An offshoot of the power series method is the asymptotic method, which takes care of the asymptotic behavior of the solutions obtained. Chapter 2 explains the power series method and presents several illustrations involving linear and nonlinear equations.

The asymptotic method also provides a solution to the problem in terms of an infinite series, but it need not always be a convergent series. The characteristic feature of these solutions is that, in spite of their divergent nature, there will be a particular partial sum that provides the best approximation to the solution function of the differential equation considered. To construct solutions valid for large values of the independent variable, a technique called the Lindstedt-Poincaré technique, also known as the method of stretched coordinates, is explained (the basic ideas for this emanated in the late nineteenth century). Chapter 3 provides the basics of the asymptotic method together with the various techniques for finding asymptotic solutions to different classes of problems.

A differential equation governing a physical phenomenon, whether linear or nonlinear, can involve certain parameters. Students of fluid mechanics, elasticity, structural mechanics, quantum mechanics, and diverse other fields are well aware of this aspect. By changing the parameters slightly (or, in other words, by perturbing the parameters), the solution may be changed slightly or otherwise. Problems in this class are termed perturbation problems. Again, there are broadly two varieties of perturbation problems: regular and singular.

One of the earliest techniques has been to express the unknown function or functions of the problem to be solved as a power series expansion in terms of this parameter (referred to as the perturbation parameter) and obtain the successive terms of the expansion with care. This is known as the perturbation technique. Using this technique and diverse improvements on it, several problems from various fields have been solved. In fact, the perturbation technique for solving problems is supposed to be one of the greatest advancements of the twentieth century. However, note that this method is aided or supplemented by a number of asymptotic methods, such as the boundary-layer method, multiple-scale method, WKB (for its developers, Gregor Wentzel, Hendrik Kramers, and Léon Brillown) method, and so on. Chapter 4 deals with the perturbation techniques for solving the so-called regular perturbation problems and details the boundary-layer method, also known as the matched asymptotic technique, for providing solutions to singularly perturbed problems. Chapters 5 and 6 throw light on two popular asymptotic techniques: the multiple-scale method and the WKB method, respectively.

In all most all the methods described, when the problem involves a linear or nonlinear differential equation, broadly we develop a sequence of linear subproblems that are again differential equations, which we successively solve and take the sum as the solution of the original problem.

Note that not every problem has a perturbation parameter involved; also, even if one exists, the subproblems constructed may become so complicated that only a few of them have a solution. Hence, to solve this category of problems, researchers have developed the so-called nonperturbation methods. Among them are Lyapunov's artificial parameter method and the δ expansion method. In these methods, again the solution to the problem is assumed to be an infinite series in terms of a small parameter that is introduced into the problem. A series of subproblems that are linear ordinary differential equations are then constructed. The advantage that is gained by introducing this so-called artificial parameter is that it can be placed in the equation in any term of our choice so that it is almost possible to find solutions to all the subproblems constructed.

Another nonperturbation technique considered to be a powerful analytical technique for solving ordinary differential equations is the Adomian decomposition method. It can be used to solve strongly nonlinear differential equations as well. Unlike the perturbation techniques and the other nonperturbation techniques, it needs neither a perturbation parameter nor an artificial parameter. The solution here is expressed in terms of "Adomian polynomials," which can be easily computed. In fact, codes are available for the computation of these polynomials in software such as Mathematica®. The solution constructed using this decomposition technique is in general convergent. Chapter 7 of the book provides an insight into these three nonperturbation techniques.

The last chapter of the book details a recent "nicely developed" homotopy analysis method proposed by Liao that overcomes the limitations of the earlier methods touched on in this book. It is also a semianalytical technique that is best suited for nonlinear ordinary and partial differential equations. It employs the concept of "homotopy" of topology to generate a convergent solution for not only weakly but also strongly nonlinear problems, going "beyond" the limitations of the perturbation/nonperturbation techniques. In fact, this method unifies Lyapunov's artificial parameter method, the δ expansion method, and the Adomian decomposition method.

This book gives a complete description of the proposed methods and illustrates them through examples. The methods are introduced and explained without going into several of the rigorous mathematical aspects. After basic exposure to the working procedures, references suggested at the end of each chapter provide greater insight. The book also includes a list of many research articles related to the applications of the techniques in various fields. This list, of course, is not exhaustive but is sufficient for an interested student.