# HANDBOOK OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS

## SECOND EDITION


Andrei D. Polyanin
Vladimir E. Nazaikinskii

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

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# PREFACE TO THE SECOND EDITION

The **Handbook of Linear Partial Differential Equations for Engineers and Scientists**, a unique reference for scientists and engineers, contains nearly 4,000 linear partial differential equations with solutions as well as analytical, symbolic, and numerical methods for solving linear equations. First-, second-, third-, fourth-, and higher-order linear equations and systems of coupled equations are considered. Equations of parabolic, hyperbolic, elliptic, mixed, and other types are discussed. A number of new linear equations, exact solutions, transformations, and methods are described. Formulas for effective construction of solutions are given. A number of specific examples where the methods described in the book are used are considered. Boundary value problems and eigenvalue problems are described. Symbolic and numerical methods for solving PDEs with Maple, Mathematica, and MATLAB® are considered. All in all, the handbook contains many more linear partial differential equations than any other book currently available.

In selecting the material, the authors have given highest priority to the following major topics:

- Equations and problems that arise in various applications (heat and mass transfer theory, wave theory, elasticity, hydrodynamics, aerodynamics, continuum mechanics, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, chemical engineering sciences, control theory, etc.).
- Systems of coupled equations that arise in various fields of continuum mechanics and physics.
- Analytical and symbolic methods for solving linear equations of mathematical physics.
- Equations of general form that depend on arbitrary functions and equations that involve many free parameters; exact solutions of such equations are of major importance for testing numerical and approximate analytical methods.

**The second edition** has been substantially updated, revised, and expanded. More than 1,500 linear equations and systems with solutions, as well some methods and many examples, have been added, which amounts to over 700 pages of new material (including 250 new pages dealing with methods).

**New to the second edition:**

- Some second-, third-, fourth-, and higher-order linear PDEs with solutions.
- Systems of coupled partial differential equations with solutions.
- First-order linear PDEs with solutions.
- Some analytical methods including decomposition methods and their applications.
- Symbolic and numerical methods with Maple, Mathematica, and MATLAB.
- Some transformations, asymptotic formulas and solutions.
- Many new examples and figures included for illustrative purposes.
- Some long tables, including tables of various integral transforms.
- Extensive table of contents and detailed index.

Note that Chapters 1–12 of the book can be used as a database of test problems for numerical, approximate analytical, and symbolic methods for solving linear partial differential equations and systems of coupled equations. To satisfy the needs of a broad audience with diverse mathematical backgrounds, the authors have done their best to avoid special terminology whenever possible. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner with necessary references made to books where these methods are considered in more detail. Many sections are written so that they can be read independently from each other. This allows the reader to get to the heart of the matter quickly.

Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics and linear PDEs.

We would like to express our keen gratitude to Alexei Zhurov for fruitful discussions and valuable remarks. We are very thankful to Inna Shingareva and Carlos Lizárraga-Celaya, who wrote three chapters (22–24) of the book at our request.

The authors hope that the handbook will prove helpful for a wide audience of researchers, university and college teachers, engineers, and students in various fields of applied mathematics, mechanics, physics, chemistry, economics, and engineering sciences.

*Andrei D. Polyanin*
*Vladimir E. Nazaikinskii*

## PREFACE TO THE FIRST EDITION

Linear partial differential equations arise in various fields of science and numerous applications, e.g., heat and mass transfer theory, wave theory, hydrodynamics, aerodynamics, elasticity, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, control theory, chemical engineering sciences, and biomechanics.

This book presents brief statements and exact solutions of more than 2000 linear equations and problems of mathematical physics. Nonstationary and stationary equations with constant and variable coefficients of parabolic, hyperbolic, and elliptic types are considered. A number of new solutions to linear equations and boundary value problems are described. Special attention is paid to equations and problems of general form that depend on arbitrary functions. Formulas for the effective construction of solutions to nonhomogeneous boundary value problems of various types are given. We consider second-order and higher-order equations as well as the corresponding boundary value problems. All in all, the handbook presents more equations and problems of mathematical physics than any other book currently available.

For the reader's convenience, the introduction outlines some definitions and basic equations, problems, and methods of mathematical physics. It also gives useful formulas that enable one to express solutions to stationary and nonstationary boundary value problems of general form in terms of the Green's function.

Two supplements are given at the end of the book. Supplement A lists properties of the most common special functions (the gamma function, Bessel functions, degenerate hypergeometric functions, Mathieu functions, etc.). Supplement B describes the methods of generalized and functional separation of variables for nonlinear partial differential equations. We give specific examples and an overview application of these methods to construct exact solutions for various classes of second-, third-, fourth-, and higher-order equations (in total, about 150 nonlinear equations with solutions are described). Special attention is paid to equations of heat and mass transfer theory, wave theory, and hydrodynamics as well as to mathematical physics equations of general form that involve arbitrary functions.

The equations in all chapters are in ascending order of complexity. Many sections can be read independently, which facilitates working with the material. An extended table of contents will help the reader find the desired equations and boundary value problems. We refer to specific equations using notation like "1.8.5.2," which means "Equation 2 in Subsection 1.8.5."

To extend the range of potential readers with diverse mathematical backgrounds, the author strove to avoid the use of special terminology wherever possible. For this reason, some results are presented schematically, in a simplified manner (without details), which is, however, quite sufficient in most applications.

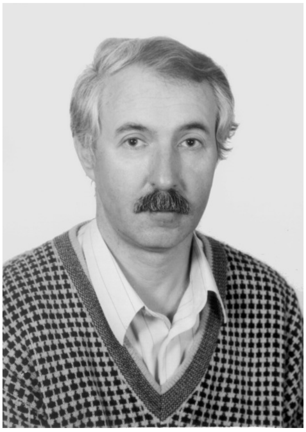
Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics.

The author thanks Alexei Zhurov for useful remarks on the manuscript.

The author hopes that the handbook will be useful for a wide range of scientists, university teachers, engineers, and students in various areas of mathematics, physics, mechanics, control, and engineering sciences.

*Andrei D. Polyanin*

# AUTHORS



**Andrei D. Polyanin, D.Sc.,** is an internationally renowned scientist of broad interests and is active in various areas of mathematics, mechanics, and chemical engineering sciences. He is one of the most prominent authors in the field of reference literature on mathematics.

Professor Polyanin graduated with honors from the Faculty of Mechanics and Mathematics at Lomonosov Moscow State University in 1974. He received his Ph.D. in 1981 and D.Sc. in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, Professor Polyanin has been working at the Institute for Problems in Mechanics of the Russian Academy of Sciences; he is also Professor of applied mathematics at Bauman Moscow State Technical University and at National Research Nuclear University MEPhI. He is a member of the Russian National Committee on Theoretical and Applied Mechanics and the Mathematics and Mechanics Expert Council of the Higher Certification Committee of the Russian Federation.

Professor Polyanin has made important contributions to the theory of differential and integral equations, mathematical physics, engineering mathematics, theory of heat and mass transfer, and chemical hydrodynamics. He has obtained exact solutions for several thousand ordinary differential, partial differential, delay partial differential, and integral equations.

Professor Polyanin has authored more than 30 books in English, Russian, German, and Bulgarian as well as over 170 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyanin and V. F. Zaitsev's *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, 1995 and 2003); A. D. Polyanin and A. V. Manzhirov's *Handbook of Integral Equations* (CRC Press, 1998 and 2008); A. D. Polyanin's *Handbook of Linear Partial Differential Equations for Engineers and Scientists* (Chapman & Hall/CRC Press, 2002); A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux's *Handbook of First Order Partial Differential Equations* (Taylor & Francis, 2002); A. D. Polyanin and V. F. Zaitsev's *Handbook of Nonlinear Partial Differential Equations* (Chapman & Hall/CRC Press, 2004 and 2012); A. D. Polyanin and A. V. Manzhirov's *Handbook of Mathematics for Engineers and Scientists* (Chapman & Hall/CRC Press, 2007), and A. D. Polyanin and A. I. Chernoutsan's (Eds.) *A Concise Handbook of Mathematics, Physics, and Engineering Sciences* (Chapman & Hall/CRC Press, 2010).

Professor Polyanin is editor-in-chief of the international scientific educational website *EqWorld—The World of Mathematical Equations* and he is editor of the book series *Differential and Integral Equations and Their Applications* (Chapman & Hall/CRC Press, London/Boca Raton). Professor Polyanin is a member of the editorial board of the journals *Theoretical Foundations of Chemical Engineering*, *Mathematical Modeling and Computational Methods* (in Russian), and *Bulletin of the National Research Nuclear University MEPhI* (in Russian).

In 1991, Professor Polyanin was awarded the Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

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**Vladimir E. Nazaikinskii, D.Sc.,** is an actively working mathematician specializing in partial differential equations, mathematical physics, and noncommutative analysis. He was born in 1955 in Moscow, graduated from the Moscow Institute of Electronic Engineering in 1977, defended his Ph.D. in 1980 and D.Sc. in 2014, and worked at the Institute for Automated Control Systems, Moscow Institute of Electronic Engineering, Potsdam University, and Moscow State University. Currently he is a senior researcher at the Institute for Problems in Mechanics, Russian Academy of Sciences. He is the author of seven monographs (V. P. Maslov and V. E. Nazaikinskii, *Asymptotics of Operator and Pseudo-Differential Equations*, Consultants Bureau, New York, 1988; V. Nazaikinskii, B. Sternin, and V. Shatalov, *Contact Geometry and Linear Differential Equations*, Walter de Gruyter, Berlin-New York, 1992; V. Nazaikinskii, B. Sternin, and V. Shatalov, *Methods of Noncommutative Analysis. Theory and Applications*, Walter de Gruyter, Berlin-New York, 1996; V. Nazaikinskii, B.-W. Schulze, and B. Sternin, *Quantization Methods in Differential Equations*, Taylor and Francis, London-New York, 2002; V. Nazaikinskii, A. Savin, B.-W. Schulze, and B. Sternin, *Elliptic Differential Equations on Singular Manifolds*, CRC Press, Boca Raton, 2005; V. Nazaikinskii, A. Savin, and B. Sternin, *Elliptic Theory and Noncommutative Geometry*, Birkhäuser, Basel, 2008; V. Nazaikinskii, B.-W. Schulze, and B. Sternin, *The Localization Problem in Index Theory of Elliptic Operators*, Springer, Basel, 2014) and more than 90 papers on various aspects of noncommutative analysis, asymptotic problems, and elliptic theory.

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# BASIC NOTATION AND REMARKS

## Latin Characters

| | |
|---|---|
| curl $\mathbf{u}$ | curl of a vector $\mathbf{u}$, sometimes also denoted by rot $\mathbf{u}$ |
| div $\mathbf{u}$ | divergence of a vector $\mathbf{u}$; div $\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$ in the two-dimensional case $\mathbf{u} = (u_1, u_2)$ |
| $\mathscr{E}$ | fundamental solution of the Cauchy problem |
| $\mathscr{E}_{\mathrm{e}}$ | fundamental solution corresponding to an operator (or fundamental solution of an equation) |
| grad $a$ | gradient of a scalar $a$, also denoted by $\nabla a$, where $\nabla$ is the nabla vector differential operator |
| $\mathrm{Im}[A]$ | imaginary part of a complex number $A$ |
| $G$ | Green function |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space, $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1, \dots, n\}$ |
| $\mathrm{Re}[A]$ | real part of a complex number $A$ |
| $r,\ \varphi,\ z$ | cylindrical coordinates, $r = \sqrt{x^2 + y^2}$ with $x = r\cos\varphi$ and $y = r\sin\varphi$ |
| $r,\ \theta,\ \varphi$ | spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ with $x = r\sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and $z = r\cos\theta$ |
| $t$ | time ($t \geq 0$) |
| $w$ | unknown function (dependent variable) |
| $x, y, z$ | space (Cartesian) coordinates |
| $x_1, \dots, x_n$ | Cartesian coordinates in $n$-dimensional space |
| $\mathbf{x}$ | $n$-dimensional vector, $\mathbf{x} = \{x_1, \dots, x_n\}$ |
| $\lvert\mathbf{x}\rvert$ | magnitude (length) of $n$-dimensional vector, $\lvert\mathbf{x}\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ |
| $\mathbf{y}$ | $n$-dimensional vector, $\mathbf{y} = \{y_1, \dots, y_n\}$ |

## Greek Characters

| | |
|---|---|
| $\Delta$ | Laplace operator |
| $\Delta_2$ | two-dimensional Laplace operator, $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ |
| $\Delta_3$ | three-dimensional Laplace operator, $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ |
| $\Delta_n$ | $n$-dimensional Laplace operator, $\Delta_n = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$ |
| $\Delta\Delta$ | biharmonic operator; $\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ in the two-dimensional case |
| $\delta(x)$ | Dirac delta function; $\displaystyle\int_{-a}^{a} f(y)\delta(x-y)\,dy = f(x)$, where $f(x)$ is any continuous function, $a > 0$, and $-a < x < a$ |
| $\delta_{nm}$ | Kronecker delta, $\delta_{nm} = \begin{cases} 1 \text{ if } n=m, \\ 0 \text{ if } n\neq m \end{cases}$ |
| $\vartheta(x)$ | Heaviside unit step function, $\vartheta(x) = \begin{cases} 1 \text{ if } x>0, \\ 0 \text{ if } x\leq 0 \end{cases}$ |

## Brief Notation for Derivatives

Partial derivatives:

$$w_x = \partial_x w = \frac{\partial w}{\partial x}, \quad w_t = \partial_t w = \frac{\partial w}{\partial t}, \quad w_{xx} = \partial_{xx} w = \frac{\partial^2 w}{\partial x^2}, \quad w_{xt} = \partial_{tx} w = \frac{\partial^2 w}{\partial x \partial t},$$
$$w_{tt} = \partial_{tt} w = \frac{\partial^2 w}{\partial t^2}, \quad w_{xxx} = \partial_{xxx} w = \frac{\partial^3 w}{\partial x^3}, \quad w_{xxt} = \partial_{xxt} w = \frac{\partial^3 w}{\partial x^2 \partial t}, \quad \dots$$

Ordinary derivatives for $f = f(x)$:

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f_x^{(n)} = \frac{d^n f}{dx^n} \quad \text{with} \quad n \ge 4.$$

## Special Functions

| | |
|---|---|
| $\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\bigl(\frac{1}{3}t^3 + xt\bigr)\,dt$ | Airy function; $\operatorname{Ai}(x) = \frac{1}{\pi}\bigl(\frac{1}{3}x\bigr)^{1/2} K_{1/3}\bigl(\frac{2}{3}x^{3/2}\bigr)$ |
| $\operatorname{Ce}_{2n+p}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k+p}^{2n+p} \cosh[(2k+p)x]$ | even modified Mathieu functions, where $p = 0, 1$; $\operatorname{Ce}_{2n+p}(x,q) = \operatorname{ce}_{2n+p}(ix,q)$ |
| $\operatorname{ce}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k}^{2n} \cos 2kx$ | even $\pi$-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n}(q)$ are eigenvalues |
| $\operatorname{ce}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k+1}^{2n+1} \cos[(2k+1)x]$ | even $2\pi$-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n+1}(q)$ are eigenvalues |
| $D_\nu = D_\nu(x)$ | parabolic cylinder function; it satisfies the equation $y'' + \bigl(\nu + \frac{1}{2} - \frac{1}{4}x^2\bigr)y = 0$ |
| $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp\bigl(-\xi^2\bigr)\,d\xi$ | error function |
| $\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\bigl(-\xi^2\bigr)\,d\xi$ | complementary error function |
| $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\bigl(e^{-x^2}\bigr)$ | Hermite polynomial |
| $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ | Hankel function of the first kind; $i^2 = -1$ |
| $H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$ | Hankel function of the second kind |
| $F(a,b,c;x) = 1 + \sum\limits_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$ | hypergeometric function, $(a)_n = a(a+1)\dots(a+n-1)$ |
| $I_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$ | modified Bessel function of the first kind |
| $J_\nu(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$ | Bessel function of the first kind |
| $K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$ | modified Bessel function of the second kind |

| | |
|---|---|
| $L_n^s(x) = \frac{1}{n!} x^{-s} e^x \frac{d^n}{dx^n}\big(x^{n+s} e^{-x}\big)$ | generalized Laguerre polynomial |
| $P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n}(x^2-1)^n$ | Legendre polynomial |
| $P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$ | associated Legendre functions |
| $\mathrm{Se}_{2n+p}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+p}^{2n+p} \sinh[(2k+p)x]$ | odd modified Mathieu functions, where $p=0,1$; $\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q)$ |
| $\mathrm{se}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k}^{2n} \sin 2kx$ | odd $\pi$-periodic Mathieu functions; these satisfy the equation $y''+(a-2q\cos 2x)y=0$, where $a=b_{2n}(q)$ are eigenvalues |
| $\mathrm{se}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+1}^{2n+1} \sin[(2k+1)x]$ | odd $2\pi$-periodic Mathieu functions; these satisfy the equation $y''+(a-2q\cos 2x)y=0$, where $a=b_{2n+1}(q)$ are eigenvalues |
| $Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$ | Bessel function of the second kind |
| $\gamma(\alpha,x) = \int_0^x e^{-\xi}\xi^{\alpha-1}\,d\xi$ | incomplete gamma function |
| $\Gamma(\alpha) = \int_0^\infty e^{-\xi}\xi^{\alpha-1}\,d\xi$ | gamma function |
| $\Phi(a,b;x) = 1 + \sum\limits_{n=1}^{\infty} \frac{(a)_n}{(b)_n}\frac{x^n}{n!}$ | degenerate hypergeometric function, $(a)_n = a(a+1)\dots(a+n-1)$ |

## Miscellaneous Remarks

1. The previous handbooks by Polyanin (2002) and Polyanin, Zaitsev, and Moussiaux (2002) were extensively used in compiling this book; references to these sources are often omitted.
2. The conventional abbreviations ODE and PDE stand for "ordinary differential equation" and "partial differential equation," respectively.
3. The conventional abbreviations 2D equation and 3D equation stand for "two-dimensional equation" and "three-dimensional equation," respectively.
4. Throughout the book, unless explicitly specified otherwise, all parameters occurring in the equations considered are assumed to be real numbers.
5. The term "exact solution" with regard to linear PDEs and systems of PDEs is used in the following cases:
   - the solution is expressible in terms of elementary functions;
   - the solution is expressible via special functions, in closed form via infinite function series, and/or via definite (indefinite) integrals; the solution may depend on arbitrary functions, which may occur in the equation itself or in the initial and boundary conditions.
6. If a formula or a solution contains derivatives of some functions, then the functions are assumed to be differentiable.
7. If a formula or a solution contains finite or definite integrals, then the integrals are supposed to be convergent.

8. If a formula or a solution contains an expression like $\frac{f(x)}{a-2}$, then the assumption that $a \neq 2$ is implied but often not stated explicitly.
9. Equations are numbered separately within each subsection. In Chapters 1–12, when referring to a particular equation, we use notation like 3.2.1.5, which denotes Eq. 5 in Section 3.2.1.
10. The symbol $\odot$ indicates references to literature sources whenever
    - at least one of the solutions was obtained in the cited source;
    - the cited source provides further information on the equations in question and their solutions.
11. The symbol ▶ marks the beginning of a small section; such sections are referred to as paragraphs.
12. The symbol $\rightrightarrows$ stands for uniform convergence.

# Part I

# Exact Solutions

# Chapter 1

# First-Order Equations with Two Independent Variables

## 1.1 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = 0$

◆ *For brevity, often only a principal integral*

$$\Xi = \Xi(x,y)$$

*of an equation will be presented in Section 1.1. The general solution of the equation is given by*

$$w = \Phi(\Xi),$$

*where* $\Phi = \Phi(\Xi)$ *is an arbitrary function.*

### 1.1.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in $x$ and $y$.**

**1.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = 0.$

General solution: $w = \Phi(bx - ay)$, where $\Phi$ is an arbitrary function.

⊙ *Literature*: E. Kamke (1965).

**2.** $a\dfrac{\partial w}{\partial x} + (bx + c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \frac{1}{2}bx^2 + cx - ay.$

**3.** $\dfrac{\partial w}{\partial x} + (ax + by + c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = (abx + b^2 y + a + bc)e^{-bx}.$

**4.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = 0.$

For $a = b$, this is a *conoid equation*. Principal integral: $\Xi = |x|^b |y|^{-a}$.

⊙ *Literature*: E. Kamke (1965).

**5.** $\displaystyle ay\frac{\partial w}{\partial x} + bx\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = bx^2 - ay^2$.

⊙ *Literature*: E. Kamke (1965).

**6.** $\displaystyle y\frac{\partial w}{\partial x} + (y + a)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - y + a\ln|y + a|$.

**7.** $\displaystyle (ay + bx + c)\frac{\partial w}{\partial x} - (by + kx + s)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = ay^2 + kx^2 + 2(bxy + cy + sx)$.

**8.** $\displaystyle (a_1x + b_1y + c_1)\frac{\partial w}{\partial x} + (a_2x + b_2y + c_2)\frac{\partial w}{\partial y} = 0.$

The principal integral is determined by solutions of the following auxiliary system of algebraic equations for the parameters $s$, $\lambda$, $\mu$, $\alpha$, $\beta$, and $\gamma$:

$$(a_1 - s)(b_2 - s) = a_2b_1, \tag{1}$$
$$a_1\lambda + a_2\mu = s\lambda, \qquad b_1\lambda + b_2\mu = s\mu, \tag{2}$$
$$c_1\alpha + c_2\beta - s\gamma = c_1\lambda + c_2\mu, \tag{3}$$
$$(a_1 - s)\alpha + a_2\beta = \lambda s, \quad b_1\alpha + (b_2 - s)\beta = \mu s. \tag{4}$$

*Case 1*: $(a_1 - b_2)^2 + 4a_2b_1 \neq 0$. Equation (1) has two different roots $s_1$ and $s_2$. To these roots there correspond two sets of solutions, $\lambda_1$, $\mu_1$ and $\lambda_2$, $\mu_2$, of system (2).

*1.1*. If $a_1b_2 - a_2b_1 \neq 0$, then $s_1 \neq 0$ and $s_2 \neq 0$. Hence the principal integral has the form

$$\Xi = \frac{|s_1(\lambda_1x + \mu_1y) + \lambda_1c_1 + \mu_1c_2|^{s_2}}{|s_2(\lambda_2x + \mu_2y) + \lambda_2c_1 + \mu_2c_2|^{s_1}}.$$

*1.2*. If $a_1b_2 - a_2b_1 = 0$, then $s_1 = s = a_1 + b_2$ and $s_2 = 0$.

Principal integral for $\lambda_2c_1 + \mu_2c_2 \neq 0$:

$$\Xi = s\frac{\lambda_2x + \mu_2y}{\lambda_2c_1 + \mu_2c_2} - \ln|s_1(\lambda_1x + \mu_1y) + \lambda_1c_1 + \mu_1c_2|.$$

Principal integral for $\lambda_2c_1 + \mu_2c_2 = 0$:

$$\Xi = \lambda_2x + \mu_2y.$$

*Case 2*: $(a_1 - b_2)^2 + 4a_2b_1 = 0$. Equation (1) has the double root $s = \frac{1}{2}(a_1 + b_2)$. System (2) gives $\lambda$ and $\mu$ not equal to zero simultaneously.

*2.1.* If $s \neq 0$, then we find $\gamma$ from (3) and take nonzero $\alpha$ and $\beta$ that satisfy relations (4). This leads to the principal integral

$$\Xi = \ln|s(\lambda x+\mu y)+c_1\lambda+c_2\mu| - \frac{s(\alpha x+\beta y+\gamma)}{s(\lambda x+\mu y)+c_1\lambda+c_2\mu}.$$

*2.2.* If $s=0$, then $b_2=-a_1$. We have

$$\Xi = a_2x^2-2a_1xy-b_1y^2+2c_2x-2c_1y.$$

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations are quadratic in $x$ and $y$.**

**9.** $\dfrac{\partial w}{\partial x}+(ax^2+bx+c)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi = \frac{1}{3}ax^3+\frac{1}{2}bx^2+cx-y.$

**10.** $\dfrac{\partial w}{\partial x}+(ay^2+by+c)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $4ac-b^2>0$:

$$\Xi = x-\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ay+b}{\sqrt{4ac-b^2}}.$$

2°. Principal integral for $4ac-b^2<0$:

$$\Xi = x-\frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{2ay+b-\sqrt{b^2-4ac}}{2ay+b+\sqrt{b^2-4ac}}\right|.$$

**11.** $\dfrac{\partial w}{\partial x}+(ay+bx^2+cx)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.7.1 with $f(x)=a$ and $g(x)=bx^2+cx$.

**12.** $\dfrac{\partial w}{\partial x}+(axy+bx^2+cx+ky+s)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi = y\exp\bigl(-\frac{1}{2}ax^2-kx\bigr)-\displaystyle\int(bx^2+cx+s)\exp\bigl(-\tfrac{1}{2}ax^2-kx\bigr)\,dx.$

**13.** $\dfrac{\partial w}{\partial x}+(y^2-a^2x^2+3a)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi = \dfrac{\exp(ax^2)}{x(xy-ax^2+1)}+\displaystyle\int\exp(ax^2)\frac{dx}{x^2}.$

**14.** $\dfrac{\partial w}{\partial x}+(y^2-a^2x^2+a)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.59 with $n=1$.

**15.** $\dfrac{\partial w}{\partial x}+(y^2+axy+a)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.60 with $n=1$.

**16.** $\dfrac{\partial w}{\partial x}+(y^2+axy-abx-b^2)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.61 with $n=1$.

**17.** $\dfrac{\partial w}{\partial x}+k(ax+by+c)^2\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.8.6 with $f(z)=kz^2$.

**18.** $x\dfrac{\partial w}{\partial x}+(ay^2+cx^2+y)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.75 with $b=1$.

**19.** $x\dfrac{\partial w}{\partial x}+(ay^2+bxy+cx^2+y)\dfrac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.1.76 with $n=1$.

**20.** $(ax+c)\dfrac{\partial w}{\partial x}+\big[\alpha(ay+bx)^2+\beta(ay+bx)-bx+\gamma\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\ln|ax+c|-\int\frac{dv}{\alpha v^2+\beta v+\gamma+bc/a},\qquad v=ay+bx.$$

**21.** $ax^2\dfrac{\partial w}{\partial x}+by^2\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{by}-\dfrac{1}{ax}.$

**22.** $(ax^2+b)\dfrac{\partial w}{\partial x}-\big[y^2-2xy+(1-a)x^2-b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=-\displaystyle\int\frac{dx}{ax^2+b}+\frac{1}{y-x}.$

**23.** $(a_1x^2+b_1x+c_1)\dfrac{\partial w}{\partial x}+(a_2y^2+b_2y+c_2)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int\frac{dx}{a_1x^2+b_1x+c_1}-\int\frac{dy}{a_2y^2+b_2y+c_2}.$

**24.** $(x-a)(x-b)\dfrac{\partial w}{\partial x}-\big[y^2+k(y+x-a)(y+x-b)\big]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $a\neq b$:

$$\Xi=\frac{y+k(y+x-a)}{y+k(y+x-b)}\left(\frac{x-a}{x-b}\right)^k,\qquad k\neq 0,\quad k\neq -1.$$

2°. Principal integral for $a=b$:

$$\Xi=\frac{(x-a)+\big[y+k(y+x-a)\big]}{\big[y+k(y+x-a)\big](x-a)},\qquad k\neq 0,\quad k\neq -1.$$

**25.** $\boldsymbol{(a_1y^2+b_1y+c_1)\dfrac{\partial w}{\partial x}+(a_2x^2+b_2x+c_2)\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=\frac{1}{3}a_1y^3+\frac{1}{2}b_1y^2+c_1y-\frac{1}{3}a_2x^3-\frac{1}{2}b_2x^2-c_2x.$

**26.** $\boldsymbol{y(ax+b)\dfrac{\partial w}{\partial x}+(ay^2-cx)\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=\dfrac{(ax+b)^2}{cx^2+by^2}.$

**27.** $\boldsymbol{(ay^2+bx)\dfrac{\partial w}{\partial x}-(cx^2+by)\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=\frac{1}{3}ay^3+\frac{1}{3}cx^3+bxy.$

**28.** $\boldsymbol{(ay^2+bx^2)\dfrac{\partial w}{\partial x}+2bx\dfrac{\partial w}{\partial y}=0.}$

This is a special case of equation 1.1.8.2 with $f(x)=bx^2$ and $g(y)=ay^2$.

**29.** $\boldsymbol{(ay^2+bx^2)\dfrac{\partial w}{\partial x}+2bxy\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=\dfrac{bx^2-ay^2}{y}.$

**30.** $\boldsymbol{(ay^2+x^2)\dfrac{\partial w}{\partial x}+(bx^2+c-2xy)\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=ay^3-bx^3+3(x^2y-cx).$

**31.** $\boldsymbol{(Ay^2+Bx^2-a^2B)\dfrac{\partial w}{\partial x}+(Cy^2+2Bxy)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=(x-a)E+2aB\int\frac{E\,dv}{v(Av^2-Cv-B)},\qquad v=\frac{y}{x-a},$$

where $E=\exp\biggl[\displaystyle\int\frac{(Av^2+B)\,dv}{v(Av^2-Cv-B)}\biggr].$

**32.** $\boldsymbol{(ay^2+bx^2+cy)\dfrac{\partial w}{\partial x}+2bx\dfrac{\partial w}{\partial y}=0.}$

This is a special case of equation 1.1.8.2 with $f(x)=bx^2$ and $g(y)=ay^2+cy$.

**33.** $\boldsymbol{(Axy+Bx^2+kx)\dfrac{\partial w}{\partial x}+(Dy^2+Exy+Fx^2+ky)\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=xV+k\int\frac{V\,dv}{(A-D)v^2+(B-E)v-F},\qquad v=\frac{y}{x},$$

where $V=\exp\biggl[\displaystyle\int\frac{(Av+B)\,dv}{(A-D)v^2+(B-E)v-F}\biggr].$

**34.** $\displaystyle (Axy + Aky + Bx^2 + Bkx)\frac{\partial w}{\partial x} + \big[Cy^2 + Dxy + k(D - B)y\big]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = (x+k)E + kB\int \frac{E\,dv}{v\big[(C-A)v + D - B\big]}, \qquad v = \frac{y}{x+k},$$

where $E = \exp\left[\int \frac{(Av+B)\,dv}{v\big[(A-C)v + B - D\big]}\right].$

**35.** $\displaystyle (Ay^2 + Bxy + Cx^2 + kx)\frac{\partial w}{\partial x} + (Dy^2 + Exy + Fx^2 + ky)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = xV + k\int \frac{V\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F}, \qquad v = \frac{y}{x},$$

where $V = \exp\left[\int \frac{(Av^2 + Bv + C)\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F}\right].$

**36.** $\displaystyle (Ay^2 + Bxy + Cx^2)\frac{\partial w}{\partial x} + (Dy^2 + Exy + Fx^2)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{(Av^2 + Bv + C)\,dv}{Av^3 + (B-D)v^2 + (C-E)v - F} + \ln|x|, \qquad v = \frac{y}{x}.$$

**37.** $\displaystyle (Ay^2 + 2Bxy + Dx^2 + a)\frac{\partial w}{\partial x} - (By^2 + 2Dxy - Ex^2 - b)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = Ay^3 - Ex^3 + 3(Bxy^2 + Dx^2y + ay - bx).$

**38.** $\displaystyle (y^2 - 2xy + x^2 + ay)\frac{\partial w}{\partial x} + ay\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{a}{x-y} + \ln|y|.$

**39.** $\displaystyle (xf_1 - f_2)\frac{\partial w}{\partial x} + (yf_1 - f_3)\frac{\partial w}{\partial y} = 0, \qquad f_n = a_n + b_n x + c_n y.$

*Hesse's equation.* The introduction of the homogeneous coordinates $x = \xi_2/\xi_1$, $y = \xi_3/\xi_1$ leads to an equation with three independent variables for $w = w(\xi_1, \xi_2, \xi_3)$:

$$g_1\frac{\partial w}{\partial \xi_1} + g_2\frac{\partial w}{\partial \xi_2} + g_3\frac{\partial w}{\partial \xi_3} = 0,$$

where $g_n = a_n\xi_1 + b_n\xi_2 + c_n\xi_3$ $(n = 1, 2, 3)$. See 2.1.1.21 for the solution of this equation.

⊙ *Literature*: E. Kamke (1965).

► **Coefficients of equations contain integer powers of $x$ and $y$.**

**40.** $\displaystyle \frac{\partial w}{\partial x}+(y^2+bx^2y-a^2-abx^2)\frac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.7.3 with $f(x)=bx^2$.

**41.** $\displaystyle \frac{\partial w}{\partial x}+(ax^2y+bx^3+c)\frac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.7.1 with $f(x)=ax^2$ and $g(x)=bx^3+c$.

**42.** $\displaystyle \frac{\partial w}{\partial x}+(ax^2y+by^3)\frac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.7.2 with $k=3$, $f(x)=ax^2$, and $g(x)=b$.

**43.** $\displaystyle \frac{\partial w}{\partial x}+(axy+b)y^2\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{v(av^2+bv+1)}-\ln|x|,\qquad v=xy.$$

**44.** $\displaystyle \frac{\partial w}{\partial x}+A(ax+by+c)^3\frac{\partial w}{\partial y}=0.$

This is a special case of equation 1.1.8.6 with $f(z)=Az^3$.

**45.** $\displaystyle x\frac{\partial w}{\partial x}+\bigl[ax^4y^3+(bx^2-1)y+cx\bigr]\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv+c}-\frac{x^2}{2},\qquad v=xy.$$

**46.** $\displaystyle x^2\frac{\partial w}{\partial x}+(ax^2y^2+bxy+c)\frac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+(b+1)v+c}-\ln|x|,\qquad v=xy.$$

**47.** $\displaystyle (ax^2y+b)\frac{\partial w}{\partial x}-(axy^2+c)\frac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\frac{1}{2}ax^2y^2+by+cx.$

**48.** $\displaystyle (ax+by^3)\frac{\partial w}{\partial x}-(cx^3+ay)\frac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=axy+\frac{1}{4}by^4+\frac{1}{4}cx^4.$

◆ *See also equations 1.1.1.56–1.1.1.111 for integer values of exponents.*

► **Coefficients of equations contain fractional powers.**

**49.** $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + b)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y\exp\bigl(-\frac{2}{3}ax^{3/2}\bigr) - b\displaystyle\int \exp\bigl(-\tfrac{2}{3}ax^{3/2}\bigr)\,dx.$

**50.** $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + b\sqrt{y}\,)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $k = \frac{1}{2}$, $f(x) = a\sqrt{x}$, and $g(x) = b$.

**51.** $\dfrac{\partial w}{\partial x} + (a\sqrt{x}\,y + bx\sqrt{y}\,)\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.7.2 with $k = \frac{1}{2}$, $f(x) = a\sqrt{x}$, and $g(x) = bx$.

**52.** $\dfrac{\partial w}{\partial x} + A\sqrt{ax + by + c}\,\dfrac{\partial w}{\partial y} = 0.$

This is a special case of equation 1.1.8.6 with $f(z) = A\sqrt{z}$.

**53.** $x\dfrac{\partial w}{\partial x} + \bigl(ay + b\sqrt{y^2 + cx^2}\,\bigr)\dfrac{\partial w}{\partial y} = 0.$

$1^\circ$. Principal integral for $a \neq 1$:

$$\Xi = \ln|x| - \int \frac{du}{(a-1)u + b\sqrt{u^2 + c}}, \qquad u = \frac{y}{x}.$$

$2^\circ$. Principal integral for $a = 1$:

$$\Xi = |x|^{-b-1}\bigl(y + \sqrt{y^2 + cx^2}\,\bigr).$$

**54.** $(ax + b\sqrt{y}\,)\dfrac{\partial w}{\partial x} - (c\sqrt{x} + ay)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = axy + \frac{2}{3}by^{3/2} + \frac{2}{3}cx^{3/2}.$

**55.** $\sqrt{f(x)}\,\dfrac{\partial w}{\partial x} + \sqrt{f(y)}\,\dfrac{\partial w}{\partial y} = 0, \qquad f(t) = \displaystyle\sum_{\nu=0}^{4} a_\nu t^\nu.$

Principal integral: $\Xi = \left[\dfrac{\sqrt{f(x)} + \sqrt{f(y)}}{x - y}\right]^2 - a_4(x + y)^2 - a_3(x + y).$

⊙ *Literature*: E. Kamke (1965).

◆ *See also equations in 1.1.1.56–1.1.1.111 for fractional values of exponents.*

► **Coefficients of equations contain arbitrary powers of $x$ and $y$.**

**56.** $\dfrac{\partial w}{\partial x} + (ay + bx^k)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = ye^{-ax} - b\int x^k e^{-ax}\,dx.$

**57.** $\dfrac{\partial w}{\partial x} + (ax^k y + bx^n)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y\exp\Bigl(-\dfrac{a}{k+1}x^{k+1}\Bigr) - b\int x^n \exp\Bigl(-\dfrac{a}{k+1}x^{k+1}\Bigr)\,dx.$

**58.** $\dfrac{\partial w}{\partial x} + (ay^2 + bx^n)\dfrac{\partial w}{\partial y} = 0.$

The principal integral $\Xi(x,y)$ can be found as the general solution $\Xi(x,y) = C$ of the special Riccati equation $y'_x = ay^2 + bx^n$, which is considered in the handbooks by G. M. Murphy (1960), E. Kamke (1977), and A. D. Polyanin and V. F. Zaitsev (2003).

**59.** $\dfrac{\partial w}{\partial x} + (y^2 + anx^{n-1} - a^2x^{2n})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$:

$$\Xi = \frac{E}{y - ax^n} + \int E\,dx, \qquad E = \exp\Bigl(\frac{2a}{n+1}x^{n+1}\Bigr).$$

2°. Principal integral for $n = -1$ and $a \neq -\frac{1}{2}$:

$$\Xi = \frac{xy + a + 1}{(2a+1)(xy - a)}x^{2a+1}.$$

3°. Principal integral for $n = -1$ and $a = -\frac{1}{2}$:

$$\Xi = \frac{2}{2xy + 1} + \ln|x|.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**60.** $\dfrac{\partial w}{\partial x} + (y^2 + ax^n y + ax^{n-1})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\Bigl(\frac{a}{n+1}x^{n+1}\Bigr).$$

2°. Principal integral for $n = -1$ and $a \neq 1$:

$$\Xi = \frac{xy + a}{(a-1)(xy+1)}x^{a-1}.$$

3°. Principal integral for $n = -1$ and $a = 1$:

$$\Xi = \frac{1}{xy+1} + \ln|x|.$$

**61.** $\dfrac{\partial w}{\partial x} + (y^2 + ax^n y - abx^n - b^2)\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n \neq -1$:

$$\Xi = \frac{1}{y-b}\exp\left(2bx + \frac{a}{n+1}x^{n+1}\right) + \int \exp\left(2bx + \frac{a}{n+1}x^{n+1}\right) dx.$$

2°. Principal integral for $n = -1$:

$$\Xi = \frac{x^a e^{2bx}}{y-b} + \int x^a e^{2bx}\,dx.$$

**62.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 + bx^{-n-2})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \ln|x| - \int \frac{dv}{av^2 + (n+1)v + b}, \qquad v = x^{n+1}y.$$

**63.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 + bmx^{m-1} - ab^2 x^{n+2m})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n + m \neq -1$:

$$\Xi = \frac{E}{y - bx^m} + a\int x^n E\,dx, \qquad E = \exp\left(\frac{2ab}{n+m+1}x^{n+m+1}\right).$$

2°. Principal integral for $n + m = -1$ and $m \neq 2ab$:

$$\Xi = \frac{x^{2ab}}{y - bx^m} + \frac{a}{2ab - m}x^{2ab-m}.$$

3°. Principal integral for $n + m = -1$ and $m = 2ab$:

$$\Xi = \frac{x^m}{y - bx^m} + a\ln x.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**64.** $\dfrac{\partial w}{\partial x} - \left[(n+1)x^n y^2 - ax^{n+m+1}y + ax^m\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{x^{-n-1}E}{x^{n+1}y - 1} - (n+1)\int x^{-n-2}E\,dx, \qquad E = \exp\left(\frac{a}{n+m+2}x^{n+m+2}\right).$$

**65.** $\dfrac{\partial w}{\partial x}+(ax^ny^2+bx^my+bcx^m-ac^2x^n)\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $m,n\neq-1$:

$$\Xi=\frac{E}{y+c}+a\int x^nE\,dx,\qquad E=\exp\biggl(\frac{b}{m+1}x^{m+1}-\frac{2ac}{n+1}x^{n+1}\biggr).$$

2°. Principal integral for $n=-1$:

$$\Xi=\frac{x^{-2ac}}{y+c}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr)+a\int x^{-2ac-1}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr)\,dx.$$

3°. Principal integral for $m=-1$:

$$\Xi=\frac{x^b}{y+c}\exp\biggl(-\frac{2ac}{n+1}x^{n+1}\biggr)+a\int x^{n+b}\exp\biggl(-\frac{2ac}{n+1}x^{n+1}\biggr)\,dx.$$

**66.** $\dfrac{\partial w}{\partial x}+\bigl[ax^ny^2-ax^n(bx^m+c)y+bmx^{m-1}\bigr]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $n\neq-1$ and $m+n\neq-1$:

$$\Xi=\frac{E}{y-bx^m-c}+a\int x^nE\,dx,\qquad E=\exp\biggl(\frac{abx^{n+m+1}}{n+m+1}+\frac{acx^{n+1}}{n+1}\biggr).$$

2°. Principal integral for $n=-1$ and $m\neq0$:

$$\Xi=\frac{x^{ac}}{y-bx^m-c}\exp\biggl(\frac{ab}{m}x^m\biggr)+a\int x^{ac-1}\exp\biggl(\frac{ab}{m}x^m\biggr)\,dx.$$

3°. Principal integral for $n\neq-1$ and $m=-1-n$:

$$\Xi=\frac{x^{ab}}{y-bx^{-n-1}-c}\exp\biggl(\frac{ac}{n+1}x^{n+1}\biggr)+a\int x^{ab+n}\exp\biggl(\frac{ac}{n+1}x^{n+1}\biggr)dx.$$

**67.** $\dfrac{\partial w}{\partial x}-\bigl[anx^{n-1}y^2-cx^m(ax^n+b)+cx^m\bigr]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $m\neq-1$ and $m+n\neq-1$:

$$\Xi=\frac{E}{(ax^n+b)\bigl[(ax^n+b)y-1\bigr]}-an\int\frac{x^{n-1}E}{(ax^n+b)^2}\,dx,$$
$$E=\exp\biggl(\frac{acx^{m+n+1}}{m+n+1}+\frac{bcx^{m+1}}{m+1}\biggr).$$

2°. Principal integral for $m=-1$ and $n\neq0$:

$$\Xi=\frac{x^{bc}}{(ax^n+b)\bigl[(ax^n+b)y-1\bigr]}\exp\biggl(\frac{ac}{n}x^n\biggr)-an\int\exp\biggl(\frac{ac}{n}x^n\biggr)\frac{x^{bc+n-1}}{(ax^n+b)^2}\,dx.$$

3°. Principal integral for $n\neq-1$ and $m=-1-n$:

$$\Xi=\frac{x^{ac}}{(ax^n+b)\bigl[(ax^n+b)y-1\bigr]}\exp\biggl(-\frac{bc}{n}x^{-n}\biggr)-an\int\frac{x^{ac+n-1}}{(ax^n+b)^2}\exp\biggl(-\frac{bc}{n}x^{-n}\biggr)\,dx.$$

**68.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 + bx^m y + ckx^{k-1} - bcx^{m+k} - ac^2 x^{n+2k})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $m \neq -1$ and $n+k \neq -1$:

$$\Xi = \frac{E}{y - cx^k} + a\int x^n E\,dx, \qquad E = \exp\biggl(\frac{2ac}{n+k+1}x^{n+k+1} + \frac{b}{m+1}x^{m+1}\biggr).$$

2°. Principal integral for $m = -1$ and $n+k \neq -1$:

$$\Xi = \frac{x^b E}{y - cx^k} + a\int x^{b+n} E\,dx, \qquad E = \exp\biggl(\frac{2ac}{n+k+1}x^{n+k+1}\biggr).$$

3°. Principal integral for $m \neq -1$ and $n+k = -1$:

$$\Xi = \frac{x^{2ac}}{y - cx^k}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr) + a\int x^{2ac+n}\exp\biggl(\frac{b}{m+1}x^{m+1}\biggr)dx.$$

4°. Principal integral for $m = -1$, $n+k = -1$, and $2ac + b \neq k$:

$$\Xi = \frac{ay + (ac+b-k)x^k}{(2ac+b-k)(y - cx^k)}x^{2ac+b-k}.$$

5°. Principal integral for $m = -1$, $n+k = -1$, and $2ac + b = k$:

$$\Xi = \frac{x^k}{y - cx^k} + a\ln x.$$

**69.** $\dfrac{\partial w}{\partial x} + (ax^{2n+1}y^3 + bx^{-n-2})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{av^3 + (n+1)v + b} - \ln|x|, \qquad v = x^{n+1}y.$$

**70.** $\dfrac{\partial w}{\partial x} + (ax^n y^3 + 3abx^{n+m}y^2 - bmx^{m-1} - 2ab^3 x^{n+3m})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $n + 2m \neq -1$:

$$\Xi = \frac{E}{(y + bx^m)^2} + 2a\int x^n E\,dx, \qquad E = \exp\biggl(-\frac{6ab^2}{n+2m+1}x^{n+2m+1}\biggr).$$

2°. Principal integral for $n = -2m - 1$:

$$\Xi = \frac{x^{-6ab^2}}{(y + bx^m)^2} + \frac{a}{3ab^2 + m}x^{-2(3ab^2+m)}.$$

3°. Principal integral for $n = -2m - 1$ and $m = -3ab^2$:

$$\Xi = \frac{x^{2m}}{(y + bx^m)^2} + 2a\ln|x|.$$

**71.** $$\frac{\partial w}{\partial x} + (ax^n y^3 + 3abx^{n+m}y^2 + cx^k y - 2ab^3x^{n+3m} + bcx^{m+k} - bmx^{m-1})\frac{\partial w}{\partial y} = 0.$$

$1^\circ$. Principal integral for $k \neq -1$ and $n+2m \neq -1$:

$$\Xi = \frac{E}{(y+bx^m)^2} + 2a\int x^n E\,dx, \qquad E = \exp\left(\frac{2c}{k+1}x^{k+1} - \frac{6ab^2}{n+2m+1}x^{n+2m+1}\right).$$

$2^\circ$. Principal integral for $k = -1$ and $n+2m \neq -1$:

$$\Xi = \frac{x^{2c}E_2}{(y+bx^m)^2} + 2a\int x^{n+2c}E_2\,dx, \qquad E_2 = \exp\left(-\frac{6ab^2}{n+2m+1}x^{n+2m+1}\right).$$

$3^\circ$. Principal integral for $k \neq -1$ and $n+2m = -1$:

$$\Xi = \frac{x^{-6ab^2}E_1}{(y+bx^m)^2} + 2a\int x^{n-6ab^2}E_1\,dx, \qquad E_1 = \exp\left(\frac{2c}{k+1}x^{k+1}\right).$$

$4^\circ$. Principal integral for $k = n+2m = -1$ and $c \neq 3ab^2 + m$:

$$\Xi = \frac{x^{2(c-3ab^2)}}{(y+bx^m)^2} + \frac{a}{c-3ab^2-m}x^{2(c-3ab^2-m)}.$$

$5^\circ$. Principal integral for $k = n+2m = -1$ and $c = 3ab^2 + m$:

$$\Xi = \frac{x^{2m}}{(y+bx^m)^2} + 2a\ln|x|.$$

**72.** $$\frac{\partial w}{\partial x} + \left(ay^n + bx^{\frac{n}{1-n}}\right)\frac{\partial w}{\partial y} = 0.$$

Principal integral:

$$\Xi = \int\frac{dv}{av^n + \frac{1}{n-1}v + b} - \ln|x|, \qquad v = yx^{\frac{1}{n-1}}.$$

**73.** $$\frac{\partial w}{\partial x} + (ax^{m-n-mn}y^n + bx^m)\frac{\partial w}{\partial y} = 0.$$

Principal integral:

$$\Xi = \ln|x| - \int\frac{dv}{av^n - (m+1)v + b}, \qquad v = yx^{-m-1}.$$

**74.** $$\frac{\partial w}{\partial x} + (ax^n y^k + bx^m y)\frac{\partial w}{\partial y} = 0.$$

This is a special case of equation 1.1.7.2 with $f(x) = bx^m$ and $g(x) = ax^n$.

**75.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+(ay^2+by+cx^{2b})\dfrac{\partial w}{\partial y}=0.}$

1°. Principal integral for $ac>0$:

$$\Xi=\frac{b}{\sqrt{ac}}\arctan\left(\sqrt{\frac{a}{c}}x^{-b}y\right)-x^b.$$

2°. Principal integral for $ac<0$:

$$\Xi=\frac{b}{2\sqrt{-ac}}\ln\frac{ax^{-b}y-\sqrt{-ac}}{ax^{-b}y+\sqrt{-ac}}-x^b.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**76.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\big[ay^2+(n+bx^n)y+cx^{2n}\big]\dfrac{\partial w}{\partial y}=0.}$

1°. Principal integral for $n\neq 0$:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\frac{1}{n}x^n,\qquad v=x^{-n}y.$$

2°. Principal integral for $n=0$:

$$\Xi=\int\frac{dy}{ay^2+by+c}-\ln|x|.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**77.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+(ax^ny^2+by+cx^{-n})\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+(b+n)v+c}-\ln x,\qquad v=x^ny.$$

**78.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+(ax^ny^2+my-ab^2x^{n+2m})\dfrac{\partial w}{\partial y}=0.}$

1°. Principal integral for $m+n\neq 0$:

$$\Xi=\frac{x^mE}{y-bx^m}+a\int x^{m+n-1}E\,dx,\qquad E=\exp\left(\frac{2ab}{m+n}x^{m+n}\right).$$

2°. Principal integral for $m=-n$:

$$\Xi=\frac{x^{2ab}(y+bx^m)}{2b(y-bx^m)}.$$

**79.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\big[x^{2n}y^2+(m-n)y+x^{2m}\big]\dfrac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=\arctan(x^{n-m}y)-\dfrac{x^{n+m}}{n+m}.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**80.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\bigl[ax^{2n}y^2+(bx^n-n)y+c\bigr]\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=n\int\frac{dv}{av^2+bv+c}-x^n,\qquad v=x^ny.$$

**81.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\bigl[ax^{2n+m}y^2+(bx^{n+m}-n)y+cx^m\bigr]\dfrac{\partial w}{\partial y}=0.}$

$1^\circ$. Principal integral for $n+m\neq 0$:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\frac{x^{n+m}}{n+m},\qquad v=x^ny.$$

$2^\circ$. Principal integral for $n+m=0$:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\ln x,\qquad v=x^ny.$$

**82.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+(ay^3+3abx^ny^2-bnx^n-2ab^3x^{3n})\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\frac{E}{(y+bx^n)^2}+2a\int x^{-1}E\,dx,\qquad E=\exp\biggl(-\frac{3ab^2}{n}x^{2n}\biggr).$$

**83.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\bigl[ax^{2n+1}y^3+(bx-n)y+cx^{1-n}\bigr]\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv+c}-x,\qquad v=x^ny.$$

**84.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+\bigl[ax^{n+2}y^3+(bx^n-1)y+cx^{n-1}\bigr]\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=\int\frac{dv}{av^3+bv+c}-\frac{1}{n}x^n,\qquad v=xy.$$

**85.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+(y+ax^{n-m}y^m+bx^{n-k}y^k)\dfrac{\partial w}{\partial y}=0.}$

$1^\circ$. Principal integral for $n\neq 1$:

$$\Xi=\int\frac{dv}{av^m+bv^k}-\frac{x^{n-1}}{n-1},\qquad v=\frac{y}{x}.$$

$2^\circ$. Principal integral for $n=1$:

$$\Xi=\int\frac{dv}{av^m+bv^k}-\ln|x|,\qquad v=\frac{y}{x}.$$

**86.** $\boldsymbol{y\frac{\partial w}{\partial x}+\left\{x^{n-1}[(1+2n)x+an]y-nx^{2n}(x+a)\right\}\frac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=(x^{n+1}+ax^n-y)^{-1/n}+\int\frac{dv}{a-v^{-n}},\qquad v=x(x^{n+1}+ax^n-y)^{-1/n}.$$

**87.** $\boldsymbol{y\frac{\partial w}{\partial x}+\left\{[a(2n+k)x^k+b]x^{n-1}y-(a^2nx^{2k}+abx^k-c)x^{2n-1}\right\}\frac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi=x^{-k}E-ak\int\frac{E\,dv}{nv^2-bv-c},\qquad v=x^{-n}y-ax^k,$$

where $E=\exp\left(-k\int\frac{v\,dv}{nv^2-bv-c}\right)$.

**88.** $\boldsymbol{x(2axy+b)\frac{\partial w}{\partial x}-\left[a(m+3)xy^2+b(m+2)y-cx^m\right]\frac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=cx^{m+2}\left[cx^m-2(m+1)y(axy+b)\right]$.

**89.** $\boldsymbol{x^2(2axy+b)\frac{\partial w}{\partial x}-(4ax^2y^2+3bxy-cx^2-k)\frac{\partial w}{\partial y}=0.}$

Principal integral: $\Xi=(cx^2+k)^2-4cx^3y(axy+b)$.

**90.** $\boldsymbol{ax^m\frac{\partial w}{\partial x}+by^n\frac{\partial w}{\partial y}=0.}$

1°. Principal integral for $m\neq 1$ and $n\neq 1$:

$$\Xi=b(n-1)x^{1-m}-a(m-1)y^{1-n}.$$

2°. Principal integral for $m=1$ and $n\neq 1$:

$$\Xi=b\ln|x|+\frac{a}{n-1}y^{1-n}.$$

3°. Principal integral for $m\neq 1$ and $n=1$:

$$\Xi=\frac{b}{m-1}x^{1-m}+a\ln|y|.$$

4°. Principal integral for $m=n=1$:

$$\Xi=b\ln|x|-a\ln|y|.$$

⊙ *Literature*: E. Kamke (1965).

**91.** $\boldsymbol{ax^n \dfrac{\partial w}{\partial x} + (by + cx^m)\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $n \neq 1$:

$$\Xi = e^{-F} y - \frac{c}{a} \int e^{-F} x^{m-n}\, dx, \qquad F = \frac{b}{a(1-n)} x^{1-n}.$$

2°. Principal integral for $n = 1$ and $am \neq b$:

$$\Xi = x^{-b/a} y - \frac{c}{am-b} x^{\frac{am-b}{a}}.$$

3°. Principal integral for $n = 1$ and $am = b$:

$$\Xi = x^{-b/a} y - \frac{c}{a} \ln|x|.$$

**92.** $\boldsymbol{ax^k \dfrac{\partial w}{\partial x} + (y^n + bx^m y)\dfrac{\partial w}{\partial y} = 0, \qquad n \neq 1.}$

1°. Principal integral for $m \neq k-1$:

$$\Xi = e^{-F} y^{1-n} + \frac{n-1}{a} \int e^{-F} x^{-k}\, dx, \qquad F = \frac{(1-n)b}{a(m+k-1)} x^{m-k+1}.$$

2°. Principal integral for $m = k-1$ and $(n-1)b \neq ma$:

$$\Xi = x^{\frac{(n-1)b}{a}} y^{1-n} + \frac{n-1}{(n-1)b - ma} x^{\frac{(n-1)b-ma}{a}}.$$

3°. Principal integral for $m = k-1$ and $(n-1)b = ma$:

$$\Xi = x^{\frac{(n-1)b}{a}} y^{1-n} + \frac{n-1}{a} \ln|x|.$$

**93.** $\boldsymbol{x(ax^k + b)\dfrac{\partial w}{\partial x} + \left[\alpha x^n y^2 + (\beta - anx^k)y + \gamma x^{-n}\right]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = x^{-k} E + ka \int \frac{E\, dv}{\alpha v^2 + (\beta + bn)v + \gamma}, \qquad v = x^n y,$$

where $E = \exp\left[kb \displaystyle\int \frac{dv}{\alpha v^2 + (\beta + bn)v + \gamma}\right]$.

**94.** $\boldsymbol{(y + Ax^n + a)\dfrac{\partial w}{\partial x} - (nAx^{n-1}y + kx^m + b)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = y^2 + \dfrac{2k}{m+1} x^{m+1} + 2(Ax^n y + ay + bx)$.

**95.** $\boldsymbol{(y + ax^{n+1} + bx^n)\dfrac{\partial w}{\partial x} + (anx^n + cx^{n-1})y\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = x^{-1}E - a\int \frac{E\,dv}{nv^2 - (bn+c)v + bc}, \qquad v = x^{-n}y + b,$$

where $E = \exp\left[-\displaystyle\int \frac{v\,dv}{nv^2 - (bn+c)v + bc}\right]$.

**96.** $\boldsymbol{x(2ax^n y+b)\dfrac{\partial w}{\partial x} - \bigl[a(3n+m)x^n y^2 + b(2n+m)y - Ax^m - Cx^{-n}\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = (Ax^{n+m} + C)^2 - 2A(n+m)x^{2n+m}y(ax^n y + b)$.

**97.** $\boldsymbol{(ax^n + bx^2 + xy)\dfrac{\partial w}{\partial x} + (cx^n + bxy + y^2)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{1}{n-2}(ay - cx)^{n-2} + \int (v+b)(av-c)^{n-3}\,dv, \qquad v = \frac{y}{x}.$$

**98.** $\boldsymbol{(ay^n + bx^2 + cxy)\dfrac{\partial w}{\partial x} + (ky^n + bxy + cy^2)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{1}{n-2}(kx - ay)^{n-2} - \int \frac{(k-av)^{n-3}(b+cv)}{v^n}\,dv, \qquad v = \frac{y}{x}.$$

**99.** $\boldsymbol{(ax^n + bx^m + c)\dfrac{\partial w}{\partial x} + (cy^2 - bx^{m-1}y + ax^{n-2})\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = c\int \frac{E\,dx}{ax^n + bx^m + c} + \frac{xE}{xy+1}, \qquad E = \exp\left[-\int \frac{(bx^m + 2c)\,dx}{x(ax^n + bx^m + c)}\right].$$

**100.** $\boldsymbol{(ax^n + bx^m + c)\dfrac{\partial w}{\partial x} + (ax^{n-2}y^2 + bx^{m-1}y + c)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = a\int \frac{x^{n-2}E\,dx}{ax^n + bx^m + c} + \frac{E}{y-x}, \qquad E = \exp\left[\int \frac{(2ax^n + bx^m)\,dx}{x(ax^n + bx^m + c)}\right].$$

**101.** $\boldsymbol{(ax^n + bx^m + c)\dfrac{\partial w}{\partial x} + (\alpha x^k y^2 + \beta x^s y - \alpha\lambda^2 x^k + \beta\lambda x^s)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \alpha\int \frac{x^k E\,dx}{ax^n + bx^m + c} + \frac{E}{y+\lambda}, \qquad E = \exp\left(\int \frac{\beta x^s - 2\alpha\lambda x^k}{ax^n + bx^m + c}\,dx\right).$$

**102.** $x(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}-\big[sx^ky^2-(ax^n+bx^m+c)y-s\lambda x^{k+2}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{y-x\sqrt{\lambda}}{y+x\sqrt{\lambda}}\exp\biggl(2s\sqrt{\lambda}\displaystyle\int\frac{x^k\,dx}{ax^n+bx^m+c}\biggr).$

**103.** $(ax^n+bx^m+c)\dfrac{\partial w}{\partial x}$

$+\big[(ax^n+bx^m+c)y^2-an(n-1)x^{n-2}-bm(m-1)x^{m-2}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{(ax^n+bx^m+c)\big[(ax^n+bx^m+c)y+anx^{n-1}+bmx^{m-1}\big]}+\int\frac{dx}{(ax^n+bx^m+c)^2}.$$

**104.** $(ax^n+by^n+x)\dfrac{\partial w}{\partial x}+(\alpha x^ky^{n-k}+\beta x^my^{n-m}+y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{n-1}x^{n-1}E-\int\frac{E\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av},\qquad v=\frac{y}{x},$$

where $E=\exp\biggl[(1-n)\displaystyle\int\frac{(bv^n+a)\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av}\biggr].$

**105.** $(ax^n+by^n+Ax^2+Bxy)\dfrac{\partial w}{\partial x}$

$+(\alpha x^ky^{n-k}+\beta x^my^{n-m}+Axy+By^2)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{1}{n-2}x^{n-2}E-\int\frac{(Bv+A)E\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av},\qquad v=\frac{y}{x},$$

where $E=\exp\biggl[(2-n)\displaystyle\int\frac{(bv^n+a)\,dv}{\alpha v^{n-k}+\beta v^{n-m}-bv^{n+1}-av}\biggr].$

**106.** $(ay^m+bx^n+s)\dfrac{\partial w}{\partial x}-(\alpha x^k+bnx^{n-1}y+\beta)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=a\varphi(y)+\alpha\psi(x)+bx^ny+sy+\beta x,$$

where

$$\varphi(y)=\begin{cases}\dfrac{y^{m+1}}{m+1} & \text{if } m\neq-1,\\ \ln|y| & \text{if } m=-1,\end{cases}\qquad \psi(x)=\begin{cases}\dfrac{x^{k+1}}{k+1} & \text{if } k\neq-1,\\ \ln|x| & \text{if } k=-1.\end{cases}$$

**107.** $\displaystyle (ax^n y^m + x)\frac{\partial w}{\partial x} + (bx^k y^{n+m-k} + y)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{n+m-1}x^{n+m-1}E - a\int \frac{E\,dv}{v^m(bv^{n-k} - av)}, \qquad v = \frac{y}{x},$$

where $E = \exp\left[a(1-n-m)\int \frac{dv}{bv^{n-k} - av}\right]$.

**108.** $\displaystyle x(ax^n y^m + \alpha)\frac{\partial w}{\partial x} - y(bx^n y^m + \beta)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{(y^a x^b)^A}{A} + \frac{(y^\alpha x^\beta)^B}{B}, \qquad \text{where} \quad A = \frac{m\beta - n\alpha}{a\beta - b\alpha}, \quad B = \frac{mb - na}{a\beta - b\alpha}.$$

**109.** $\displaystyle x(anx^k y^{n+k} + s)\frac{\partial w}{\partial x} - y(bmx^{m+k} y^k + s)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = aky^n + bkx^m - s(xy)^{-k}$.

**110.** $\displaystyle (ax^n y^m + Ax^2 + Bxy)\frac{\partial w}{\partial x} + (bx^k y^{n+m-k} + Axy + By^2)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{n+m-2}x^{n+m-2}E - a\int \frac{(Bv + A)E\,dv}{v^m(bv^{n-k} - av)}, \qquad v = \frac{y}{x},$$

where $E = \exp\left[a(2-n-m)\int \frac{dv}{bv^{n-k} - av}\right]$.

**111.** $\displaystyle (ax^n y^m + bxy^k)\frac{\partial w}{\partial x} + (\alpha y^s + \beta)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int \frac{y^m E}{\alpha y^s + \beta}\,dy, \qquad E = \exp\left[b(n-1)\int \frac{y^k\,dy}{\alpha y^s + \beta}\right].$$

## 1.1.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

**1.** $\displaystyle \frac{\partial w}{\partial x} + ae^{\lambda x}\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - ae^{\lambda x}$.

**2.** $\displaystyle \frac{\partial w}{\partial x} + \big(ae^{\lambda x} + b\big)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda(bx - y) + ae^{\lambda x}$.

**3.** $\dfrac{\partial w}{\partial x} + \big(ae^{\lambda y} + b\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda(bx - y) + \ln\big|b + ae^{\lambda y}\big|.$

**4.** $\dfrac{\partial w}{\partial x} + \big(ae^{\lambda y+\beta x} + b\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = e^{b\lambda x-\lambda y} + \dfrac{a\lambda}{\beta + b\lambda}e^{(\beta+b\lambda)x}.$

**5.** $\dfrac{\partial w}{\partial x} + \big(ae^{\lambda y+\beta x} + be^{\gamma x}\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = e^{-\lambda y}E + a\lambda\displaystyle\int e^{\beta x}E\,dx,$ where $E = \exp\left(\dfrac{b\lambda}{\gamma}e^{\gamma x}\right).$

**6.** $ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{\beta b}e^{-\beta y} - \dfrac{1}{\lambda a}e^{-\lambda x}.$

**7.** $\big(ae^{\lambda x} + b\big)\dfrac{\partial w}{\partial x} + \big(ce^{\beta x} + d\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - \displaystyle\int\frac{ce^{\beta x} + d}{ae^{\lambda x} + b}\,dx.$

**8.** $\big(ae^{\lambda x} + b\big)\dfrac{\partial w}{\partial x} + \big(ce^{\beta y} + d\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda\beta(dx - by) - d\beta\ln\big|ae^{\lambda x} + b\big| + b\lambda\ln\big|ce^{\beta y} + d\big|.$

**9.** $\big(ae^{\lambda y} + b\big)\dfrac{\partial w}{\partial x} + \big(ce^{\beta x} + d\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \beta\lambda(dx - by) + c\lambda e^{\beta x} - a\beta e^{\lambda y}.$

**10.** $\big(ae^{\lambda x} + be^{\beta y}\big)\dfrac{\partial w}{\partial x} + a\lambda e^{\lambda x}\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = ae^{\lambda x-y} - \dfrac{b}{\beta - 1}e^{(\beta-1)y}.$

**11.** $\big(ae^{\lambda x+\beta y} + c\mu\big)\dfrac{\partial w}{\partial x} - \big(be^{\gamma x+\mu y} + c\lambda\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{a}{\beta - \mu}e^{(\beta-\mu)y} + \dfrac{b}{\gamma - \lambda}e^{(\gamma-\lambda)x} - ce^{-\lambda x-\mu y}.$

► **Coefficients of equations contain exponential and power-law functions.**

**12.** $\dfrac{\partial w}{\partial x} + (y^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int E\,dx, \qquad E = \exp\left(\frac{2a}{\lambda}e^{\lambda x}\right).$$

**13.** $\dfrac{\partial w}{\partial x} + \left[y^2 + by + a(\lambda - b)e^{\lambda x} - a^2 e^{2\lambda x}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int E\,dx, \qquad E = \exp\left(\frac{2a}{\lambda}e^{\lambda x} + bx\right).$$

**14.** $\dfrac{\partial w}{\partial x} + (y^2 + ae^{\lambda x}y - abe^{\lambda x} - b^2)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - b} + \int E\,dx, \qquad E = \exp\left(2bx + \frac{a}{\lambda}e^{\lambda x}\right).$$

**15.** $\dfrac{\partial w}{\partial x} - (y^2 - axe^{\lambda x}y + ae^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy - 1)} - \int \frac{E}{x^2}\,dx, \qquad E = \exp\left[\frac{a}{\lambda^2}(\lambda x - 1)e^{\lambda x}\right].$$

**16.** $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + be^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + \lambda v + b} - x, \qquad v = e^{\lambda x}y.$$

**17.** $\dfrac{\partial w}{\partial x} + \left[ae^{\lambda x}y^2 + b\mu e^{\mu x} - ab^2 e^{(\lambda + 2\mu)x}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\mu x}} + a\int e^{\lambda x}E\,dx, \qquad E = \exp\left[\frac{2ab}{\lambda + \mu}e^{(\lambda + \mu)x}\right].$$

**18.** $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + by + ce^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + (b + \lambda)v + c} - x, \qquad v = e^{\lambda x}y.$$

**19.** $\dfrac{\partial w}{\partial x}+\big[ae^{\lambda x}y^2+\mu y-ab^2e^{(\lambda+2\mu)x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-be^{\mu x}}+a\int e^{\lambda x}E\,dx,\qquad E=\exp\left[\frac{2ab}{\lambda+\mu}e^{(\lambda+\mu)x}+\mu x\right].$$

**20.** $\dfrac{\partial w}{\partial x}+\big[e^{\lambda x}y^2+ae^{\mu x}y+a\lambda e^{(\mu-\lambda)x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+\lambda e^{-\lambda x}}+\int e^{\lambda x}E\,dx,\qquad E=\exp\left(\frac{a}{\mu}e^{\mu x}-2\lambda x\right).$$

**21.** $\dfrac{\partial w}{\partial x}-\big[\lambda e^{\lambda x}y^2-ae^{\mu x}y+ae^{(\mu-\lambda)x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-e^{-\lambda x}}-\lambda\int e^{\lambda x}E\,dx,\qquad E=\exp\left(\frac{a}{\mu}e^{\mu x}-2\lambda x\right).$$

**22.** $\dfrac{\partial w}{\partial x}+\big[ae^{\lambda x}y^2+abe^{(\lambda+\mu)x}y-b\mu e^{\mu x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+be^{\mu x}}+a\int e^{\lambda x}E\,dx,\qquad E=\exp\left[-\frac{ab}{\lambda+\mu}e^{(\lambda+\mu)x}\right].$$

**23.** $\dfrac{\partial w}{\partial x}+\big[ae^{(2\lambda+\mu)x}y^2+(be^{(\lambda+\mu)x}-\lambda)y+ce^{\mu x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\frac{1}{\mu+\lambda}e^{(\mu+\lambda)x},\qquad v=e^{\lambda x}y.$$

**24.** $\dfrac{\partial w}{\partial x}+\big[e^{\lambda x}(y-be^{\mu x})^2+b\mu e^{\mu x}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y-be^{\mu x}}+\dfrac{1}{\lambda}e^{\lambda x}.$

**25.** $\dfrac{\partial w}{\partial x}+(ae^{\lambda x}y^2+bnx^{n-1}-ab^2e^{\lambda x}x^{2n})\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^n}+a\int e^{\lambda x}E\,dx,\qquad E=\exp\left(2ab\int x^ne^{\lambda x}\,dx\right).$$

**26.** $\dfrac{\partial w}{\partial x} + (e^{\lambda x}y^2 + ax^n y + a\lambda x^n e^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + \lambda e^{-\lambda x}} + \int e^{\lambda x} E\,dx, \qquad E = \exp\left(\frac{a}{n+1}x^{n+1} - 2\lambda x\right).$$

**27.** $\dfrac{\partial w}{\partial x} + (\lambda e^{\lambda x}y^2 + ax^n e^{\lambda x} y - ax^n e^{2\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{2\lambda x}E}{y - e^{\lambda x}} + \lambda\int e^{\lambda x} E\,dx, \qquad E = \exp\left(a\int x^n e^{-\lambda x}\,dx\right).$$

**28.** $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 - abx^n e^{\lambda x} y + bnx^{n-1})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^n} + a\int e^{\lambda x} E\,dx, \qquad E = \exp\left(ab\int x^n e^{\lambda x}\,dx\right).$$

**29.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 + b\lambda e^{\lambda x} - ab^2 x^n e^{2\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\lambda x}} + a\int x^n E\,dx, \qquad E = \exp\left(2ab\int x^n e^{\lambda x}\,dx\right).$$

**30.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 + \lambda y - ab^2 x^n e^{2\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\lambda x}} + a\int x^n E\,dx, \qquad E = \exp\left(\lambda x + 2ab\int x^n e^{\lambda x}\,dx\right).$$

**31.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 - abx^n e^{\lambda x} y + b\lambda e^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\lambda x}} + a\int x^n E\,dx, \qquad E = \exp\left(ab\int x^n e^{\lambda x}\,dx\right).$$

**32.** $\dfrac{\partial w}{\partial x} + \bigl[ax^n y^2 - ax^n(be^{\lambda x} + c)y + b\lambda e^{\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - be^{\lambda x} - c} + a\int x^n E\,dx,$$

$$\text{where } E = \begin{cases} \exp\left(\dfrac{ac}{n+1}x^{n+1} + ab\displaystyle\int x^n e^{\lambda x}\,dx\right) & \text{if } n \neq -1, \\ x^{ac}\exp\left(ab\displaystyle\int \dfrac{e^{\lambda x}}{x}\,dx\right) & \text{if } n = -1. \end{cases}$$

**33.** $\dfrac{\partial w}{\partial x} + \big[ax^n e^{2\lambda x} y^2 + (bx^n e^{\lambda x} - \lambda)y + cx^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + bv + c} - \int x^n e^{\lambda x}\, dx, \qquad v = e^{\lambda x} y.$$

**34.** $\dfrac{\partial w}{\partial x} + \big[ae^{\lambda x}(y - bx^n - c)^2 + bnx^{n-1}\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y - bx^n - c} + \dfrac{a}{\lambda} e^{\lambda x}.$

**35.** $\dfrac{\partial w}{\partial x} + \big(y^2 + 2a\lambda x e^{\lambda x^2} - a^2 e^{2\lambda x^2}\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x^2}} + \int E\, dx, \qquad E = \exp\biggl(2a \int e^{\lambda x^2}\, dx\biggr).$$

**36.** $\dfrac{\partial w}{\partial x} + \big(ae^{-\lambda x^2} y^2 + \lambda xy + ab^2\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\left[\dfrac{1}{b} y \exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\right] - ab \displaystyle\int \exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\, dx.$

**37.** $\dfrac{\partial w}{\partial x} + \big(ax^n y^2 + \lambda xy + ab^2 x^n e^{\lambda x^2}\big)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\left[\dfrac{y}{b} \exp\bigl(-\tfrac{1}{2}\lambda x^2\bigr)\right] - ab \displaystyle\int x^n \exp\bigl(\tfrac{1}{2}\lambda x^2\bigr)\, dx.$

**38.** $\dfrac{\partial w}{\partial x} + (ae^{2\lambda x} y^3 + be^{\lambda x} y^2 + cy + de^{-\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^3 + bv^2 + (c + \lambda)v + d} - x, \qquad v = e^{\lambda x} y.$$

**39.** $\dfrac{\partial w}{\partial x} + (ae^{\lambda x} y^3 + 3abe^{\lambda x} y^2 + cy - 2ab^3 e^{\lambda x} + bc)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{2cx} E}{(y + b)^2} + 2a \int e^{(\lambda + 2c)x} E\, dx, \qquad E = \exp\biggl(-\frac{6ab^2}{\lambda} e^{\lambda x}\biggr).$$

**40.** $x\dfrac{\partial w}{\partial x} + (ae^{\lambda x} y^2 + ky + ab^2 x^{2k} e^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan \dfrac{y}{bx^k} - ab \displaystyle\int x^{k-1} e^{\lambda x}\, dx.$

**41.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + \left[ax^{2n}e^{\lambda x}y^2 + (bx^n e^{\lambda x} - n)y + ce^{\lambda x}\right]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + bv + c} - \int x^{n-1}e^{\lambda x}\,dx, \qquad v = x^n y.$$

**42.** $\boldsymbol{y\dfrac{\partial w}{\partial x} + e^{\lambda x}\left[(2a\lambda x + a + b)y - e^{\lambda x}(a^2\lambda x^2 + abx - c)\right]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = xE + \int \frac{vE\,dv}{\lambda v^2 - bv - c}, \qquad v = e^{-\lambda x}y - ax,$$

where $E = \exp\left(a\displaystyle\int \frac{dv}{\lambda v^2 - bv - c}\right)$.

**43.** $\boldsymbol{ae^{\lambda x}\dfrac{\partial w}{\partial x} + by^m\dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $m \neq 1$: $\Xi = \dfrac{1}{b(1-m)}y^{1-m} + \dfrac{1}{\lambda a}e^{-\lambda x}$.

2°. Principal integral for $m = 1$: $\Xi = \dfrac{1}{b}\ln y + \dfrac{1}{\lambda a}e^{-\lambda x}$.

**44.** $\boldsymbol{(ae^y + bx)\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} = 0.}$

1°. Principal integral for $b \neq 1$: $\Xi = xe^{-by} - \dfrac{a}{1-b}e^{(1-b)y}$.

2°. Principal integral for $b = 1$: $\Xi = xe^{-y} - ay$.

**45.** $\boldsymbol{(ax^n e^{\lambda y} + bxy^m)\dfrac{\partial w}{\partial x} + e^{\mu y}\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int e^{(\lambda-\mu)y}E\,dy, \qquad E = \exp\left[b(n-1)\int y^m e^{-\mu y}\,dy\right].$$

**46.** $\boldsymbol{(ax^n y^m + bxe^{\lambda y})\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{1}{n-1}x^{n-1}E + a\int y^{m-k}E\,dy, \qquad E = \exp\left[b(n-1)\int y^{-k}e^{\lambda y}\,dy\right].$$

**47.** $\boldsymbol{(ax^n y^m + bxy^k)\dfrac{\partial w}{\partial x} + e^{\lambda y}\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{1}{1-n}x^{1-n}E - a\int y^m e^{-\lambda y}E\,dy, \qquad E = \exp\left[b(n-1)\int y^k e^{-\lambda y}\,dy\right].$$

### 1.1.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

**1.** $\dfrac{\partial w}{\partial x} + a\sinh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - a\cosh(\lambda x)$.

**2.** $\dfrac{\partial w}{\partial x} + a\sinh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\mu x - \ln\bigl|\tanh(\tfrac{1}{2}\mu y)\bigr|$.

**3.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 - a^2 + a\lambda\sinh(\lambda x) - a^2\sinh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\cosh(\lambda x)} + \int E\,dx, \qquad E = \exp\biggl[\frac{2a}{\lambda}\sinh(\lambda x)\biggr].$$

**4.** $\dfrac{\partial w}{\partial x} + \lambda\bigl[\sinh(\lambda x)y^2 - \sinh^3(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\exp\bigl[\frac{1}{2}\cosh(2\lambda x)\bigr]}{y - \cosh(\lambda x)} + \lambda\int \sinh(\lambda x)\exp\bigl[\tfrac{1}{2}\cosh(2\lambda x)\bigr]\,dx.$$

**5.** $\dfrac{\partial w}{\partial x} + \bigl\{[a\sinh^2(\lambda x) - \lambda]y^2 - a\sinh^2(\lambda x) + \lambda - a\bigr\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\sinh(\lambda x)\bigl[\sinh(\lambda x)y - \cosh(\lambda x)\bigr]} + \int\biggl[a - \frac{\lambda}{\sinh^2(\lambda x)}\biggr]E\,dx,$$

$$E = \exp\biggl[\frac{a}{2\lambda}\cosh(2\lambda x)\biggr].$$

**6.** $\sinh(\lambda x)\dfrac{\partial w}{\partial x} + a\sinh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\mu\ln\bigl|\tanh(\tfrac{1}{2}\lambda x)\bigr| - \lambda\ln\bigl|\tanh(\tfrac{1}{2}\mu y)\bigr|$.

**7.** $\sinh(\mu y)\dfrac{\partial w}{\partial x} + a\sinh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda\cosh(\mu y) - a\mu\cosh(\lambda x)$.

► **Coefficients of equations contain hyperbolic cosine.**

**8.** $\dfrac{\partial w}{\partial x} + a\cosh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\sinh(\lambda x) - \lambda y$.

**9.** $\dfrac{\partial w}{\partial x} + a\cosh(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\lambda x - 2\arctan(e^{\lambda y})$.

**10.** $\dfrac{\partial w}{\partial x} + \bigl\{[a\cosh^2(\lambda x) - \lambda]y^2 - a\cosh^2(\lambda x) + \lambda + a\bigr\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cosh(\lambda x)\bigl[\cosh(\lambda x)y - \sinh(\lambda x)\bigr]} + \int\biggl[a - \frac{\lambda}{\cosh^2(\lambda x)}\biggr]E\,dx,$$
$$E = \exp\biggl[\frac{a}{2\lambda}\cosh(2\lambda x)\biggr].$$

**11.** $2\dfrac{\partial w}{\partial x} + \bigl\{[a - \lambda + a\cosh(\lambda x)]y^2 + a + \lambda - a\cosh(\lambda x)\bigr\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - \tanh\bigl(\frac{1}{2}\lambda x\bigr)} + \frac{1}{2}\int\bigl[a - \lambda + a\cosh(\lambda x)\bigr]E\,dx,$$

where

$$E = \bigl[\cosh(\tfrac{1}{2}\lambda x)\bigr]^{\frac{2(a-\lambda)}{\lambda}}\exp\biggl[a\int\cosh(\lambda x)\tanh\bigl(\tfrac{1}{2}\lambda x\bigr)\,dx\biggr].$$

**12.** $(ax^n + bx\cosh^m y)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int y^{-k}E\,dy, \qquad E = \exp\biggl[b(n-1)\int y^{-k}\cosh^m y\,dy\biggr].$$

**13.** $(ax^n + bx\cosh^m y)\dfrac{\partial w}{\partial x} + \cosh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int\frac{E\,dy}{\cosh^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int\frac{\cosh^m y\,dy}{\cosh^k(\lambda y)}\biggr].$$

**14.** $(ax^ny^m + bx)\dfrac{\partial w}{\partial x} + \cosh^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int\frac{y^mE\,dy}{\cosh^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int\frac{dy}{\cosh^k(\lambda y)}\biggr].$$

**15.** $\cosh(\mu y)\dfrac{\partial w}{\partial x} + a\cosh(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \mu a\sinh(\lambda x) - \lambda\sinh(\mu y)$.

► **Coefficients of equations contain hyperbolic tangent.**

**16.** $\dfrac{\partial w}{\partial x}+a\tanh(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda y-a\ln\bigl[\cosh(\lambda x)\bigr].$

**17.** $\dfrac{\partial w}{\partial x}+a\tanh(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\lambda x-\ln\bigl|\sinh(\lambda y)\bigr|.$

**18.** $\dfrac{\partial w}{\partial x}+\bigl[y^2+a\lambda-a(a+\lambda)\tanh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{y-a\tanh(\lambda x)}+\displaystyle\int\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}\,dx.$

**19.** $\dfrac{\partial w}{\partial x}+\bigl[y^2+3a\lambda-\lambda^2-a(a+\lambda)\tanh^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{\sinh^2(\lambda x)\bigl[y-a\tanh(\lambda x)+\lambda\coth(\lambda x)\bigr]}+\int\frac{\bigl[\cosh(\lambda x)\bigr]^{2a/\lambda}}{\sinh^2(\lambda x)}\,dx.$$

**20.** $(ax^n+bx\tanh^m y)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int y^{-k}E\,dy,\qquad E=\exp\biggl[b(n-1)\int y^{-k}\tanh^m y\,dy\biggr].$$

**21.** $(ax^n+bx\tanh^m y)\dfrac{\partial w}{\partial x}+\tanh^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{E\,dy}{\tanh^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{\tanh^m y\,dy}{\tanh^k(\lambda y)}\biggr].$$

**22.** $(ax^ny^m+bx)\dfrac{\partial w}{\partial x}+\tanh^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{y^mE\,dy}{\tanh^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{dy}{\tanh^k(\lambda y)}\biggr].$$

**23.** $(ax^n\tanh^m y+bx)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $k\neq 1$:

$$\Xi=x^{1-n}E+(n-1)a\int y^{-k}E\tanh^m y\,dy,\qquad E=\exp\biggl[\frac{b(n-1)}{1-k}y^{1-k}\biggr].$$

2°. Principal integral for $k=1$:

$$\Xi=(xy^{-b})^{1-n}+(n-1)a\int y^{(n-1)b-1}\tanh^m y\,dy.$$

► **Coefficients of equations contain hyperbolic cotangent.**

**24.** $\dfrac{\partial w}{\partial x} + a\coth(\lambda x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda y - a\ln\bigl|\sinh(\lambda x)\bigr|.$

**25.** $\dfrac{\partial w}{\partial x} + a\coth(\lambda y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\lambda x - \ln\bigl[\cosh(\lambda y)\bigr].$

**26.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 + a\lambda - a(a+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{y - a\coth(\lambda x)} + \displaystyle\int\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}\,dx.$

**27.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 + 3a\lambda - \lambda^2 - a(a+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{\cosh^2(\lambda x)\bigl[y - a\coth(\lambda x) + \lambda\tanh(\lambda x)\bigr]} + \int\frac{\bigl[\sinh(\lambda x)\bigr]^{2a/\lambda}}{\cosh^2(\lambda x)}\,dx.$$

► **Coefficients of equations contain different hyperbolic functions.**

**28.** $\dfrac{\partial w}{\partial x} + a\sinh(\lambda x)\cosh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = 2\lambda\arctan\bigl(e^{\mu y}\bigr) - a\mu\cosh(\lambda x).$

**29.** $\dfrac{\partial w}{\partial x} + a\cosh(\lambda x)\sinh(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \lambda\ln\bigl|\tanh\bigl(\tfrac{1}{2}\mu y\bigr)\bigr| - a\mu\sinh(\lambda x).$

**30.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 - 2\lambda^2\tanh^2(\lambda x) - 2\lambda^2\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\sinh^2(\lambda x)\cosh^2(\lambda x)}{y - \lambda\tanh(\lambda x) - \lambda\coth(\lambda x)} + \int\sinh^2(\lambda x)\cosh^2(\lambda x)\,dx.$$

**31.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 + \lambda(a+b) - 2ab - a(a+\lambda)\tanh^2(\lambda x)$

$\qquad - b(b+\lambda)\coth^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\bigl[\sinh(\lambda x)\bigr]^{\frac{2b}{\lambda}}\bigl[\cosh(\lambda x)\bigr]^{\frac{2a}{\lambda}}}{y - a\tanh(\lambda x) - b\coth(\lambda x)} + \int\bigl[\sinh(\lambda x)\bigr]^{\frac{2b}{\lambda}}\bigl[\cosh(\lambda x)\bigr]^{\frac{2a}{\lambda}}\,dx.$$

**32.** $\sinh(\lambda y)\dfrac{\partial w}{\partial x}+a\cosh(\beta x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\beta\cosh(\lambda y)-a\lambda\sinh(\beta x).$

**33.** $\big[ax^n\cosh^m(\lambda y)+bx\big]\dfrac{\partial w}{\partial x}+\sinh^k(\beta y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{\cosh^m(\lambda y)E\,dy}{\sinh^k(\beta y)},\qquad E=\exp\biggl[b(n-1)\int\frac{dy}{\sinh^k(\beta y)}\biggr].$$

## 1.1.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

**1.** $\dfrac{\partial w}{\partial x}+\big[a\ln^k(\lambda x)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=y-bx-a\displaystyle\int\ln^k(\lambda x)\,dx.$

**2.** $\dfrac{\partial w}{\partial x}+\big[a\ln^k(\lambda y)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=x-\displaystyle\int\frac{dy}{a\ln^k(\lambda y)+b}.$

**3.** $\dfrac{\partial w}{\partial x}+a\ln^k(\lambda x)\ln^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\ln^k(\lambda x)\,dx-\int\frac{dy}{\ln^n(\mu y)}.$

**4.** $\dfrac{\partial w}{\partial x}+a\ln^k(x+\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x-\int\frac{dz}{1+a\lambda\ln^k z},\qquad z=x+\lambda y.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

**5.** $\dfrac{\partial w}{\partial x}+ax^n\ln^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{a}{n+1}x^{n+1}-\displaystyle\int\frac{dy}{\ln^k(\lambda y)}.$

**6.** $\dfrac{\partial w}{\partial x}+ay^n\ln^k(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{1-n}y^{1-n}-a\displaystyle\int\ln^k(\lambda x)\,dx.$

**7.** $\dfrac{\partial w}{\partial x} + \left[y^2 + a\ln(\beta x)y - ab\ln(\beta x) - b^2\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{(2b-a)x}E}{y-b} + \int e^{(2b-a)x}E\,dx, \qquad E = \exp\left[ax\ln(\beta x)\right].$$

**8.** $\dfrac{\partial w}{\partial x} + \left[y^2 + ax\ln^m(bx)y + a\ln^m(bx)\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[a\int x\ln^m(bx)\,dx\right].$$

**9.** $\dfrac{\partial w}{\partial x} + (ax^n y^2 - abx^{n+1}y\ln x + b\ln x + b)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx\ln x} + a\int x^n E\,dx, \qquad E = \exp\left[\frac{ab}{n+2}x^{n+2}\left(\ln x - \frac{1}{n+2}\right)\right].$$

**10.** $\dfrac{\partial w}{\partial x} - \left[(n+1)x^n y^2 - ax^{n+1}(\ln x)^m y + a(\ln x)^m\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{x^{-2(n+1)}E}{y-x^{-n-1}} - (n+1)\int x^{-n-2}E\,dx, \qquad E = \exp\left[a\int x^{n+1}(\ln x)^m\,dx\right].$$

**11.** $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^n y^2 + bmx^{m-1} - ab^2x^{2m}(\ln x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx^m} + a\int (\ln x)^n E\,dx, \qquad E = \exp\left[2ab\int x^m(\ln x)^n\,dx\right].$$

**12.** $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^n y^2 - abx(\ln x)^{n+1}y + b\ln x + b\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx\ln x} + a\int (\ln x)^n E\,dx, \qquad E = \exp\left[ab\int x(\ln x)^{n+1}\,dx\right].$$

**13.** $\dfrac{\partial w}{\partial x} + \left[a(\ln x)^k(y-bx^n-c)^2 + bnx^{n-1}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y-bx^n-c} + a\displaystyle\int (\ln x)^k\,dx.$

**14.** $\dfrac{\partial w}{\partial x}+\bigl[a(\ln x)^n y^2+b(\ln x)^m y+bc(\ln x)^m-ac^2(\ln x)^n\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+c}+a\int(\ln x)^n E\,dx,\qquad E=\exp\biggl\{\int\bigl[b(\ln x)^m-2ac(\ln x)^n\bigr]\,dx\biggr\}.$$

**15.** $x\dfrac{\partial w}{\partial x}+(ay+b\ln x)^2\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\ln x-\int\frac{dv}{av^2+b},\qquad v=ay+b\ln x.$$

**16.** $x\dfrac{\partial w}{\partial x}+\bigl[xy^2-A^2x\ln^2(\beta x)+A\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{-2Ax}E}{y-A\ln(\beta x)}+\int e^{-2Ax}E\,dx,\qquad E=\exp\bigl[2Ax\ln(\beta x)\bigr].$$

**17.** $x\dfrac{\partial w}{\partial x}+\bigl[xy^2-A^2x\ln^{2k}(\beta x)+kA\ln^{k-1}(\beta x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-A\ln^k(\beta x)}+\int E\,dx,\qquad E=\exp\biggl[2A\int\ln^k(\beta x)\,dx\biggr].$$

**18.** $x\dfrac{\partial w}{\partial x}+(ax^ny^2+b-ab^2x^n\ln^2 x)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-b\ln x}+\int ax^{n-1}E\,dx,\qquad E=\exp\biggl[\frac{2abx^n}{n^2}(n\ln x-1)\biggr].$$

**19.** $x\dfrac{\partial w}{\partial x}+\bigl[a\ln^m(\lambda x)y^2+ky+ab^2x^{2k}\ln^m(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=\arctan\biggl(\frac{y}{bx^k}\biggr)-ab\int x^{k-1}\ln^m(\lambda x)\,dx.$

**20.** $x\dfrac{\partial w}{\partial x}+\bigl[ax^n(y+b\ln x)^2-b\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\displaystyle\Xi=\frac{1}{y+b\ln x}+\frac{a}{n}x^n.$

**21.** $x\dfrac{\partial w}{\partial x}+\bigl[ax^{2n}(\ln x)y^2+(bx^n\ln x-n)y+c\ln x\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{av^2+bv+c}-\int x^{n-1}\ln x\,dx,\qquad v=x^ny.$$

**22.** $\displaystyle x^k \frac{\partial w}{\partial x} + (ay^n \ln^m x + by \ln^s x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a \int x^{-k} E \ln^m x \, dx, \qquad E = \exp\left[b(n-1)\int x^{-k} \ln^s x \, dx\right].$$

**23.** $\displaystyle (a \ln x + b)\frac{\partial w}{\partial x} + \left[y^2 + c(\ln x)^n y - \lambda^2 + \lambda c(\ln x)^n\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y+\lambda} + \int \frac{E\,dx}{a\ln x + b}, \qquad E = \exp\left[\int \frac{c(\ln x)^n - 2\lambda}{a\ln x + b}\right] dx.$$

**24.** $\displaystyle (a \ln x + b)\frac{\partial w}{\partial x} + \left[(\ln x)^n y^2 + cy - \lambda^2(\ln x)^n + c\lambda\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y+\lambda} + \int \frac{(\ln x)^n E\,dx}{a\ln x + b}, \qquad E = \exp\left[\int \frac{c - 2\lambda(\ln x)^n}{a\ln x + b}\,dx\right].$$

**25.** $\displaystyle x^2 \ln(ax)\frac{\partial w}{\partial x} - \left[x^2y^2\ln(ax) + 1\right]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \frac{x}{\ln(ax)\left[xy\ln(ax) - 1\right]} - \int \frac{dx}{\ln^2(ax)}.$

**26.** $\displaystyle \ln^k(\lambda x)\frac{\partial w}{\partial x} + (ay^n + by\ln^m x)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a\int \frac{E\,dx}{\ln^k(\lambda x)}, \qquad E = \exp\left[b(n-1)\int \frac{\ln^m x\,dx}{\ln^k(\lambda x)}\right].$$

**27.** $\displaystyle \ln^k(\lambda x)\frac{\partial w}{\partial x} + (ay^n\ln^m x + by)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{1-n}E + (n-1)a\int \frac{E\ln^m x\,dx}{\ln^k(\lambda x)}, \qquad E = \exp\left[b(n-1)\int \frac{dx}{\ln^k(\lambda x)}\right].$$

### 1.1.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

**1.** $\displaystyle \frac{\partial w}{\partial x} + \left[a\sin^k(\lambda x) + b\right]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = y - bx - a\int \sin^k(\lambda x)\,dx.$

**2.** $\dfrac{\partial w}{\partial x}+\big[a\sin^k(\lambda y)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=x-\displaystyle\int\frac{dy}{a\sin^k(\lambda y)+b}.$

**3.** $\dfrac{\partial w}{\partial x}+a\sin^k(\lambda x)\sin^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\sin^k(\lambda x)\,dx-\int\frac{dy}{\sin^n(\mu y)}.$

**4.** $\dfrac{\partial w}{\partial x}+a\sin^k(x+\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x-\int\frac{dz}{1+a\lambda\sin^k z},\qquad z=x+\lambda y.$$

**5.** $\dfrac{\partial w}{\partial x}+\big[y^2-a^2+a\lambda\sin(\lambda x)+a^2\sin^2(\lambda x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+a\cos(\lambda x)}+\int E\,dx,\qquad E=\exp\left[-\frac{2a}{\lambda}\sin(\lambda x)\right].$$

**6.** $\dfrac{\partial w}{\partial x}+\big[y^2+a\sin(\beta x)y+ab\sin(\beta x)-b^2\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+b}+\int E\,dx,\qquad E=\exp\left[-2bx-\frac{a}{\beta}\cos(\beta x)\right].$$

**7.** $\dfrac{\partial w}{\partial x}+\big[y^2+ax\sin^m(bx)y+a\sin^m(bx)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x(xy+1)}+\int x^{-2}E\,dx,\qquad E=\exp\left[a\int x\sin^m(bx)\,dx\right].$$

**8.** $\dfrac{\partial w}{\partial x}+\big[\lambda\sin(\lambda x)y^2+\lambda\sin^3(\lambda x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+\cos(\lambda x)}+\lambda\int E\sin(\lambda x)\,dx,\qquad E=\exp\big[\tfrac{1}{2}\cos(2\lambda x)\big].$$

**9.** $2\dfrac{\partial w}{\partial x}+\big\{[\lambda+a-a\sin(\lambda x)]y^2+\lambda-a-a\sin(\lambda x)\big\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan\left(\frac{1}{2}\lambda x+\frac{1}{4}\pi\right)}+\frac{1}{2}\int\big[\lambda+a-a\sin(\lambda x)\big]E\,dx,$$
$$E=\frac{1}{1-\sin(\lambda x)}\exp\left[\frac{a}{\lambda}\sin(\lambda x)\right].$$

**10.** $\dfrac{\partial w}{\partial x} + \big\{[\lambda + a\sin^2(\lambda x)]y^2 + \lambda - a + a\sin^2(\lambda x)\big\}\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + \cot(\lambda x)} + \int \big[\lambda + a\sin^2(\lambda x)\big] E\,dx, \qquad E = \frac{1}{\sin^2(\lambda x)} \exp\left[\frac{a}{2\lambda}\cos(2\lambda x)\right].$$

**11.** $\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - a x^{k+1}(\sin x)^m y + a(\sin x)^m\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y - 1)} - (k+1)\int \frac{E\,dx}{x^{k+2}}, \qquad E = \exp\left[a\int x^{k+1}(\sin x)^m\,dx\right].$$

**12.** $\dfrac{\partial w}{\partial x} + \big[a\sin^k(\lambda x + \mu)(y - bx^n - c)^2 + y - bx^n + bnx^{n-1} - c\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{e^x}{y - bx^n - c} + a\displaystyle\int e^x \sin^k(\lambda x + \mu)\,dx.$

**13.** $x\dfrac{\partial w}{\partial x} + \big[a\sin^m(\lambda x)y^2 + ky + ab^2x^{2k}\sin^m(\lambda x)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\left(\dfrac{y}{bx^k}\right) - ab\displaystyle\int x^{k-1}\sin^m(\lambda x)\,dx.$

**14.** $\big[a\sin(\lambda x) + b\big]\dfrac{\partial w}{\partial x} + \big[y^2 + c\sin(\mu x)y - k^2 + ck\sin(\mu x)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + k} + \int \frac{E\,dx}{a\sin(\lambda x) + b}, \qquad E = \exp\left[\int \frac{c\sin(\mu x) - 2k}{a\sin(\lambda x) + b}\,dx\right].$$

► **Coefficients of equations contain cosine.**

**15.** $\dfrac{\partial w}{\partial x} + \big[a\cos^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \cos^k(\lambda x)\,dx.$

**16.** $\dfrac{\partial w}{\partial x} + \big[a\cos^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\cos^k(\lambda y) + b}.$

**17.** $\dfrac{\partial w}{\partial x} + a\cos^k(\lambda x)\cos^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int \cos^k(\lambda x)\,dx - \int \frac{dy}{\cos^n(\mu y)}.$

**18.** $\dfrac{\partial w}{\partial x}+a\cos^k(x+\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x-\int\frac{dz}{1+a\lambda\cos^k z},\qquad z=x+\lambda y.$$

**19.** $\dfrac{\partial w}{\partial x}+\bigl[y^2-a^2+a\lambda\cos(\lambda x)+a^2\cos^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\sin(\lambda x)}+\int E\,dx,\qquad E=\exp\biggl[-\frac{2a}{\lambda}\cos(\lambda x)\biggr].$$

**20.** $\dfrac{\partial w}{\partial x}+\bigl[\lambda\cos(\lambda x)y^2+\lambda\cos^3(\lambda x)\bigr]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\sin(\lambda x)}+\lambda\int E\cos(\lambda x)\,dx,\qquad E=\exp\bigl[-\tfrac{1}{2}\cos(2\lambda x)\bigr].$$

**21.** $2\dfrac{\partial w}{\partial x}+\bigl\{[\lambda+a+a\cos(\lambda x)]y^2+\lambda-a+a\cos(\lambda x)\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan(\frac{1}{2}\lambda x)}+\frac{1}{2}\int\bigl[\lambda+a+a\cos(\lambda x)\bigr]E\,dx,$$
$$E=\frac{1}{1+\cos(\lambda x)}\exp\biggl[-\frac{a}{\lambda}\cos(\lambda x)\biggr].$$

**22.** $\dfrac{\partial w}{\partial x}+\bigl\{[\lambda+a\cos^2(\lambda x)]y^2+\lambda-a+a\cos^2(\lambda x)\bigr\}\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-\tan(\lambda x)}+\int\bigl[\lambda+a\cos^2(\lambda x)\bigr]E\,dx,\qquad E=\frac{1}{\cos^2(\lambda x)}\exp\biggl[-\frac{a}{2\lambda}\cos(2\lambda x)\biggr].$$

**23.** $(ax^ny^m+bx)\dfrac{\partial w}{\partial x}+\cos^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+a(n-1)\int\frac{y^mE\,dy}{\cos^k(\lambda y)},\qquad E=\exp\biggl[b(n-1)\int\frac{dy}{\cos^k(\lambda y)}\biggr].$$

**24.** $(ax^n+bx\cos^m y)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+a(n-1)\int y^{-k}\,E\,dy,\qquad E=\exp\biggl[b(n-1)\int\frac{\cos^m y\,dy}{y^k}\biggr].$$

**25.** $\displaystyle (ax^n + bx\cos^m y)\frac{\partial w}{\partial x} + \cos^k(\lambda y)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + a(n-1)\int \frac{E\,dy}{\cos^k(\lambda y)}, \qquad E = \exp\left[b(n-1)\int \frac{\cos^m y\,dy}{\cos^k(\lambda y)}\right].$$

**26.** $\displaystyle (ax^n\cos^m y + bx)\frac{\partial w}{\partial x} + \cos^k(\lambda y)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + a(n-1)\int \frac{\cos^m y\,E\,dy}{\cos^k(\lambda y)}, \qquad E = \exp\left[b(n-1)\int \frac{dy}{\cos^k(\lambda y)}\right].$$

► **Coefficients of equations contain tangent.**

**27.** $\displaystyle \frac{\partial w}{\partial x} + \left[a\tan^k(\lambda x) + b\right]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = y - bx - a\int \tan^k(\lambda x)\,dx.$

**28.** $\displaystyle \frac{\partial w}{\partial x} + \left[a\tan^k(\lambda y) + b\right]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = x - \int \frac{dy}{a\tan^k(\lambda y) + b}.$

**29.** $\displaystyle \frac{\partial w}{\partial x} + a\tan^k(\lambda x)\tan^n(\mu y)\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = a\int \tan^k(\lambda x)\,dx - \int \cot^n(\mu y)\,dy.$

**30.** $\displaystyle \frac{\partial w}{\partial x} + \left[y^2 + a\lambda + a(\lambda - a)\tan^2(\lambda x)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \frac{\left[\cos(\lambda x)\right]^{-2a/\lambda}}{y - a\tan(\lambda x)} + \int \left[\cos(\lambda x)\right]^{-2a/\lambda}\,dx.$

**31.** $\displaystyle \frac{\partial w}{\partial x} + \left[y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\tan^2(\lambda x)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\left[\cos(\lambda x)\right]^{-2a/\lambda}}{\sin^2(\lambda x)\left[y - a\tan(\lambda x) + \lambda\cot(\lambda x)\right]} + \int \frac{\left[\cos(\lambda x)\right]^{-2a/\lambda}}{\sin^2(\lambda x)}\,dx.$$

**32.** $\displaystyle \frac{\partial w}{\partial x} + \left[y^2 + ax\tan^k(bx)y + a\tan^k(bx)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[a\int x\tan^k(bx)\,dx\right].$$

**33.** $\dfrac{\partial w}{\partial x}-\left[(k+1)x^k y^2 - ax^{k+1}(\tan x)^m y + a(\tan x)^m\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{x^{k+1}(x^{k+1}y-1)}-(k+1)\int\frac{E\,dx}{x^{k+2}},\qquad E=\exp\left[a\int x^{k+1}(\tan x)^m\,dx\right].$$

**34.** $\dfrac{\partial w}{\partial x}+\left[a\tan^n(\lambda x)y^2 - ab^2\tan^{n+2}(\lambda x)+b\lambda\tan^2(\lambda x)+b\lambda\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-b\tan(\lambda x)}+a\int E\tan^n(\lambda x)\,dx,\qquad E=\exp\left[2ab\int\tan^{n+1}(\lambda x)\,dx\right].$$

**35.** $\dfrac{\partial w}{\partial x}+\left[a\tan^k(\lambda x+\mu)(y-bx^n-c)^2+y-bx^n+bnx^{n-1}-c\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{e^x}{y-bx^n-c}+a\displaystyle\int e^x\tan^k(\lambda x+\mu)\,dx.$

**36.** $x\dfrac{\partial w}{\partial x}+\left[a\tan^m(\lambda x)y^2+ky+ab^2x^{2k}\tan^m(\lambda x)\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\left(\dfrac{1}{b}x^{-k}y\right)-ab\displaystyle\int x^{k-1}\tan^m(\lambda x)\,dx.$

**37.** $\left[a\tan(\lambda x)+b\right]\dfrac{\partial w}{\partial x}+\left[y^2+c\tan(\mu x)y-k^2+ck\tan(\mu x)\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y+k}+\int\frac{E\,dx}{a\tan(\lambda x)+b},\qquad E=\exp\left[\int\frac{c\tan(\mu x)-2k}{a\tan(\lambda x)+b}\,dx\right].$$

**38.** $(ax^n y^m+bx)\dfrac{\partial w}{\partial x}+\tan^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{y^m E\,dy}{\tan^k(\lambda y)},\qquad E=\exp\left[b(n-1)\int\frac{dy}{\tan^k(\lambda y)}\right].$$

**39.** $(ax^n+bx\tan^m y)\dfrac{\partial w}{\partial x}+y^k\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int y^{-k}E\,dy,\qquad E=\exp\left[b(n-1)\int y^{-k}\tan^m y\,dy\right].$$

**40.** $(ax^n+bx\tan^m y)\dfrac{\partial w}{\partial x}+\tan^k(\lambda y)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=x^{1-n}E+(n-1)a\int\frac{E\,dy}{\tan^k(\lambda y)},\qquad E=\exp\left[b(n-1)\int\frac{\tan^m y\,dy}{\tan^k(\lambda y)}\right].$$

**41.** $\boldsymbol{(ax^n \tan^m y + bx)\dfrac{\partial w}{\partial x} + \tan^k(\lambda y)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)a\int \frac{\tan^m yE\,dy}{\tan^k(\lambda y)}, \qquad E = \exp\biggl[b(n-1)\int \frac{dy}{\tan^k(\lambda y)}\biggr].$$

► **Coefficients of equations contain cotangent.**

**42.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[a\cot^k(\lambda x) + b\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = y - bx - a\displaystyle\int \cot^k(\lambda x)\,dx.$

**43.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[a\cot^k(\lambda y) + b\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\cot^k(\lambda y) + b}.$

**44.** $\boldsymbol{\dfrac{\partial w}{\partial x} + a\cot^k(x + \lambda y)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = x - \int \frac{dz}{1 + a\lambda\cot^k z}, \qquad z = x + \lambda y.$$

**45.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[y^2 + a\lambda + a(\lambda - a)\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \dfrac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{y + a\cot(\lambda x)} + \displaystyle\int \bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}\,dx.$

**46.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[y^2 + \lambda^2 + 3a\lambda + a(\lambda - a)\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{\cos^2(\lambda x)\bigl[y - \lambda\tan(\lambda x) + a\cot(\lambda x)\bigr]} + \int \frac{\bigl[\sin(\lambda x)\bigr]^{-2a/\lambda}}{\cos^2(\lambda x)}\,dx.$$

**47.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[y^2 - 2a\cot(ax)y + b^2 - a^2\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \dfrac{\sin^{-2}(bx)}{y - a\cot(ax) + b\cot(bx)} - \dfrac{1}{b}\cot(bx).$

**48.** $\boldsymbol{\cot(\lambda x)\dfrac{\partial w}{\partial x} + a\cot(\mu y)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = a\mu\ln\bigl|\cos(\lambda x)\bigr| - \lambda\ln\bigl|\cos(\mu y)\bigr|.$

**49.** $\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu\ln\bigl|\sin(\lambda x)\bigr|-\lambda\ln\bigl|\sin(\mu y)\bigr|$.

**50.** $\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot^2(\lambda x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\ln\bigl|\sin(\mu y)\bigr|+a\mu\cot(\lambda x)+a\lambda\mu x$.

**51.** $\cot(y+a)\dfrac{\partial w}{\partial x}+c\cot(x+b)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=c\ln\bigl|\sin(x+b)\bigr|-\ln\bigl|\sin(y+a)\bigr|$.

**52.** $\cot(\lambda x)\cot(\mu y)\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\lambda\ln\bigl|\sin(\mu y)\bigr|+a\mu\ln\bigl|\cos(\lambda x)\bigr|$.

**53.** $\cot(\lambda x)\cot(\mu y)\dfrac{\partial w}{\partial x}+a\cot(\nu x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\mu\displaystyle\int\frac{\cot(\nu x)}{\cot(\lambda x)}\,dx-\ln\bigl|\sin(\mu y)\bigr|$.

► **Coefficients of equations contain different trigonometric functions.**

**54.** $\dfrac{\partial w}{\partial x}+a\sin^k(\lambda x)\cos^n(\mu y)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=a\displaystyle\int\sin^k(\lambda x)\,dx-\int\frac{dy}{\cos^n(\mu y)}$. In the special case $a=1$, $k=1$, and $n=-1$ we have $\Xi=\mu\cos(\lambda x)+\lambda\sin(\mu y)$.

**55.** $\dfrac{\partial w}{\partial x}+\bigl[y^2-y\tan x+a(1-a)\cot^2 x\bigr]\dfrac{\partial w}{\partial y}=0.$

1°. Principal integral for $a\neq\frac{1}{2}$:

$$\Xi=\frac{(\sin x)^{-2a}\cos x}{y+a\cot x}+\frac{1}{1-2a}(\sin x)^{1-2a}.$$

2°. Principal integral for $a=\frac{1}{2}$:

$$\Xi=\frac{\cos x}{y\sin x+\frac{1}{2}\cos x}+\ln|\sin x|.$$

**56.** $\dfrac{\partial w}{\partial x}+(y^2-my\tan x+b^2\cos^{2m}x)\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\left(\dfrac{1}{b}y\cos^{-m}x\right)-b\displaystyle\int\cos^m x\,dx.$

**57.** $\dfrac{\partial w}{\partial x} + (y^2 + my\cot x + b^2\sin^m x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\left(\dfrac{1}{b}y\sin^{-m} x\right) - b\displaystyle\int \sin^m x\,dx.$

**58.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 - 2\lambda^2\tan^2(\lambda x) - 2\lambda^2\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{\sin^2(\lambda x)\cos^2(\lambda x)}{y - \lambda\cot(\lambda x) + \lambda\tan(\lambda x)} + \frac{1}{8}x - \frac{1}{8\lambda}\sin(\lambda x)\cos(\lambda x)\cos(2\lambda x).$$

**59.** $\dfrac{\partial w}{\partial x} + \bigl[y^2 + \lambda(a+b) + 2ab + a(\lambda-a)\tan^2(\lambda x) + b(\lambda-b)\cot^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\tan(\lambda x) + b\cot(\lambda x)} + \int E\,dx, \qquad E = \bigl[\cos(\lambda x)\bigr]^{-\frac{2a}{\lambda}}\bigl[\sin(\lambda x)\bigr]^{-\frac{2b}{\lambda}}.$$

**60.** $\dfrac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + a\cos^n(\lambda x)y - a\cos^{n-1}(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[a\int \cos^n(\lambda x)\,dx\right].$$

**61.** $\dfrac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + a\sin(\lambda x)y - a\tan(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[-\frac{a}{\lambda}\cos(\lambda x)\right].$$

**62.** $\dfrac{\partial w}{\partial x} + \bigl[\lambda\sin(\lambda x)y^2 + ax^n\cos(\lambda x)y - ax^n\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)\bigl[y\cos(\lambda x) - 1\bigr]} + \lambda\int \frac{E\sin(\lambda x)}{\cos^2(\lambda x)}\,dx, \qquad E = \exp\left[a\int x^n\cos(\lambda x)\,dx\right].$$

**63.** $\dfrac{\partial w}{\partial x} + \bigl[Ae^{\lambda x}\cos(ay) + Be^{\mu x}\sin(ay) + Ae^{\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \tan\dfrac{ay}{2}\exp\left(-\dfrac{aB}{\mu}e^{\mu x}\right) - aA\displaystyle\int \exp\left(\lambda x - \dfrac{aB}{\mu}e^{\mu x}\right)dx.$

**64.** $\sin^{n+1}(2x)\dfrac{\partial w}{\partial x} + (ay^2\sin^{2n} x + b\cos^{2n} x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^2 + n2^{n+1}v + b} - 2^{-n-1}\ln\tan x, \qquad v = y\tan^n x.$$

### 1.1.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

**1.** $\dfrac{\partial w}{\partial x} + \big[a \arcsin^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int \arcsin^k(\lambda x)\,dx.$

**2.** $\dfrac{\partial w}{\partial x} + \big[a \arcsin^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\arcsin^k(\lambda y) + b}.$

**3.** $\dfrac{\partial w}{\partial x} + k \arcsin^n(ax + by + c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{a + bk\arcsin^n v} - x, \qquad v = ax + by + c.$$

**4.** $\dfrac{\partial w}{\partial x} + a \arcsin^k(\lambda x)\arcsin^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int \arcsin^k(\lambda x)\,dx - \int \frac{dy}{\arcsin^n(\mu y)}.$

**5.** $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda(\arcsin x)^n y - a^2 + a\lambda(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2ax}E}{y + a} + \int e^{-2ax}E\,dx, \qquad E = \exp\left[\lambda\int(\arcsin x)^n\,dx\right].$$

**6.** $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda x(\arcsin x)^n y + \lambda(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy + 1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[\lambda\int x(\arcsin x)^n\,dx\right].$$

**7.** $\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - \lambda(\arcsin x)^n(x^{k+1}y - 1)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y - 1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\left[\lambda\int x^{k+1}(\arcsin x)^n\,dx\right].$$

**8.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\arcsin x)^n y^2 + ay + ab - b^2\lambda(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y+b} + \lambda\int e^{ax}(\arcsin x)^n E\,dx, \qquad E = \exp\left[-2b\lambda\int(\arcsin x)^n\,dx\right].$$

**9.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\arcsin x)^n y^2 - b\lambda x^m(\arcsin x)^n y + bmx^{m-1}\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx^m} + \lambda\int(\arcsin x)^n E\,dx, \qquad E = \exp\left[b\lambda\int x^m(\arcsin x)^n\,dx\right].$$

**10.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\arcsin x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-bx^m} + \lambda\int(\arcsin x)^n E\,dx, \qquad E = \exp\left[2b\lambda\int x^m(\arcsin x)^n\,dx\right].$$

**11.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\arcsin x)^n(y - ax^m - b)^2 + amx^{m-1}\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y-ax^m-b} + \lambda\displaystyle\int(\arcsin x)^n\,dx.$

**12.** $x\dfrac{\partial w}{\partial x} + \big[\lambda(\arcsin x)^n y^2 + ky + \lambda b^2 x^{2k}(\arcsin x)^n\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \arctan\left(\dfrac{y}{bx^k}\right) - \lambda b\displaystyle\int x^{k-1}(\arcsin x)^n\,dx.$

► **Coefficients of equations contain arccosine.**

**13.** $\dfrac{\partial w}{\partial x} + \big[a\arccos^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = y - bx - a\displaystyle\int\arccos^k(\lambda x)\,dx.$

**14.** $\dfrac{\partial w}{\partial x} + \big[a\arccos^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = x - \displaystyle\int\frac{dy}{a\arccos^k(\lambda y) + b}.$

**15.** $\dfrac{\partial w}{\partial x} + k\arccos^n(ax+by+c)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{a + bk\arccos^n v} - x, \qquad v = ax + by + c.$$

**16.** $\dfrac{\partial w}{\partial x} + a\arccos^k(\lambda x)\arccos^n(\mu y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a\displaystyle\int \arccos^k(\lambda x)\,dx - \int \frac{dy}{\arccos^n(\mu y)}.$

**17.** $\dfrac{\partial w}{\partial x} + \left[y^2 + \lambda(\arccos x)^n y - a^2 + a\lambda(\arccos x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2ax}E}{y+a} + \int e^{-2ax}E\,dx, \qquad E = \exp\left[\lambda\int(\arccos x)^n\,dx\right].$$

**18.** $\dfrac{\partial w}{\partial x} + \left[y^2 + \lambda x(\arccos x)^n y + \lambda(\arccos x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[\lambda\int x(\arccos x)^n\,dx\right].$$

**19.** $\dfrac{\partial w}{\partial x} - \left[(k+1)x^k y^2 - \lambda(\arccos x)^n(x^{k+1}y - 1)\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y-1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\left[\lambda\int x^{k+1}(\arccos x)^n\,dx\right].$$

**20.** $\dfrac{\partial w}{\partial x} + \left[\lambda(\arccos x)^n y^2 + ay + ab - b^2\lambda(\arccos x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y+b} + \lambda\int e^{ax}(\arccos x)^n E\,dx, \qquad E = \exp\left[-2b\lambda\int(\arccos x)^n\,dx\right].$$

**21.** $\dfrac{\partial w}{\partial x} + \left[\lambda(\arccos x)^n y^2 - b\lambda x^m(\arccos x)^n y + bmx^{m-1}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^m} + \lambda\int(\arccos x)^n E\,dx, \qquad E = \exp\left[b\lambda\int x^m(\arccos x)^n\,dx\right].$$

**22.** $\dfrac{\partial w}{\partial x} + \left[\lambda(\arccos x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\arccos x)^n\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^m} + \lambda\int(\arccos x)^n E\,dx, \qquad E = \exp\left[2b\lambda\int x^m(\arccos x)^n\,dx\right].$$

**23.** $\dfrac{\partial w}{\partial x} + \left[\lambda(\arccos x)^n(y - ax^m - b)^2 + amx^{m-1}\right]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y - ax^m - b} + \lambda\displaystyle\int(\arccos x)^n\,dx.$

**24.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + \big[\lambda(\arccos x)^n y^2 + ky + \lambda b^2 x^{2k}(\arccos x)^n\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = \arctan\left(\dfrac{y}{bx^k}\right) - \lambda b \displaystyle\int x^{k-1}(\arccos x)^n\,dx.$

► **Coefficients of equations contain arctangent.**

**25.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \big[a \arctan^k(\lambda x) + b\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = y - bx - a\displaystyle\int \arctan^k(\lambda x)\,dx.$

**26.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \big[a \arctan^k(\lambda y) + b\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = x - \displaystyle\int \frac{dy}{a\arctan^k(\lambda y) + b}.$

**27.** $\boldsymbol{\dfrac{\partial w}{\partial x} + k\arctan^n(ax + by + c)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{a + bk\arctan^n v} - x, \qquad v = ax + by + c.$$

**28.** $\boldsymbol{\dfrac{\partial w}{\partial x} + a\arctan^k(\lambda x)\arctan^n(\mu y)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral: $\Xi = a\displaystyle\int \arctan^k(\lambda x)\,dx - \int\frac{dy}{\arctan^n(\mu y)}.$

**29.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda(\arctan x)^n y - a^2 + a\lambda(\arctan x)^n\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{e^{-2ax}E}{y + a} + \int e^{-2ax}E\,dx, \qquad E = \exp\left[\lambda\int(\arctan x)^n\,dx\right].$$

**30.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda x(\arctan x)^n y + \lambda(\arctan x)^n\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{E}{x(xy + 1)} + \int x^{-2}E\,dx, \qquad E = \exp\left[\lambda\int x(\arctan x)^n\,dx\right].$$

**31.** $\boldsymbol{\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - \lambda(\arctan x)^n(x^{k+1}y - 1)\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y - 1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\left[\lambda\int x^{k+1}(\arctan x)^n\,dx\right].$$

**32.** $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+ay+ab-b^2\lambda(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{e^{ax}E}{y+b}+\lambda\int e^{ax}(\arctan x)^n E\,dx,\qquad E=\exp\left[-2b\lambda\int(\arctan x)^n\,dx\right].$$

**33.** $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2-b\lambda x^m(\arctan x)^n y+bmx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arctan x)^n E\,dx,\qquad E=\exp\left[b\lambda\int x^m(\arctan x)^n\,dx\right].$$

**34.** $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+bmx^{m-1}-\lambda b^2x^{2m}(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-bx^m}+\lambda\int(\arctan x)^n E\,dx,\qquad E=\exp\left[2b\lambda\int x^m(\arctan x)^n\,dx\right].$$

**35.** $\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n(y-ax^m-b)^2+amx^{m-1}\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\dfrac{1}{y-ax^m-b}+\lambda\displaystyle\int(\arctan x)^n\,dx.$

**36.** $x\dfrac{\partial w}{\partial x}+\big[\lambda(\arctan x)^n y^2+ky+\lambda b^2x^{2k}(\arctan x)^n\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\arctan\left(\dfrac{y}{bx^k}\right)-\lambda b\displaystyle\int x^{k-1}(\arctan x)^n\,dx.$

► **Coefficients of equations contain arccotangent.**

**37.** $\dfrac{\partial w}{\partial x}+\big[a\,\mathrm{arccot}^k(\lambda x)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=y-bx-a\displaystyle\int\mathrm{arccot}^k(\lambda x)\,dx.$

**38.** $\dfrac{\partial w}{\partial x}+\big[a\,\mathrm{arccot}^k(\lambda y)+b\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=x-\displaystyle\int\frac{dy}{a\,\mathrm{arccot}^k(\lambda y)+b}.$

**39.** $\dfrac{\partial w}{\partial x}+k\,\mathrm{arccot}^n(ax+by+c)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+bk\,\mathrm{arccot}^n v}-x,\qquad v=ax+by+c.$$

**40.** $\dfrac{\partial w}{\partial x} + a \operatorname{arccot}^k(\lambda x) \operatorname{arccot}^n(\mu y) \dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = a \displaystyle\int \operatorname{arccot}^k(\lambda x)\, dx - \int \frac{dy}{\operatorname{arccot}^n(\mu y)}.$

**41.** $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda(\operatorname{arccot} x)^n y - a^2 + a\lambda(\operatorname{arccot} x)^n\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2ax}E}{y+a} + \int e^{-2ax}E\, dx, \qquad E = \exp\left[\lambda \int (\operatorname{arccot} x)^n\, dx\right].$$

**42.** $\dfrac{\partial w}{\partial x} + \big[y^2 + \lambda x(\operatorname{arccot} x)^n y + \lambda(\operatorname{arccot} x)^n\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\, dx, \qquad E = \exp\left[\lambda \int x(\operatorname{arccot} x)^n\, dx\right].$$

**43.** $\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - \lambda(\operatorname{arccot} x)^n (x^{k+1}y - 1)\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y-1)} - (k+1)\int x^{-k-2}E\, dx, \qquad E = \exp\left[\lambda \int x^{k+1}(\operatorname{arccot} x)^n\, dx\right].$$

**44.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\operatorname{arccot} x)^n y^2 + ay + ab - b^2\lambda(\operatorname{arccot} x)^n\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y+b} + \lambda \int e^{ax}(\operatorname{arccot} x)^n E\, dx, \qquad E = \exp\left[-2b\lambda \int (\operatorname{arccot} x)^n\, dx\right].$$

**45.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\operatorname{arccot} x)^n y^2 - b\lambda x^m (\operatorname{arccot} x)^n y + bmx^{m-1}\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^m} + \lambda \int (\operatorname{arccot} x)^n E\, dx, \qquad E = \exp\left[b\lambda \int x^m(\operatorname{arccot} x)^n\, dx\right].$$

**46.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\operatorname{arccot} x)^n y^2 + bmx^{m-1} - \lambda b^2 x^{2m}(\operatorname{arccot} x)^n\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - bx^m} + \lambda \int (\operatorname{arccot} x)^n E\, dx, \qquad E = \exp\left[2b\lambda \int x^m(\operatorname{arccot} x)^n\, dx\right].$$

**47.** $\dfrac{\partial w}{\partial x} + \big[\lambda(\operatorname{arccot} x)^n (y - ax^m - b)^2 + amx^{m-1}\big] \dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y - ax^m - b} + \lambda \displaystyle\int (\operatorname{arccot} x)^n\, dx.$

**48.** $\displaystyle x\frac{\partial w}{\partial x} + \big[\lambda(\operatorname{arccot} x)^n y^2 + ky + \lambda b^2 x^{2k}(\operatorname{arccot} x)^n\big]\frac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \operatorname{arccot}\left(\frac{y}{bx^k}\right) - \lambda b \int x^{k-1}(\operatorname{arccot} x)^n\, dx.$

### 1.1.7 Equations Containing Arbitrary Functions of $x$

◆ *Notation:* $f = f(x)$, $g = g(x)$, *and* $h = h(x)$ *are arbitrary functions, and* $a$, $b$, $k$, $n$, *and* $\lambda$ *are arbitrary parameters.*

► **Equations contain arbitrary and power-law functions.**

**1.** $\displaystyle \frac{\partial w}{\partial x} + \big[f(x)y + g(x)\big]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = e^{-F}y - \int e^{-F}g(x)\,dx, \qquad F = \int f(x)\,dx.$$

**2.** $\displaystyle \frac{\partial w}{\partial x} + \big[f(x)y + g(x)y^k\big]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = e^{-F}y^{1-k} - (1-k)\int e^{-F}g(x)\,dx, \qquad F = (1-k)\int f(x)\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**3.** $\displaystyle \frac{\partial w}{\partial x} + (y^2 + fy - a^2 - af)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{2ax}E}{y-a} + \int e^{2ax}E\,dx, \qquad E = \exp\left(\int f\,dx\right).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**4.** $\displaystyle \frac{\partial w}{\partial x} + (y^2 + xfy + f)\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(xy+1)} + \int x^{-2}E\,dx, \qquad E = \exp\left(\int xf\,dx\right).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**5.** $\dfrac{\partial w}{\partial x} - \big[(k+1)x^k y^2 - x^{k+1} f y + f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x^{k+1}(x^{k+1}y-1)} - (k+1)\int x^{-k-2}E\,dx, \qquad E = \exp\biggl(\int x^{k+1} f\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**6.** $\dfrac{\partial w}{\partial x} + (fy^2 + ay - ab - b^2 f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{ax}E}{y-b} + \int e^{ax} f E\,dx, \qquad E = \exp\biggl(2b\int f\,dx\biggr).$$

**7.** $\dfrac{\partial w}{\partial x} + (fy^2 - ax^n f y + anx^{n-1})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl(a\int x^n f\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**8.** $\dfrac{\partial w}{\partial x} + (fy^2 + anx^{n-1} - a^2x^{2n} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl(2a\int x^n f\,dx\biggr).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**9.** $\dfrac{\partial w}{\partial x} + (fy^2 + gy - a^2 f - ag)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-a} + \int fE\,dx, \qquad E = \exp\int (2af+g)\,dx.$$

**10.** $\dfrac{\partial w}{\partial x} + (fy^2 + gy + anx^{n-1} - ax^n g - a^2x^{2n} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y-ax^n} + \int fE\,dx, \qquad E = \exp\biggl[\int (2ax^n f + g)\,dx\biggr].$$

**11.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - ax^n gy + anx^{n-1} + a^2x^{2n}(g-f)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ax^n} + \int fE\,dx, \qquad E = \exp\biggl[a\int x^n(2f-g)\,dx\biggr].$$

**12.** $x\dfrac{\partial w}{\partial x} + (fy^2 + ny + ax^{2n}f)\dfrac{\partial w}{\partial y} = 0.$

$1^\circ$. Principal integral for $a > 0$:

$$\Xi = \arctan\biggl(\frac{y}{\sqrt{a}\,x^n}\biggr) - \sqrt{a}\int x^{n-1}f\,dx.$$

$2^\circ$. Principal integral for $a < 0$:

$$\Xi = \operatorname{arctanh}\biggl(\frac{y}{\sqrt{|a|}\,x^n}\biggr) + \sqrt{|a|}\int x^{n-1}f\,dx.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**13.** $x\dfrac{\partial w}{\partial x} + \bigl[x^{2n}fy^2 + (ax^nf - n)y + bf\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int\frac{dv}{v^2 + av + b} - \int x^{n-1}f\,dx, \qquad v = x^n y.$$

► **Equations contain arbitrary and exponential functions.**

**14.** $\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y^2 + ae^{\lambda x}fy + \lambda f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{e^{-2\lambda x}E}{ay + \lambda e^{-\lambda x}} + \int e^{-\lambda x}E\,dx, \quad E = \exp\biggl(a\int e^{\lambda x}f\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**15.** $\dfrac{\partial w}{\partial x} + (fy^2 - ae^{\lambda x}fy + a\lambda e^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl(a\int e^{\lambda x}f\,dx\biggr).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**16.** $\dfrac{\partial w}{\partial x} + (fy^2 + a\lambda e^{\lambda x} - a^2 e^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl(2a \int e^{\lambda x} f\,dx\biggr).$$

**17.** $\dfrac{\partial w}{\partial x} + (fy^2 + \lambda y + ae^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

$1^\circ$. Principal integral for $a > 0$:

$$\Xi = \arctan\biggl(\frac{e^{-\lambda x} y}{\sqrt{a}}\biggr) - \sqrt{a} \int e^{\lambda x} f\,dx.$$

$2^\circ$. Principal integral for $a < 0$:

$$\Xi = \operatorname{arctanh}\biggl(\frac{e^{-\lambda x} y}{\sqrt{|a|}}\biggr) + \sqrt{|a|} \int e^{\lambda x} f\,dx.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**18.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - (ae^{\lambda x} + b)fy + a\lambda e^{\lambda x}\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x} - b} + \int fE\,dx, \qquad E = \exp\biggl[\int (ae^{\lambda x} + b) f\,dx\biggr].$$

**19.** $\dfrac{\partial w}{\partial x} + \bigl[e^{\lambda x} fy^2 + (af - \lambda)y + be^{-\lambda x} f\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v^2 + av + b} - \int f(x)\,dx, \qquad v = e^{\lambda x} y.$$

**20.** $\dfrac{\partial w}{\partial x} + (fy^2 + gy + a\lambda e^{\lambda x} - ae^{\lambda x} g - a^2 e^{2\lambda x} f)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl[\int (2ae^{\lambda x} f + g)\,dx\biggr].$$

**21.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - ae^{\lambda x} gy + a\lambda e^{\lambda x} + a^2 e^{2\lambda x}(g - f)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ae^{\lambda x}} + \int fE\,dx, \qquad E = \exp\biggl[a \int e^{\lambda x}(2f - g)\,dx\biggr].$$

**22.** $\dfrac{\partial w}{\partial x}+\big(fy^2+2a\lambda xe^{\lambda x^2}-a^2fe^{2\lambda x^2}\big)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-ae^{\lambda x^2}}+\int fE\,dx,\qquad E=\exp\biggl(2a\int e^{\lambda x^2}f\,dx\biggr).$$

**23.** $\dfrac{\partial w}{\partial x}+\big(fy^2+2\lambda xy+afe^{2\lambda x^2}\big)\dfrac{\partial w}{\partial y}=0.$

$1^\circ$. Principal integral for $a>0$:

$$\Xi=\arctan\biggl(\frac{e^{-\lambda x^2}y}{\sqrt{a}}\biggr)-\sqrt{a}\int e^{\lambda x^2}f\,dx.$$

$2^\circ$. Principal integral for $a<0$:

$$\Xi=\operatorname{arctanh}\biggl(\frac{e^{-\lambda x^2}y}{\sqrt{|a|}}\biggr)+\sqrt{|a|}\int e^{\lambda x^2}f\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**24.** $\dfrac{\partial w}{\partial x}+\big[f(x)e^{\lambda y}+g(x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=e^{-\lambda y}E+\lambda\int f(x)E\,dx,\qquad E=\exp\biggl[\lambda\int g(x)\,dx\biggr].$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

### ► Equations contain arbitrary and hyperbolic functions.

**25.** $\dfrac{\partial w}{\partial x}+\big[fy^2-a^2f+a\lambda\sinh(\lambda x)-a^2f\sinh^2(\lambda x)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\cosh(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\cosh(\lambda x)\,dx\biggr].$$

**26.** $\dfrac{\partial w}{\partial x}+\big[fy^2-a(af+\lambda)\tanh^2(\lambda x)+a\lambda\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\tanh(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\tanh(\lambda x)\,dx\biggr].$$

**27.** $\dfrac{\partial w}{\partial x}+\big[fy^2-a(af+\lambda)\coth^2(\lambda x)+a\lambda\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\frac{E}{y-a\coth(\lambda x)}+\int fE\,dx,\qquad E=\exp\biggl[2a\int f\coth(\lambda x)\,dx\biggr].$$

► **Equations contain arbitrary and logarithmic functions.**

**28.** $\dfrac{\partial w}{\partial x} - \big[ay^2 \ln x - axy(\ln x - 1)f + f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{x(\ln x - 1)[axy(\ln x - 1) - 1]} - \int \frac{E \ln x\, dx}{x^2(\ln x - 1)^2},$$
$$E = \exp\left[a \int xf(\ln x - 1)\, dx\right].$$

**29.** $\dfrac{\partial w}{\partial x} + \big[fy^2 - ax(\ln x)fy + a\ln x + a\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - ax\ln x} + \int fE\, dx, \qquad E = \exp\left(a \int xf \ln x\, dx\right).$$

**30.** $x\dfrac{\partial w}{\partial x} + \big[fy^2 + a - a^2(\ln x)^2 f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\ln x} + \int x^{-1} fE\, dx, \qquad E = \exp\left(2a \int x^{-1} f \ln x\, dx\right).$$

**31.** $x\dfrac{\partial w}{\partial x} + \big[(y + a\ln x)^2 f - a\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{y + a\ln x} + \displaystyle\int \frac{f(x)}{x}\, dx.$

► **Equations contain arbitrary and trigonometric functions.**

**32.** $\dfrac{\partial w}{\partial x} + \big[\lambda \sin(\lambda x)y^2 + f\cos(\lambda x)y - f\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{\cos(\lambda x)[\cos(\lambda x)y - 1]} + \lambda \int \frac{\sin(\lambda x)}{\cos^2(\lambda x)} E\, dx, \qquad E = \exp\left[\int f\cos(\lambda x)\, dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**33.** $\dfrac{\partial w}{\partial x} + \big[fy^2 - a^2 f + a\lambda\sin(\lambda x) + a^2 f\sin^2(\lambda x)\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + a\cos(\lambda x)} + \int fE\, dx, \qquad E = \exp\left[-2a \int f\cos(\lambda x)\, dx\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**34.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - a^2 f + a\lambda\cos(\lambda x) + a^2 f\cos^2(\lambda x)\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\sin(\lambda x)} + \int fE\,dx, \qquad E = \exp\biggl[2a\int f\sin(\lambda x)\,dx\biggr].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**35.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - a(af - \lambda)\tan^2(\lambda x) + a\lambda\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - a\tan(\lambda x)} + \int fE\,dx, \qquad E = \exp\biggl[2a\int f\tan(\lambda x)\,dx\biggr].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**36.** $\dfrac{\partial w}{\partial x} + \bigl[fy^2 - a(af - \lambda)\cot^2(\lambda x) + a\lambda\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y + a\cot(\lambda x)} + \int fE\,dx, \qquad E = \exp\biggl[-2a\int f\cot(\lambda x)\,dx\biggr].$$

► **Equations contain arbitrary functions and their derivatives.**

**37.** $\dfrac{\partial w}{\partial x} + (fy^2 - fgy + g'_x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{y - g} + \int fE\,dx, \qquad E = \exp\biggl(\int fg\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**38.** $\dfrac{\partial w}{\partial x} - (f'_x y^2 - fgy + g)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{f(fy - 1)} - \int \frac{f'_x E}{f^2}\,dx, \qquad E = \exp\biggl(\int fg\,dx\biggr).$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**39.** $\dfrac{\partial w}{\partial x} + \bigl[g(y - f)^2 + f'_x\bigr]\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle\Xi = \frac{1}{y - f} + \int g\,dx.$

**40.** $\dfrac{\partial w}{\partial x} + \left(\dfrac{f'_x}{g} y^2 - \dfrac{g'_x}{f}\right) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{1}{f(fy+g)} + \int \frac{f'_x\,dx}{f^2 g}.$$

**41.** $f^2 \dfrac{\partial w}{\partial x} + \left[f'_x y^2 - g(y-f)\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{f^2 E}{y-f} + \int f'_x E\,dx, \qquad E = \exp\biggl(-\int \frac{g\,dx}{f^2}\biggr).$$

**42.** $\dfrac{\partial w}{\partial x} + \left(y^2 - \dfrac{f''_{xx}}{f}\right) \dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\Xi = \dfrac{1}{f(fy+f'_x)} + \displaystyle\int \frac{dx}{f^2}.$

**43.** $g \dfrac{\partial w}{\partial x} + \left[afgy^3 + (bfg^3 + g'_x)y + cfg^4\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{av^3 + bv + c} - \int fg^2\,dx, \qquad v = \frac{y}{g}.$$

**44.** $\dfrac{\partial w}{\partial x} + \left[fy^3 + 3fhy^2 + (g + 3fh^2)y + fh^3 + gh - h'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{(y+h)^2} + 2\int fE\,dx, \qquad E = \exp\biggl(2\int g\,dx\biggr).$$

**45.** $\dfrac{\partial w}{\partial x} + \left[\dfrac{g'_x}{f^2(ag+b)^3} y^3 + \dfrac{f'_x}{f} y + fg'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v^3 - av + 1} - \frac{1}{a} \ln|ag+b|, \qquad v = \frac{y}{f(ag+b)}.$$

**46.** $\dfrac{\partial w}{\partial x} + \left[(y-f)(y-g)\left(y - \dfrac{af+bg}{a+b}\right)h + \dfrac{y-g}{f-g} f'_x + \dfrac{y-f}{g-f} g'_x\right] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = E|y-f|^a |y-g|^b \left| y - \frac{af+bg}{a+b} \right|^{-a-b}, \qquad E = \exp\left[-\frac{ab}{a+b} \int (f-g)^2 h\,dx\right].$$

**47.** $\dfrac{\partial w}{\partial x} + (fy^2 + g'_x y + afe^{2g})\dfrac{\partial w}{\partial y} = 0.$

1°. Principal integral for $a > 0$:

$$\Xi = \arctan\left(\frac{e^{-g}y}{\sqrt{a}}\right) - \sqrt{a}\int fe^g\,dx.$$

2°. Principal integral for $a < 0$:

$$\Xi = \operatorname{arctanh}\left(\frac{e^{-g}y}{\sqrt{|a|}}\right) + \sqrt{|a|}\int fe^g\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**48.** $\dfrac{\partial w}{\partial x} + (f'_x y^2 + ae^{\lambda x}fy + ae^{\lambda x})\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \frac{E}{f(fy+1)} + \int \frac{f'_x E}{f^2}\,dx, \qquad E = \exp\left(a\int e^{\lambda x} f\,dx\right).$$

### 1.1.8 Equations Containing Arbitrary Functions of Different Arguments

► **Equations contain arbitrary functions of $x$ and arbitrary functions of $y$.**

**1.** $f(x)\dfrac{\partial w}{\partial x} + g(y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = \int \frac{dx}{f(x)} - \int \frac{dy}{g(y)}.$

⊙ *Literature*: E. Kamke (1965).

**2.** $[f(x) + g(y)]\dfrac{\partial w}{\partial x} + f'_x(x)\dfrac{\partial w}{\partial y} = 0.$

Principal integral: $\displaystyle \Xi = f(x)e^{-y} - \int e^{-y}g(y)\,dy.$

**3.** $[x^n f(y) + xg(y)]\dfrac{\partial w}{\partial x} + h(y)\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{1-n}E + (n-1)\int \frac{f(y)E}{h(y)}\,dy, \qquad E = \exp\left[(n-1)\int \frac{g(y)}{h(y)}\,dy\right].$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**4.** $\left[f(y)+amx^n y^{m-1}\right]\dfrac{\partial w}{\partial x}-\left[g(x)+anx^{n-1}y^m\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int f(y)\,dy+\int g(x)\,dx+ax^n y^m.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**5.** $\left[e^{\alpha x}f(y)+c\beta\right]\dfrac{\partial w}{\partial x}-\left[e^{\beta y}g(x)+c\alpha\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral: $\Xi=\displaystyle\int e^{-\beta y}f(y)\,dy+\int e^{-\alpha x}g(x)\,dx-ce^{-\alpha x-\beta y}.$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

► **Equations contain one arbitrary function of complicated argument.**

**6.** $\dfrac{\partial w}{\partial x}+f(ax+by+c)\dfrac{\partial w}{\partial y}=0,\qquad b\neq 0.$

Principal integral:

$$\Xi=\int\frac{dv}{a+bf(v)}-x,\qquad v=ax+by+c.$$

**7.** $\dfrac{\partial w}{\partial x}+f\Bigl(\dfrac{y}{x}\Bigr)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{f(v)-v}-\ln|x|,\qquad v=\frac{y}{x}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**8.** $\dfrac{\partial w}{\partial x}+\left[f(y+ax^n+b)-anx^{n-1}\right]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{f(v)}-x,\qquad v=y+ax^n+b.$$

**9.** $x\dfrac{\partial w}{\partial x}+yf(x^n y^m)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{v\left[mf(v)+n\right]}-\ln|x|,\qquad v=x^n y^m.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**10.** $y^{m-1}\dfrac{\partial w}{\partial x}+x^{n-1}f(ax^n+by^m)\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{an+bmf(v)}-\frac{1}{n}x^n,\qquad v=ax^n+by^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**11.** $\dfrac{\partial w}{\partial x} + e^{-\lambda x} f(e^{\lambda x} y) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{f(v) + \lambda v} - x, \qquad v = e^{\lambda x} y.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**12.** $\dfrac{\partial w}{\partial x} + e^{\lambda y} f(e^{\lambda y} x) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v\bigl[\lambda v f(v) + 1\bigr]} - \ln|x|, \qquad v = e^{\lambda y} x.$$

**13.** $\dfrac{\partial w}{\partial x} + y f(e^{\alpha x} y^m) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v\bigl[\alpha + m f(v)\bigr]} - x, \qquad v = e^{\alpha x} y^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**14.** $x\dfrac{\partial w}{\partial x} + f(x^n e^{\alpha y}) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{v\bigl[n + \alpha f(v)\bigr]} - \ln|x|, \qquad v = x^n e^{\alpha y}.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**15.** $\dfrac{\partial w}{\partial x} + e^{\lambda x - \beta y} f(a e^{\lambda x} + b e^{\beta y}) \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{a\lambda + b\beta f(v)} - \frac{1}{\lambda} e^{\lambda x}, \qquad v = a e^{\lambda x} + b e^{\beta y}.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**16.** $\dfrac{\partial w}{\partial x} + \bigl[f(y + a e^{\lambda x} + b) - a\lambda e^{\lambda x}\bigr] \dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{f(v)} - x, \qquad v = y + a e^{\lambda x} + b.$$

**17.** $\boldsymbol{\alpha xy\dfrac{\partial w}{\partial x} + \bigl[\alpha f(x^n e^{\alpha y}) - ny\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = yE - \frac{1}{\alpha}\int v^{-1}E\,dv, \qquad v = x^n e^{\alpha y}, \quad E = \exp\biggl[\frac{n}{\alpha^2}\int\frac{dv}{vf(v)}\biggr].$$

**18.** $\boldsymbol{mx(\ln y)\dfrac{\partial w}{\partial x} + \bigl[yf(x^n y^m) - ny\ln y\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = E\ln y - \frac{1}{m}\int v^{-1}E\,dv, \qquad v = x^n y^m, \quad E = \exp\biggl[\frac{n}{m}\int\frac{dv}{vf(v)}\biggr].$$

**19.** $\boldsymbol{\dfrac{\partial w}{\partial x} + \bigl[f(y + a\tan x) - a\tan^2 x\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int\frac{dv}{a + f(v)} - x, \qquad v = y + a\tan x.$$

**20.** $\boldsymbol{e^{\lambda x}\dfrac{\partial w}{\partial x} + f(\lambda x + \ln y)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int\frac{e^v\,dv}{f(v) + \lambda e^v} - x, \qquad v = \lambda x + \ln y.$$

**21.** $\boldsymbol{\dfrac{\partial w}{\partial x} + e^{\lambda y}f(\lambda y + \ln x)\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int\frac{dv}{\lambda e^v f(v) + 1} - \ln x, \qquad v = \lambda y + \ln x.$$

### ► Equations contain several arbitrary functions.

**22.** $\boldsymbol{mx\dfrac{\partial w}{\partial x} - \bigl[ny - xy^k f(x)g(x^n y^m)\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int v^{\frac{1-k-m}{m}}\frac{dv}{g(v)} - \int x^{\frac{n(1-k)}{m}}f(x)\,dx, \qquad v = x^n y^m.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**23.** $\boldsymbol{y^n\dfrac{\partial w}{\partial x} - \bigl[ax^n + g(x)f(y^{n+1} + ax^{n+1})\bigr]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int\frac{dv}{f(v)} + (n+1)\int g(x)\,dx, \qquad v = y^{n+1} + ax^{n+1}.$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**24.** $\left[f\left(\frac{y}{x}\right) + x^a h\left(\frac{y}{x}\right)\right]\frac{\partial w}{\partial x} + \left[g\left(\frac{y}{x}\right) + yx^{a-1}h\left(\frac{y}{x}\right)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{-a}E + a\int \frac{h(v)E\,dv}{g(v) - vf(v)}, \qquad v = \frac{y}{x},$$

where $E = \exp\left[a\int \frac{f(v)\,dv}{g(v) - vf(v)}\right].$

**25.** $\left[f(ax+by) + bxg(ax+by)\right]\frac{\partial w}{\partial x} + \left[h(ax+by) - axg(ax+by)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = xE - \int \frac{f(v)E\,dv}{af(v) + bh(v)}, \qquad v = ax + by,$$

where $E = \exp\left[-b\int \frac{g(v)\,dv}{af(v) + bh(v)}\right].$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**26.** $\left[f(ax+by) + byg(ax+by)\right]\frac{\partial w}{\partial x} + \left[h(ax+by) - ayg(ax+by)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = yE - \int \frac{h(v)E\,dv}{af(v) + bh(v)}, \qquad v = ax + by,$$

where $E = \exp\left[a\int \frac{g(v)\,dv}{af(v) + bh(v)}\right].$

**27.** $x\left[f(x^n y^m) + mx^k g(x^n y^m)\right]\frac{\partial w}{\partial x} + y\left[h(x^n y^m) - nx^k g(x^n y^m)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = x^{-k}E + km\int \frac{g(v)E\,dv}{v\left[nf(v) + mh(v)\right]}, \qquad v = x^n y^m,$$

where $E = \exp\left\{k\int \frac{f(v)\,dv}{v\left[nf(v) + mh(v)\right]}\right\}.$

**28.** $x\left[f(x^n y^m) + my^k g(x^n y^m)\right]\frac{\partial w}{\partial x} + y\left[h(x^n y^m) - ny^k g(x^n y^m)\right]\frac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = y^{-k}E - kn\int \frac{g(v)E\,dv}{v\left[nf(v) + mh(v)\right]}, \qquad v = x^n y^m,$$

where $E = \exp\left\{k\int \frac{h(v)\,dv}{v\left[nf(v) + mh(v)\right]}\right\}.$

**29.** $\boldsymbol{x\big[sf(x^n y^m) - mg(x^k y^s)\big]\dfrac{\partial w}{\partial x} + y\big[ng(x^k y^s) - kf(x^n y^m)\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{vg(v)} - \int \frac{dz}{zf(z)}, \qquad v = x^k y^s, \quad z = x^n y^m.$$

**30.** $\boldsymbol{f_y \dfrac{\partial w}{\partial x} - f_x \dfrac{\partial w}{\partial y} = 0.}$

Here $f_x$ and $f_y$ are the partial derivatives of the function $f = f(x, y)$ with respect to $x$ and $y$.

General solution: $w = \Phi(f(x, y))$, where $\Phi = \Phi(\xi)$ is an arbitrary function.

⊙ *Literature*: E. Kamke (1965).

**31.** $\boldsymbol{f(x,\, y)\dfrac{\partial w}{\partial x} - g(x,\, y)\dfrac{\partial w}{\partial y} = 0,}$ **where** $\boldsymbol{\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y}.}$

Principal integral:

$$\Xi = \int_{y_0}^{y} f(x_0,\, t)\, dt + \int_{x_0}^{x} g(t,\, y)\, dt,$$

where $x_0$ and $y_0$ are arbitrary constants.

**32.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + \big[xf(x)g(x^n e^y) - n\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int \frac{dv}{vg(v)} - \int f(x)\, dx, \qquad v = x^n e^y.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**33.** $\boldsymbol{m\dfrac{\partial w}{\partial x} + \big[my^k f(x)g(e^{\alpha x} y^m) - \alpha y\big]\dfrac{\partial w}{\partial y} = 0.}$

Principal integral:

$$\Xi = \int v^{\frac{1-k-m}{m}} \frac{dv}{g(v)} - m\int f(x)\exp\left[\frac{\alpha(1-k)}{m}x\right] dx, \qquad v = e^{\alpha x} y^m.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (1996).

**34.** $\boldsymbol{\big[f(ax+by)+be^{\lambda y}g(ax+by)\big]\dfrac{\partial w}{\partial x}+\big[h(ax+by)-ae^{\lambda y}g(ax+by)\big]\dfrac{\partial w}{\partial y}=0.}$

Principal integral:

$$\Xi = e^{-\lambda y}E - \lambda a\int \frac{g(v)E\, dv}{af(v) + bh(v)}, \qquad v = ax + by,$$

where $E = \exp\left[\lambda \displaystyle\int \frac{h(v)\, dv}{af(v) + bh(v)}\right]$.

**35.** $\big[f(ax+by)+be^{\alpha x}g(ax+by)\big]\dfrac{\partial w}{\partial x}+\big[h(ax+by)-ae^{\alpha x}g(ax+by)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=e^{-\alpha x}E+\alpha b\int\frac{g(v)E\,dv}{af(v)+bh(v)},\qquad v=ax+by,$$

where $E=\exp\left[\alpha\displaystyle\int\frac{f(v)\,dv}{af(v)+bh(v)}\right]$.

**36.** $x\big[f(x^n e^{\alpha y})+\alpha y g(x^n e^{\alpha y})\big]\dfrac{\partial w}{\partial x}+\big[h(x^n e^{\alpha y})-nyg(x^n e^{\alpha y})\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=yE-\int\frac{h(v)E\,dv}{v\big[nf(v)+\alpha h(v)\big]},\qquad v=x^n e^{\alpha y},$$

where $E=\exp\left\{n\displaystyle\int\frac{g(v)\,dv}{v\big[nf(v)+\alpha h(v)\big]}\right\}$.

**37.** $\big[f(e^{\alpha x}y^m)+mxg(e^{\alpha x}y^m)\big]\dfrac{\partial w}{\partial x}+y\big[h(e^{\alpha x}y^m)-\alpha xg(e^{\alpha x}y^m)\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=xE-\int\frac{f(v)E\,dv}{v\big[\alpha f(v)+mh(v)\big]},\qquad v=e^{\alpha x}y^m,$$

where $E=\exp\left\{-m\displaystyle\int\frac{g(v)\,dv}{v\big[\alpha f(v)+mh(v)\big]}\right\}$.

**38.** $x\dfrac{\partial w}{\partial x}+\big[xyf(x)g(x^n\ln y)-ny\ln y\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=\int\frac{dv}{g(v)}-\int x^n f(x)\,dx,\qquad v=x^n\ln y.$$

**39.** $x\big[f(x^n y^m)+mg(x^n y^m)\ln y\big]\dfrac{\partial w}{\partial x}+y\big[h(x^n y^m)-ng(x^n y^m)\ln y\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi=E\ln y-\int\frac{h(v)E\,dv}{v\big[nf(v)+mh(v)\big]},\qquad v=x^n y^m,$$

where $E=\exp\left\{n\displaystyle\int\frac{g(v)\,dv}{v\big[nf(v)+mh(v)\big]}\right\}$.

**40.** $x\big[f(x^n y^m)+mg(x^n y^m)\ln x\big]\dfrac{\partial w}{\partial x}+y\big[h(x^n y^m)-ng(x^n y^m)\ln x\big]\dfrac{\partial w}{\partial y}=0.$

Principal integral:

$$\Xi = E\ln x - \int \frac{f(v)E\,dv}{v\big[nf(v)+mh(v)\big]}, \qquad v = x^n y^m,$$

where $E = \exp\biggl\{-m\displaystyle\int \frac{g(v)\,dv}{v\big[nf(v)+mh(v)\big]}\biggr\}.$

**41.** $\cos y\,\dfrac{\partial w}{\partial x} + \big[f(x)g(\sin x\sin y) - \cot x\sin y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int f(x)\sin x\,dx, \qquad v = \sin x\sin y.$$

**42.** $\sin 2x\,\dfrac{\partial w}{\partial x} + \big[\sin 2x\cos^2 y\,f(x)g(\tan x\tan y) - \sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int f(x)\tan x\,dx, \qquad v = \tan x\tan y.$$

**43.** $x\dfrac{\partial w}{\partial x} + \big[x\cos^2 y\,f(x)g(x^{2n}\tan y) - n\sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int x^{2n}f(x)\,dx, \qquad v = x^{2n}\tan y.$$

**44.** $\dfrac{\partial w}{\partial x} + \big[\cos^2 y\,f(x)g(e^{2x}\tan y) - \sin 2y\big]\dfrac{\partial w}{\partial y} = 0.$

Principal integral:

$$\Xi = \int \frac{dv}{g(v)} - \int e^{2x}f(x)\,dx, \qquad v = e^{2x}\tan y.$$

## 1.2 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} = h(x,y)$

◆ *The solutions given below contain an arbitrary function* $\Phi = \Phi(z)$.

### 1.2.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in $x$ and $y$.**

**1.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c.$

The *equation of a cylindrical surface*. Two forms of the general solution:

$$w=\frac{c}{a}x+\Phi(bx-ay),\qquad w=\frac{c}{b}y+\Phi(bx-ay).$$

⊙ *Literature*: E. Kamke (1965).

**2.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\alpha x+\beta y+\gamma.$

General solution: $w=\dfrac{\alpha}{2a}x^2+\dfrac{\gamma}{a}x+\dfrac{\beta}{2b}y^2+\Phi(bx-ay).$

**3.** $ax\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\alpha x+\beta y+\gamma.$

General solution: $w=\dfrac{\alpha}{a}x+\dfrac{\gamma}{a}\ln|x|+\dfrac{\beta}{2b}y^2+\Phi\big(ay-b\ln|x|\big).$

**4.** $ax\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=c.$

General solution: $w=\dfrac{c}{a}\ln|x|+\Phi(bx-ay).$

**5.** $(ax+b)\dfrac{\partial w}{\partial x}+(cy+d)\dfrac{\partial w}{\partial y}=\alpha x+\beta y+\gamma.$

General solution:

$$w=\frac{\alpha}{a}x+\frac{a\gamma-b\alpha}{a^2}\ln|ax+b|+\frac{\beta}{c}y-\frac{d\beta}{c^2}\ln|cy+d|+\Phi\big(|ax+b|^c|cy+d|^{-a}\big).$$

**6.** $ay\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\alpha x+\beta y+\gamma.$

General solution: $w=\dfrac{\beta}{a}x+\dfrac{\alpha x+\gamma}{b}y-\dfrac{a\alpha}{3b^2}y^3+\Phi\big(2bx-ay^2\big).$

**7.** $ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=c.$

General solution: $w=\dfrac{c}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x+ay\big|+\Phi\big(ay^2-bx^2\big).$

**8.** $ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=cx+ky.$

General solution: $w=\dfrac{bkx+acy}{ab}+\Phi\big(ay^2-bx^2\big).$

► **Coefficients of equations are quadratic in $x$ and $y$.**

**9.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + n.$

General solution:

$$w = \frac{1}{6a^2 b}\big[b(2ac - bk)x^3 + 2a^2 dy^3 + 3abx(kxy + 2n)\big] + \Phi(bx - ay).$$

**10.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + n.$

General solution:

$$w = \begin{cases} \dfrac{1}{2ab}(bcx^2 + ady^2) + \dfrac{n}{a}\ln|x| + \dfrac{k}{a+b}xy + \Phi\big(|x|^{-b/a}y\big) & \text{if } a + b \neq 0, \\ \dfrac{1}{2a}(cx^2 - dy^2) + \dfrac{1}{a}(kxy + n)\ln|x| + \Phi(xy) & \text{if } a + b = 0. \end{cases}$$

**11.** $\displaystyle ay\frac{\partial w}{\partial x} + bx\frac{\partial w}{\partial y} = cxy + d.$

General solution: $\displaystyle w = \frac{c}{2a}x^2 + \frac{d}{\sqrt{ab}}\ln\big|\sqrt{ab}\,x + ay\big| + \Phi(ay^2 - bx^2).$

**12.** $\displaystyle ax^2\frac{\partial w}{\partial x} + by^2\frac{\partial w}{\partial y} = cx^2 + dy^2 + kxy + nx + my + s.$

General solution:

$$w = \frac{c}{a}x - \frac{s}{ax} - \frac{dy^2}{ax - by} + \frac{kxy}{ax - by}\ln\left|\frac{ax}{y}\right| + \frac{n}{a}\ln|x| + \frac{m}{b}\ln|y| + \Phi\left(\frac{ax - by}{xy}\right).$$

**13.** $\displaystyle x^2\frac{\partial w}{\partial x} + axy\frac{\partial w}{\partial y} = by^2.$

General solution:

$$w = \begin{cases} \dfrac{b}{2a - 1}\dfrac{y^2}{x} + \Phi(|x|^{-a}y) & \text{if } a \neq \frac{1}{2}, \\ b\dfrac{y^2}{x}\ln|x| + \Phi(|x|^{-1/2}y) & \text{if } a = \frac{1}{2}. \end{cases}$$

**14.** $\displaystyle ay^2\frac{\partial w}{\partial x} + bx^2\frac{\partial w}{\partial y} = cx^2 + d.$

This is a special case of equation 1.2.7.14 with $k = 2$, $f(x) = a$, $g(x) = bx^2$, and $h(x) = cx^2 + d$.

**15.** $\displaystyle ay^2\frac{\partial w}{\partial x} + bxy\frac{\partial w}{\partial y} = cx^2 + dy^2.$

General solution:

$$w = \frac{ac + bd}{ab}x - \frac{c}{b}\sqrt{\frac{ay^2 - bx^2}{b}}\arctan\left(x\sqrt{\frac{b}{ay^2 - bx^2}}\right) + \Phi(ay^2 - bx^2).$$

► **Coefficients of equations contain other power-law functions.**

**16.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a\sqrt{x^2+y^2}}$**.**

General solution: $w = a\sqrt{x^2+y^2} + \Phi\left(\dfrac{y}{x}\right)$.

**17.** $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cxy^2 + dx^2y + k}$**.**

General solution: $w = \dfrac{cxy^2}{a+2b} + \dfrac{dx^2y}{2a+b} + \dfrac{k}{a}\ln|x| + \Phi\bigl(|x|^{-b/a}y\bigr)$.

**18.** $\boldsymbol{ay\dfrac{\partial w}{\partial x} + bx\dfrac{\partial w}{\partial y} = cx^2y + d}$**.**

General solution: $w = \dfrac{c}{3a}x^3 + \dfrac{d}{\sqrt{ab}}\ln\bigl|\sqrt{ab}\,x + ay\bigr| + \Phi(ay^2 - bx^2)$.

**19.** $\boldsymbol{(ax+b)\dfrac{\partial w}{\partial x} + (cy+d)\dfrac{\partial w}{\partial y} = kx^3 + ny^3}$**.**

General solution:

$$w = \frac{k}{a}\left(\frac{1}{3}x^3 - \frac{b}{2a}x^2 + \frac{b^2}{a^2}x - \frac{b^3}{a^3}\ln|ax+b|\right) + \frac{n}{c}\left(\frac{1}{3}y^3 - \frac{d}{2c}y^2 + \frac{d^2}{c^2}y - \frac{d^3}{c^3}\ln|cy+d|\right) + \Phi\bigl(|ax+b|^c|cy+d|^{-a}\bigr).$$

**20.** $\boldsymbol{x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^2(ax+by)}$**.**

General solution: $w = \dfrac{(ax+by)y^2}{2x} + \Phi\left(\dfrac{y}{x}\right)$.

**21.** $\boldsymbol{ax^3\dfrac{\partial w}{\partial x} + by^3\dfrac{\partial w}{\partial y} = cx + d}$**.**

General solution: $w = -\dfrac{2cx+d}{2ax^2} + \Phi\left(\dfrac{ax^2-by^2}{x^2y^2}\right)$.

► **Coefficients of equations contain arbitrary powers of $x$ and $y$.**

**22.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n + dy^m}$**.**

General solution: $w = \Phi(bx-ay) + \dfrac{c}{a(n+1)}x^{n+1} + \dfrac{d}{b(m+1)}y^{m+1}$.

**23.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n y.$

General solution:

$$w = \begin{cases} \dfrac{c[a(n+2)y - bx]x^{n+1}}{a^2(n+1)(n+2)} + \Phi(bx - ay) & \text{if } n \neq -1, -2; \\ \dfrac{bc}{a^2}x\bigl(1 - \ln|x|\bigr) + \dfrac{c}{a}y\ln|x| + \Phi(bx - ay) & \text{if } n = -1; \\ \dfrac{bc}{a^2}\bigl(1 + \ln|x|\bigr) - \dfrac{cy}{ax} + \Phi(bx - ay) & \text{if } n = -2. \end{cases}$$

**24.** $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a(x^2 + y^2)^k.$

General solution: $w = \dfrac{a}{2k}(x^2 + y^2)^k + \Phi\left(\dfrac{y}{x}\right).$

**25.** $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n y^m.$

General solution:

$$w = \begin{cases} \dfrac{c}{an + bm}x^n y^m + \Phi\bigl(|y|^a|x|^{-b}\bigr) & \text{if } an + bm \neq 0, \\ \dfrac{c}{a}x^n y^m \ln|x| + \Phi\bigl(|y|^a|x|^{-b}\bigr) & \text{if } an + bm = 0. \end{cases}$$

**26.** $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n + dy^m.$

General solution: $w = \dfrac{c}{an}x^n + \dfrac{d}{bm}y^m + \Phi\bigl(y^a x^{-b}\bigr).$

**27.** $mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = (ax^n + by^m)^k.$

General solution: $w = \dfrac{1}{mnk}(ax^n + by^m)^k + \Phi\bigl(y^m x^{-n}\bigr).$

**28.** $ax^n\dfrac{\partial w}{\partial x} + by^m\dfrac{\partial w}{\partial y} = cx^k + dy^s.$

This is a special case of equation 1.2.7.20. General solution:

$$w = \frac{c}{a(k-n+1)}x^{k-n+1} + \frac{d}{b(s-m+1)}y^{s-m+1} + \Phi(u),$$
$$u = \frac{1}{a(1-n)}x^{1-n} - \frac{1}{b(1-m)}y^{1-m}.$$

**29.** $ax^n\dfrac{\partial w}{\partial x} + bx^m y\dfrac{\partial w}{\partial y} = cx^k y^s + d.$

This is a special case of equation 1.2.7.34 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x, y) = cx^k y^s + d$.

**30.** $\displaystyle ax^n \frac{\partial w}{\partial x} + \bigl(bx^m y + cx^k\bigr)\frac{\partial w}{\partial y} = sx^p y^q + d.$

This is a special case of equation 1.2.7.35 with $f(x) = ax^n$, $g_1(x) = bx^m$, $g_0(x) = cx^k$, and $h(x,y) = sx^p y^q + d$.

**31.** $\displaystyle ax^n \frac{\partial w}{\partial x} + \bigl(bx^m y^k + cx^l y\bigr)\frac{\partial w}{\partial y} = sx^p y^q + d.$

This is a special case of equation 1.2.7.36 with $f(x) = ax^n$, $g_1(x) = cx^l$, $g_0(x) = bx^m$, and $h(x,y) = sx^p y^q + d$.

**32.** $\displaystyle ay^k \frac{\partial w}{\partial x} + bx^n \frac{\partial w}{\partial y} = cx^m + d.$

This is a special case of equation 1.2.7.14 with $f(x) = a$, $g(x) = bx^n$, and $h(x) = cx^m + d$.

## 1.2.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

**1.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = ce^{\lambda x} + de^{\mu y}.$

General solution: $\displaystyle w = \frac{c}{a\lambda}e^{\lambda x} + \frac{d}{b\mu}e^{\mu y} + \Phi(bx - ay).$

**2.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = ce^{\alpha x + \beta y}.$

General solution:

$$w = \begin{cases} \dfrac{c}{a\alpha + b\beta}e^{\alpha x + \beta y} + \Phi(bx - ay) & \text{if } a\alpha + b\beta \neq 0, \\ \dfrac{c}{a}xe^{\alpha x + \beta y} + \Phi(bx - ay) & \text{if } a\alpha + b\beta = 0. \end{cases}$$

**3.** $\displaystyle ae^{\lambda x}\frac{\partial w}{\partial x} + be^{\beta y}\frac{\partial w}{\partial y} = c.$

General solution: $\displaystyle w = -\frac{c}{a\lambda}e^{-\lambda x} + \Phi(u)$, where $u = b\beta e^{-\lambda x} - a\lambda e^{-\beta y}$.

**4.** $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x} + be^{\beta x}\frac{\partial w}{\partial y} = c.$

General solution: $\displaystyle w = \frac{c(\beta x - \lambda y)}{u} + \Phi(u)$, where $u = a\beta e^{\lambda y} - b\lambda e^{\beta x}$.

**5.** $\displaystyle ae^{\alpha x}\frac{\partial w}{\partial x} + be^{\beta y}\frac{\partial w}{\partial y} = ce^{\gamma x - \beta y}.$

Introduce the notation $\displaystyle u = \frac{1}{\beta b}e^{\beta y} - \frac{1}{\alpha a}e^{\alpha x}.$

General solution:

$$w = \begin{cases} \dfrac{c}{a(\gamma-\alpha)} e^{(\gamma-\alpha)x}\left[e^{-\beta y} + \dfrac{b\beta e^{-\alpha x}}{a(\gamma-2\alpha)}\right] + \Phi(u) & \text{if } \gamma \neq \alpha,\, 2\alpha, \\ \dfrac{c}{a}\left[xe^{-\beta y} - \dfrac{b\beta}{a\alpha^2}(\alpha x+1)e^{-\alpha x}\right] + \Phi(u) & \text{if } \gamma = \alpha, \\ \dfrac{c}{a\alpha}\left[e^{\alpha x-\beta y} + \dfrac{b\beta}{a\alpha}(\alpha x-1)\right] + \Phi(u) & \text{if } \gamma = 2\alpha. \end{cases}$$

**6.** $\boldsymbol{ae^{\alpha x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = ce^{\gamma x - 2\beta y}}$**.**

Introduce the notation $u = \dfrac{1}{\beta b}e^{\beta y} - \dfrac{1}{\alpha a}e^{\alpha x}$.

1°. General solution for $\gamma \neq \alpha$, $\gamma \neq 2\alpha$, and $\gamma \neq 3\alpha$:

$$w = \frac{c}{a(\gamma-\alpha)}\left[e^{-2\beta y} + \frac{2b\beta}{a(\gamma-2\alpha)}e^{-\alpha x-\beta y} + \frac{2b^2\beta^2}{a^2(\gamma-2\alpha)(\gamma-3\alpha)}e^{-2\alpha x}\right]e^{(\gamma-\alpha)x} + \Phi(u).$$

2°. General solution for $\gamma = \alpha$:

$$w = \frac{c}{a}\left[xe^{-2\beta y} - \frac{2b\beta}{a\alpha^2}(\alpha x+1)e^{-\alpha x-\beta y} + \frac{b^2\beta^2}{a^2\alpha^3}\left(\alpha x + \frac{3}{2}\right)e^{-2\alpha x}\right] + \Phi(u).$$

3°. General solution for $\gamma = 2\alpha$:

$$w = \frac{c}{a\alpha}\left[e^{\alpha x-\beta y} + \frac{2b\beta}{a\alpha}(\alpha x-1)\right]e^{-\beta y} + \Phi(u).$$

4°. General solution for $\gamma = 3\alpha$:

$$w = \frac{c}{a\alpha}\left[\frac{1}{2}e^{2(\alpha x-\beta y)} + \frac{b\beta}{a\alpha}e^{\alpha x-\beta y} + \frac{b^2\beta^2}{a^2\alpha^2}\left(\alpha x - \frac{3}{2}\right)\right] + \Phi(u).$$

**7.** $\boldsymbol{ae^{\alpha x}\dfrac{\partial w}{\partial x} + be^{\beta y}\dfrac{\partial w}{\partial y} = ce^{\gamma x} + se^{\mu y}}$**.**

This is a special case of equation 1.2.7.20 with $f(x) = ae^{\alpha x}$, $g(y) = be^{\beta y}$, $h_1(x) = ce^{\gamma x}$, and $h_2(y) = se^{\mu y}$.

**8.** $\boldsymbol{ae^{\beta x}\dfrac{\partial w}{\partial x} + \left(be^{\gamma x} + ce^{\lambda y}\right)\dfrac{\partial w}{\partial y} = se^{\mu x} + ke^{\delta y} + p}$**.**

This is a special case of equation 1.2.7.37 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, and $h(x,y) = se^{\mu x} + ke^{\delta y} + p$.

**9.** $\boldsymbol{ae^{\beta x}\dfrac{\partial w}{\partial x} + \left(be^{\gamma x} + ce^{\lambda y}\right)\dfrac{\partial w}{\partial y} = se^{\mu x+\delta y} + k}$**.**

This is a special case of equation 1.2.7.37 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, and $h(x,y) = se^{\mu x+\delta y} + k$.

**10.** $\displaystyle ae^{\beta x}\frac{\partial w}{\partial x}+be^{\gamma x+\lambda y}\frac{\partial w}{\partial y}=ce^{\mu x+\delta y}+k.$

This is a special case of equation 1.2.7.37 with $f(x)=ae^{\beta x}$, $g_1(x)\equiv 0$, $g_0(x)=be^{\gamma x}$, and $h(x,y)=ce^{\mu x+\delta y}+k$.

**11.** $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x}+be^{\beta x}\frac{\partial w}{\partial y}=ce^{\gamma x}+d.$

This is a special case of equation 1.2.7.16 with $f(x)=a$, $g(x)=be^{\beta x}$, and $h(x)=ce^{\gamma x}+d$.

► **Coefficients of equations contain exponential and power-law functions.**

**12.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=cye^{\lambda x}+kxe^{\mu y}.$

General solution: $\displaystyle w=\frac{c}{a\lambda}e^{\lambda x}\left(y-\frac{b}{a\lambda}\right)+\frac{k}{b\mu}e^{\mu y}\left(x-\frac{a}{b\mu}\right)+\Phi(bx-ay).$

**13.** $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}=ax^k e^{\lambda y}.$

This is a special case of equation 1.2.7.5 with $f(x)=ax^k$.

**14.** $\displaystyle \frac{\partial w}{\partial x}+(ay+be^{\lambda x})\frac{\partial w}{\partial y}=ce^{\beta x}.$

This is a special case of equation 1.2.7.6 with $f(x)=be^{\lambda x}$ and $g(x)=ce^{\beta x}$.

**15.** $\displaystyle \frac{\partial w}{\partial x}+(ae^{\lambda x}y+be^{\beta x}y^k)\frac{\partial w}{\partial y}=ce^{\mu x}.$

This is a special case of equation 1.2.7.12 with $f(x)=1$, $g_1(x)=ae^{\lambda x}$, $g_2(x)=be^{\beta x}$, and $h(x)=ce^{\mu x}$.

**16.** $\displaystyle \frac{\partial w}{\partial x}+(ax^k+bx^n e^{\lambda y})\frac{\partial w}{\partial y}=ce^{\beta x}.$

This is a special case of equation 1.2.7.13 with $f(x)=1$, $g_1(x)=ax^k$, $g_2(x)=bx^n$, and $h(x)=ce^{\beta x}$.

**17.** $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=axe^{\lambda x+\mu y}.$

General solution: $\displaystyle w=\frac{ax}{\lambda x+\mu y}e^{\lambda x+\mu y}+\Phi\left(\frac{y}{x}\right).$

**18.** $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=aye^{\lambda x}+bxe^{\mu y}.$

General solution: $\displaystyle w=\frac{ay}{\lambda x}e^{\lambda x}+\frac{bx}{\mu y}e^{\mu y}+\Phi\left(\frac{y}{x}\right).$

**19.** $\displaystyle ax^k\frac{\partial w}{\partial x}+be^{\lambda y}\frac{\partial w}{\partial y}=cx^n+s.$

This is a special case of equation 1.2.7.13 with $f(x)=ax^k$, $g_1(x)=0$, $g_2(x)=b$, and $h(x)=cx^n+s$.

**20.** $\boldsymbol{ay^k \dfrac{\partial w}{\partial x} + be^{\lambda x} \dfrac{\partial w}{\partial y} = ce^{\mu x} + s.}$

This is a special case of equation 1.2.7.14 with $f(x) = a$, $g(x) = be^{\lambda x}$, and $h(x) = ce^{\mu x} + s$.

**21.** $\boldsymbol{ae^{\lambda x} \dfrac{\partial w}{\partial x} + by^k \dfrac{\partial w}{\partial y} = cx^n + s.}$

This is a special case of equation 1.2.7.12 with $f(x) = ae^{\lambda x}$, $g_1(x) = 0$, $g_2(x) = b$, and $h(x) = cx^n + s$.

**22.** $\boldsymbol{ae^{\lambda y} \dfrac{\partial w}{\partial x} + bx^k \dfrac{\partial w}{\partial y} = ce^{\mu x} + s.}$

This is a special case of equation 1.2.7.16 with $f(x) = a$, $g(x) = bx^k$, and $h(x) = ce^{\mu x} + s$.

## 1.2.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

**1.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = c \sinh(\lambda x) + k \sinh(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda} \cosh(\lambda x) + \dfrac{k}{b\mu} \cosh(\mu y) + \Phi(bx - ay)$.

**2.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = c \sinh(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu} \cosh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a} x \sinh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**3.** $\boldsymbol{x \dfrac{\partial w}{\partial x} + y \dfrac{\partial w}{\partial y} = ax \sinh(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y} \cosh(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right)$.

**4.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \sinh^n(\lambda x) \dfrac{\partial w}{\partial y} = c \sinh^m(\mu x) + s \sinh^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b \sinh^n(\lambda x)$, and $h(x, y) = c \sinh^m(\mu x) + s \sinh^k(\beta y)$.

**5.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \sinh^n(\lambda y) \dfrac{\partial w}{\partial y} = c \sinh^m(\mu x) + s \sinh^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b \sinh^n(\lambda y)$, $h_1(x) = c \sinh^m(\mu x)$, and $h_2(y) = s \sinh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cosine.**

**6.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cosh(\lambda x) + k\cosh(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda}\sinh(\lambda x) + \dfrac{k}{b\mu}\sinh(\mu y) + \Phi(bx - ay)$.

**7.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cosh(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\sinh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\cosh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**8.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\cosh(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\sinh(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right)$.

**9.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cosh^n(\lambda x)\dfrac{\partial w}{\partial y} = c\cosh^m(\mu x) + s\cosh^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^n(\lambda x)$, and $h(x,y) = c\cosh^m(\mu x) + s\cosh^k(\beta y)$.

**10.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cosh^n(\lambda y)\dfrac{\partial w}{\partial y} = c\cosh^m(\mu x) + s\cosh^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\cosh^n(\lambda y)$, $h_1(x) = c\cosh^m(\mu x)$, and $h_2(y) = s\cosh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic tangent.**

**11.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tanh(\lambda x) + k\tanh(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda}\ln\bigl[\cosh(\lambda x)\bigr] + \dfrac{k}{b\mu}\ln\bigl[\cosh(\mu y)\bigr] + \Phi(bx - ay)$.

**12.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tanh(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\ln\bigl[\cosh(\lambda x + \mu y)\bigr] + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\tanh(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**13.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\tanh(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\ln\bigl[\cosh(\lambda x + \mu y)\bigr] + \Phi\left(\dfrac{y}{x}\right)$.

**14.** $a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda x)\dfrac{\partial w}{\partial y} = c\tanh^m(\mu x) + s\tanh^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^n(\lambda x)$, and $h(x,y) = c\tanh^m(\mu x) + s\tanh^k(\beta y)$.

**15.** $a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda y)\dfrac{\partial w}{\partial y} = c\tanh^m(\mu x) + s\tanh^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\tanh^n(\lambda y)$, $h_1(x) = c\tanh^m(\mu x)$, and $h_2(y) = s\tanh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cotangent.**

**16.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\coth(\lambda x) + k\coth(\mu y).$

General solution: $w = \dfrac{c}{a\lambda}\ln\bigl|\sinh(\lambda x)\bigr| + \dfrac{k}{b\mu}\ln\bigl|\sinh(\mu y)\bigr| + \Phi(bx - ay).$

**17.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\coth(\lambda x + \mu y).$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\ln\bigl|\sinh(\lambda x + \mu y)\bigr| + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a}x\coth(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**18.** $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\coth(\lambda x + \mu y).$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\ln\bigl|\sinh(\lambda x + \mu y)\bigr| + \Phi\left(\dfrac{y}{x}\right).$

**19.** $a\dfrac{\partial w}{\partial x} + b\coth^n(\lambda x)\dfrac{\partial w}{\partial y} = c\coth^m(\mu x) + s\coth^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^n(\lambda x)$, and $h(x,y) = c\coth^m(\mu x) + s\coth^k(\beta y)$.

**20.** $a\dfrac{\partial w}{\partial x} + b\coth^n(\lambda y)\dfrac{\partial w}{\partial y} = c\coth^m(\mu x) + s\coth^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\coth^n(\lambda y)$, $h_1(x) = c\coth^m(\mu x)$, and $h_2(y) = s\coth^k(\beta y)$.

► **Coefficients of equations contain different hyperbolic functions.**

**21.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\sinh(\lambda x) + k\cosh(\mu y).$

General solution: $w = \dfrac{c}{a\lambda}\cosh(\lambda x) + \dfrac{k}{b\mu}\sinh(\mu y) + \Phi(bx - ay).$

**22.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \tanh(\lambda x) + k\coth(\mu y).}$

General solution: $w = \dfrac{1}{a\lambda}\ln\bigl|\cosh(\lambda x)\bigr| + \dfrac{k}{b\mu}\ln\bigl|\sinh(\mu y)\bigr| + \Phi(bx - ay).$

**23.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \sinh(\lambda x) + k\tanh(\mu y).}$

General solution: $w = \dfrac{1}{a\lambda}\cosh(\lambda x) + \dfrac{k}{b\mu}\ln\bigl|\cosh(\mu y)\bigr| + \Phi(bx - ay).$

**24.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cosh(\mu y)\dfrac{\partial w}{\partial y} = \sinh(\lambda x).}$

General solution: $w = \dfrac{1}{a\lambda}\cosh(\lambda x) + \Phi(u)$, where $u = b\mu x - 2a\arctan\Bigl(\tanh\dfrac{\mu x}{2}\Bigr).$

**25.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sinh(\mu y)\dfrac{\partial w}{\partial y} = \cosh(\lambda x).}$

General solution: $w = \dfrac{1}{a\lambda}\sinh(\lambda x) + \Phi(u)$, where $u = b\mu x - a\ln\Bigl|\tanh\dfrac{\mu x}{2}\Bigr|.$

### 1.2.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

**1.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(\lambda x + \beta y).}$

General solution:

$$w = \begin{cases} \dfrac{c(\lambda x + \beta y)}{a\lambda + b\beta}\bigl[\ln(\lambda x + \beta y) - 1\bigr] + \Phi(bx - ay) & \text{if } a\lambda \neq -b\beta, \\ \dfrac{c}{a}x\ln(\lambda x + \beta y) + \Phi(bx - ay) & \text{if } a\lambda = -b\beta. \end{cases}$$

**2.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln(\lambda x) + k\ln(\beta y).}$

General solution: $w = \dfrac{c}{a}x\bigl[\ln(\lambda x) - 1\bigr] + \dfrac{k}{b}y\bigl[\ln(\beta y) - 1\bigr] + \Phi(bx - ay).$

**3.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln(\lambda x)\ln(\beta y)\dfrac{\partial w}{\partial y} = c\ln(\gamma x).}$

General solution:

$$w = \frac{c}{a}x\bigl[\ln(\gamma x) - 1\bigr] + \Phi(u), \qquad \text{where} \quad u = bx\bigl[\ln(\lambda x) - 1\bigr] - a\int\frac{dy}{\ln(\beta y)}.$$

**4.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\ln^n(\lambda x)\dfrac{\partial w}{\partial y} = c\ln^m(\mu x) + s\ln^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^n(\lambda x)$, and $h(x,y) = c\ln^m(\mu x) + s\ln^k(\beta y)$.

**5.** $\displaystyle a\frac{\partial w}{\partial x} + b\ln^n(\lambda y)\frac{\partial w}{\partial y} = c\ln^m(\mu x) + s\ln^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\ln^n(\lambda y)$, $h_1(x) = c\ln^m(\mu x)$, and $h_2(y) = s\ln^k(\beta y)$.

**6.** $\displaystyle a\ln^n(\lambda x)\frac{\partial w}{\partial x} + b\ln^k(\beta y)\frac{\partial w}{\partial y} = c\ln^m(\gamma x).$

General solution:

$$w = \frac{c}{a}\int \frac{\ln^m(\gamma x)}{\ln^n(\lambda x)}\,dx + \Phi(u), \qquad \text{where} \quad u = b\int \frac{dx}{\ln^n(\lambda x)} - a\int \frac{dy}{\ln^k(\beta y)}.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cx^n + s\ln^k(\lambda y).$

General solution: $\displaystyle w = \frac{c}{a(n+1)}x^{n+1} + \frac{s}{b}\int \ln^k(\lambda y)\,dy + \Phi(bx - ay).$

**8.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = by^2 + cx^n y + s\ln^k(\lambda x).$

This is a special case of equation 1.2.7.3 with $f(x) = b$, $g(x) = cx^n$, and $h(x) = s\ln^k(\lambda x)$.

**9.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = b\ln^k(\lambda x)\ln^n(\beta y).$

This is a special case of equation 1.2.7.18 with $f(x) = b\ln^k(\lambda x)$ and $g(y) = \ln^n(\beta y)$.

**10.** $\displaystyle \frac{\partial w}{\partial x} + (ay + bx^n)\frac{\partial w}{\partial y} = c\ln^k(\lambda x).$

This is a special case of equation 1.2.7.6 with $f(x) = bx^n$ and $g(x) = c\ln^k(\lambda x)$.

**11.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = x^k(n\ln x + m\ln y).$

This is a special case of equation 1.2.7.28 with $f(u) = \ln u$.

**12.** $\displaystyle ax^k\frac{\partial w}{\partial x} + by^n\frac{\partial w}{\partial y} = c\ln^m(\lambda x) + s\ln^l(\beta y).$

General solution:

$$w = \frac{c}{a}\int x^{-k}\ln^m(\lambda x)\,dx + \frac{s}{b}\int y^{-n}\ln^l(\beta y)\,dy + \Phi(u), \qquad u = \frac{b}{1-k}x^{1-k} - \frac{a}{1-n}y^{1-n}.$$

## 1.2.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

**1.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\sin(\lambda x) + k\sin(\mu y).$

General solution: $\displaystyle w = -\frac{c}{a\lambda}\cos(\lambda x) - \frac{k}{b\mu}\cos(\mu y) + \Phi(bx - ay).$

**2.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\sin(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} -\dfrac{c}{a\lambda + b\mu}\cos(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a} x\sin(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**3.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\sin(\lambda x + \mu y).}$

General solution: $w = -\dfrac{ax}{\lambda x + \mu y}\cos(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

**4.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sin^n(\lambda x)\dfrac{\partial w}{\partial y} = c\sin^m(\mu x) + s\sin^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^n(\lambda x)$, and $h(x,y) = c\sin^m(\mu x) + s\sin^k(\beta y)$.

**5.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sin^n(\lambda y)\dfrac{\partial w}{\partial y} = c\sin^m(\mu x) + s\sin^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\sin^n(\lambda y)$, $h_1(x) = c\sin^m(\mu x)$, and $h_2(y) = s\sin^k(\beta y)$.

► **Coefficients of equations contain cosine.**

**6.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos(\lambda x) + k\cos(\mu y).}$

General solution: $w = \dfrac{c}{a\lambda}\sin(\lambda x) + \dfrac{k}{b\mu}\sin(\mu y) + \Phi(bx - ay).$

**7.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\cos(\lambda x + \mu y).}$

General solution:

$$w = \begin{cases} \dfrac{c}{a\lambda + b\mu}\sin(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \dfrac{c}{a} x\cos(\lambda x + \mu y) + \Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**8.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\cos(\lambda x + \mu y).}$

General solution: $w = \dfrac{ax}{\lambda x + \mu y}\sin(\lambda x + \mu y) + \Phi\left(\dfrac{y}{x}\right).$

**9.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda x)\dfrac{\partial w}{\partial y} = c\cos^m(\mu x) + s\cos^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^n(\lambda x)$, and $h(x,y) = c\cos^m(\mu x) + s\cos^k(\beta y)$.

**10.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\cos^n(\lambda y)\dfrac{\partial w}{\partial y} = c\cos^m(\mu x) + s\cos^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\cos^n(\lambda y)$, $h_1(x) = c\cos^m(\mu x)$, and $h_2(y) = s\cos^k(\beta y)$.

▶ **Coefficients of equations contain tangent.**

**11.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\tan(\lambda x)+k\tan(\mu y).}$

General solution: $w=-\dfrac{c}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr|-\dfrac{k}{b\mu}\ln\bigl|\cos(\mu y)\bigr|+\Phi(bx-ay).$

**12.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\tan(\lambda x+\mu y).}$

General solution:

$$w=\begin{cases}-\dfrac{c}{a\lambda+b\mu}\ln\bigl|\cos(\lambda x+\mu y)\bigr|+\Phi(bx-ay) & \text{if }\ a\lambda+b\mu\neq 0,\\ \dfrac{c}{a}x\tan(\lambda x+\mu y)+\Phi(bx-ay) & \text{if }\ a\lambda+b\mu=0.\end{cases}$$

**13.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\tan(\lambda x+\mu y).}$

General solution: $w=-\dfrac{ax}{\lambda x+\mu y}\ln\bigl|\cos(\lambda x+\mu y)\bigr|+\Phi\Bigl(\dfrac{y}{x}\Bigr).$

**14.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\tan^n(\lambda x)\dfrac{\partial w}{\partial y}=c\tan^m(\mu x)+s\tan^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\tan^n(\lambda x)$, and $h(x,y)=c\tan^m(\mu x)+s\tan^k(\beta y)$.

**15.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\tan^n(\lambda y)\dfrac{\partial w}{\partial y}=c\tan^m(\mu x)+s\tan^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x)=a$, $g(y)=b\tan^n(\lambda y)$, $h_1(x)=c\tan^m(\mu x)$, and $h_2(y)=s\tan^k(\beta y)$.

▶ **Coefficients of equations contain cotangent.**

**16.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\cot(\lambda x)+k\cot(\mu y).}$

General solution: $w=\dfrac{c}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr|+\dfrac{k}{b\mu}\ln\bigl|\sin(\mu y)\bigr|+\Phi(bx-ay).$

**17.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\cot(\lambda x+\mu y).}$

General solution:

$$w=\begin{cases}\dfrac{c}{a\lambda+b\mu}\ln\bigl|\sin(\lambda x+\mu y)\bigr|+\Phi(bx-ay) & \text{if }\ a\lambda+b\mu\neq 0,\\ \dfrac{c}{a}x\cot(\lambda x+\mu y)+\Phi(bx-ay) & \text{if }\ a\lambda+b\mu=0.\end{cases}$$

**18.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\cot(\lambda x+\mu y).}$

General solution: $w=\dfrac{ax}{\lambda x+\mu y}\ln\bigl|\sin(\lambda x+\mu y)\bigr|+\Phi\Bigl(\dfrac{y}{x}\Bigr).$

**19.** $\displaystyle a\frac{\partial w}{\partial x}+b\cot^n(\lambda x)\frac{\partial w}{\partial y}=c\cot^m(\mu x)+s\cot^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\cot^n(\lambda x)$, and $h(x,y)=c\cot^m(\mu x)+s\cot^k(\beta y)$.

**20.** $\displaystyle a\frac{\partial w}{\partial x}+b\cot^n(\lambda y)\frac{\partial w}{\partial y}=c\cot^m(\mu x)+s\cot^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x)=a$, $g(y)=b\cot^n(\lambda y)$, $h_1(x)=c\cot^m(\mu x)$, and $h_2(y)=s\cot^k(\beta y)$.

► **Coefficients of equations contain different trigonometric functions.**

**21.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\sin(\lambda x)+c\cos(\mu y)+k.$

General solution: $\displaystyle w=\frac{k}{a}x-\frac{1}{a\lambda}\cos(\lambda x)+\frac{c}{b\mu}\sin(\mu y)+\Phi(bx-ay).$

**22.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\tan(\lambda x)+c\sin(\mu y)+k.$

General solution: $\displaystyle w=\frac{k}{a}x-\frac{1}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr|-\frac{c}{b\mu}\cos(\mu y)+\Phi(bx-ay).$

**23.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\sin(\lambda x)\cos(\mu y)+c.$

General solution:

$$w=\begin{cases}\dfrac{c}{a}x-\dfrac{\cos(\lambda x-\mu y)}{2(a\lambda-b\mu)}-\dfrac{\cos(\lambda x+\mu y)}{2(a\lambda+b\mu)}+\Phi(bx-ay) & \text{if } a\lambda\pm b\mu\neq 0,\\ \dfrac{c}{a}x+\dfrac{x}{2a}\sin\Bigl[\dfrac{\mu}{a}(bx-ay)\Bigr]-\dfrac{\cos(\lambda x+\mu y)}{2(a\lambda+b\mu)}+\Phi(bx-ay) & \text{if } a\lambda-b\mu=0,\\ \dfrac{c}{a}x-\dfrac{x}{2a}\sin\Bigl[\dfrac{\mu}{a}(bx-ay)\Bigr]-\dfrac{\cos(\lambda x-\mu y)}{2(a\lambda-b\mu)}+\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

**24.** $\displaystyle a\frac{\partial w}{\partial x}+b\sin(\mu y)\frac{\partial w}{\partial y}=\cos(\lambda x)+c.$

General solution: $\displaystyle w=\frac{c}{a}x+\frac{1}{a\lambda}\sin(\lambda x)+\Phi(u),$ where $u=b\mu x-a\ln\bigl|\tan\bigl(\tfrac{1}{2}\mu y\bigr)\bigr|.$

**25.** $\displaystyle a\frac{\partial w}{\partial x}+b\tan(\mu y)\frac{\partial w}{\partial y}=\sin(\lambda x)+c.$

General solution: $\displaystyle w=\frac{c}{a}x-\frac{1}{a\lambda}\cos(\lambda x)+\Phi(u),$ where $u=b\mu x-a\ln\bigl|\sin(\mu y)\bigr|.$

**26.** $\displaystyle a\frac{\partial w}{\partial x}+b\tan(\mu y)\frac{\partial w}{\partial y}=\cot(\lambda x)+c.$

General solution: $\displaystyle w=\frac{c}{a}x-\frac{1}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr|+\Phi(u),$ where $u=b\mu x-a\ln\bigl|\sin(\mu y)\bigr|.$

### 1.2.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

**1.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arcsin\dfrac{x}{\lambda}+k\arcsin\dfrac{y}{\beta}.}$

General solution:

$$w=\frac{c}{a}\Big(x\arcsin\frac{x}{\lambda}+\sqrt{\lambda^2-x^2}\Big)+\frac{k}{b}\Big(y\arcsin\frac{y}{\beta}+\sqrt{\beta^2-y^2}\Big)+\Phi(bx-ay).$$

**2.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arcsin(\lambda x+\beta y).}$

$1^\circ$. General solution for $a\lambda+b\beta\neq 0$:

$$w=\frac{c}{a\lambda+b\beta}\Big[(\lambda x+\beta y)\arcsin(\lambda x+\beta y)+\sqrt{1-(\lambda x+\beta y)^2}\Big]+\Phi(bx-ay).$$

$2^\circ$. General solution for $a\lambda+b\beta=0$:

$$w=\frac{c}{a}x\arcsin(\lambda x+\beta y)+\Phi(bx-ay).$$

**3.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\arcsin(\lambda x+\beta y).}$

General solution: $w=ax\left[\arcsin(\lambda x+\beta y)+\dfrac{\sqrt{1-(\lambda x+\beta y)^2}}{\lambda x+\beta y}\right]+\Phi\Big(\dfrac{y}{x}\Big).$

**4.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y}=c\arcsin^m(\mu x)+s\arcsin^k(\beta y).}$

This is a special case of equation 1.2.7.35 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arcsin^n(\lambda x)$, and $h(x,y)=c\arcsin^m(\mu x)+s\arcsin^k(\beta y)$.

**5.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\arcsin^n(\lambda y)\dfrac{\partial w}{\partial y}=c\arcsin^m(\mu x)+s\arcsin^k(\beta y).}$

This is a special case of equation 1.2.7.20 with $f(x)=a$, $g(y)=b\arcsin^n(\lambda y)$, $h_1(x)=c\arcsin^m(\mu x)$, and $h_2(y)=s\arcsin^k(\beta y)$.

► **Coefficients of equations contain arccosine.**

**6.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arccos\dfrac{x}{\lambda}+k\arccos\dfrac{y}{\beta}.}$

General solution:

$$w=\frac{c}{a}\Big(x\arccos\frac{x}{\lambda}-\sqrt{\lambda^2-x^2}\Big)+\frac{k}{b}\Big(y\arccos\frac{y}{\beta}-\sqrt{\beta^2-y^2}\Big)+\Phi(bx-ay).$$

**7.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arccos(\lambda x+\beta y).$

1°. General solution for $a\lambda+b\beta\neq 0$:

$$w=\frac{c}{a\lambda+b\beta}\Big[(\lambda x+\beta y)\arccos(\lambda x+\beta y)-\sqrt{1-(\lambda x+\beta y)^2}\,\Big]+\Phi(bx-ay).$$

2°. General solution for $a\lambda+b\beta=0$:

$$w=\frac{c}{a}x\arccos(\lambda x+\beta y)+\Phi(bx-ay).$$

**8.** $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=ax\arccos(\lambda x+\beta y).$

General solution: $w=ax\left[\arccos(\lambda x+\beta y)-\dfrac{\sqrt{1-(\lambda x+\beta y)^2}}{\lambda x+\beta y}\right]+\Phi\left(\dfrac{y}{x}\right).$

**9.** $a\dfrac{\partial w}{\partial x}+b\arccos^n(\lambda x)\dfrac{\partial w}{\partial y}=c\arccos^m(\mu x)+s\arccos^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\arccos^n(\lambda x)$, and $h(x,y)=c\arccos^m(\mu x)+s\arccos^k(\beta y)$.

**10.** $a\dfrac{\partial w}{\partial x}+b\arccos^n(\lambda y)\dfrac{\partial w}{\partial y}=c\arccos^m(\mu x)+s\arccos^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x)=a$, $g(y)=b\arccos^n(\lambda y)$, $h_1(x)=c\arccos^m(\mu x)$, and $h_2(y)=s\arccos^k(\beta y)$.

### ► Coefficients of equations contain arctangent.

**11.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arctan\dfrac{x}{\lambda}+k\arctan\dfrac{y}{\beta}.$

General solution:

$$w=\frac{c}{a}\left[x\arctan\frac{x}{\lambda}-\frac{\lambda}{2}\ln(\lambda^2+x^2)\right]+\frac{k}{b}\left[y\arctan\frac{y}{\beta}-\frac{\beta}{2}\ln(\beta^2+y^2)\right]+\Phi(bx-ay).$$

**12.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=c\arctan(\lambda x+\beta y).$

1°. General solution for $a\lambda+b\beta\neq 0$:

$$w=\frac{c}{a\lambda+b\beta}\left\{(\lambda x+\beta y)\arctan(\lambda x+\beta y)-\frac{1}{2}\ln\big[1+(\lambda x+\beta y)^2\big]\right\}+\Phi(bx-ay).$$

2°. General solution for $a\lambda+b\beta=0$:

$$w=\frac{c}{a}x\arctan(\lambda x+\beta y)+\Phi(bx-ay).$$

**13.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\arctan(\lambda x + \beta y).$

General solution:

$$w = ax\left\{\arctan(\lambda x + \beta y) - \frac{1}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\} + \Phi\left(\frac{y}{x}\right).$$

**14.** $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda x)\frac{\partial w}{\partial y} = c\arctan^m(\mu x) + s\arctan^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arctan^n(\lambda x)$, and $h(x,y) = c\arctan^m(\mu x) + s\arctan^k(\beta y)$.

**15.** $\displaystyle a\frac{\partial w}{\partial x} + b\arctan^n(\lambda y)\frac{\partial w}{\partial y} = c\arctan^m(\mu x) + s\arctan^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\arctan^n(\lambda y)$, $h_1(x) = c\arctan^m(\mu x)$, and $h_2(y) = s\arctan^k(\beta y)$.

### ► Coefficients of equations contain arccotangent.

**16.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\,\mathrm{arccot}\,\frac{x}{\lambda} + k\,\mathrm{arccot}\,\frac{y}{\beta}.$

General solution:

$$w = \frac{c}{a}\left[x\,\mathrm{arccot}\,\frac{x}{\lambda} + \frac{\lambda}{2}\ln(\lambda^2 + x^2)\right] + \frac{k}{b}\left[y\,\mathrm{arccot}\,\frac{y}{\beta} + \frac{\beta}{2}\ln(\beta^2 + y^2)\right] + \Phi(bx - ay).$$

**17.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\,\mathrm{arccot}(\lambda x + \beta y).$

$1^\circ$. General solution for $a\lambda + b\beta \neq 0$:

$$w = \frac{c}{a\lambda + b\beta}\left\{(\lambda x + \beta y)\,\mathrm{arccot}(\lambda x + \beta y) + \frac{1}{2}\ln\left[1 + (\lambda x + \beta y)^2\right]\right\} + \Phi(bx - ay).$$

$2^\circ$. General solution for $a\lambda + b\beta = 0$:

$$w = \frac{c}{a}x\,\mathrm{arccot}(\lambda x + \beta y) + \Phi(bx - ay).$$

**18.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\,\mathrm{arccot}(\lambda x + \beta y).$

General solution:

$$w = ax\left\{\mathrm{arccot}(\lambda x + \beta y) + \frac{1}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\} + \Phi\left(\frac{y}{x}\right).$$

**19.** $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y).$

This is a special case of equation 1.2.7.35 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\,\mathrm{arccot}^n(\lambda x)$, and $h(x,y) = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)$.

**20.** $\displaystyle a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda y)\frac{\partial w}{\partial y} = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y).$

This is a special case of equation 1.2.7.20 with $f(x) = a$, $g(y) = b\,\mathrm{arccot}^n(\lambda y)$, $h_1(x) = c\,\mathrm{arccot}^m(\mu x)$, and $h_2(y) = s\,\mathrm{arccot}^k(\beta y)$.

### 1.2.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of $x$.**

**1.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=f(x).$

General solution: $w=\dfrac{1}{a}\displaystyle\int f(x)\,dx+\Phi(bx-ay).$

⊙ *Literature*: E. Kamke (1965).

**2.** $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}=f(x)y.$

General solution: $w=\displaystyle\int_{x_0}^{x}(y-ax+at)f(t)\,dt+\Phi(y-ax),$ where $x_0$ may be taken as arbitrary.

**3.** $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}=f(x)y^2+g(x)y+h(x).$

General solution:

$$w=\varphi(x)y^2+\psi(x)y+\chi(x)+\Phi(y-ax),$$

where

$$\varphi(x)=\int f(x)\,dx,\quad \psi(x)=\int\bigl[g(x)-2a\varphi(x)\bigr]\,dx,\quad \chi(x)=\int\bigl[h(x)-a\psi(x)\bigr]\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**4.** $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}=f(x)y^k.$

General solution: $w=\displaystyle\int_{x_0}^{x}(y-ax+at)^k f(t)\,dt+\Phi(y-ax),$ where $x_0$ may be taken as arbitrary.

**5.** $\displaystyle \frac{\partial w}{\partial x}+a\frac{\partial w}{\partial y}=f(x)e^{\lambda y}.$

General solution: $w=e^{\lambda(y-ax)}\displaystyle\int f(x)e^{a\lambda x}\,dx+\Phi(y-ax).$

**6.** $\displaystyle \frac{\partial w}{\partial x}+[ay+f(x)]\frac{\partial w}{\partial y}=g(x).$

General solution: $w=\displaystyle\int g(x)\,dx+\Phi(u),$ where $u=e^{-ax}y-\displaystyle\int f(x)e^{-ax}\,dx.$

**7.** $\displaystyle \frac{\partial w}{\partial x}+\bigl[ay+f(x)\bigr]\frac{\partial w}{\partial y}=g(x)y^k.$

This is a special case of equation 1.2.7.19 with $h(y)=y^k$.

**8.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + y^k \dfrac{\partial w}{\partial y} = g(x).}$

General solution:

$$w = \int \frac{g(x)}{f(x)}\,dx + \Phi(u), \qquad \text{where} \quad u = \begin{cases} \dfrac{1}{k-1} y^{1-k} + \displaystyle\int \frac{dx}{f(x)} & \text{if} \quad k \neq 1, \\ y \exp\left[-\displaystyle\int \frac{dx}{f(x)}\right] & \text{if} \quad k = 1. \end{cases}$$

**9.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + (y+a)\dfrac{\partial w}{\partial y} = by + c.}$

General solution: $w = by + (c - ab)\ln|y+a| + \Phi(u)$, where $u = (y+a)\exp\left[-\displaystyle\int \frac{dx}{f(x)}\right]$.

**10.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + (y+ax)\dfrac{\partial w}{\partial y} = g(x).}$

General solution:

$$w = \int \frac{g(x)}{f(x)}\,dx + \Phi\left(e^{-z}y - a\int \frac{xe^{-z}}{f(x)}\,dx\right), \qquad \text{where} \quad z = \int \frac{dx}{f(x)}.$$

**11.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_0(x)\big]\dfrac{\partial w}{\partial y} = h_2(x)y^2 + h_1(x)y + h_0(x).}$

General solution:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x) + \Phi(u), \qquad u = e^{-G}y - \int e^{-G}\frac{g_0}{f}\,dx,$$

where

$$\varphi(x) = e^{-2G}\int e^{2G}\frac{h_2}{f}\,dx, \qquad G = G(x) = \int \frac{g_1}{f}\,dx,$$
$$\psi(x) = e^{-G}\int e^{G}\frac{h_1 - 2g_0\varphi}{f}\,dx, \quad \chi(x) = \int \frac{h_0 - g_0\psi}{f}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**12.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x)y + g_2(x)y^k\big]\dfrac{\partial w}{\partial y} = h(x).}$

General solution: $w = \displaystyle\int \frac{h(x)}{f(x)}\,dx + \Phi(u)$, where

$$u = e^{-G}y^{1-k} - (1-k)\int e^{-G}\frac{g_2(x)}{f(x)}\,dx, \qquad G = (1-k)\int \frac{g_1(x)}{f(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**13.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x} + \big[g_1(x) + g_2(x)e^{\lambda y}\big]\dfrac{\partial w}{\partial y} = h(x).}$

General solution: $w = \displaystyle\int \frac{h(x)}{f(x)}\,dx + \Phi(u)$, where

$$u = e^{-\lambda y}E(x) + \lambda\int \frac{g_2(x)}{f(x)}E(x)\,dx, \qquad E(x) = \exp\left[\lambda\int \frac{g_1(x)}{f(x)}\,dx\right].$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**14.** $\boldsymbol{f(x)y^k\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}=h(x).}$

General solution: $w=\Phi(u)+\displaystyle\int_{x_0}^{x}\frac{h(t)}{f(t)}\big[u+E(t)\big]^{-\frac{k}{k+1}}\,dt,$ where

$$u=y^{k+1}-E(x),\qquad E(x)=(k+1)\int\frac{g(x)}{f(x)}\,dx,\qquad x_0 \text{ may be taken as arbitrary.}$$

**15.** $\boldsymbol{f(x)y^k\dfrac{\partial w}{\partial x}+\big[g_1(x)y^{k+1}+g_0(x)\big]\dfrac{\partial w}{\partial y}}$
$\boldsymbol{=h_2(x)y^{3k+2}+h_1(x)y^{2k+1}+h_0(x)y^k.}$

The substitution $z=y^{k+1}$ leads to an equation of the form 1.2.7.11:

$$f(x)\frac{\partial w}{\partial x}+(k+1)\big[g_1(x)z+g_0(x)\big]\frac{\partial w}{\partial z}=h_2(x)z^2+h_1(x)z+h_0(x).$$

**16.** $\boldsymbol{f(x)e^{\lambda y}\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}=h(x).}$

General solution:

$$w=\Phi(u)+\int_{x_0}^{x}\frac{h(t)\,dt}{f(t)\big[u+E(t)\big]},\qquad u=e^{\lambda y}-E(x),\quad E(x)=\lambda\int\frac{g(x)}{f(x)}\,dx,$$

where $x_0$ may be taken as arbitrary.

### ► Equations contain arbitrary functions of $x$ and arbitrary functions of $y$.

**17.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(x)+g(y).}$

General solution: $w=\dfrac{1}{a}\displaystyle\int f(x)\,dx+\dfrac{1}{b}\int g(y)\,dy+\Phi(bx-ay).$

**18.** $\boldsymbol{\dfrac{\partial w}{\partial x}+a\dfrac{\partial w}{\partial y}=f(x)g(y).}$

General solution: $w=\displaystyle\int_{x_0}^{x}f(t)g(y-ax+at)\,dt+\Phi(y-ax),$ where $x_0$ may be taken as arbitrary.

**19.** $\boldsymbol{\dfrac{\partial w}{\partial x}+\big[ay+f(x)\big]\dfrac{\partial w}{\partial y}=g(x)h(y).}$

General solution:

$$w=\int g(x)\,h\biggl(e^{ax}u+e^{ax}\int f(x)e^{-ax}\,dx\biggr)\,dx+\Phi(u),\qquad u=e^{-ax}y-\int f(x)e^{-ax}\,dx.$$

In the integration, $u$ is considered a parameter.

**20.** $f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}=h_1(x)+h_2(y).$

General solution:

$$w=\int\frac{h_1(x)}{f(x)}dx+\int\frac{h_2(y)}{g(y)}dy+\Phi\biggl(\int\frac{dx}{f(x)}-\int\frac{dy}{g(y)}\biggr).$$

**21.** $f_1(x)\dfrac{\partial w}{\partial x}+\bigl[f_2(x)y+f_3(x)y^k\bigr]\dfrac{\partial w}{\partial y}=g(x)h(y).$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=y^{1-k}$ leads to an equation of the form 1.2.7.19:

$$\frac{\partial w}{\partial \xi}+\bigl[(1-k)\eta+F(\xi)\bigr]\frac{\partial w}{\partial \eta}=G(\xi)H(\eta),$$

where $F(\xi)=(1-k)\dfrac{f_3(x)}{f_2(x)},\ G(\xi)=\dfrac{g(x)}{f_2(x)},$ and $H(\eta)=h(y).$

**22.** $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h_1(x)h_2(y).$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.2.7.18:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi)G(\eta),\qquad \text{where}\quad F(\xi)=\frac{h_1(x)}{f_2(x)},\quad G(\eta)=\frac{h_2(y)}{g_1(y)}.$$

**23.** $f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h_1(x)+h_2(y).$

This is a special case of equation 1.2.7.38 with $h(x,y)=h_1(x)+h_2(y)$.

### ► Equations contain arbitrary functions of complicated arguments.

**24.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(\alpha x+\beta y).$

General solution:

$$w=\begin{cases}\dfrac{1}{a\alpha+b\beta}\displaystyle\int f(z)\,dz+\Phi(bx-ay) & \text{if } a\alpha+b\beta\neq 0,\\ \dfrac{1}{a}xf(\alpha x+\beta y)+\Phi(bx-ay) & \text{if } a\alpha+b\beta=0,\end{cases}$$

where $z=\alpha x+\beta y$.

**25.** $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=xf\Bigl(\dfrac{y}{x}\Bigr)+yg\Bigl(\dfrac{y}{x}\Bigr).$

General solution: $w=xf\Bigl(\dfrac{y}{x}\Bigr)+yg\Bigl(\dfrac{y}{x}\Bigr)+\Phi\Bigl(\dfrac{y}{x}\Bigr).$

**26.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = f(x^2+y^2).$

General solution: $\displaystyle w = \Phi\Big(\frac{y}{x}\Big) + \frac{1}{2}\int f(\xi)\frac{d\xi}{\xi}$, where $\xi = x^2+y^2$.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**27.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xf\Big(\frac{y}{x}\Big) + g(x^2+y^2).$

General solution: $\displaystyle w = \Phi\Big(\frac{y}{x}\Big) + xf\Big(\frac{y}{x}\Big) + \frac{1}{2}\int g(\xi)\frac{d\xi}{\xi}$, where $\xi = x^2+y^2$.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**28.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = x^k f(x^n y^m).$

General solution:

$$w = \begin{cases} \dfrac{1}{a}\displaystyle\int x^{k-1} f\Big(x^{\frac{an+bm}{a}} u^{\frac{m}{a}}\Big)\,dx + \Phi(u) & \text{if } an \neq -bm, \\ \dfrac{1}{ak} x^k f(x^n y^m) + \Phi(u) & \text{if } an = -bm,\ k\neq 0, \\ \dfrac{1}{a} f(x^n y^m)\ln|x| + \Phi(u) & \text{if } an = -bm,\ k = 0, \end{cases}$$

where $u = y^a x^{-b}$. In the integration, $u$ is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**29.** $\displaystyle mx\frac{\partial w}{\partial x} + ny\frac{\partial w}{\partial y} = f(ax^n + by^m).$

General solution: $\displaystyle w = \Phi\big(y^m x^{-n}\big) + \frac{1}{nm}\int f(\xi)\frac{d\xi}{\xi}$, where $\xi = ax^n + by^m$.

**30.** $\displaystyle x^2\frac{\partial w}{\partial x} + xy\frac{\partial w}{\partial y} = y^k f(\alpha x + \beta y).$

General solution: $\displaystyle w = \frac{y^k}{x(\alpha x+\beta y)^{k-1}}\int z^{k-2} f(z)\,dz + \Phi\Big(\frac{y}{x}\Big)$, where $z = \alpha x + \beta y$.

**31.** $\displaystyle \frac{f(x)}{f'(x)}\frac{\partial w}{\partial x} + \frac{g(y)}{g'(y)}\frac{\partial w}{\partial y} = h\big(f(x)+g(y)\big).$

General solution:

$$w = \Phi(u) + \int h(\xi)\frac{d\xi}{\xi}, \qquad \text{where} \quad u = \frac{g(y)}{f(x)}, \quad \xi = f(x) + g(y).$$

### ► Equations contain arbitrary functions of two variables.

**32.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x,y).$

General solution: $\displaystyle w = \int_{x_0}^{x} f(t, y - ax + at)\,dt + \Phi(y - ax)$, where $x_0$ may be taken as arbitrary.

**33.** $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(x,y).}$

General solution:

$$w=\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx+\Phi(u),\qquad \text{where}\quad u=y^a x^{-b}.$$

In the integration, $u$ is considered a parameter.

**34.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+g(x)y\dfrac{\partial w}{\partial y}=h(x,y).}$

General solution:

$$w=\Phi(u)+\int\frac{h(x,\,uG)}{f}\,dx,\qquad \text{where}\quad u=\frac{y}{G},\quad G=\exp\biggl(\int\frac{g}{f}\,dx\biggr).$$

In the integration, $u$ is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**35.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

General solution:

$$w=\Phi(u)+\int\frac{h(x,\,uG+Q)}{f}\,dx,\qquad u=\frac{y-Q}{G},$$

where $G=\exp\biggl(\displaystyle\int\frac{g_1}{f}\,dx\biggr)$ and $Q=G\displaystyle\int\frac{g_0\,dx}{fG}$. In the integration, $u$ is considered a parameter.

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**36.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

For $k=1$, see equation 1.2.7.34. For $k\neq 1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 1.2.7.35:

$$f(x)\frac{\partial w}{\partial x}+(1-k)\bigl[g_1(x)\xi+g_0(x)\bigr]\frac{\partial w}{\partial \xi}=h\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr).$$

⊙ *Literature*: V. F. Zaitsev and A. D. Polyanin (1996).

**37.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x,y).}$

The substitution $z=e^{-\lambda y}$ leads to an equation of the form 1.2.7.35:

$$f(x)\frac{\partial w}{\partial x}-\lambda\bigl[g_1(x)z+g_0(x)\bigr]\frac{\partial w}{\partial z}=h\biggl(x,\,-\frac{1}{\lambda}\ln z\biggr).$$

**38.** $\boldsymbol{f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h(x,y).}$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.2.7.32:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi,\eta),\qquad \text{where}\quad F(\xi,\eta)=\frac{h(x,y)}{f_2(x)g_1(y)}.$$

# 1.3 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h(x,y)w$

◆ *The solutions given below contain an arbitrary function* $\Phi=\Phi(z)$.

## 1.3.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in $x$ and $y$.**

**1.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw.$

Two forms of the representation of the general solution:

$$w=\exp\Bigl(\frac{c}{a}x\Bigr)\Phi(bx-ay),\qquad w=\exp\Bigl(\frac{c}{b}y\Bigr)\Phi(bx-ay).$$

**2.** $a\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=bw.$

General solution: $w=|y|^b\Phi\bigl(|y|^a e^{-x}\bigr)$.

⊙ *Literature*: E. Kamke (1965).

**3.** $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=aw.$

Differential equation for homogeneous functions of order $a$ with two independent variables. General solution: $w=x^a\Phi(y/x)$.

⊙ *Literature*: E. Kamke (1965).

**4.** $x\Bigl(a\dfrac{\partial w}{\partial x}-b\dfrac{\partial w}{\partial y}\Bigr)=cyw.$

General solution: $w=\exp\Bigl\{\dfrac{c}{a^2}\bigl[(bx+ay)\ln x-bx\bigr]\Bigr\}\Phi(bx+ay).$

**5.** $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=axw.$

General solution: $w=e^{ax}\Phi\Bigl(\dfrac{y}{x}\Bigr).$

**6.** $(x-a)\dfrac{\partial w}{\partial x}+(y-b)\dfrac{\partial w}{\partial y}=w.$

*Differential equation of a conic surface* with the vertex at the point $(a,b,0)$.

General solution: $w=(x-a)\Phi\Bigl(\dfrac{y-b}{x-a}\Bigr).$

**7.** $(y+ax)\dfrac{\partial w}{\partial x}+(y-ax)\dfrac{\partial w}{\partial y}=bw.$

General solution:

$$w=\xi^{\frac{b}{a+1}}\Phi\biggl(\ln\sqrt{\xi}+\frac{a+1}{2}\int\frac{dv}{v^2+(a-1)v+a}\biggr),$$

where $\xi=y^2+(a-1)xy+ax^2$ and $v=y/x$.

► **Coefficients of equations are quadratic in $x$ and $y$.**

**8.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = (x^2 - y^2)w.$

General solution: $\displaystyle w = \exp\left[\frac{1}{3ab}(bx^3 - ay^3)\right]\Phi(bx - ay).$

**9.** $\displaystyle x^2\frac{\partial w}{\partial x} + axy\frac{\partial w}{\partial y} = by^2 w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{b}{2a-1}\dfrac{y^2}{x}\right)\Phi(x^{-a}y) & \text{if } a \neq \frac{1}{2}, \\ \exp\left(b\dfrac{y^2}{x}\ln x\right)\Phi(x^{-1/2}y) & \text{if } a = \frac{1}{2}. \end{cases}$$

**10.** $\displaystyle ax^2\frac{\partial w}{\partial x} + by^2\frac{\partial w}{\partial y} = (x + cy)w.$

General solution: $\displaystyle w = x^{1/a}y^{c/b}\Phi\left(\frac{b}{x} - \frac{a}{y}\right).$

**11.** $\displaystyle x^2\frac{\partial w}{\partial x} + ay^2\frac{\partial w}{\partial y} = (bx^2 + cxy + dy^2)w.$

General solution: $\displaystyle w = \exp\left(\frac{dy^2 + abxy - bx^2}{ay - x} - \frac{cxy}{ay - x}\ln\left|\frac{x}{y}\right|\right)\Phi\left(\frac{x - ay}{xy}\right).$

**12.** $\displaystyle y^2\frac{\partial w}{\partial x} + ax^2\frac{\partial w}{\partial y} = (bx^2 + cy^2)w.$

General solution: $\displaystyle w = \exp\left(cx + \frac{b}{a}y\right)\Phi\bigl(ax^3 - y^3\bigr).$

**13.** $\displaystyle xy\frac{\partial w}{\partial x} + ay^2\frac{\partial w}{\partial y} = (bx + cy + d)w.$

General solution:

$$w = \begin{cases} x^c \exp\left[\dfrac{(1-a)d - abx}{a(a-1)y}\right]\Phi\bigl(x^{-a}y\bigr) & \text{if } a \neq 1, \\ \exp\left[\left(\dfrac{bx}{y} + c\right)\ln|x| - \dfrac{d}{y}\right]\Phi\left(\dfrac{y}{x}\right) & \text{if } a = 1. \end{cases}$$

**14.** $\displaystyle x(ay + b)\frac{\partial w}{\partial x} + (ay^2 - bx)\frac{\partial w}{\partial y} = ayw.$

General solution: $\displaystyle w = (x + y)\Phi\left(\frac{ax - b}{x + y} + a\ln\left|\frac{x + y}{x}\right|\right).$

**15.** $\displaystyle x(ky - x + a)\frac{\partial w}{\partial x} - y(kx - y + a)\frac{\partial w}{\partial y} = b(y - x)w.$

General solution: $\displaystyle w = (x + y - a)^b\Phi\left(\frac{(x + y - a)^k}{xy}\right).$

► **Coefficients of equations contain other power-law functions.**

**16.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = (cx^3 + dy^3)w.$

General solution: $w = \exp\left(\dfrac{bcx^4 + ady^4}{4ab}\right)\Phi(bx - ay).$

**17.** $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a\sqrt{x^2 + y^2}\,w.$

General solution: $w = \exp\left(a\sqrt{x^2 + y^2}\,\right)\Phi\left(\dfrac{y}{x}\right).$

**18.** $x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^2(ax + by)w.$

General solution: $w = \exp\left[\dfrac{(ax + by)y^2}{2x}\right]\Phi\left(\dfrac{y}{x}\right).$

**19.** $x^2y\dfrac{\partial w}{\partial x} + axy^2\dfrac{\partial w}{\partial y} = (bxy + cx + dy + k)w.$

General solution:

$$w = \begin{cases} x^b \exp\left[-\dfrac{k}{(a+1)xy} - \dfrac{d}{x} - \dfrac{c}{ay}\right]\Phi\left(x^{-a}y\right) & \text{if } a \neq -1, \\ \exp\left[\left(\dfrac{k}{xy} + b\right)\ln|x| + \dfrac{c}{y} - \dfrac{d}{y}\right]\Phi(xy) & \text{if } a = -1. \end{cases}$$

**20.** $axy^2\dfrac{\partial w}{\partial x} + bx^2y\dfrac{\partial w}{\partial y} = (any^2 + bmx^2)w.$

General solution: $w = x^n y^m \Phi(ay^2 - bx^2).$

**21.** $x^3\dfrac{\partial w}{\partial x} + ay^3\dfrac{\partial w}{\partial y} = x^2(bx + cy)w.$

General solution:

$$w = \exp\left(c\sqrt{\frac{x^2y^2}{x^2 - ay^2}}\,\ln\left|\sqrt{\frac{x^2}{y^2} - a} + \frac{x}{y}\right| + bx\right)\Phi\left(\frac{x^2 - ay^2}{x^2y^2}\right).$$

► **Coefficients of equations contain arbitrary powers of $x$ and $y$.**

**22.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = (cx^n + dy^m)w.$

General solution: $w = \Phi(bx - ay)\exp\left[\dfrac{c}{a(n+1)}x^{n+1} + \dfrac{d}{b(m+1)}y^{m+1}\right].$

**23.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cx^n yw.$

General solution:

$$w = \begin{cases} \exp\left\{\dfrac{c[a(n+2)y - bx]x^{n+1}}{a^2(n+1)(n+2)}\right\}\Phi(bx - ay) & \text{if } n \neq -1, -2; \\ \exp\left[\dfrac{bc}{a^2}x(1 - \ln x) + \dfrac{c}{a}y \ln x\right]\Phi(bx - ay) & \text{if } n = -1; \\ \exp\left[\dfrac{bc}{a^2}(1 + \ln x) - \dfrac{cy}{ax}\right]\Phi(bx - ay) & \text{if } n = -2. \end{cases}$$

**24.** $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = a(x^2 + y^2)^k w.$

General solution: $w = \exp\left[\dfrac{a}{2k}(x^2 + y^2)^k\right]\Phi\left(\dfrac{y}{x}\right).$

**25.** $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cx^n y^m w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{an + bm}x^n y^m\right)\Phi\bigl(y^a x^{-b}\bigr) & \text{if } an + bm \neq 0, \\ \exp\left(\dfrac{c}{a}x^n y^m \ln x\right)\Phi\bigl(y^a x^{-b}\bigr) & \text{if } an + bm = 0. \end{cases}$$

**26.** $ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = (cx^n + ky^m)w.$

General solution: $w = \exp\left(\dfrac{c}{an}x^n + \dfrac{k}{bm}y^m\right)\Phi\bigl(y^a x^{-b}\bigr).$

**27.** $mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = (ax^n + by^m)^k w.$

General solution: $w = \exp\left[\dfrac{1}{mnk}(ax^n + by^m)^k\right]\Phi\bigl(y^m x^{-n}\bigr).$

**28.** $ax^n\dfrac{\partial w}{\partial x} + by^m\dfrac{\partial w}{\partial y} = (cx^k + dy^s)w.$

This is a special case of equation 1.3.7.19. General solution:

$$w = \exp\left[\frac{cx^{k-n+1}}{a(k-n+1)} + \frac{dy^{s-m+1}}{b(s-m+1)}\right]\Phi(u), \qquad u = \frac{x^{1-n}}{a(1-n)} - \frac{y^{1-m}}{b(1-m)}.$$

**29.** $ax^n\dfrac{\partial w}{\partial x} + bx^m y\dfrac{\partial w}{\partial y} = \bigl(cx^k y^s + d\bigr)w.$

This is a special case of equation 1.3.7.32 with $f(x) = ax^n$, $g(x) = bx^m$, and $h(x,y) = cx^k y^s + d$.

**30.** $\boldsymbol{ax^n\frac{\partial w}{\partial x}+\bigl(bx^my+cx^k\bigr)\frac{\partial w}{\partial y}=\bigl(sx^py^q+d\bigr)w.}$

This is a special case of equation 1.3.7.33 with $f(x)=ax^n$, $g_1(x)=bx^m$, $g_0(x)=cx^k$, and $h(x,y)=sx^py^q+d$.

**31.** $\boldsymbol{ax^n\frac{\partial w}{\partial x}+bx^my^k\frac{\partial w}{\partial y}=\bigl(cx^py^q+s\bigr)w.}$

This is a special case of equation 1.3.7.34 with $f(x)=ax^n$, $g_1(x)\equiv 0$, $g_0(x)=bx^m$, and $h(x,y)=cx^py^q+s$.

**32.** $\boldsymbol{ay^k\frac{\partial w}{\partial x}+bx^n\frac{\partial w}{\partial y}=\bigl(cx^m+s\bigr)w.}$

This is a special case of equation 1.3.7.14 with $f(x)=a$, $g(x)=bx^n$, and $h(x)=cx^m+s$.

**33.** $\boldsymbol{x\bigl[x^n+(2n-1)y^n\bigr]\frac{\partial w}{\partial x}+y\bigl[y^n+(2n-1)x^n\bigr]\frac{\partial w}{\partial y}=kn(x^n+y^n)w.}$

General solution: $w=(x^n-y^n)^k\Phi\left(\dfrac{(x^n-y^n)^2}{xy}\right)$.

**34.** $\boldsymbol{x\bigl[(n-2)y^n-2x^n\bigr]\frac{\partial w}{\partial x}+y\bigl[2y^n-(n-2)x^n\bigr]\frac{\partial w}{\partial y}}$
$\boldsymbol{=\bigl\{[a(n-2)+2b]y^n-[2a+b(n-2)]x^n\bigr\}w.}$

General solution: $w=x^ay^b\Phi\left(\dfrac{x^n+y^n}{x^2y^2}\right)$.

## 1.3.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

**1.** $\boldsymbol{a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=ce^{\alpha x+\beta y}w.}$

General solution:

$$w=\begin{cases}\exp\left(\dfrac{c}{a\alpha+b\beta}e^{\alpha x+\beta y}\right)\Phi(bx-ay) & \text{if } a\alpha+b\beta\neq 0,\\ \exp\left(\dfrac{c}{a}xe^{\alpha x+\beta y}\right)\Phi(bx-ay) & \text{if } a\alpha+b\beta=0.\end{cases}$$

**2.** $\boldsymbol{a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\bigl(ce^{\lambda x}+ke^{\mu y}\bigr)w.}$

General solution: $w=\exp\left(\dfrac{c}{a\lambda}e^{\lambda x}+\dfrac{k}{b\mu}e^{\mu y}\right)\Phi(bx-ay)$.

**3.** $\boldsymbol{ae^{\lambda x}\frac{\partial w}{\partial x}+be^{\beta y}\frac{\partial w}{\partial y}=cw.}$

General solution: $w=\exp\left(-\dfrac{c}{a\lambda}e^{-\lambda x}\right)\Phi\bigl(b\beta e^{-\lambda x}-a\lambda e^{-\beta y}\bigr)$.

**4.** $\boldsymbol{ae^{\lambda y}\dfrac{\partial w}{\partial x}+be^{\beta x}\dfrac{\partial w}{\partial y}=cw.}$

General solution: $w=\exp\biggl[\dfrac{c(\beta x-\lambda y)}{a\beta e^{\lambda y}-b\lambda e^{\beta x}}\biggr]\Phi\bigl(a\beta e^{\lambda y}-b\lambda e^{\beta x}\bigr).$

**5.** $\boldsymbol{ae^{\lambda x}\dfrac{\partial w}{\partial x}+be^{\beta x}\dfrac{\partial w}{\partial y}=ce^{\gamma y}w.}$

This is a special case of equation 1.3.7.33 with $f(x)=ae^{\lambda x}$, $g_1(x)\equiv 0$, $g_0(x)=be^{\beta x}$, and $h(x,y)=ce^{\gamma y}$.

**6.** $\boldsymbol{ae^{\lambda x}\dfrac{\partial w}{\partial x}+be^{\beta y}\dfrac{\partial w}{\partial y}=\bigl(ce^{\gamma x}+se^{\delta y}\bigr)w.}$

This is a special case of equation 1.3.7.19 with $f(x)=ae^{\lambda x}$, $g(y)=be^{\beta y}$, $h_1(x)=ce^{\gamma x}$, and $h_2(y)=se^{\delta y}$.

**7.** $\boldsymbol{ae^{\beta x}\dfrac{\partial w}{\partial x}+\bigl(be^{\gamma x}+ce^{\lambda y}\bigr)\dfrac{\partial w}{\partial y}=\bigl(se^{\mu x}+ke^{\delta y}+p\bigr)w.}$

This is a special case of equation 1.3.7.35 with $f(x)=ae^{\beta x}$, $g_1(x)=be^{\gamma x}$, $g_0(x)=c$, and $h(x,y)=se^{\mu x}+ke^{\delta y}+p$.

**8.** $\boldsymbol{ae^{\beta x}\dfrac{\partial w}{\partial x}+\bigl(be^{\gamma x}+ce^{\lambda y}\bigr)\dfrac{\partial w}{\partial y}=\bigl(se^{\mu x+\delta y}+k\bigr)w.}$

This is a special case of equation 1.3.7.35 with $f(x)=ae^{\beta x}$, $g_1(x)=be^{\gamma x}$, $g_0(x)=c$, and $h(x,y)=se^{\mu x+\delta y}+k$.

**9.** $\boldsymbol{ae^{\beta x}\dfrac{\partial w}{\partial x}+be^{\gamma x+\lambda y}\dfrac{\partial w}{\partial y}=\bigl(ce^{\mu x+\delta y}+k\bigr)w.}$

This is a special case of equation 1.3.7.35 with $f(x)=ae^{\beta x}$, $g_1(x)\equiv 0$, $g_0(x)=be^{\gamma x}$, and $h(x,y)=ce^{\mu x+\delta y}+k$.

**10.** $\boldsymbol{ae^{\lambda y}\dfrac{\partial w}{\partial x}+be^{\beta x}\dfrac{\partial w}{\partial y}=\bigl(ce^{\mu x}+k\bigr)w.}$

This is a special case of equation 1.3.7.15 with $f(x)=a$, $g(x)=be^{\beta x}$, and $h(x)=ce^{\mu x}+k$.

► **Coefficients of equations contain exponential and power-law functions.**

**11.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=\bigl(cye^{\lambda x}+kxe^{\mu y}\bigr)w.}$

General solution: $w=\exp\biggl[\dfrac{c}{a\lambda}e^{\lambda x}\biggl(y-\dfrac{b}{a\lambda}\biggr)+\dfrac{k}{b\mu}e^{\mu y}\biggl(x-\dfrac{a}{b\mu}\biggr)\biggr]\Phi(bx-ay).$

**12.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=axe^{\lambda x+\mu y}w.}$

General solution: $w=\exp\biggl(\dfrac{ax}{\lambda x+\mu y}e^{\lambda x+\mu y}\biggr)\Phi\biggl(\dfrac{y}{x}\biggr).$

**13.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \big(aye^{\lambda x} + bxe^{\mu y}\big)w.$

General solution: $\displaystyle w = \exp\left(\frac{ay}{\lambda x}e^{\lambda x} + \frac{bx}{\mu y}e^{\mu y}\right)\Phi\left(\frac{y}{x}\right).$

**14.** $\displaystyle ax^k\frac{\partial w}{\partial x} + be^{\lambda y}\frac{\partial w}{\partial y} = \big(cx^n + s\big)w.$

This is a special case of equation 1.3.7.13 with $f(x) = ax^k$, $g_1(x) = 0$, $g_2(x) = b$, and $h(x) = cx^n + s$.

**15.** $\displaystyle ay^k\frac{\partial w}{\partial x} + be^{\lambda x}\frac{\partial w}{\partial y} = \big(ce^{\mu x} + s\big)w.$

This is a special case of equation 1.3.7.14 with $f(x) = a$, $g(x) = be^{\lambda x}$, and $h(x) = ce^{\mu x} + s$.

**16.** $\displaystyle ae^{\lambda x}\frac{\partial w}{\partial x} + by^k\frac{\partial w}{\partial y} = \big(cx^n + s\big)w.$

This is a special case of equation 1.3.7.12 with $f(x) = ae^{\lambda x}$, $g_1(x) = 0$, $g_2(x) = b$, and $h(x) = cx^n + s$.

**17.** $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x} + bx^k\frac{\partial w}{\partial y} = \big(ce^{\mu x} + s\big)w.$

This is a special case of equation 1.3.7.15 with $f(x) = a$, $g(x) = bx^k$, and $h(x) = ce^{\mu x} + s$.

### 1.3.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

**1.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c\sinh(\lambda x) + k\sinh(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[\frac{c}{a\lambda}\cosh(\lambda x) + \frac{k}{b\mu}\cosh(\mu y)\right]\Phi(bx - ay).$

**2.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\sinh(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c}{a\lambda + b\mu}\cosh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\sinh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**3.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\sinh(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left[\frac{ax}{\lambda x + \mu y}\cosh(\lambda x + \mu y)\right]\Phi\left(\frac{y}{x}\right).$

**4.** $\displaystyle a\frac{\partial w}{\partial x} + b\sinh^n(\lambda x)\frac{\partial w}{\partial y} = \big[c\sinh^m(\mu x) + s\sinh^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\sinh^n(\lambda x)$, and $h(x,y) = c\sinh^m(\mu x) + s\sinh^k(\beta y)$.

**5.** $\displaystyle a\frac{\partial w}{\partial x} + b\sinh^n(\lambda y)\frac{\partial w}{\partial y} = \big[c\sinh^m(\mu x) + s\sinh^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\sinh^n(\lambda y)$, $h_1(x) = c\sinh^m(\mu x)$, and $h_2(y) = s\sinh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic cosine.**

**6.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c\cosh(\lambda x) + k\cosh(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[\frac{c}{a\lambda}\sinh(\lambda x) + \frac{k}{b\mu}\sinh(\mu y)\right]\Phi(bx - ay).$

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cosh(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c}{a\lambda + b\mu}\sinh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\cosh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**8.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cosh(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left[\frac{ax}{\lambda x + \mu y}\sinh(\lambda x + \mu y)\right]\Phi\left(\frac{y}{x}\right).$

**9.** $\displaystyle a\frac{\partial w}{\partial x} + b\cosh^n(\lambda x)\frac{\partial w}{\partial y} = \big[c\cosh^m(\mu x) + s\cosh^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^n(\lambda x)$, and $h(x,y) = c\cosh^m(\mu x) + s\cosh^k(\beta y)$.

**10.** $\displaystyle a\frac{\partial w}{\partial x} + b\cosh^n(\lambda y)\frac{\partial w}{\partial y} = \big[c\cosh^m(\mu x) + s\cosh^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\cosh^n(\lambda y)$, $h_1(x) = c\cosh^m(\mu x)$, and $h_2(y) = s\cosh^k(\beta y)$.

► **Coefficients of equations contain hyperbolic tangent.**

**11.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c\tanh(\lambda x) + k\tanh(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[\frac{c}{a\lambda}\ln\cosh(\lambda x) + \frac{k}{b\mu}\ln\cosh(\mu y)\right]\Phi(bx - ay).$

**12.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\tanh(\lambda x + \mu y)w.}$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c}{a\lambda + b\mu}\ln\cosh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\tanh(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**13.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\tanh(\lambda x + \mu y)w.}$

General solution: $w = \exp\left[\dfrac{ax}{\lambda x + \mu y}\ln\cosh(\lambda x + \mu y)\right]\Phi\left(\dfrac{y}{x}\right).$

**14.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda x)\dfrac{\partial w}{\partial y} = \left[c\tanh^m(\mu x) + s\tanh^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^n(\lambda x)$, and $h(x,y) = c\tanh^m(\mu x) + s\tanh^k(\beta y)$.

**15.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\tanh^n(\lambda y)\dfrac{\partial w}{\partial y} = \left[c\tanh^m(\mu x) + s\tanh^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\tanh^n(\lambda y)$, $h_1(x) = c\tanh^m(\mu x)$, and $h_2(y) = s\tanh^k(\beta y)$.

### ► Coefficients of equations contain hyperbolic cotangent.

**16.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \left[c\coth(\lambda x) + k\coth(\mu y)\right]w.}$

General solution: $w = \exp\left(\dfrac{c}{a\lambda}\ln\left|\sinh(\lambda x)\right| + \dfrac{k}{b\mu}\ln\left|\sinh(\mu y)\right|\right)\Phi(bx - ay).$

**17.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\coth(\lambda x + \mu y)w.}$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{a\lambda + b\mu}\ln\left|\sinh(\lambda x + \mu y)\right|\right)\Phi(bx - ay) & \text{if } a\lambda + b\mu \neq 0, \\ \exp\left[\dfrac{c}{a}x\coth(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda + b\mu = 0. \end{cases}$$

**18.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\coth(\lambda x + \mu y)w.}$

General solution: $w = \exp\left(\dfrac{ax}{\lambda x + \mu y}\ln\left|\sinh(\lambda x + \mu y)\right|\right)\Phi\left(\dfrac{y}{x}\right).$

**19.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\coth^n(\lambda x)\dfrac{\partial w}{\partial y} = \left[c\coth^m(\mu x) + s\coth^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^n(\lambda x)$, and $h(x,y) = c\coth^m(\mu x) + s\coth^k(\beta y)$.

**20.** $\displaystyle a\frac{\partial w}{\partial x} + b\coth^n(\lambda y)\frac{\partial w}{\partial y} = \big[c\coth^m(\mu x) + s\coth^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\coth^n(\lambda y)$, $h_1(x) = c\coth^m(\mu x)$, and $h_2(y) = s\coth^k(\beta y)$.

► **Coefficients of equations contain different hyperbolic functions.**

**21.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[c\sinh(\lambda x) + k\cosh(\mu y)\big]w.$

General solution: $\displaystyle w = \exp\left[\frac{c}{a\lambda}\cosh(\lambda x) + \frac{k}{b\mu}\sinh(\mu y)\right]\Phi(bx - ay).$

**22.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \big[\tanh(\lambda x) + k\coth(\mu y)\big]w.$

General solution: $w = \cosh^{1/a\lambda}(\lambda x)\sinh^{k/b\mu}(\mu y)\Phi(bx - ay).$

**23.** $\displaystyle \frac{\partial w}{\partial x} + a\sinh(\mu y)\frac{\partial w}{\partial y} = b\cosh(\lambda x)w.$

General solution: $\displaystyle w = \exp\left[\frac{b}{\lambda}\sinh(\lambda x)\right]\Phi\left(a\mu x - \ln\left|\tanh\frac{\mu y}{2}\right|\right).$

**24.** $\displaystyle \frac{\partial w}{\partial x} + a\sinh(\mu y)\frac{\partial w}{\partial y} = b\tanh(\lambda x)w.$

General solution: $\displaystyle w = \cosh^{b/\lambda}(\lambda x)\Phi\left(a\mu x - \ln\left|\tanh\frac{\mu y}{2}\right|\right).$

**25.** $\displaystyle a\sinh(\lambda x)\frac{\partial w}{\partial x} + b\cosh(\mu y)\frac{\partial w}{\partial y} = w.$

General solution: $\displaystyle w = \tanh^{1/a\lambda}\left(\frac{\lambda x}{2}\right)\Phi\left(2a\arctan\left(\tanh\frac{\mu y}{2}\right) + \frac{b\mu}{\lambda}\ln\left|\coth\frac{\lambda x}{2}\right|\right).$

**26.** $\displaystyle a\tanh(\lambda x)\frac{\partial w}{\partial x} + b\coth(\mu y)\frac{\partial w}{\partial y} = w.$

General solution: $w = \sinh^{1/a\lambda}(\lambda x)\Phi\big(\cosh^{a\lambda}(\mu y)\sinh^{-b\mu}(\lambda x)\big).$

## 1.3.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

**1.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\ln(\lambda x + \beta y)w.$

General solution:

$$w = \begin{cases} \exp\left[\dfrac{c(\lambda x + \beta y)}{a\lambda + b\beta}\big(\ln(\lambda x + \beta y) - 1\big)\right]\Phi(bx - ay) & \text{if } a\lambda \neq -b\beta, \\ \exp\left[\dfrac{c}{a}x\ln(\lambda x + \beta y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\beta. \end{cases}$$

**2.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c\ln(\lambda x) + k\ln(\beta y)\bigr]w.$

General solution: $w = \exp\left[\dfrac{c}{a}x\bigl(\ln(\lambda x) - 1\bigr) + \dfrac{k}{b}y\bigl(\ln(\beta y) - 1\bigr)\right]\Phi(bx - ay).$

**3.** $\displaystyle a\frac{\partial w}{\partial x} + b\ln^n(\lambda x)\frac{\partial w}{\partial y} = \bigl[c\ln^m(\mu x) + s\ln^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^n(\lambda x)$, and $h(x,y) = c\ln^m(\mu x) + s\ln^k(\beta y)$.

**4.** $\displaystyle a\frac{\partial w}{\partial x} + b\ln^n(\lambda y)\frac{\partial w}{\partial y} = \bigl[c\ln^m(\mu x) + s\ln^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\ln^n(\lambda y)$, $h_1(x) = c\ln^m(\mu x)$, and $h_2(y) = s\ln^k(\beta y)$.

**5.** $\displaystyle \ln(\beta y)\frac{\partial w}{\partial x} + a\ln(\lambda x)\frac{\partial w}{\partial y} = bw\ln(\beta y).$

General solution: $w = e^{bx}\Phi(u)$, where $u = ax\bigl[1 - \ln(\lambda x)\bigr] + y\bigl[\ln(\beta y) - 1\bigr]$.

**6.** $\displaystyle a\ln^n(\lambda x)\frac{\partial w}{\partial x} + b\ln^k(\beta y)\frac{\partial w}{\partial y} = c\ln^m(\gamma x)w.$

General solution:

$$w = \Phi(u)\exp\left[\frac{c}{a}\int\frac{\ln^m(\gamma x)}{\ln^n(\lambda x)}\,dx\right], \qquad \text{where } u = b\int\frac{dx}{\ln^n(\lambda x)} - a\int\frac{dy}{\ln^k(\beta y)}.$$

► **Coefficients of equations contain logarithmic and power-law functions.**

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[cx^n + s\ln^k(\lambda y)\bigr]w.$

General solution: $w = \Phi(bx - ay)\exp\left[\dfrac{c}{a(n+1)}x^{n+1} + \dfrac{s}{b}\displaystyle\int\ln^k(\lambda y)\,dy\right].$

**8.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \bigl[by^2 + cx^n y + s\ln^k(\lambda x)\bigr]w.$

This is a special case of equation 1.3.7.3 with $f(x) = b$, $g(x) = cx^n$, and $h(x) = s\ln^k(\lambda x)$.

**9.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = b\ln^k(\lambda x)\ln^n(\beta y)w.$

This is a special case of equation 1.3.7.17 with $f(x) = b\ln^k(\lambda x)$ and $g(y) = \ln^n(\beta y)$.

**10.** $\displaystyle \frac{\partial w}{\partial x} + (ay + bx^n)\frac{\partial w}{\partial y} = c\ln^k(\lambda x)w.$

This is a special case of equation 1.3.7.6 with $f(x) = bx^n$ and $g(x) = c\ln^k(\lambda x)$.

**11.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = x^k(n\ln x + m\ln y)w.$

This is a special case of equation 1.3.7.26 with $f(u) = \ln u$.

**12.** $\displaystyle ax^k\frac{\partial w}{\partial x}+by^n\frac{\partial w}{\partial y}=\big[c\ln^m(\lambda x)+s\ln^l(\beta y)\big]w.$

General solution:

$$w=\Phi(u)\exp\left[\frac{c}{a}\int x^{-k}\ln^m(\lambda x)\,dx+\frac{s}{b}\int y^{-n}\ln^l(\beta y)\,dy\right],$$
$$u=\frac{b}{1-k}x^{1-k}-\frac{a}{1-n}y^{1-n}.$$

### 1.3.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

**1.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\sin(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[-\dfrac{c}{a\lambda+b\mu}\cos(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\sin(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

**2.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=\big[c\sin(\lambda x)+k\sin(\mu y)\big]w.$

General solution: $\displaystyle w=\exp\left[-\frac{c}{a\lambda}\cos(\lambda x)-\frac{k}{b\mu}\cos(\mu y)\right]\Phi(bx-ay).$

**3.** $\displaystyle x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=ax\sin(\lambda x+\mu y)w.$

General solution: $\displaystyle w=\exp\left[-\frac{ax}{\lambda x+\mu y}\cos(\lambda x+\mu y)\right]\Phi\left(\frac{y}{x}\right).$

**4.** $\displaystyle a\frac{\partial w}{\partial x}+b\sin^n(\lambda x)\frac{\partial w}{\partial y}=\big[c\sin^m(\mu x)+s\sin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x)=a$, $g_1(x)\equiv 0$, $g_0(x)=b\sin^n(\lambda x)$, and $h(x,y)=c\sin^m(\mu x)+s\sin^k(\beta y)$.

**5.** $\displaystyle a\frac{\partial w}{\partial x}+b\sin^n(\lambda y)\frac{\partial w}{\partial y}=\big[c\sin^m(\mu x)+s\sin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x)=a$, $g(y)=b\sin^n(\lambda y)$, $h_1(x)=c\sin^m(\mu x)$, and $h_2(y)=s\sin^k(\beta y)$.

► **Coefficients of equations contain cosine.**

**6.** $\displaystyle a\frac{\partial w}{\partial x}+b\frac{\partial w}{\partial y}=c\cos(\lambda x+\mu y)w.$

General solution:

$$w=\begin{cases}\exp\left[\dfrac{c}{a\lambda+b\mu}\sin(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu\neq 0,\\ \exp\left[\dfrac{c}{a}x\cos(\lambda x+\mu y)\right]\Phi(bx-ay) & \text{if } a\lambda+b\mu=0.\end{cases}$$

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c\cos(\lambda x) + k\cos(\mu y)\bigr]w.$

General solution: $\displaystyle w = \exp\left[\frac{c}{a\lambda}\sin(\lambda x) + \frac{k}{b\mu}\sin(\mu y)\right]\Phi(bx - ay).$

**8.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cos(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left[\frac{ax}{\lambda x + \mu y}\sin(\lambda x + \mu y)\right]\Phi\left(\frac{y}{x}\right).$

**9.** $\displaystyle a\frac{\partial w}{\partial x} + b\cos^n(\lambda x)\frac{\partial w}{\partial y} = \bigl[c\cos^m(\mu x) + s\cos^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cos^n(\lambda x)$, and $h(x,y) = c\cos^m(\mu x) + s\cos^k(\beta y)$.

**10.** $\displaystyle a\frac{\partial w}{\partial x} + b\cos^n(\lambda y)\frac{\partial w}{\partial y} = \bigl[c\cos^m(\mu x) + s\cos^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\cos^n(\lambda y)$, $h_1(x) = c\cos^m(\mu x)$, and $h_2(y) = s\cos^k(\beta y)$.

► **Coefficients of equations contain tangent.**

**11.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\tan(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left(-\dfrac{c}{a\lambda + b\mu}\ln\bigl|\cos(\lambda x + \mu y)\bigr|\right)\Phi(bx - ay) & \text{if } a\lambda \neq -b\mu, \\ \exp\left[\dfrac{c}{a}x\tan(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\mu. \end{cases}$$

**12.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c\tan(\lambda x) + k\tan(\mu y)\bigr]w.$

General solution: $\displaystyle w = \exp\left(-\frac{c}{a\lambda}\ln\bigl|\cos(\lambda x)\bigr| - \frac{k}{b\mu}\ln\bigl|\cos(\mu y)\bigr|\right)\Phi(bx - ay).$

**13.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\tan(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left(-\frac{ax}{\lambda x + \mu y}\ln\bigl|\cos(\lambda x + \mu y)\bigr|\right)\Phi\left(\frac{y}{x}\right).$

**14.** $\displaystyle a\frac{\partial w}{\partial x} + b\tan^n(\lambda x)\frac{\partial w}{\partial y} = \bigl[c\tan^m(\mu x) + s\tan^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\tan^n(\lambda x)$, and $h(x,y) = c\tan^m(\mu x) + s\tan^k(\beta y)$.

**15.** $\displaystyle a\frac{\partial w}{\partial x} + b\tan^n(\lambda y)\frac{\partial w}{\partial y} = \bigl[c\tan^m(\mu x) + s\tan^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\tan^n(\lambda y)$, $h_1(x) = c\tan^m(\mu x)$, and $h_2(y) = s\tan^k(\beta y)$.

► **Coefficients of equations contain cotangent.**

**16.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cot(\lambda x + \mu y)w.$

General solution:

$$w = \begin{cases} \exp\left(\dfrac{c}{a\lambda + b\mu}\ln\bigl|\sin(\lambda x + \mu y)\bigr|\right)\Phi(bx - ay) & \text{if } a\lambda \neq -b\mu, \\ \exp\left[\dfrac{c}{a}x\cot(\lambda x + \mu y)\right]\Phi(bx - ay) & \text{if } a\lambda = -b\mu. \end{cases}$$

**17.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c\cot(\lambda x) + k\cot(\mu y)\bigr]w.$

General solution: $\displaystyle w = \exp\left(\frac{c}{a\lambda}\ln\bigl|\sin(\lambda x)\bigr| + \frac{k}{b\mu}\ln\bigl|\sin(\mu y)\bigr|\right)\Phi(bx - ay).$

**18.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cot(\lambda x + \mu y)w.$

General solution: $\displaystyle w = \exp\left(\frac{ax}{\lambda x + \mu y}\ln\bigl|\sin(\lambda x + \mu y)\bigr|\right)\Phi\left(\frac{y}{x}\right).$

**19.** $\displaystyle a\frac{\partial w}{\partial x} + b\cot^n(\lambda x)\frac{\partial w}{\partial y} = \bigl[c\cot^m(\mu x) + s\cot^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\cot^n(\lambda x)$, and $h(x, y) = c\cot^m(\mu x) + s\cot^k(\beta y)$.

**20.** $\displaystyle a\frac{\partial w}{\partial x} + b\cot^n(\lambda y)\frac{\partial w}{\partial y} = \bigl[c\cot^m(\mu x) + s\cot^k(\beta y)\bigr]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\cot^n(\lambda y)$, $h_1(x) = c\cot^m(\mu x)$, and $h_2(y) = s\cot^k(\beta y)$.

► **Coefficients of equations contain different trigonometric functions.**

**21.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \bigl[b\sin(\lambda x) + k\cos(\mu y)\bigr]w.$

General solution: $\displaystyle w = \exp\left[\frac{k}{a\mu}\sin(\mu y) - \frac{b}{\lambda}\cos(\lambda x)\right]\Phi(ax - y).$

**22.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \bigl[b\sin(\lambda x) + k\tan(\mu y)\bigr]w.$

General solution: $\displaystyle w = \exp\left[-\frac{b}{\lambda}\cos(\lambda x)\right]\cos^{k/a\mu}(\mu y)\Phi(ax - y).$

**23.** $\displaystyle \frac{\partial w}{\partial x} + a\sin(\mu y)\frac{\partial w}{\partial y} = bw\tan(\lambda x).$

General solution: $\displaystyle w = \cos^{-b/\lambda}(\lambda x)\Phi\left(a\mu x - \ln\left|\tan\frac{\mu y}{2}\right|\right).$

**24.** $\dfrac{\partial w}{\partial x} + a\tan(\mu y)\dfrac{\partial w}{\partial y} = bw\sin(\lambda x).$

General solution: $w = \exp\left[-\dfrac{b}{\lambda}\cos(\lambda x)\right]\Phi\Big(a\mu x - \ln\big|\sin(\mu y)\big|\Big).$

**25.** $\sin(\lambda x)\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = bw\cos(\mu y).$

General solution: $w = \exp\left[\dfrac{b}{a\mu}\sin(\mu y)\right]\Phi\left(\lambda y + b\ln\left|\cot\dfrac{\lambda x}{2}\right|\right).$

**26.** $\cot(\lambda x)\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = bw\tan(\mu y).$

General solution: $w = \cos^{-b/a\mu}(\mu y)\Phi\Big(\lambda y + b\ln\big|\cos(\lambda x)\big|\Big).$

### 1.3.6 Equations Containing Inverse Trigonometric Functions

► **Coefficients of equations contain arcsine.**

**1.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \left(c\arcsin\dfrac{x}{\lambda} + k\arcsin\dfrac{y}{\beta}\right)w.$

General solution:

$$w = \exp\left[\frac{c}{a}\left(x\arcsin\frac{x}{\lambda} + \sqrt{\lambda^2 - x^2}\right) + \frac{k}{b}\left(y\arcsin\frac{y}{\beta} + \sqrt{\beta^2 - y^2}\right)\right]\Phi(bx - ay).$$

**2.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arcsin(\lambda x + \beta y)w.$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\left[\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\arcsin(\lambda x + \beta y) + \frac{\sqrt{1 - (\lambda x + \beta y)^2}}{a\lambda + b\beta}\right]\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\left[\frac{c}{a}x\arcsin(\lambda x + \beta y)\right]\Phi(bx - ay).$$

**3.** $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\arcsin(\lambda x + \beta y)w.$

General solution: $w = \exp\left[ax\arcsin(\lambda x + \beta y) + ax\dfrac{\sqrt{1 - (\lambda x + \beta y)^2}}{\lambda x + \beta y}\right]\Phi\left(\dfrac{y}{x}\right).$

**4.** $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda x)\dfrac{\partial w}{\partial y} = \big[c\arcsin^m(\mu x) + s\arcsin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arcsin^n(\lambda x)$, and $h(x,y) = c\arcsin^m(\mu x) + s\arcsin^k(\beta y)$.

**5.** $a\dfrac{\partial w}{\partial x} + b\arcsin^n(\lambda y)\dfrac{\partial w}{\partial y} = \big[c\arcsin^m(\mu x) + s\arcsin^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\arcsin^n(\lambda y)$, $h_1(x) = c\arcsin^m(\mu x)$, and $h_2(y) = s\arcsin^k(\beta y)$.

► **Coefficients of equations contain arccosine.**

**6.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \Big(c \arccos\frac{x}{\lambda} + k \arccos\frac{y}{\beta}\Big)w.$

General solution:

$$w = \exp\left[\frac{c}{a}\Big(x \arccos\frac{x}{\lambda} - \sqrt{\lambda^2 - x^2}\Big) + \frac{k}{b}\Big(y \arccos\frac{y}{\beta} - \sqrt{\beta^2 - y^2}\Big)\right]\Phi(bx - ay).$$

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c \arccos(\lambda x + \beta y)w.$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\left[\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\arccos(\lambda x + \beta y) - \frac{\sqrt{1 - (\lambda x + \beta y)^2}}{a\lambda + b\beta}\right]\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\left[\frac{c}{a}x \arccos(\lambda x + \beta y)\right]\Phi(bx - ay).$$

**8.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax \arccos(\lambda x + \beta y)w.$

General solution: $\displaystyle w = \exp\left[ax \arccos(\lambda x + \beta y) - ax\frac{\sqrt{1 - (\lambda x + \beta y)^2}}{\lambda x + \beta y}\right]\Phi\Big(\frac{y}{x}\Big).$

**9.** $\displaystyle a\frac{\partial w}{\partial x} + b \arccos^n(\lambda x)\frac{\partial w}{\partial y} = \big[c \arccos^m(\mu x) + s \arccos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b \arccos^n(\lambda x)$, and $h(x, y) = c \arccos^m(\mu x) + s \arccos^k(\beta y)$.

**10.** $\displaystyle a\frac{\partial w}{\partial x} + b \arccos^n(\lambda y)\frac{\partial w}{\partial y} = \big[c \arccos^m(\mu x) + s \arccos^k(\beta y)\big]w.$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b \arccos^n(\lambda y)$, $h_1(x) = c \arccos^m(\mu x)$, and $h_2(y) = s \arccos^k(\beta y)$.

► **Coefficients of equations contain arctangent.**

**11.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \Big(c \arctan\frac{x}{\lambda} + k \arctan\frac{y}{\beta}\Big)w.$

General solution:

$$w = \exp\left\{\frac{c}{a}\Big[x \arctan\frac{x}{\lambda} - \frac{\lambda}{2}\ln\big(\lambda^2 + x^2\big)\Big] + \frac{k}{b}\Big[y \arctan\frac{y}{\beta} - \frac{\beta}{2}\ln\big(\beta^2 + y^2\big)\Big]\right\}\Phi(bx - ay).$$

**12.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\arctan(\lambda x + \beta y)w.}$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\left\{\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\arctan(\lambda x + \beta y) - \frac{\ln[1 + (\lambda x + \beta y)^2]}{2(a\lambda + b\beta)}\right\}\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\left[\frac{c}{a}x\arctan(\lambda x + \beta y)\right]\Phi(bx - ay).$$

**13.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\arctan(\lambda x + \beta y)w.}$

General solution:

$$w = \exp\left\{ax\arctan(\lambda x + \beta y) - \frac{ax}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\}\Phi\left(\frac{y}{x}\right).$$

**14.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arctan^n(\lambda x)\dfrac{\partial w}{\partial y} = \left[c\arctan^m(\mu x) + s\arctan^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\arctan^n(\lambda x)$, and $h(x,y) = c\arctan^m(\mu x) + s\arctan^k(\beta y)$.

**15.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\arctan^n(\lambda y)\dfrac{\partial w}{\partial y} = \left[c\arctan^m(\mu x) + s\arctan^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\arctan^n(\lambda y)$, $h_1(x) = c\arctan^m(\mu x)$, and $h_2(y) = s\arctan^k(\beta y)$.

► **Coefficients of equations contain arccotangent.**

**16.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \left(c\,\mathrm{arccot}\,\frac{x}{\lambda} + k\,\mathrm{arccot}\,\frac{y}{\beta}\right)w.}$

General solution:

$$w = \exp\left\{\frac{c}{a}\left[x\,\mathrm{arccot}\,\frac{x}{\lambda} + \frac{\lambda}{2}\ln\left(\lambda^2 + x^2\right)\right] + \frac{k}{b}\left[y\,\mathrm{arccot}\,\frac{y}{\beta} + \frac{\beta}{2}\ln\left(\beta^2 + y^2\right)\right]\right\}\Phi(bx - ay).$$

**17.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\,\mathrm{arccot}(\lambda x + \beta y)w.}$

1°. General solution for $a\lambda + b\beta \neq 0$:

$$w = \exp\left\{\frac{c(\lambda x + \beta y)}{a\lambda + b\beta}\mathrm{arccot}(\lambda x + \beta y) + \frac{\ln[1 + (\lambda x + \beta y)^2]}{2(a\lambda + b\beta)}\right\}\Phi(bx - ay).$$

2°. General solution for $a\lambda + b\beta = 0$:

$$w = \exp\left[\frac{c}{a}x\,\mathrm{arccot}(\lambda x + \beta y)\right]\Phi(bx - ay).$$

**18.** $\boldsymbol{x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\,\mathrm{arccot}(\lambda x + \beta y)w.}$

General solution:

$$w = \exp\left\{ax\,\mathrm{arccot}(\lambda x + \beta y) + \frac{ax}{2(\lambda x + \beta y)}\ln\left[x^2 + \frac{x^2}{(\lambda x + \beta y)^2}\right]\right\}\Phi\left(\frac{y}{x}\right).$$

**19.** $\boldsymbol{a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda x)\frac{\partial w}{\partial y} = \left[c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.33 with $f(x) = a$, $g_1(x) \equiv 0$, $g_0(x) = b\,\mathrm{arccot}^n(\lambda x)$, and $h(x,y) = c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)$.

**20.** $\boldsymbol{a\frac{\partial w}{\partial x} + b\,\mathrm{arccot}^n(\lambda y)\frac{\partial w}{\partial y} = \left[c\,\mathrm{arccot}^m(\mu x) + s\,\mathrm{arccot}^k(\beta y)\right]w.}$

This is a special case of equation 1.3.7.19 with $f(x) = a$, $g(y) = b\,\mathrm{arccot}^n(\lambda y)$, $h_1(x) = c\,\mathrm{arccot}^m(\mu x)$, and $h_2(y) = s\,\mathrm{arccot}^k(\beta y)$.

## 1.3.7 Equations Containing Arbitrary Functions

► **Coefficients of equations contain arbitrary functions of $x$.**

**1.** $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = f(x)w.}$

General solution: $w = \exp\left[\frac{1}{a}\int f(x)\,dx\right]\Phi(bx - ay)$.

**2.** $\boldsymbol{\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x)yw.}$

General solution: $w = \exp\left[\int_{x_0}^{x}(y - ax + at)f(t)\,dt\right]\Phi(y - ax)$, where $x_0$ may be chosen arbitrarily.

**3.** $\boldsymbol{\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = \left[f(x)y^2 + g(x)y + h(x)\right]w.}$

General solution:

$$w = \exp\left[\varphi(x)y^2 + \psi(x)y + \chi(x)\right]\Phi(y - ax),$$

where

$$\varphi(x) = \int f(x)\,dx, \quad \psi(x) = \int\left[g(x) - 2a\varphi(x)\right]dx, \quad \chi(x) = \int\left[h(x) - a\psi(x)\right]dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**4.** $\boldsymbol{\frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x)y^k w.}$

General solution: $w = \exp\left[\int_{x_0}^{x}(y - ax + at)^k f(t)\,dt\right]\Phi(y - ax)$, where $x_0$ can be chosen arbitrarily.

**5.** $\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x)e^{\lambda y}w.$

General solution: $w = \exp\left[e^{\lambda(y-ax)}\int f(x)e^{a\lambda x}\,dx\right]\Phi(y-ax).$

**6.** $\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)w.$

General solution: $w = \exp\left[\int g(x)\,dx\right]\Phi(u)$, where $u = e^{-ax}y - \int f(x)e^{-ax}\,dx.$

**7.** $\dfrac{\partial w}{\partial x} + \bigl[ay + f(x)\bigr]\dfrac{\partial w}{\partial y} = g(x)y^k w.$

This is a special case of equation 1.3.7.18 with $h(y) = y^k$.

**8.** $f(x)\dfrac{\partial w}{\partial x} + y^k\dfrac{\partial w}{\partial y} = g(x)w.$

General solution:

$$w = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right]\Phi(u), \qquad \text{where} \quad u = \begin{cases} \dfrac{1}{k-1}y^{1-k} + \displaystyle\int \frac{dx}{f(x)} & \text{if } k \neq 1, \\ y\exp\left[-\displaystyle\int \frac{dx}{f(x)}\right] & \text{if } k = 1. \end{cases}$$

**9.** $f(x)\dfrac{\partial w}{\partial x} + (y+a)\dfrac{\partial w}{\partial y} = (by+c)w.$

General solution: $w = (y+a)^{c-ab}e^{by}\Phi(u)$, where $u = (y+a)\exp\left[-\int \frac{dx}{f(x)}\right].$

**10.** $f(x)\dfrac{\partial w}{\partial x} + (y+ax)\dfrac{\partial w}{\partial y} = g(x)w.$

General solution:

$$w = \exp\left[\int \frac{g(x)}{f(x)}\,dx\right]\Phi\left(e^{-z}y - a\int \frac{xe^{-z}}{f(x)}\,dx\right), \qquad \text{where} \quad z = \int \frac{dx}{f(x)}.$$

**11.** $f(x)\dfrac{\partial w}{\partial x} + \bigl[g_1(x)y + g_0(x)\bigr]\dfrac{\partial w}{\partial y} = \bigl[h_2(x)y^2 + h_1(x)y + h_0(x)\bigr]w.$

General solution:

$$w = \exp\bigl[\varphi(x)y^2 + \psi(x)y + \chi(x)\bigr]\Phi(u), \qquad u = e^{-G}y - \int e^{-G}\frac{g_0}{f}\,dx,$$

where

$$\varphi(x) = e^{-2G}\int e^{2G}\frac{h_2}{f}\,dx, \qquad G = G(x) = \int \frac{g_1}{f}\,dx,$$
$$\psi(x) = e^{-G}\int e^{G}\frac{h_1 - 2g_0\varphi}{f}\,dx, \quad \chi(x) = \int \frac{h_0 - g_0\psi}{f}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**12.** $\displaystyle f(x)\frac{\partial w}{\partial x} + \left[g_1(x)y + g_2(x)y^k\right]\frac{\partial w}{\partial y} = h(x)w.$

General solution: $\displaystyle w = \exp\left[\int \frac{h(x)}{f(x)}\,dx\right]\Phi(u)$, where

$$u = e^{-G}y^{1-k} - (1-k)\int e^{-G}\frac{g_2(x)}{f(x)}\,dx, \qquad G = (1-k)\int \frac{g_1(x)}{f(x)}\,dx.$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**13.** $\displaystyle f(x)\frac{\partial w}{\partial x} + \left[g_1(x) + g_2(x)e^{\lambda y}\right]\frac{\partial w}{\partial y} = h(x)w.$

General solution: $\displaystyle w = \exp\left[\int \frac{h(x)}{f(x)}\,dx\right]\Phi(u)$, where

$$u = e^{-\lambda y}E(x) + \lambda\int \frac{g_2(x)}{f(x)}E(x)\,dx, \qquad E(x) = \exp\left[\lambda\int \frac{g_1(x)}{f(x)}\,dx\right].$$

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**14.** $\displaystyle f(x)y^k\frac{\partial w}{\partial x} + g(x)\frac{\partial w}{\partial y} = h(x)w.$

General solution: $\displaystyle w = \Phi(u)\exp\left\{\int_{x_0}^{x} \frac{h(t)}{f(t)}\left[u + E(t)\right]^{-\frac{k}{k+1}}\,dt\right\}$, where

$\displaystyle u = y^{k+1} - E(x), \qquad E(x) = (k+1)\int \frac{g(x)}{f(x)}\,dx,$ where $x_0$ may be chosen arbitrarily.

**15.** $\displaystyle f(x)e^{\lambda y}\frac{\partial w}{\partial x} + g(x)\frac{\partial w}{\partial y} = h(x)w.$

General solution:

$$w = \Phi(u)\exp\left\{\int_{x_0}^{x} \frac{h(t)\,dt}{f(t)\left[u + E(t)\right]}\right\}, \qquad u = e^{\lambda y} - E(x), \qquad E(x) = \lambda\int \frac{g(x)}{f(x)}\,dx,$$

where $x_0$ may be chosen arbitrarily.

► **Equations contain arbitrary functions of $x$ and arbitrary functions of $y$.**

**16.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \left[f(x) + g(y)\right]w.$

General solution: $\displaystyle w = \exp\left[\frac{1}{a}\int f(x)\,dx + \frac{1}{b}\int g(y)\,dy\right]\Phi(bx - ay).$

**17.** $\displaystyle \frac{\partial w}{\partial x} + a\frac{\partial w}{\partial y} = f(x)g(y)w.$

General solution: $\displaystyle w = \exp\left[\int_{x_0}^{x} f(t)g(y - ax + at)\,dt\right]\Phi(y - ax)$, where $x_0$ may be taken as arbitrary.

**18.** $\boldsymbol{\dfrac{\partial w}{\partial x}+[ay+f(x)]\dfrac{\partial w}{\partial y}=g(x)h(y)w.}$

The substitutions $w=\pm e^u$ lead to an equation of the form 1.2.7.19:

$$\frac{\partial u}{\partial x}+[ay+f(x)]\frac{\partial u}{\partial y}=g(x)h(y).$$

**19.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+g(y)\dfrac{\partial w}{\partial y}=\big[h_1(x)+h_2(y)\big]w.}$

General solution:

$$w=\exp\biggl[\int\frac{h_1(x)}{f(x)}\,dx+\int\frac{h_2(y)}{g(y)}\,dy\biggr]\Phi\biggl(\int\frac{dx}{f(x)}\,dx-\int\frac{dy}{g(y)}\,dy\biggr).$$

**20.** $\boldsymbol{f_1(x)\dfrac{\partial w}{\partial x}+\big[f_2(x)y+f_3(x)y^k\big]\dfrac{\partial w}{\partial y}=g(x)h(y)w.}$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=y^{1-k}$ leads to an equation of the form 1.3.7.18:

$$\frac{\partial w}{\partial \xi}+\big[(1-k)\eta+F(\xi)\big]\frac{\partial w}{\partial \eta}=G(\xi)H(\eta)w,$$

where $F(\xi)=(1-k)\dfrac{f_3(x)}{f_2(x)},\ G(\xi)=\dfrac{g(x)}{f_2(x)},$ and $H(\eta)=h(y).$

**21.** $\boldsymbol{f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h_1(x)h_2(y)w.}$

The transformation $\xi=\displaystyle\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.3.7.17:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi)G(\eta)w,\qquad \text{where}\quad F(\xi)=\frac{h_1(x)}{f_2(x)},\quad G(\eta)=\frac{h_2(y)}{g_1(y)}.$$

**22.** $\boldsymbol{f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=\big[h_1(x)+h_2(y)\big]w.}$

This is a special case of equation 1.3.7.36 with $h(x,y)=h_1(x)+h_2(y)$.

### ► Equations contain arbitrary functions of complicated arguments.

**23.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=f(\alpha x+\beta y)w.}$

General solution:

$$w=\begin{cases}\exp\biggl[\dfrac{1}{a\alpha+b\beta}\displaystyle\int f(u)\,du\biggr]\Phi(bx-ay) & \text{if } a\alpha+b\beta\neq 0,\\ \exp\biggl[\dfrac{1}{a}xf(\alpha x+\beta y)\biggr]\Phi(bx-ay) & \text{if } a\alpha+b\beta=0,\end{cases}$$

where $u=\alpha x+\beta y$.

**24.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = xf\Big(\dfrac{y}{x}\Big)w.}$

General solution: $w = \exp\Big[xf\Big(\dfrac{y}{x}\Big)\Big]\Phi\Big(\dfrac{y}{x}\Big).$

**25.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = f(x^2+y^2)w.}$

General solution: $w = \Phi\Big(\dfrac{y}{x}\Big)\exp\Big[\dfrac{1}{2}\displaystyle\int f(\xi)\dfrac{d\xi}{\xi}\Big]$, where $\xi = x^2+y^2$.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**26.** $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = x^k f(x^n y^m)w.}$

General solution:

$$w = \begin{cases} \exp\Big[\dfrac{1}{a}\displaystyle\int x^{k-1} f\Big(x^{\frac{an+bm}{a}}u^{\frac{m}{a}}\Big)\,dx\Big]\Phi(u) & \text{if } an \neq -bm; \\ \exp\Big[\dfrac{1}{ak}x^k f(x^n y^m)\Big]\Phi(u) & \text{if } an = -bm,\ k\neq 0; \\ \exp\Big[\dfrac{1}{a}f(x^n y^m)\ln x\Big]\Phi(u) & \text{if } an = -bm,\ k = 0, \end{cases}$$

where $u = y^a x^{-b}$. In the integration, $u$ is considered a parameter.

**27.** $\boldsymbol{mx\dfrac{\partial w}{\partial x} + ny\dfrac{\partial w}{\partial y} = f(ax^n+by^m)w.}$

General solution: $w = \Phi\big(y^m x^{-n}\big)\exp\Big[\dfrac{1}{nm}\displaystyle\int f(\xi)\dfrac{d\xi}{\xi}\Big]$, where $\xi = ax^n+by^m$.

**28.** $\boldsymbol{x^2\dfrac{\partial w}{\partial x} + xy\dfrac{\partial w}{\partial y} = y^k f(\alpha x+\beta y)w.}$

General solution:

$$w = \exp\Big[\frac{y^k}{x(\alpha x+\beta y)^{k-1}}\int z^{k-2}f(z)\,dz\Big]\Phi\Big(\frac{y}{x}\Big), \qquad \text{where} \quad z = \alpha x+\beta y.$$

**29.** $\boldsymbol{\dfrac{f(x)}{f'(x)}\dfrac{\partial w}{\partial x} + \dfrac{g(y)}{g'(y)}\dfrac{\partial w}{\partial y} = h\big(f(x)+g(y)\big)w.}$

General solution:

$$w = \Phi(u)\exp\Big[\int h(\xi)\frac{d\xi}{\xi}\Big], \qquad \text{where} \quad u = \frac{g(y)}{f(x)}, \quad \xi = f(x)+g(y).$$

► **Equations contain arbitrary functions of two variables.**

**30.** $\boldsymbol{\dfrac{\partial w}{\partial x} + a\dfrac{\partial w}{\partial y} = f(x,y)w.}$

General solution: $w = \exp\Big[\displaystyle\int_{x_0}^{x} f(t, y-ax+at)\,dt\Big]\Phi(y-ax)$, where $x_0$ may be taken as arbitrary.

**31.** $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=f(x,y)w.}$

General solution:

$$w=\exp\left[\frac{1}{a}\int\frac{1}{x}f\bigl(x,\,u^{1/a}x^{b/a}\bigr)\,dx\right]\Phi(u),\qquad \text{where}\quad u=y^a x^{-b}.$$

In the integration, $u$ is considered a parameter.

**32.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+g(x)y\dfrac{\partial w}{\partial y}=h(x,y)w.}$

General solution:

$$w=\Phi(u)\exp\left[\int\frac{h(x,\,uG)}{f(x)}\,dx\right],\qquad \text{where}\quad u=\frac{y}{G},\quad G=\exp\left(\int\frac{g}{f}\,dx\right).$$

In the integration, $u$ is considered a parameter.

**33.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.}$

General solution:

$$w=\Phi(u)\exp\left[\int\frac{h(x,\,uG+Q)}{f(x)}\,dx\right],\quad u=\frac{y-Q}{G},$$

where $G=\exp\left(\int\frac{g_1}{f}\,dx\right)$ and $Q=G\int\frac{g_0\,dx}{fG}$. In the integration, $u$ is considered a parameter.

⊙ *Literature*: A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

**34.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)y+g_0(x)y^k\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.}$

For $k=1$, see equation 1.3.7.32. For $k\neq 1$, the substitution $\xi=y^{1-k}$ leads to an equation of the form 1.3.7.33:

$$f(x)\frac{\partial w}{\partial x}+(1-k)\bigl[g_1(x)\xi+g_0(x)\bigr]\frac{\partial w}{\partial \xi}=h\Bigl(x,\,\xi^{\frac{1}{1-k}}\Bigr)w.$$

**35.** $\boldsymbol{f(x)\dfrac{\partial w}{\partial x}+\bigl[g_1(x)+g_0(x)e^{\lambda y}\bigr]\dfrac{\partial w}{\partial y}=h(x,y)w.}$

The substitution $z=e^{-\lambda y}$ leads to an equation of the form 1.3.7.33:

$$f(x)\frac{\partial w}{\partial x}-\lambda\bigl[g_1(x)z+g_0(x)\bigr]\frac{\partial w}{\partial z}=h\Bigl(x,\,-\frac{1}{\lambda}\ln z\Bigr)w.$$

**36.** $\boldsymbol{f_1(x)g_1(y)\dfrac{\partial w}{\partial x}+f_2(x)g_2(y)\dfrac{\partial w}{\partial y}=h(x,y)w.}$

The transformation $\xi=\int\frac{f_2(x)}{f_1(x)}\,dx,\ \eta=\int\frac{g_1(y)}{g_2(y)}\,dy$ leads to an equation of the form 1.3.7.30:

$$\frac{\partial w}{\partial \xi}+\frac{\partial w}{\partial \eta}=F(\xi,\eta)w,\qquad \text{where}\quad F(\xi,\eta)=\frac{h(x,y)}{f_2(x)g_1(y)}.$$

## 1.4 Equations of the Form $f(x,y)\dfrac{\partial w}{\partial x}+g(x,y)\dfrac{\partial w}{\partial y}=h_1(x,y)w+h_0(x,y)$

◆ *The solutions given below contain an arbitrary function* $\Phi=\Phi(z)$.

### 1.4.1 Equations Containing Power-Law Functions

► **Coefficients of equations are linear in $x$ and $y$.**

**1.** $a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+d.$

General solution: $w=-\dfrac{d}{c}+e^{cx/a}\Phi(bx-ay)$.

**2.** $(x-a)\dfrac{\partial w}{\partial x}+(y-b)\dfrac{\partial w}{\partial y}=w-c.$

*Differential equation of a conic surface* with the vertex at the point $(a,b,c)$.

General solution: $w=c+(x-a)\Phi\Bigl(\dfrac{y-b}{x-a}\Bigr)$.

⊙ *Literature*: E. Kamke (1965).

**3.** $(ax+b)\dfrac{\partial w}{\partial x}+(cx+d)\dfrac{\partial w}{\partial y}=\alpha w+\beta.$

General solution:

$$w=\begin{cases}-\dfrac{\beta}{\alpha}+(ax+b)^{\alpha/a}\Phi\bigl(a(cx-ay)+(ad-bc)\ln|ax+b|\bigr) & \text{if } a\neq 0,\\ -\dfrac{\beta}{\alpha}+e^{\alpha x/b}\Phi\bigl(x(cx+2d)-2by\bigr) & \text{if } a=0.\end{cases}$$

**4.** $(ax+b)\dfrac{\partial w}{\partial x}+(cy+d)\dfrac{\partial w}{\partial y}=\alpha w+\beta.$

General solution:

$$w=\begin{cases}-\dfrac{\beta}{\alpha}+(ax+b)^{\alpha/a}\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr) & \text{if } a\neq 0,\\ -\dfrac{\beta}{\alpha}+e^{\alpha x/b}\Phi\bigl((cy+d)e^{-cx/b}\bigr) & \text{if } a=0.\end{cases}$$

**5.** $(ax+b)\dfrac{\partial w}{\partial x}+(cy+d)\dfrac{\partial w}{\partial y}=\alpha w+\beta y+\gamma x.$

1°. General solution for $a\neq 0$, $a\neq\alpha$, and $c\neq\alpha$:

$$w=\frac{\gamma(\alpha x+b)}{\alpha(a-\alpha)}-\frac{\beta(\alpha y+d)}{\alpha(\alpha-c)}+(ax+b)^{\alpha/a}\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr).$$

2°. General solution for $a\neq 0$, $a=\alpha$, and $c\neq\alpha$:

$$w=\frac{\gamma\bigl[b+(ax+b)\ln|ax+b|\bigr]}{a^2}-\frac{\beta(ay+d)}{a(a-c)}+(ax+b)\Phi\bigl((ax+b)^{-c/a}(cy+d)\bigr).$$

$3^\circ$. General solution for $a \neq 0$ and $a = c = \alpha$:

$$w = \frac{b\gamma + d\beta[\gamma(ax+b) + \beta(ay+d)]\ln|ax+b|}{a^2} + (ax+b)\Phi\left(\frac{ay+d}{ax+b}\right).$$

$4^\circ$. General solution for $a = 0$ and $c \neq \alpha$:

$$w = -\frac{\gamma(\alpha x+b)}{\alpha^2} - \frac{\beta(\alpha y+d)}{\alpha(\alpha-c)} + e^{\alpha x/b}\Phi\big((cy+d)e^{-cx/b}\big).$$

$5^\circ$. General solution for $a = 0$ and $c = \alpha$:

$$w = \frac{(d\beta - b\gamma)(cx+b)}{bc^2} + \frac{\beta}{b}xy + e^{cx/b}\Phi\big((cy+d)e^{-cx/b}\big).$$

**6.** $\boldsymbol{(ax+b)\frac{\partial w}{\partial x} + (cx+dy)\frac{\partial w}{\partial y} = \alpha w + \beta.}$

$1^\circ$. General solution for $a \neq 0$ and $a \neq d$:

$$w = -\frac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\big(\big[c(dx+b) + d(d-a)y\big](ax+b)^{-d/a}\big).$$

$2^\circ$. General solution for $a \neq 0$ and $a = d$:

$$w = -\frac{\beta}{\alpha} + (ax+b)^{\alpha/a}\Phi\left(\frac{bc - a^2y}{ax+b} + c\ln|ax+b|\right).$$

$3^\circ$. General solution for $a = 0$:

$$w = -\frac{\beta}{\alpha} + e^{\alpha x/b}\Phi\big(\big[bc + d(cx+dy)\big]e^{-dx/b}\big).$$

**7.** $\boldsymbol{(a_1x+a_0)\frac{\partial w}{\partial x} + (b_2y+b_1x+b_0)\frac{\partial w}{\partial y} = (c_2y+c_1x+c_0)w + k_2y+k_1x+k_0.}$

This is a special case of equation 1.4.7.22 with $f(x) = a_1x + a_0$, $g_1(x) = b_2$, $g_0(x) = b_1x + b_0$, $h(x,y) = c_2y + c_1x + c_0$, and $F(x,y) = k_2y + k_1x + k_0$.

**8.** $\boldsymbol{ay\frac{\partial w}{\partial x} + (b_1x+b_0)\frac{\partial w}{\partial y} = (c_1x+c_0)w + s_1x + s_0.}$

This is a special case of equation 1.4.7.11 with $k = 1$, $f_1(x) = a$, $f_2(x) = b_1x + b_0$, $g(x) = c_1x + c_0$, and $h(x) = s_1x + s_0$.

### ► Coefficients of equations are quadratic in $x$ and $y$.

**9.** $\boldsymbol{a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \beta xy + \gamma.}$

General solution: $w = -\dfrac{\gamma}{c} - \dfrac{\beta}{c^3}\big[(cx+a)(cy+b) + ab\big] + e^{cx/a}\Phi(bx - ay).$

**10.** $\boldsymbol{a\dfrac{\partial w}{\partial x}+b\dfrac{\partial w}{\partial y}=cw+x(\beta x+\gamma y)+\delta.}$

General solution:

$$w=-\frac{\delta}{c}-\frac{1}{c^3}\big[\beta(cx+a)^2+\gamma(cx+a)(cy+b)+a(a\beta+b\gamma)\big]+e^{cx/a}\Phi(bx-ay).$$

**11.** $\boldsymbol{x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}=w+ax^2+by^2+c.}$

General solution: $w=ax^2+by^2-c+x\Phi(y/x)$.

**12.** $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=cw+x(\beta x+\gamma y)+\delta.}$

1°. General solution for $c\neq 2a$ and $c\neq a+b$:

$$w=-\frac{\delta}{c}+\frac{\beta}{2a-c}x^2+\frac{\gamma}{a+b-c}xy+x^{c/a}\Phi\big(y|x|^{-b/a}\big).$$

2°. General solution for $c=2a$ and $a\neq b$:

$$w=-\frac{\delta}{c}+\frac{\beta}{a}x^2\ln|x|-\frac{\gamma}{a-b}xy+x^2\Phi\big(y|x|^{-b/a}\big).$$

3°. General solution for $c=a+b$ and $a\neq b$:

$$w=-\frac{\delta}{c}+\frac{\beta}{a-b}x^2+\frac{\gamma}{a}xy\ln|x|+x\Phi\big(y|x|^{-b/a}\big).$$

4°. General solution for $c=2a$ and $a=b$:

$$w=-\frac{\delta}{c}+\frac{1}{a}x(\beta x+\gamma y)\ln|x|+\Phi\Big(\frac{y}{x}\Big).$$

**13.** $\boldsymbol{ay\dfrac{\partial w}{\partial x}+(b_2x^2+b_1x+b_0)\dfrac{\partial w}{\partial y}=(c_2x^2+c_1x+c_0)w+s_2x^2+s_1x+s_0.}$

This is a special case of equation 1.4.7.11 with $k=1$, $f_1(x)=a$, $f_2(x)=b_2x^2+b_1x+b_0$, $g(x)=c_2x^2+c_1x+c_0$, and $h(x)=s_2x^2+s_1x+s_0$.

**14.** $\boldsymbol{ay^2\dfrac{\partial w}{\partial x}+(b_1x^2+b_0)\dfrac{\partial w}{\partial y}=(c_1x^2+c_0)w+s_1x^2+s_0.}$

This is a special case of equation 1.4.7.11 with $k=2$, $f_1(x)=a$, $f_2(x)=b_1x^2+b_0$, $g(x)=c_1x^2+c_0$, and $h(x)=s_1x^2+s_0$.

**15.** $\boldsymbol{(a_1x^2+a_0)\dfrac{\partial w}{\partial x}+(y+b_2x^2+b_1x+b_0)\dfrac{\partial w}{\partial y}}$
$\boldsymbol{=(c_2y+c_1x+c_0)w+k_{22}y^2+k_{12}xy+k_{11}x^2+k_0.}$

This is a special case of equation 1.4.7.22 with $f(x)=a_1x^2+a_0$, $g_1(x)=1$, $g_0(x)=b_2x^2+b_1x+b_0$, $h(x,y)=c_2y+c_1x+c_0$, and $F(x,y)=k_{22}y^2+k_{12}xy+k_{11}x^2+k_0$.

**16.** $\boldsymbol{(a_1x^2+a_0)\dfrac{\partial w}{\partial x}+(b_2y^2+b_1xy)\dfrac{\partial w}{\partial y}}$
$\boldsymbol{=(c_2y^2+c_1x^2)w+s_{22}y^2+s_{12}xy+s_{11}x^2+s_0.}$

This is a special case of equation 1.4.7.23 with $k=2$, $f(x)=a_1x^2+a_0$, $g_1(x)=b_1x$, $g_0(x)=b_2$, $h(x,y)=c_2y^2+c_1x^2$, and $F(x,y)=s_{22}y^2+s_{12}xy+s_{11}x^2+s_0$.

► **Coefficients of equations contain square roots.**

**17.** $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{xy}+\gamma.}$

1°. General solution for $2\alpha\neq a+b$:

$$w=\frac{2\beta}{a+b-2\alpha}\sqrt{xy}-\frac{\gamma}{\alpha}+x^{\alpha/a}\Phi\bigl(y|x|^{-b/a}\bigr).$$

2°. General solution for $2\alpha=a+b$:

$$w=\frac{\beta}{a}\sqrt{xy}\,\ln|x|-\frac{2\gamma}{a+b}+\sqrt{xy}\,\Phi\bigl(y|x|^{-b/a}\bigr).$$

3°. General solution for $\alpha=a=-b$:

$$w=\frac{1}{a}\bigl(\beta\sqrt{xy}+\gamma\bigr)+x\Phi(xy).$$

**18.** $\boldsymbol{ax\dfrac{\partial w}{\partial x}+by\dfrac{\partial w}{\partial y}=\lambda\sqrt{xy}\,w+\beta xy+\gamma.}$

1°. General solution for $b\neq -a$:

$$w=-\frac{\beta}{\lambda}\sqrt{xy}-\frac{\beta(a+b)}{2\lambda^2}+\exp\left(\frac{2\lambda}{a+b}\sqrt{xy}\right)\Phi\bigl(x^{-b/a}y\bigr).$$

2°. General solution for $b=-a$:

$$w=-\frac{\beta}{\lambda}\sqrt{xy}+\exp\left(\frac{\lambda}{a}\sqrt{xy}\,\ln|x|\right)\Phi(xy).$$

**19.** $\boldsymbol{ay\dfrac{\partial w}{\partial x}+bx\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1$, $f_1(x)=a$, $f_2(x)=bx$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

**20.** $\boldsymbol{ay\dfrac{\partial w}{\partial x}+b\sqrt{x}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1$, $f_1(x)=a$, $f_2(x)=b\sqrt{x}$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

**21.** $\boldsymbol{a\sqrt{x}\dfrac{\partial w}{\partial x}+b\sqrt{y}\dfrac{\partial w}{\partial y}=\alpha w+\beta x+\gamma y+\delta.}$

General solution:

$$w=-\frac{a\beta\sqrt{x}+b\gamma\sqrt{y}}{\alpha^2}-\frac{\beta x+\gamma y+\delta}{\alpha}-\frac{a^2\beta+b^2\gamma}{2\alpha^3}+\exp\left(\frac{2\alpha}{a}\sqrt{x}\right)\Phi\Bigl(b\sqrt{x}-a\sqrt{y}\Bigr).$$

**22.** $\boldsymbol{a\sqrt{x}\dfrac{\partial w}{\partial x}+b\sqrt{y}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

General solution: $w=-\dfrac{\beta\sqrt{x}+\gamma}{\alpha}-\dfrac{a\beta}{2\alpha^2}+\exp\left(\dfrac{2\alpha}{a}\sqrt{x}\right)\Phi\Bigl(b\sqrt{x}-a\sqrt{y}\Bigr).$

**23.** $\boldsymbol{a\sqrt{y}\dfrac{\partial w}{\partial x}+b\sqrt{x}\dfrac{\partial w}{\partial y}=\alpha w+\beta\sqrt{x}+\gamma.}$

This is a special case of equation 1.4.7.11 with $k=1/2$, $f_1(x)=a$, $f_2(x)=b\sqrt{x}$, $g(x)=\alpha$, and $h(x)=\beta\sqrt{x}+\gamma$.

► **Coefficients of equations contain arbitrary powers of $x$ and $y$.**

**24.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + kx^n y^m.$

Two forms of the representation of the general solution:

$$w = \exp\Bigl(\frac{c}{a}x\Bigr)\biggl[\Phi(bx-ay) + \frac{k}{a^{m+1}}\int x^n(bx-u)^m \exp\Bigl(-\frac{c}{a}x\Bigr)\,dx\biggr],$$

$$w = \exp\Bigl(\frac{c}{b}y\Bigr)\biggl[\Phi(bx-ay) + \frac{k}{b^{n+1}}\int y^m(ay+u)^n \exp\Bigl(-\frac{c}{b}y\Bigr)\,dy\biggr],$$

where $u = bx - ay$. In the integration, $u$ is considered a parameter.

**25.** $\displaystyle a\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = bw + cx^n y^m.$

General solution:

$$w = y^b\biggl[\Phi(y^a e^{-x}) + c\int y^{m-b-1}(a\ln y - \ln u)^n\,dy\biggr], \quad \text{where} \quad u = y^a e^{-x}.$$

In the integration, $u$ is considered a parameter.

**26.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = axw + bx^n y^m.$

General solution: $\displaystyle w = e^{ax}\biggl[\Phi\Bigl(\frac{y}{x}\Bigr) + bx^{-m}y^m\int x^{m+n-1}e^{-ax}\,dx\biggr].$

**27.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = a\sqrt{x^2+y^2}\,w + bx^n y^m.$

General solution:

$$w = \exp\Bigl(a\sqrt{x^2+y^2}\Bigr)\biggl[\Phi\Bigl(\frac{y}{x}\Bigr) + bx^{-m}y^m\int x^{m+n-1}\exp\Bigl(-ax\sqrt{1+u^2}\Bigr)\,dx\biggr], \quad u = \frac{y}{x}.$$

In the integration, $u$ is considered a parameter.

**28.** $\displaystyle ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = cx^n y^m w + px^k y^s.$

1°. General solution for $an + bm \neq 0$:

$$w = \exp\biggl(\frac{c}{an+bm}x^n y^m\biggr)\Bigl[\Phi\bigl(y^a x^{-b}\bigr) + \psi(x,y)\Bigr],$$

$$\psi(x,y) = px^{-\frac{bs}{a}}y^s\int x^{\frac{bs+ak-a}{a}}\exp\biggl(-\frac{c}{an+bm}u^{\frac{m}{a}}x^{\frac{an+bm}{a}}\biggr)\,dx,$$

where $u = y^a x^{-b}$. In the integration, $u$ is considered a parameter.

2°. General solution for $an + bm = 0$:

$$w = \exp\biggl(\frac{c}{a}x^n y^m\ln x\biggr)\Bigl[\Phi\bigl(y^a x^{-b}\bigr) + \psi(x,y)\Bigr],$$

$$\psi(x,y) = \begin{cases} pk^{-2}x^{\frac{ak-bs}{a}}y^s\exp\biggl(-\dfrac{c}{a}x^{-\frac{bm}{a}}y^m\biggr)(k\ln x - 1) & \text{if } k \neq 0,\\ \frac{1}{2}px^{-\frac{bs}{a}}y^s\exp\biggl(-\dfrac{c}{a}x^{-\frac{bm}{a}}y^m\biggr)(\ln x)^2 & \text{if } k = 0.\end{cases}$$

**29.** $\boldsymbol{ax\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = (cx^n + py^m)w + qx^k y^s.}$

General solution:

$$w = \exp\biggl(\frac{cx^n}{an} + \frac{py^m}{bm}\biggr)\biggl[\Phi\bigl(y^a x^{-b}\bigr) + qx^{-\frac{bs}{a}} y^s \int x^{\frac{ak-a+bs}{a}} \exp\biggl(-\frac{cx^n}{an} - \frac{p}{bm}u^{\frac{m}{a}} x^{\frac{bm}{a}}\biggr)\,dx\biggr],$$

where $u = y^a x^{-b}$. In the integration, $u$ is considered a parameter.

**30.** $\boldsymbol{x^2\frac{\partial w}{\partial x} + axy\frac{\partial w}{\partial y} = by^2 w + cx^n y^m.}$

$1^\circ$. General solution for $a \neq 1/2$:

$$w = \exp\biggl(\frac{b}{2a-1}\frac{y^2}{x}\biggr)\biggl[\Phi(x^{-a}y) + cx^{-am}y^m \int x^{am+n-2} \exp\biggl(-\frac{b}{2a-1}u^2 x^{2a-1}\biggr)\,dx\biggr],$$

where $u = x^{-a}y$. In the integration, $u$ is considered a parameter.

$2^\circ$. General solution for $a = 1/2$:

$$w = \exp\biggl(b\frac{y^2}{x}\ln x\biggr)\Phi(x^{-1/2}y) + \frac{2cx^n y^m}{(m+2n-2)x - by^2}.$$

**31.** $\boldsymbol{x^2\frac{\partial w}{\partial x} + xy\frac{\partial w}{\partial y} = y^2(ax+by)w + cx^n y^m.}$

General solution:

$$w = \exp\biggl[\frac{(ax+by)y^2}{2x}\biggr]\biggl\{\Phi\Bigl(\frac{y}{x}\Bigr) + cx^{-m}y^m \int x^{m+n-2} \exp\biggl[-\frac{(a+bu)u^2x^2}{2}\biggr]\,dx\biggr\},$$

where $u = y/x$. In the integration, $u$ is considered a parameter.

**32.** $\boldsymbol{ax^n\frac{\partial w}{\partial x} + bx^m y\frac{\partial w}{\partial y} = cx^p y^q w + sx^\gamma y^\delta + d.}$

This is a special case of equation 1.4.7.21 with $f(x) = ax^n$, $g(x) = bx^m$, $h(x,y) = cx^p y^q$, and $F(x,y) = sx^\gamma y^\delta + d$.

**33.** $\boldsymbol{ax^n\frac{\partial w}{\partial x} + \bigl(bx^m y + cx^k\bigr)\frac{\partial w}{\partial y} = sx^p y^q w + d.}$

This is a special case of equation 1.4.7.22 with $f(x) = ax^n$, $g_1(x) = bx^m$, $g_0(x) = cx^k$, $h(x,y) = sx^p y^q$, and $F(x,y) = d$.

**34.** $\boldsymbol{ax^n\frac{\partial w}{\partial x} + bx^m y^k\frac{\partial w}{\partial y} = cw + sx^p y^q + d.}$

This is a special case of equation 1.4.7.23 with $f(x) = ax^n$, $g_1(x) \equiv 0$, $g_0(x) = bx^m$, $h(x,y) = c$, and $F(x,y) = sx^p y^q + d$.

**35.** $\boldsymbol{ay^k\frac{\partial w}{\partial x} + bx^n\frac{\partial w}{\partial y} = cw + sx^m.}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = bx^n$, $g(x) = c$, and $h(x) = sx^m$.

### 1.4.2 Equations Containing Exponential Functions

► **Coefficients of equations contain exponential functions.**

**1.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \left(ce^{\lambda x} + se^{\mu y}\right)w + ke^{\nu x}.$

General solution:

$$w = \exp\Big(\frac{c}{a\lambda}e^{\lambda x} + \frac{s}{b\mu}e^{\mu y}\Big)\left[\Phi(bx-ay) + \frac{k}{a}\int \exp\Big(\nu x - \frac{c}{a\lambda}e^{\lambda x} - \frac{s}{b\mu}e^{\frac{\mu bx-\mu u}{a}}\Big)\,dx\right],$$

where $u = bx - ay$. In the integration, $u$ is considered a parameter.

**2.** $a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = ce^{\alpha x+\beta y}w + ke^{\gamma x}.$

1°. General solution for $a\alpha + b\beta \neq 0$:

$$w = \exp\left(\frac{c}{a\alpha+b\beta}e^{\alpha x+\beta y}\right)\left\{\Phi(bx-ay) + \frac{k}{a}\int \exp\left[\gamma x - \frac{c}{a\alpha+b\beta}e^{\frac{(a\alpha+b\beta)x-\beta u}{a}}\right]dx\right\},$$

where $u = bx - ay$. In the integration, $u$ is considered a parameter.

2°. General solution for $a\alpha + b\beta = 0$:

$$w = \exp\left(\frac{c}{a}xe^{\alpha x+\beta y}\right)\Phi(bx-ay) + \frac{ke^{\gamma x}}{a\gamma - ce^{\alpha x+\beta y}}.$$

**3.** $ae^{\lambda x}\dfrac{\partial w}{\partial x} + be^{\beta x}\dfrac{\partial w}{\partial y} = ce^{\gamma y}w + se^{\mu x+\delta y}.$

This is a special case of equation 1.4.7.22 with $f(x) = ae^{\lambda x}$, $g_1(x) \equiv 0$, $g_0(x) = be^{\beta x}$, $h(x,y) = ce^{\gamma y}$, and $F(x,y) = se^{\mu x+\delta y}$.

**4.** $ae^{\beta x}\dfrac{\partial w}{\partial x} + \left(be^{\gamma x} + ce^{\lambda y}\right)\dfrac{\partial w}{\partial y} = sw + ke^{\mu x+\delta y}.$

This is a special case of equation 1.4.7.24 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, $h(x,y) = s$, and $F(x,y) = ke^{\mu x+\delta y}$.

**5.** $ae^{\beta x}\dfrac{\partial w}{\partial x} + \left(be^{\gamma x} + ce^{\lambda y}\right)\dfrac{\partial w}{\partial y} = se^{\mu x+\delta y}w + k.$

This is a special case of equation 1.4.7.24 with $f(x) = ae^{\beta x}$, $g_1(x) = be^{\gamma x}$, $g_0(x) = c$, $h(x,y) = se^{\mu x+\delta y}$, and $F(x,y) = k$.

**6.** $ae^{\beta x}\dfrac{\partial w}{\partial x} + be^{\gamma x+\lambda y}\dfrac{\partial w}{\partial y} = ce^{\sigma y}w + ke^{\mu x+\delta y} + d.$

This is a special case of equation 1.4.7.24 with $f(x) = ae^{\beta x}$, $g_1(x) \equiv 0$, $g_0(x) = be^{\gamma x}$, $h(x,y) = ce^{\sigma y}$, and $F(x,y) = ke^{\mu x+\delta y} + d$.

**7.** $ae^{\lambda y}\dfrac{\partial w}{\partial x} + bx^{\beta x}\dfrac{\partial w}{\partial y} = cw + se^{\gamma x}.$

This is a special case of equation 1.4.7.12 with $f_1(x) = a$, $f_2(x) = bx^{\beta x}$, $g(x) = c$, and $h(x) = se^{\gamma x}$.

**8.** $\boldsymbol{ae^{\lambda y}\dfrac{\partial w}{\partial x} + bx^{\beta x}\dfrac{\partial w}{\partial y} = ce^{\gamma x}w + s.}$

This is a special case of equation 1.4.7.12 with $f_1(x) = a$, $f_2(x) = bx^{\beta x}$, $g(x) = ce^{\gamma x}$, and $h(x) = s$.

► **Coefficients of equations contain exponential and power-law functions.**

**9.** $\boldsymbol{\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y + bx^n)\dfrac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = bx^n$, $h_1(x) = c$, and $h_0(x) = ke^{\gamma x}$.

**10.** $\boldsymbol{\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y + be^{\beta x})\dfrac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = be^{\beta x}$, $h_1(x) = c$, and $h_0(x) = ke^{\gamma x}$.

**11.** $\boldsymbol{\dfrac{\partial w}{\partial x} + (ae^{\lambda x}y + be^{\beta x})\dfrac{\partial w}{\partial y} = cw + kx^n.}$

This is a special case of equation 1.4.7.7 with $f(x) = 1$, $g_1(x) = ae^{\lambda x}$, $g_0(x) = be^{\beta x}$, $h_1(x) = c$, and $h_0(x) = kx^n$.

**12.** $\boldsymbol{\dfrac{\partial w}{\partial x} + (ae^{\lambda y} + bx^k)\dfrac{\partial w}{\partial y} = cw + ke^{\gamma x}.}$

This is a special case of equation 1.4.7.10 with $f(x)=1$, $g_1(x)=bx^k$, $g_0(x)=a$, $h_2(x)=c$, $h_1(x) = 0$, and $h_0(x) = ke^{\gamma x}$.

**13.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = axe^{\lambda x+\mu y}w + be^{\nu x}.}$

General solution:

$$w = \exp\left(\frac{ax}{\lambda x + \mu y}e^{\lambda x+\mu y}\right)\left\{\Phi\left(\frac{y}{x}\right) + b\int \exp\left[\nu x - \frac{a}{\lambda + \mu u}e^{(\lambda+\mu u)x}\right]\frac{dx}{x}\right\},$$

where $u = y/x$. In the integration, $u$ is considered a parameter.

**14.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = \bigl(aye^{\lambda x} + bxe^{\mu y}\bigr)w + ce^{\nu x}.}$

General solution:

$$w = \exp\left(\frac{ay}{\lambda x}e^{\lambda x} + \frac{bx}{\mu y}e^{\mu y}\right)\left[\Phi\left(\frac{y}{x}\right) + c\int \exp\left(\nu x - \frac{au}{\lambda}e^{\lambda x} - \frac{b}{\mu u}e^{\mu u x}\right)\frac{dx}{x}\right],$$

where $u = y/x$. In the integration, $u$ is considered a parameter.

**15.** $\boldsymbol{ay^k\dfrac{\partial w}{\partial x} + be^{\lambda x}\dfrac{\partial w}{\partial y} = w + ce^{\beta x}.}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = be^{\lambda x}$, $g(x) = 1$, and $h(x) = ce^{\beta x}$.

**16.** $\displaystyle ae^{\lambda x}\frac{\partial w}{\partial x} + by\frac{\partial w}{\partial y} = w + ce^{\lambda x}.$

This is a special case of equation 1.4.7.7 with $f(x)=ae^{\lambda x}$, $g_1(x)=b$, $g_0(x)=0$, $h_1(x)=1$, and $h_0(x) = ce^{\lambda x}$.

**17.** $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x} + bx^k\frac{\partial w}{\partial y} = w + ce^{\beta x}.$

This is a special case of equation 1.4.7.12 with $f_1(x) = a$, $f_2(x) = bx^k$, $g(x) = 1$, and $h(x) = ce^{\beta x}$.

**18.** $\displaystyle ae^{\lambda y}\frac{\partial w}{\partial x} + be^{\beta x}\frac{\partial w}{\partial y} = w + cx^k.$

This is a special case of equation 1.4.7.12 with $f_1(x) = a$, $f_2(x) = be^{\beta x}$, $g(x) = 1$, and $h(x) = cx^k$.

## 1.4.3 Equations Containing Hyperbolic Functions

► **Coefficients of equations contain hyperbolic sine.**

**1.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \sinh^k(\lambda x)\sinh^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x) = \sinh^k(\lambda x)$ and $g(y) = \sinh^n(\beta y)$.

**2.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\sinh^k(\lambda x)w + s\sinh^n(\beta x).$

This is a special case of equation 1.4.7.1 with $f(x) = \sinh^k(\lambda x)$ and $g(y) = \sinh^n(\beta x)$.

**3.** $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c_1\sinh^{n_1}(\lambda_1 x) + c_2\sinh^{n_2}(\lambda_2 y)\bigr]w + s_1\sinh^{k_1}(\beta_1 x) + s_2\sinh^{k_2}(\beta_2 y).$$

This is a special case of equation 1.4.7.16 with $f(x)=c_1\sinh^{n_1}(\lambda_1 x)$, $g(y)=c_2\sinh^{n_2}(\lambda_2 y)$, $p(x) = s_1\sinh^{k_1}(\beta_1 x)$, and $q(y) = s_2\sinh^{k_2}(\beta_2 y)$.

**4.** $\displaystyle a\sinh^n(\lambda x)\frac{\partial w}{\partial x} + b\sinh^m(\mu x)\frac{\partial w}{\partial y} = c\sinh^k(\nu x)w + p\sinh^s(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x) = a\sinh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sinh^m(\mu x)$, $h(x,y) = c\sinh^k(\nu x)$, and $F(x,y) = p\sinh^s(\beta y)$.

**5.** $\displaystyle a\sinh^n(\lambda x)\frac{\partial w}{\partial x} + b\sinh^m(\mu x)\frac{\partial w}{\partial y} = c\sinh^k(\nu y)w + p\sinh^s(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x) = a\sinh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sinh^m(\mu x)$, $h(x,y) = c\sinh^k(\nu y)$, and $F(x,y) = p\sinh^s(\beta x)$.

► **Coefficients of equations contain hyperbolic cosine.**

**6.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \cosh^k(\lambda x)\cosh^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x) = \cosh^k(\lambda x)$ and $g(y) = \cosh^n(\beta y)$.

**7.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\cosh^k(\lambda x)w + s\cosh^n(\beta x).$

This is a special case of equation 1.4.7.1 with $f(x) = \cosh^k(\lambda x)$ and $g(y) = \cosh^n(\beta x)$.

**8.** $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c_1\cosh^{n_1}(\lambda_1 x) + c_2\cosh^{n_2}(\lambda_2 y)\bigr]w + s_1\cosh^{k_1}(\beta_1 x) + s_2\cosh^{k_2}(\beta_2 y).$$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1\cosh^{n_1}(\lambda_1 x)$, $g(y) = c_2\cosh^{n_2}(\lambda_2 y)$, $p(x) = s_1\cosh^{k_1}(\beta_1 x)$, and $q(y) = s_2\cosh^{k_2}(\beta_2 y)$.

**9.** $\displaystyle x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = ax\cosh(\lambda x + \mu y)w + b\cosh(\nu x).$

General solution:

$$w = \exp\left[\frac{ax\sinh(\lambda x + \mu y)}{\lambda x + \mu y}\right]\left[\Phi\left(\frac{y}{x}\right) + b\int\cosh(\nu x)\exp\left(-\frac{a\sinh[(\lambda + \mu u)x]}{\lambda + \mu u}\right)\frac{dx}{x}\right],$$

where $u = y/x$. In the integration, $u$ is considered a parameter.

**10.** $\displaystyle a\cosh^n(\lambda x)\frac{\partial w}{\partial x} + b\cosh^m(\mu x)\frac{\partial w}{\partial y} = c\cosh^k(\nu x)w + p\cosh^s(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x) = a\cosh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu x)$, and $F(x,y) = p\cosh^s(\beta y)$.

**11.** $\displaystyle a\cosh^n(\lambda x)\frac{\partial w}{\partial x} + b\cosh^m(\mu x)\frac{\partial w}{\partial y} = c\cosh^k(\nu y)w + p\cosh^s(\beta x).$

This is a special case of equation 1.4.7.22 with $f(x) = a\cosh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu y)$, and $F(x,y) = p\cosh^s(\beta x)$.

► **Coefficients of equations contain hyperbolic tangent.**

**12.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = cw + \tanh^k(\lambda x)\tanh^n(\beta y).$

This is a special case of equation 1.4.7.13 with $f(x) = \tanh^k(\lambda x)$ and $g(y) = \tanh^n(\beta y)$.

**13.** $\displaystyle a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = c\tanh^k(\lambda x)w + s\tanh^n(\beta x).$

This is a special case of equation 1.4.7.1 with $f(x) = \tanh^k(\lambda x)$ and $g(y) = \tanh^n(\beta x)$.

**14.** $$a\frac{\partial w}{\partial x} + b\frac{\partial w}{\partial y} = \bigl[c_1\tanh^{n_1}(\lambda_1 x) + c_2\tanh^{n_2}(\lambda_2 y)\bigr]w + s_1\tanh^{k_1}(\beta_1 x) + s_2\tanh^{k_2}(\beta_2 y).$$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1\tanh^{n_1}(\lambda_1 x)$, $g(y) = c_2\tanh^{n_2}(\lambda_2 y)$, $p(x) = s_1\tanh^{k_1}(\beta_1 x)$, and $q(y) = s_2\tanh^{k_2}(\beta_2 y)$.

**15.** $\displaystyle a\tanh^n(\lambda x)\frac{\partial w}{\partial x} + b\tanh^m(\mu x)\frac{\partial w}{\partial y} = c\tanh^k(\nu x)w + p\tanh^s(\beta y).$

This is a special case of equation 1.4.7.22 with $f(x) = a\tanh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\tanh^m(\mu x)$, $h(x,y) = c\tanh^k(\nu x)$, and $F(x,y) = p\tanh^s(\beta y)$.

**16.** $\boldsymbol{a \tanh^n(\lambda x) \dfrac{\partial w}{\partial x} + b \tanh^m(\mu x) \dfrac{\partial w}{\partial y} = c \tanh^k(\nu y) w + p \tanh^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a \tanh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b \tanh^m(\mu x)$, $h(x, y) = c \tanh^k(\nu y)$, and $F(x, y) = p \tanh^s(\beta x)$.

► **Coefficients of equations contain hyperbolic cotangent.**

**17.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = cw + \coth^k(\lambda x) \coth^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \coth^k(\lambda x)$ and $g(y) = \coth^n(\beta y)$.

**18.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = c \coth^k(\lambda x) w + s \coth^n(\beta x).}$

This is a special case of equation 1.4.7.1 with $f(x) = \coth^k(\lambda x)$ and $g(y) = \coth^n(\beta x)$.

**19.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = \big[c_1 \coth^{n_1}(\lambda_1 x) + c_2 \coth^{n_2}(\lambda_2 y)\big] w}$

$\boldsymbol{+ s_1 \coth^{k_1}(\beta_1 x) + s_2 \coth^{k_2}(\beta_2 y).}$

This is a special case of equation 1.4.7.16 in which $f(x) = c_1 \coth^{n_1}(\lambda_1 x)$, $g(y) = c_2 \coth^{n_2}(\lambda_2 y)$, $p(x) = s_1 \coth^{k_1}(\beta_1 x)$, and $q(y) = s_2 \coth^{k_2}(\beta_2 y)$.

**20.** $\boldsymbol{a \coth^n(\lambda x) \dfrac{\partial w}{\partial x} + b \coth^m(\mu x) \dfrac{\partial w}{\partial y} = c \coth^k(\nu x) w + p \coth^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a \coth^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b \coth^m(\mu x)$, $h(x, y) = c \coth^k(\nu x)$, and $F(x, y) = p \coth^s(\beta y)$.

**21.** $\boldsymbol{a \coth^n(\lambda x) \dfrac{\partial w}{\partial x} + b \coth^m(\mu x) \dfrac{\partial w}{\partial y} = c \coth^k(\nu y) w + p \coth^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a \coth^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b \coth^m(\mu x)$, $h(x, y) = c \coth^k(\nu y)$, and $F(x, y) = p \coth^s(\beta x)$.

► **Coefficients of equations contain different hyperbolic functions.**

**22.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = w + c_1 \sinh^k(\lambda x) + c_2 \cosh^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \sinh^k(\lambda x)$, and $q(y) = c_2 \cosh^n(\beta y)$.

**23.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = cw + \sinh^k(\lambda x) \cosh^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \sinh^k(\lambda x)$ and $g(y) = \cosh^n(\beta y)$.

**24.** $\boldsymbol{a \dfrac{\partial w}{\partial x} + b \dfrac{\partial w}{\partial y} = cw + k \tanh(\lambda x) + s \coth(\mu y).}$

General solution:

$$w = e^{cx/a} \biggl\{ \Phi(bx - ay) - \frac{1}{a} \int_0^x \biggl[ s \coth\biggl( \frac{b\mu}{a}(x - t) - \mu y \biggr) - k \tanh(\lambda t) \biggr] e^{-ct/a}\, dt \biggr\}.$$

**25.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\sinh(\lambda x)\dfrac{\partial w}{\partial y} = cw + k\cosh(\mu y).}$

General solution:

$$w = e^{cx/a}\left\{\int_0^x \cosh\left[\mu y + \frac{b\mu}{a\lambda}\Big(\cosh(\lambda t) - \cosh(\lambda x)\Big)\right]e^{-ct/a}\,dt + \Phi\big(a\lambda y - b\cosh(\lambda x)\big)\right\}.$$

**26.** $\boldsymbol{a\sinh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\cosh^m(\mu x)\dfrac{\partial w}{\partial y} = c\cosh^k(\nu x)w + p\sinh^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sinh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\cosh^m(\mu x)$, $h(x,y) = c\cosh^k(\nu x)$, and $F(x,y) = p\sinh^s(\beta y)$.

**27.** $\boldsymbol{a\tanh^n(\lambda x)\dfrac{\partial w}{\partial x} + b\coth^m(\mu x)\dfrac{\partial w}{\partial y} = c\tanh^k(\nu y)w + p\coth^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\tanh^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\coth^m(\mu x)$, $h(x,y) = c\tanh^k(\nu y)$, and $F(x,y) = p\coth^s(\beta x)$.

### 1.4.4 Equations Containing Logarithmic Functions

► **Coefficients of equations contain logarithmic functions.**

**1.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \ln^k(\lambda x)\ln^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \ln^k(\lambda x)$ and $g(y) = \ln^n(\beta y)$.

**2.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = c\ln^k(\lambda x)w + s\ln^n(\beta x).}$

This is a special case of equation 1.4.7.1 with $f(x) = \ln^k(\lambda x)$ and $g(y) = \ln^n(\beta x)$.

**3.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \big[c_1\ln^{n_1}(\lambda_1 x) + c_2\ln^{n_2}(\lambda_2 y)\big]w + s_1\ln^{k_1}(\beta_1 x) + s_2\ln^{k_2}(\beta_2 y).}$

This is a special case of equation 1.4.7.16 with $f(x) = c_1\ln^{n_1}(\lambda_1 x)$, $g(y) = c_2\ln^{n_2}(\lambda_2 y)$, $p(x) = s_1\ln^{k_1}(\beta_1 x)$, and $q(y) = s_2\ln^{k_2}(\beta_2 y)$.

**4.** $\boldsymbol{a\ln(\lambda x)\dfrac{\partial w}{\partial x} + b\ln(\mu y)\dfrac{\partial w}{\partial y} = cw + k.}$

General solution:

$$w = -\frac{k}{c} + \Phi(u)\exp\left[\frac{c}{a}\int\frac{dx}{\ln(\lambda x)}\right], \qquad u = b\int\frac{dx}{\ln(\lambda x)} - a\int\frac{dy}{\ln(\mu y)}.$$

**5.** $\boldsymbol{a\ln^n(\lambda x)\dfrac{\partial w}{\partial x} + b\ln^m(\mu x)\dfrac{\partial w}{\partial y} = c\ln^k(\nu x)w + p\ln^s(\beta y) + q.}$

This is a special case of equation 1.4.7.22 with $f(x) = a\ln^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^m(\mu x)$, $h(x,y) = c\ln^k(\nu x)$, and $F(x,y) = p\ln^s(\beta y) + q$.

**6.** $\boldsymbol{a\ln^n(\lambda x)\dfrac{\partial w}{\partial x} + b\ln^m(\mu x)\dfrac{\partial w}{\partial y} = c\ln^k(\nu y)w + p\ln^s(\beta x) + q.}$

This is a special case of equation 1.4.7.22 with $f(x) = a\ln^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\ln^m(\mu x)$, $h(x,y) = c\ln^k(\nu y)$, and $F(x,y) = p\ln^s(\beta x) + q$.

► **Coefficients of equations contain logarithmic and power-law functions.**

7. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1 x^k + c_2 \ln^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 x^k$, and $q(y) = c_2 \ln^n(\beta y)$.

8. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + x^k \ln^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = x^k$ and $g(y) = \ln^n(\beta y)$.

9. $\boldsymbol{ax^k\dfrac{\partial w}{\partial x} + bx^n\dfrac{\partial w}{\partial y} = cw + s\ln^m(\beta x).}$

This is a special case of equation 1.4.7.7 with $f(x) = ax^k$, $g_1(x) = 0$, $g_0(x) = bx^n$, $h_1(x) = c$, and $h_0(x) = s\ln^m(\beta x)$.

10. $\boldsymbol{ax^n\dfrac{\partial w}{\partial x} + by^k\dfrac{\partial w}{\partial y} = cw + s\ln^m(\beta x).}$

This is a special case of equation 1.4.7.23 with $f(x) = ax^n$, $g_1(x) = 0$, $g_0(x) = b$, $h(x, y) = c$, and $F(x, y) = s\ln^m(\beta x)$.

11. $\boldsymbol{ax^k\dfrac{\partial w}{\partial x} + b\ln^n(\lambda x)\dfrac{\partial w}{\partial y} = cw + sx^m.}$

This is a special case of equation 1.4.7.7 with $f(x) = ax^k$, $g_1(x) = 0$, $g_0(x) = b\ln^n(\lambda x)$, $h_1(x) = c$, and $h_0(x) = sx^m$.

12. $\boldsymbol{ay^k\dfrac{\partial w}{\partial x} + bx^n\dfrac{\partial w}{\partial y} = cw + s\ln^m(\beta x).}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = bx^n$, $g(x) = c$, and $h(x) = s\ln^m(\beta x)$.

13. $\boldsymbol{ay^k\dfrac{\partial w}{\partial x} + b\ln^n(\lambda x)\dfrac{\partial w}{\partial y} = cw + sx^m.}$

This is a special case of equation 1.4.7.11 with $f_1(x) = a$, $f_2(x) = b\ln^n(\lambda x)$, $g(x) = c$, and $h(x) = sx^m$.

## 1.4.5 Equations Containing Trigonometric Functions

► **Coefficients of equations contain sine.**

1. $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + k\sin(\lambda x + \mu y).}$

General solution:

$$w = e^{cx/a}\Phi(bx - ay) - \frac{k}{c^2 + (a\lambda + b\mu)^2}\bigl[(a\lambda + b\mu)\cos(\lambda x + \mu y) + c\sin(\lambda x + \mu y)\bigr].$$

**2.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1 \sin^k(\lambda x) + c_2 \sin^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1 \sin^k(\lambda x)$, and $q(y) = c_2 \sin^n(\beta y)$.

**3.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + \sin^k(\lambda x)\sin^n(\beta y).}$

This is a special case of equation 1.4.7.13 with $f(x) = \sin^k(\lambda x)$ and $g(y) = \sin^n(\beta y)$.

**4.** $\boldsymbol{ax\dfrac{\partial w}{\partial x} + by\dfrac{\partial w}{\partial y} = cw + k\sin(\lambda x + \mu y).}$

General solution:

$$w = x^{c/a}\left[\frac{k}{a}\int_0^x t^{-(a+c)/a}\sin\bigl(\lambda t + \mu t^{b/a}x^{-b/a}y\bigr)\,dt + \Phi\bigl(x^{-b/a}y\bigr)\right].$$

**5.** $\boldsymbol{x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = ax\sin(\lambda x + \mu y)w + b\sin(\nu x).}$

General solution:

$$w = \exp\left[-\frac{ax}{\lambda x + \mu y}\cos(\lambda x + \mu y)\right]\left\{\Phi\Bigl(\frac{y}{x}\Bigr)\right.$$
$$\left. + b\int \sin(\nu x)\exp\left(\frac{a}{\lambda + \mu u}\cos\bigl[(\lambda + \mu u)x\bigr]\right)dx\right\},$$

where $u = y/x$. In the integration, $u$ is considered a parameter.

**6.** $\boldsymbol{a\sin^n(\lambda x)\dfrac{\partial w}{\partial x} + b\sin^m(\mu x)\dfrac{\partial w}{\partial y} = c\sin^k(\nu x)w + p\sin^s(\beta y).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sin^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^m(\mu x)$, $h(x,y) = c\sin^k(\nu x)$, and $F(x,y) = p\sin^s(\beta y)$.

**7.** $\boldsymbol{a\sin^n(\lambda x)\dfrac{\partial w}{\partial x} + b\sin^m(\mu x)\dfrac{\partial w}{\partial y} = c\sin^k(\nu y)w + p\sin^s(\beta x).}$

This is a special case of equation 1.4.7.22 with $f(x) = a\sin^n(\lambda x)$, $g_1(x) \equiv 0$, $g_0(x) = b\sin^m(\mu x)$, $h(x,y) = c\sin^k(\nu y)$, and $F(x,y) = p\sin^s(\beta x)$.

► **Coefficients of equations contain cosine.**

**8.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = cw + k\cos(\lambda x + \mu y).}$

General solution:

$$w = e^{cx/a}\Phi(bx - ay) + \frac{k}{c^2 + (a\lambda + b\mu)^2}\bigl[(a\lambda + b\mu)\sin(\lambda x + \mu y) - c\cos(\lambda x + \mu y)\bigr].$$

**9.** $\boldsymbol{a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = w + c_1\cos^k(\lambda x) + c_2\cos^n(\beta y).}$

This is a special case of equation 1.4.7.16 with $f(x) = 0$, $g(y) = 1$, $p(x) = c_1\cos^k(\lambda x)$, and $q(y) = c_2\cos^n(\beta y)$.