LINEAR SYSTEMS

Non-Fragile Control and Filtering

Guang-Hong Yang • Xiang-Gui Guo
Wei-Wei Che • Wei Guan

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Preface

Digital control systems design has become an important field in electrical engineering and in systems and control theory. One of the important and fundamental issues in digital control systems design is the filter or controller coefficient sensitivity because even vanishingly small perturbations in controller or filter coefficients may destabilize the resulting systems. In the actual engineering systems, the controllers or filters realized by microprocessors/microcontrollers do have some uncertainties due to limitations in available microprocessor/microcontroller memory, effects of finite word length (FWL) of digital processors, quantization of the A/D and D/A converters, and so on. Therefore, non-fragile (insensitive) control is becoming popular in many fields of engineering and science, and there is a vast amount of literature on design and analysis of non-fragile control problems using rigorous methods based on different performance criteria.

In order to obtain non-fragile (insensitive) controllers, numerous works in the filtering and control theory are devoted to solving such problems. The previous results were mainly developed in the framework of robust control theory, that is, non-fragile controller/filter design methods have been proposed to obtain the non-fragile controllers/filters which can be insensitive or non-fragile with respect to controller/filter gain uncertainties by considering controller/filter gain uncertainties directly. There are two main types of gain uncertainties considered in the design methods. One is known as norm-bounded gain uncertainty, the other is known as interval-bounded coefficient variations. It is worth mentioning that the type of norm-bounded uncertainty cannot exactly reflect the uncertain information due to the FWL effects, while the type of interval-bounded coefficient variations may result in numerical problems because the number of linear matrix inequalities (LMIs) involved in the design conditions grows *exponentially* with the number of uncertain parameters. On the other hand, sensitivity analysis techniques in performance assessments are important in operations research as well as in the practical design of control systems because sensitivity analysis provides valuable insights into the influence of parameter variations on the dynamic behavior of systems. However, they mainly consider the optimal realization of a controller or filter via minimizing the coefficient sensitivity.

In this book, the aim is to present our recent research results in designing non-fragile controllers/filters for linear systems. The main feature of this text is that the algebraic Riccati equation technique is successfully introduced to solve the type of additive/multiplicative norm-bounded controller/filter gain uncertainty, while a structured vertex separator is proposed to approach the numerical problem by considering interval-bounded coefficient variations. Moreover, sensitivity theory is always used to characterize the phenomenon of trivial deviations, which motivates us to design insensitive controllers/filters in the framework of coefficient sensitivity theory because the controller/filter coefficient variations resulting from limitations of the available computer memory are of trivial deviations. This book provides a coherent approach and contains valuable reference materials for researchers wishing to explore the area of non-fragile control/filtering. Its contents are also suitable for a one-semester graduate course.

The text focuses exclusively on the issues of non-fragile control/filtering in the framework of algebraic Riccati equations, LMI techniques, structured vertex separator methods, and coefficient sensitivity methods. The book begins with the development and main research methods in non-fragile control, while offering a systematic presentation of the newly proposed methods for non-fragile control/filtering of linear systems with respect to additive/ multiplicative controller/filter gain uncertainties. The tools for design and analysis presented in the book will be valuable in understanding and analyzing parameter uncertainties.

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Symbol Description

$\in$	belongs to
R	field of real numbers
R^n	n-dimensional real Euclidean space
$R^{n\times m}$	set of $n \times m$ real matrices
I	identity matrix
I_n	$n \times n$ identity matrix
X^T	transpose of matrix X
$P \geq 0$	symmetric positive semi-definite matrix $P \in R^{n\times n}$
$P > 0$	symmetric positive definite matrix $P \in R^{n\times n}$
$P \leq 0$	symmetric negative semi-definite matrix $P \in R^{n\times n}$
$P < 0$	symmetric negative definite matrix $P \in R^{n\times n}$
P^{-1}	the inverse of matrix P
$\mathrm{rank}(\cdot)$	rank of a matrix
$\mathrm{trace}(\cdot)$	trace of a matrix
$\\|\cdot\\|$	Euclidean matrix norm
$\\|e\\|_2$	L_2-norm of signal e
$L_2[0,\infty)$	space of square integrable functions on $[0,\infty)$
$l_2[0,\infty)$	space of square summable infinite vector sequences over $[0,\infty)$
$*$	symmetric terms in a symmetric matrix
$He\{M\}$	$He\{M\} := M + M^T$
$\bigotimes$	Kronecker product
$\mathrm{diag}\{...\}$	block diagonal matrix
$\bigoplus_{i=1}^{n} A_i$	block diagonal matrix with blocks $A_1,\ A_2, \cdots,\ A_n$, i.e., $\mathrm{diag}\{A_1, A_2, \cdots, A_n\}$
CCLM	cone complementarity linearization method
SLPMM	sequential linear programming matrix method
LMI	linear matrix inequality
BLMI	bilinear matrix inequality

1

Introduction

With the rapid development of computer and automation technologies, more and more attention is paid to the *digital control system* which has been considered as one of the most important and active fields in the research. A typical configuration of the digital control system is shown in Figure 1.1, in which limitation in available microprocessor memory, effects of *finite word length* (FWL) of the digital processor, errors for truncation and quantization of the A/D and D/A converters, and so on, always cause the controller parameters *trivial deviations* from the original design values [47]. Keel and Bhattacharyya [77], by means of numerical examples, have shown that the controllers designed by using weighted H_∞, μ, and L_1 synthesis techniques may be very sensitive, or fragile with respect to relatively small perturbations in controller parameters. Therefore, a significant issue is how to design a filter or controller for a given plant such that the filter or controller is insensitive to some errors with respect to its coefficients, that is, the designed filter or controller is *insensitive* or *non-fragile*.

The configuration shown in Figure 1.2 simply describes the *robust control*. Figure 1.3 shows the non-fragile control, while the robust *non-fragile control* is shown in Figure 1.4, where P denotes the plant, ΔP stands for the uncertainties of the plant P, K denotes the controller, and ΔK denotes the inaccuracies or uncertainties in the implementation of a designed controller. There are two main types of coefficient uncertainties considered in the designed methods. One is of a *norm-bounded* type, the other is of an *interval-bounded* type. Furthermore, the above two types also can be divided into *additive* case and *multiplicative* case. Then, the models of the uncertainty ΔK are given as follows:

- Norm-Bounded Uncertainty:

$$\begin{cases} \Delta K = H_a \Delta_a E_a, \Delta_a^T \Delta_a \leq I & \text{Additive Case} \\ \Delta K = H_m \Delta_m E_m K, \Delta_m^T \Delta_m \leq I & \text{Multiplicative Case} \end{cases} \tag{1.1}$$

where H_a, E_a, H_m, and E_m are known constant matrices of appropriate dimensions, and Δ_a and Δ_m are the uncertain parameter matrices.

- Interval-Bounded Uncertainty:

$$\begin{cases} \Delta K = [\theta_{ij}] & \text{Additive Case} \\ \Delta K = [\theta_{ij} k_{ij}] & \text{Multiplicative Case} \end{cases} \tag{1.2}$$

where k_{ij} denotes the (i, j)th element of the matrix K, and θ_{ij} $(|\theta_{ij}| \leq \theta)$ is

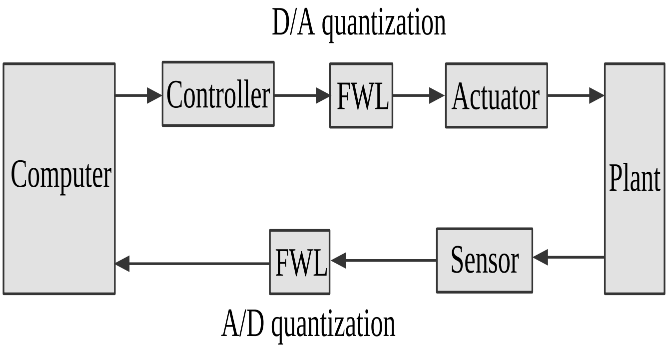


FIGURE 1.1
A typical digital control system configuration.

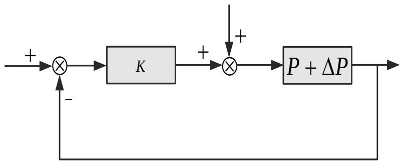


FIGURE 1.2
A robust control configuration.

used to describe the magnitude of the deviation of the matrix coefficient k_{ij}, where θ denotes the maximum possible deviation.

In recent years, the type of norm-bounded uncertainty (1.1), which is investigated in Chapters 3–5, has received wide attention, however, it cannot exactly reflect the uncertain information due to implementation imprecision. Therefore, the type of interval-bounded uncertainty (1.2) is introduced in Chapters 6 and 7. Yet, it has a numerical problem because the number of the linear matrix inequalities (LMIs) involved in the design conditions grows *exponentially* with the number of uncertain parameters, which make it difficult to apply the results to systems with high orders. Although the *structured vertex separator method* is proposed to deal with the numerical problem, the

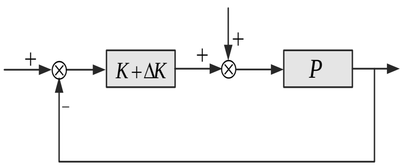


FIGURE 1.3
A non-fragile control configuration.

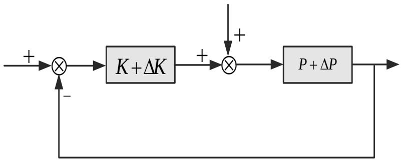


FIGURE 1.4
A robust non-fragile control configuration.

number of LMI constraints involved in the design conditions is still large. Furthermore, another important problem is that the computational efficiency is critical in real-time applications, so it is highly desirable for a controller to have a *sparse structure*, namely containing many trivial parameters (trivial parameters mean that they are 0 and ± 1, which can be digitally implemented exactly and cause no rounding errors). Other parameters are, therefore, referred to as nontrivial parameters [92]. The problem of finding sparse controller realizations has been considered by several researchers [3, 57, 58, 92]. These results consider the sparse structure problem from the point of view the controller realization. How to design a controller with sparse structure is a valuable problem.

On the other hand, *sensitivity analysis* allows us to assess the effects of changes in the parameter values [12, 13, 23, 80]. Hence, it is very useful to understand how changes in the parameter values influence the design [12, 13, 23, 50, 80]. After the hard work of many researchers in more than one decade, fundamental results have been obtained for the study of sensitivity analysis and performance limitations in automatic control systems ([see, for example, 21, 59, 60, 129, 140, 141], and the references therein), and many different definitions of sensitivity have been used for sensitivity analysis [61–64, 87, 91, 100, 125, 131, 132, 136, 152]. One of the effective synthesis methods is the *coefficient sensitivity method*, which describes the variations in performance due to variations in the parameters that affect the system dynamics [see 90, 92, 99, 121]. It is well known that very small perturbations in the coefficient of the designed controller or filter may result in the serious deterioration of the system performance, including instability. Therefore, the controller or filter should be designed to be insensitive to some amount of error with respect to its coefficients. Sensitivity theory is always used to characterize the phenomenon of *trivial deviations*, which motivates us to design insensitive controllers and filters in the framework of coefficient sensitivity theory in Chapters 8–11 because the coefficient variations resulted from the limitation of the available computer memory are of trivial deviations.

The main contribution of this book is that the algebraic Riccati technique, the *linear matrix inequality* technique, and the sensitivity analysis method have been successfully combined to establish a set of new non-fragile (insensitive) control methods [19, 47–49, 142–144, 147–150]. The proposed method can optimize the closed-loop system performances and simultaneously make the designed controllers or filters tolerant of coefficient variations in controller or filter gain matrices. Parts of the developed theories are applied to the simulation studies of the F-404 engine model and the F-18 aircraft model, which show intuitively the feasibility and superiority of the newly proposed methods.

A summary of the rest of the chapters of this book is given below.

Chapter 2 presents some preliminaries about the considered problem. Some lemmas to be used to derive the main results of this book are also given.

Chapter 3 investigates the problem of *guaranteed cost control* of discrete-time linear systems subject to *additive/multiplicative* controller gain uncertainties, respectively. First, an optimal guaranteed cost control design method is presented by using the algebraic Riccati equation technique. It is worth mentioning that the standard optimal control design for the same system can be obtained by modifying the cost function. Under a bound condition for the gain uncertainties, an optimal guaranteed cost control design method is also given for the case of the multiplicative gain uncertainties. The numerical example has shown the effectiveness of the proposed design procedures.

Based on the results in Chapter 3, Chapters 4 and 5 deal with the corresponding non-fragile controller and filter design problems. The procedures of designing non-fragile dynamic output-feedback controllers that can tolerate some *additive/multiplicative* controller gain uncertainties are presented in Chapter 4 in terms of symmetric positive-definite solutions of algebraic Riccati inequalities. Chapter 5 presents a robust *non-fragile Kalman filter* design method corresponding to the filter gain uncertainties in terms of solutions to *algebraic Riccati* equations, which depend on two design parameters, one from the system uncertainty and another from the state estimator gain uncertainty. When the controller/filter gain uncertainties are not considered, the results are reduced to those for the standard control. Finally, the effectiveness of the proposed methods is validated by numerical examples.

Chapter 6 studies the full parameterized and sparse structured non-fragile H_∞ controller design problems. The type of the additive interval-bounded coefficient variations, which less conservative than the type of norm-bounded controller gain uncertainties, is considered. First, a *two-step procedure* is adopted to solve the full parameterized controller design problem for the discrete-time and continuous-time systems, respectively. In addition, a structured vertex separator is proposed to approach the numerical computational problem resulting from the interval type of coefficient variations, and exploited to develop sufficient conditions for the non-fragile H_∞ controller design in terms of solutions to a set of LMIs. Second, for the sparse structured controller design problem, a class of sparse structures is specified. Then, a three-step procedure for non-fragile H_∞ controller design under the restriction of the sparse structure is provided. The contribution of this method is that it not only reduces the number of nontrivial parameters but also designs the *sparse structured controllers* with non-fragility. The resulting designs of the two cases guarantee that the closed-loop system is *asymptotically stable* and the H_∞ performance from the disturbance to the regulated output is less than a prescribed level. Finally, the effectiveness of the proposed design methods is illustrated by numerical examples.

Based on the results of Chapter 6, Chapter 7 deals with the problem of non-fragile H_∞ filter design subject to the additive interval-bounded filter coefficient variations. The full parameterized and sparse structured filter design problems are investigated simultaneously. For the full parameter filter design, the structured vertex separator proposed in the previous chapter is exploited to solve the numerical computational problem and to further develop sufficient conditions for the non-fragile H_∞ filter design in terms of solutions to a set of LMIs. For the sparse structured filter design, first, a class of *sparse structures* is specified. Then, an LMI-based procedure for non-fragile H_∞ filters design under the restriction of the sparse structure is provided. The effectiveness of the proposed methods are illustrated via some numerical examples and their simulations.

Chapter 8 investigates the problem of designing multi-objective coefficient insensitive H_∞ filters for linear continuous-time systems. Parameter sensitivity functions of transfer functions with respect to filter *additive/multiplicative* parameter variations are defined first, and the H_∞ norms of the sensitivity functions are used to measure the sensitivity of the transfer functions with respect to filter parameters. In addition, in order to deal with the filter design problem for the multiplicative filter coefficient variation case, new measures based on the average of the sensitivity functions are also defined. Based on the above two types of sensitivity measures, novel methods for designing insensitive H_∞ filters subjected to *additive/multiplicative* filter coefficient variations, respectively, are given in terms of LMI techniques. Furthermore, an indirect method for solving the multiplicative variations is also proposed. In comparison with the existing method, the new proposed method has less computational burden. In addition, it is difficult to use the techniques developed in Chapter 7 to obtain convex conditions for the filter design problem with respect to the interval multiplicative parameter variation case, while this problem can be resolved well by using the new proposed method. The simulation examples have also shown the effectiveness of the proposed method.

Based on the results in Chapter 8, Chapters 9 and 10 focus on the problems of designing multi-objective coefficient insensitive H_∞ filters and an *output tracking controller* for delta operator discrete-time systems, respectively. The designed filters/controllers are insensitive to the filter/controller parameter variations. Being different from using a common Lyapunov matrix of Chapter 8, the design conservatism is reduced by introducing slack variables in these two chapters. It is worth mentioning that the delta operator approach offers better parameter sensitivity than the traditional shift operator approach at a high sampling rate. Finally, some numerical examples including a linearized model of an F-404 *engine* and an F-18 *aircraft* are given to show the effectiveness and superiority of the proposed approaches in the above two chapters.

Chapter 11 studies the problem of designing multi-objective coefficient insensitive H_∞ *dynamic output feedback* controllers for linear discrete-time systems. Two different design methods with different degrees of conservativeness and computational complexity are proposed for this problem. The designed controllers are insensitive to the controller parameter variations. The first method presents a necessary and sufficient condition for the existence of the insensitive controller. The problem of designing multi-objective dynamic output feedback controllers is a *non-convex problem* itself, an LMI-based procedure which is a *sequential linear programming matrix method* (SLPMM) is proposed to solve this non-convex problem. However, the search for satisfactory solutions may be difficult when the SLPMM algorithm acts on a module of very high dimension. To overcome the above difficulty, the non-fragile controller design method is adopted to obtain an initial solution for the SLPMM algorithm for the first time. In the second method, a sufficient condition is provided for the multiplicative parameter variation case based on a new type of sensitivity measures. Finally, the effectiveness of the proposed method is validated by numerical examples.

2

Preliminaries

In this chapter, *non-fragile control* and *filtering* problems for *linear systems* are investigated under both H_∞ and *guaranteed cost performance index*, using the *linear matrix inequality (LMI)* technique and the *coefficient sensitivity method*. For the convenience of discussion in the rest of this chapter, some preliminaries, including a few definitions, notions, and lemmas, are presented in this chapter.

2.1 Delta Operator Definition

A definition of *delta operator* or Euler operator is introduced as follows:

Definition 2.1 *[44, 45] For a continuous-time signal $x(t)$, the discrete-time sequence by sampling the continuous-time signal is $x(nh)$ where h is the sampling period and $n = 0, 1, 2, \cdots$. We assume that $h = 0$ signifies $x(nh) = x(t)$. For $h \neq 0$ we denote $x(nh) = x_q(k), x((n+1)h) = x_q(k+1), k = 0, 1, 2, \cdots$. Then, the definition of an incremental difference operator (or delta operator for short) is given out as follows:*

$$\delta x(k) \triangleq \begin{cases} \frac{d}{dt}x(t) & h = 0 \\ (x(k+1) - x(k))/h & h \neq 0 \end{cases}$$

where the delta representation converges to the continuous-time representation as $h = 0$, and it converges to the discrete-time representation as $h \neq 0$. Obviously, the delta operator provides a theoretically unified formulation of continuous-time and discrete-time systems.

In addition, from the above definition, the delta operator and the traditional forward *shift operator* (q operator) are related as

$$\delta x(k) = \delta[x(nh)] = \frac{x(nh+h) - x(nh)}{h} = \frac{q[x(nh)] - x(nh)}{h}, \text{ for } h \neq 0 \quad (2.1)$$

where q is a forward shift operator ($qx_q(k) = q[x(nh)] = x_q(k+1)$) with $x_q(k)$ being sampled by using the forward shift operator approach. The above equation can be rewritten as

$$\delta = \frac{q-1}{h}.$$

In view of this, δ as a dynamic operator provides the same flexibility and implementability as a shift operator [44, 45]. However, it is well known that the usual shift operator approach suffers from numerical ill-conditioning at sufficiently small sampling periods. Therefore, in order to solve this problem, the delta operator instead of the traditional *shift operator* was constructed to study sampling continuous-time systems by Goodwin et al. [44, 45]. Two major advantages are known for the use of delta operator parameterization: a theoretically unified formulation of continuous-time and discrete-time systems, and better numerical properties in FWL implementations when compared with traditional z-transform at high sampling periods [90, 92]. Therefore, the delta operator is widely applied in many fields such as high-speed digital signal processing [36], system modeling [35,81], robust control/filtering [135], reliable control [116], and non-fragile control/filtering [47, 96].

2.2 H_∞ Performance Index

A popular performance measure of a stable linear time-invariant system is the H_∞ *norm* of its *transfer function*. It is defined as follows.

Definition 2.2 *[154] Consider a linear time-invariant continuous-time system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t) \\ z(t) &= Cx(t) + D_1\omega(t) \end{aligned} \tag{2.2}$$

where $x(t) \in R^n$ is the state, $\omega(t) \in R^s$ is an exogenous disturbance in $L_2[0,\infty]$, that is,

$$\|\omega(t)\|_2^2 = \int_0^\infty \omega^T(t)\omega(t)dt < \infty$$

and $z(t) \in R^r$ is the regulated output, respectively. A, B_1, C, D_1 are known constant matrices of appropriate dimensions.

Let $\gamma > 0$ be a given constant, then the system (2.2) is said to be with an H_∞ performance index no larger than γ, if the following conditions hold:
(1) Systems (2.2) are asymptotically stable
(2) Subject to initial conditions $x(0) = 0$, the transfer function matrix $T_{\omega z}(s)$ satisfies

$$\|T_{\omega z}(s)\|_\infty := \sup_{\|\omega\|_2 \le 1} \frac{\|z\|_2}{\|\omega\|_2} \le \gamma \tag{2.3}$$

Equation (2.3) is equivalent to

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt, \quad \forall \omega(t) \in L_2[0,\infty) \tag{2.4}$$

It is easy to see that the inequality (2.4) describes the restraint disturbance ability. Moreover, the smaller the value of γ is, the better the system performance is.

In addition, the definition of the H_∞ performance index for the z-domain or δ-domain is similar to Definition 2.2, therefore it is omitted here.

2.3 Operations on Systems

In this section, some facts about system interconnection are introduced, which will be used to obtain the sensitivity functions.

For brevity, the state-space models in the s-, z-, and δ-domains are unified as

$$\begin{aligned} \rho x(t) &= A_\rho x(t) + B_\rho u(t) \\ y(t) &= C_\rho x(t) + D_\rho u(t) \end{aligned} \tag{2.5}$$

where

$$\begin{cases} \rho x(t) = \dot{x}(t) & s\text{-domain} \\ \rho x(t) = x(t+1) & z\text{-domain} \\ \rho x(t) = \delta x(t) & \delta\text{-domain} \end{cases}$$

The state-space of the transfer function is described by

$$T(\rho) = \left[\begin{array}{c|c} A_\rho & B_\rho \\ \hline C_\rho & D_\rho \end{array}\right] = C_\rho(\rho I - A_\rho)^{-1}B_\rho + D_\rho$$

Then, the transpose of the transfer matrix $T(\rho)$ (or the dual system) is defined as

$$T^T(\rho) = \left[\begin{array}{c|c} A_\rho^T & C_\rho^T \\ \hline B_\rho^T & D_\rho^T \end{array}\right] = B_\rho^T(\rho I - A_\rho^T)^{-1}C_\rho^T + D_\rho^T$$

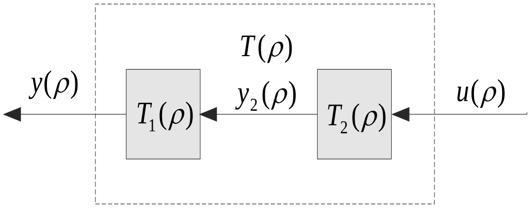
or equivalently

$$\left[\begin{array}{c|c} A_\rho & B_\rho \\ \hline C_\rho & D_\rho \end{array}\right] \longmapsto \left[\begin{array}{c|c} A_\rho^T & C_\rho^T \\ \hline B_\rho^T & D_\rho^T \end{array}\right]$$

Further, suppose that $T_1(\rho)$ and $T_2(\rho)$ are two subsystems with state-space representations:

$$\begin{cases} \rho x_1(t) = A_\rho^1 x_1(t) + B_\rho^1 u_1(t) \\ y_1(t) = C_\rho^1 x_1(t) + D_\rho^1 u_1(t) \end{cases}$$

$$\begin{cases} \rho x_2(t) = A_\rho^2 x_2(t) + B_\rho^2 u_2(t) \\ y_2(t) = C_\rho^2 x_2(t) + D_\rho^2 u_2(t) \end{cases}$$



$u(\rho)=u_2(\rho), u_1(\rho)=y_2(\rho), y(\rho)=y_1(\rho)$

FIGURE 2.1
Two subsystems in a series.

The state-space of their transfer functions can be described by

$$\begin{aligned} T_1(\rho) &= \left[\begin{array}{c|c} A_\rho^1 & B_\rho^1 \\ \hline C_\rho^1 & D_\rho^1 \end{array}\right] = C_\rho^1(\rho I - A_\rho^1)^{-1}B_\rho^1 + D_\rho^1 \\ T_2(\rho) &= \left[\begin{array}{c|c} A_\rho^2 & B_\rho^2 \\ \hline C_\rho^2 & D_\rho^2 \end{array}\right] = C_\rho^2(\rho I - A_\rho^2)^{-1}B_\rho^2 + D_\rho^2 \end{aligned}$$

Then the series or cascade connection of these two subsystems is a system with the output of the second subsystem as the input of the first subsystem as shown in the following.

$$u(\rho) = u_2(\rho), u_1(\rho) = y_2(\rho), y(\rho) = y_1(\rho)$$

The diagram is shown in Figure 2.1.

This operation in terms of the transfer matrices of the two subsystems is essentially the product of two transfer matrices. Hence, the representation for the series system can be obtained as

$$\begin{aligned} T(\rho) &= T_1(\rho)T_2(\rho) \\ &= \left[\begin{array}{c|c} A_\rho^1 & B_\rho^1 \\ \hline C_\rho^1 & D_\rho^1 \end{array}\right] \left[\begin{array}{c|c} A_\rho^2 & B_\rho^2 \\ \hline C_\rho^2 & D_\rho^2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\rho^1 & B_\rho^1 C_\rho^2 & B_\rho^1 D_\rho^2 \\ 0 & A_\rho^2 & B_\rho^2 \\ \hline C_\rho^1 & D_\rho^1 C_\rho^2 & D_\rho^1 D_\rho^2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\rho^2 & 0 & B_\rho^2 \\ B_\rho^1 C_2 & A_\rho^1 & B_\rho^1 D_\rho^2 \\ \hline D_\rho^1 C_\rho^2 & C_\rho^1 & D_\rho^1 D_\rho^2 \end{array}\right] \end{aligned}$$

Similarly, the parallel connection or the addition of $T_1(\rho)$ and $T_2(\rho)$ can

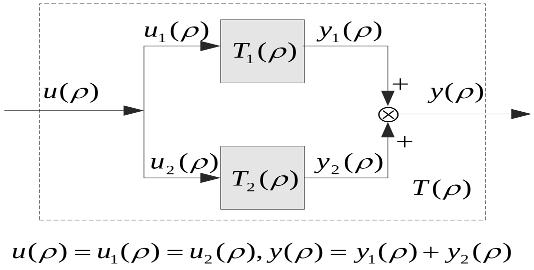


FIGURE 2.2
Two subsystems in parallel.

be obtained as

$$\begin{aligned} T(\rho) &= T_1(\rho) + T_2(\rho) \\ &= \left[\begin{array}{c|c} A_\rho^1 & B_\rho^1 \\ \hline C_\rho^1 & D_\rho^1 \end{array}\right] + \left[\begin{array}{c|c} A_\rho^2 & B_\rho^2 \\ \hline C_\rho^2 & D_\rho^2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\rho^1 & 0 & B_\rho^1 \\ 0 & A_\rho^2 & B_\rho^2 \\ \hline C_\rho^1 & C_\rho^2 & D_\rho^1 + D_\rho^2 \end{array}\right] \end{aligned}$$

The diagram is shown in Figure 2.2.
More system operations can be found in Zhou, Doyle, and Glover [154].

2.4 Some Other Definitions and Lemmas

Some other definitions and lemmas that will be used in this chapter are presented as follows.

Definition 2.3 *[152] For a matrix $M \in R^{n\times m}$, m_{ij} denotes the (i,j)th element of the matrix M. Then, $\frac{\partial M^{-1}}{\partial m_{ij}}$ can be evaluated by*

$$\frac{\partial M^{-1}}{\partial m_{ij}} = -M^{-1}\frac{\partial M}{\partial m_{ij}}M^{-1}$$

Definition 2.4 *[43] Let m_{ij} denote the (i,j)th element of the matrix M with M being an $m \times n$ real matrix and let $f(M)$ be a matrix function of M. Then, the coefficient sensitivity function of f with respect to the (i,j)th element of M is given by*

$$S_{ij} = \frac{\partial f}{\partial m_{ij}}$$

Definition 2.5 *[111, 151] Let $V(x_\delta(k))$ be a Lyapunov functional in the delta-domain. A delta operator system is asymptotically stable, if the following conditions hold:*

(i) $V(x_\delta(k)) \geq 0$, *with equality if and only if* $x_\delta(k) = 0$;

(ii) $\delta V(x_\delta(k)) = [V(x_\delta(k+1)) - V(x_\delta(k))]/h < 0$.

Remark 2.1 *For Lyapunov functional $V(\bullet)$ both in the z-domain and the s-domain, the condition (i) in Definition 2.5 can always be given. In condition (ii), when $h = 1$, there exists*

$$\delta V(x_\delta(k))|_{h=1} = \tfrac{V(x(nh+h))-V(x(nh))}{h}|_{h=1} = \Delta V(x_q(k)) < 0.$$

On the other hand, when $h \to 0$, referring to Equation (2.1) there exists

$$\lim_{h\to 0} \delta V(x_\delta(k)) = \lim_{h\to 0} \tfrac{V(x(nh+h))-V(x(nh))}{h} = \tfrac{dV(x(t))}{dt} < 0.$$

The above results imply that the Lyapunov functional in the δ-domain can be reduced to the traditional Lyapunov functional in the z-domain and s-domain when the sampling period is 1 or tends to be 0.

Now, some important lemmas are introduced, which will be useful in this chapter.

Lemma 2.1 *[11] (Schur Complement Lemma) For any given symmetric matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, where $S_{11} \in R^{r\times r}$. Then the following three conditions are equivalent:*

(i) $S < 0$

(ii) $S_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$

(ii) $S_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$

Lemma 2.2 *[113] Let matrices $Q = Q^T$, G, and a compact subset of real matrices $\mathbf{H}$ be given. Then the following statements are equivalent:*

(i) for each $H \in \mathbf{H}$

$$\xi^T Q \xi < 0 \text{ for all } \xi \neq 0 \text{ such that } HG\xi = 0;$$

(ii) there exists $\Theta = \Theta^T$ such that

$$Q + G^T \Theta G < 0, \mathbf{N}_H^T \Theta \mathbf{N}_H \geq 0 \text{ for all } H \in \mathbf{H}.$$

Lemma 2.3 *[154] Let $T_{az\omega} = C_a(sI - A_a)^{-1}B_a$, then A_a is Hurwitz and $\|T_{az\omega}\| < \gamma$ for some constant $\gamma > 0$ if and only if there exists a symmetric matrix $X > 0$ such that*

$$A_a^T X + XA_a + \frac{1}{\gamma^2} XB_a B_a^T X + C_a^T C_a < 0.$$

Lemma 2.4 *[46] Let $G_{az\omega}(z) = C_a(zI - A_a)^{-1}B_a$, then A_a is Shur stable and $\|G_{az\omega}(z)\| < \gamma$ for some constant $\gamma > 0$ if and only if there exists a symmetric matrix $X > 0$, such that*

$$\begin{bmatrix} -X & 0 & XA_a & XB_a \\ * & -I & C_a & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.6}$$

Denote

$$G_{0z\omega}(z) = C_{e0}(zI - A_{e0})^{-1}B_{e0}, \tag{2.7}$$

where

$$A_{e0} = \begin{bmatrix} A & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}, \quad B_{e0} = \begin{bmatrix} B_1 \\ B_kD_{21} \end{bmatrix}, C_{e0} = \begin{bmatrix} C_1 & D_{12}C_k \end{bmatrix}, \tag{2.8}$$

with $A_k \in R^{n\times n}$.

Then, we have the following lemma.

Lemma 2.5 *Let $\gamma > 0$ be a given constant. Then the following statements are equivalent:*

(i) *A_{e0} is Shur stable, and $\|G_{0z\omega}(z)\| < \gamma$;*

(ii) *there exists a symmetric positive matrix $X > 0$ such that*

$$\begin{bmatrix} -X & 0 & XA_{e0} & XB_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.9}$$

(iii) *there exists a symmetric positive matrix $X > 0$ and a matrix G such that*

$$\begin{bmatrix} X - G - G^T & 0 & G^TA_{e0} & G^TB_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.10}$$

(iv) *there exists a nonsingular matrix T and a symmetric matrix $P > 0$ with*

$$P = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \tag{2.11}$$

such that

$$\begin{bmatrix} -P & 0 & PA_{ea} & PB_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.12}$$

where

$$A_{ea} = \begin{bmatrix} A & B_2C_{ka} \\ B_{ka}C_2 & A_{ka} \end{bmatrix}, \quad B_{ea} = \begin{bmatrix} B_1 \\ B_{ka}D_{21} \end{bmatrix}, \quad C_{ea} = \begin{bmatrix} C_1 & D_{12}C_{ka} \end{bmatrix} \tag{2.13}$$

and

$$A_{ka} = T^{-1}A_kT, \quad B_{ka} = T^{-1}B_k, \quad C_{ka} = C_kT.$$

(v) there exist a symmetric matrix $X > 0$ and a matrix G with structure

$$G = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \tag{2.14}$$

such that

$$\begin{bmatrix} X - G - G^T & 0 & G^TA_{ea} & G^TB_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{2.15}$$

holds, where A_{ea}, B_{ea}, and C_{ea} are defined by (2.13).

Proof 2.1 *(i) $\Longleftrightarrow$ (ii). From Lemma 2.3, the equivalence of (i) and (ii) is immediate.*
(ii) $\Longleftrightarrow$ (iii). On the one hand, let $G = G^T = X$, then (iii) holds if (ii) holds. On the other hand, if (iii) holds, we have $X - G - G^T < 0$; obviously, G is invertible. It is known to all that

$$G^TX^{-1}G > G^T + G - X. \tag{2.16}$$

In fact, $(G^T - X)X^{-1}(G - X) > 0$. Now, according to (2.16), if (2.10) holds, then the following inequality holds:

$$\begin{bmatrix} -G^TX^{-1}G & 0 & G^TA_{e0} & G^TB_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0. \tag{2.17}$$

Let $T = diag\{G^{-1}X, I, I, I\}$ perform a transformation with T on (2.17), resulting in (2.9), which establishes that (iii) implies (ii).
(ii) $\Longleftrightarrow$ (iv). Notice the fact that, for any square matrix E and scalar $\eta > 0$, there exists an $\epsilon > 0$ with $\epsilon < \eta$ such that $E + \epsilon I$ is nonsingular, which implies statement (ii) if and only if there exists a symmetric matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} > 0$ with X_{12} nonsingular such that (2.9) holds. Let $A_{ka} = (X_{12}^{-1})^TX_{22}A_kX_{22}^{-1}X_{12}^T$, $B_{ka} = -(X_{12}^{-1})^TX_{22}B_k$, $C_{ka} = -C_kX_{22}^{-1}X_{12}^T$, $Y = X_{11}$, and $N = -X_{12}X_{22}^{-1}X_{12}^T$.

Denote $\bar{\Gamma} = diag\{\Gamma, I, \Gamma, I\}$, *where* $\Gamma = \begin{bmatrix} I & 0 \\ 0 & -X_{12}X_{22}^{-1} \end{bmatrix}$. *Then*

$$P = \Gamma X \Gamma^T = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}$$

and

$$\begin{bmatrix} -P & 0 & PA_{ea} & PB_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} \\ = \bar{\Gamma} \begin{bmatrix} -X & 0 & XA_{e0} & XB_{e0} \\ * & -I & C_{e0} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} \bar{\Gamma}^T < 0,$$

so inequalities $X > 0$ *and (2.9) are equivalent to* $P > 0$ *and (2.12), respectively.*
(iv) $\Longleftrightarrow$ *(v). On the one hand, let* $G = G^T = X = P > 0$ *with the structure (2.11), then (2.15) holds if (2.12) holds.*
On the other hand, let $X = P$ *with the structure (2.11), according to (2.16), then we have*

$$\begin{bmatrix} -G^T P^{-1} G & 0 & G^T A_{ea} & G^T B_{ea} \\ * & -I & C_{ea} & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{2.18}$$

which holds if (2.15) holds. Let $\Upsilon = diag\{G^{-1}P, I, I, I\}$ *perform a transformation with* Υ *on (2.18), resulting in (2.12), which establishes that (v) implies (iv).*
Thus, the proof is complete.

Denote

$$T_{0z\omega} = C_{e0}(sI - A_{e0})^{-1} B_{e0}.$$

Let controller gain matrices A_k, B_k, and C_k be given, and such that

$$\|T_{0z\omega}\| = \|C_{e0}(sI - A_{e0})^{-1} B_{e0}\| < \gamma. \tag{2.19}$$

Lemma 2.6 *Let* $T_{az\omega} = C_a(sI - A_a)^{-1} B_a$, *then* A_a *is stable and* $\|T_{az\omega}\| < \gamma$ *for some constant* $\gamma > 0$ *if and only if there exists a symmetric matrix* $X > 0$, *such that*

$$A_a^T X + X A_a + \frac{1}{\gamma^2} X B_a B_a^T X + C_a^T C_a < 0.$$