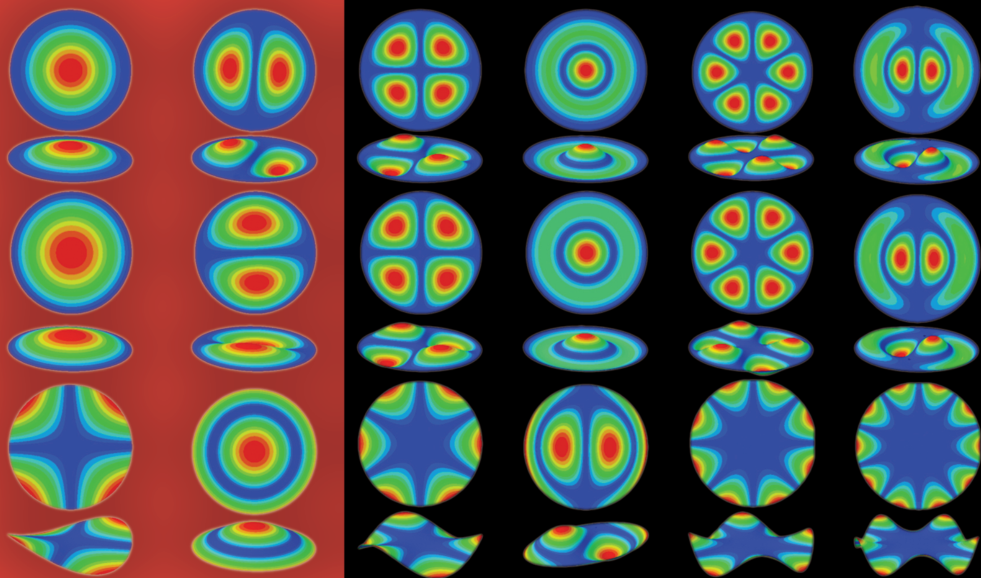
STRUCTURAL VIBRATION

Exact Solutions for Strings, Membranes, Beams, and Plates



C.Y. Wang and C.M. Wang

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Preface

There is a staggering number of research studies on the vibration of structures. Based on a simple search using the Science Citation Index, the numbers of references associated with the following words are 1,000 for “vibration and string,” 2,000 for “vibration and membrane,” 7,000 for “vibration and plate,” and 16,000 for “vibration and beam, bar or rod.” This clearly illustrates the importance of the subject of free and forced vibrations for analysis and design of structures and machines.

The free vibration of a structural member eventually ceases due to energy dissipation, either from the material strains or from the resistance of the surrounding fluid. The frequency of such a system will be lowered by damping. But since damping also causes the amplitude to decay, the resonance with a forced excitation of a strongly damped system will not be as important as the weakly damped system. In this book, we shall consider the undamped system, which models the weakly damped system, and only focus on the exact solutions for free transverse vibration of strings, bars, membranes, and plates because these solutions elucidate the intrinsic, fundamental, and unexpected features of the solutions. They also serve as benchmarks to assess the validity, convergence, and accuracy of numerical methods and approximate analytical methods. We define exact solutions to mean solutions in terms of known functions as well as those solutions determined from exact characteristic equations. However, this book will not cover longitudinal in-plane/translational vibrations, shear waves, torsional oscillations, infinite domains (wave propagation), discrete systems (such as linked masses), and frames. The exact solutions for a wide range of differential equations are useful to academics teaching differential equations, as they may draw the practical problems associated with the differential equations.

There are seven chapters in this book. Chapter 1 gives the introduction to structural vibration and the importance of the natural frequencies in design. Chapter 2 presents the vibration solutions for strings. Chapter 3 presents the vibration solutions for membranes. Chapter 4 deals with vibration of bars and beams. Chapter 5 gives the vibration solutions for isotropic plates with uniform thickness. Chapter 6 deals with plates with complicating effects such as the presence of in-plane forces, internal spring support, internal hinge, elastic foundation, and nonuniform thickness distribution. Chapter 7 presents vibration solutions for nonisotropic plates, such as orthotropic, sandwich, laminated, and functionally graded plates.

Owing to the vastness of the literature, there may be relevant papers that escaped our search in the Science Citation Index. To these authors, we offer our sincere apology. Such omissions shall be rectified in a future edition.

Finally, we wish to express our thanks to Dr. Tay Zhi Yung and Mr. Ding Zhiwei of the National University of Singapore for checking the manuscript and plotting the vibration mode shapes and also to Dr. Liu Bo of The Solid Mechanics Research Centre, Beihang University, China, for contributing the sections on rectangular isotropic and orthotropic Mindlin plates.

C. Y. Wang and C. M. Wang

About the Authors

C. Y. Wang is a professor in the Department of Mathematics with a joint appointment in mechanical engineering at the Michigan State University, East Lansing, Michigan. He obtained his BS from Taiwan University and PhD from Massachusetts Institute of Technology. Prof. Wang has published about 170 papers in solid mechanics (elastica, torsion, buckling, and vibrations of structural members), 170 papers in fluid mechanics (exact Navier-Stokes solutions, Stokes flow, unsteady viscous flow), and 120 papers in other areas (biological, thermal, electromechanics). Prof. Wang wrote a monograph *Perturbation Methods* (Taiwan University Press) and is a coauthor of *Exact Solutions for Buckling of Structural Members* (CRC Press). He has served as a technical editor for *Applied Mechanics Reviews.*

C. M. Wang is a professor in the Department of Civil and Environmental Engineering and the director of the Engineering Science Programme, Faculty of Engineering, National University of Singapore. He is a chartered structural engineer, a fellow of the Singapore Academy of Engineering, a fellow of the Institution of Engineers Singapore, and a fellow of the Institution of Structural Engineers. His research interests are in the areas of structural stability, vibration, optimization, plated structures, and Mega-Floats. He has published more than 400 scientific publications, co-edited three books, *Analysis and Design of Plated Structures: Stability and Dynamics: Volumes 1 and 2* (Woodhead Publishing) and *Very Large Floating Structures* (Taylor & Francis) and co-authored three books: *Vibration of Mindlin Plates* (Elsevier), *Shear Deformable Beams and Plates: Relationships with Classical Solutions* (Elsevier), and *Exact Solutions for Buckling of Structural Members* (CRC Press). He is the editor-in-chief of the *International Journal of Structural Stability and Dynamics* and the *IES Journal Part A: Civil and Structural Engineering* and an editorial board member of *Engineering Structures, Advances in Applied Mathematics and Mechanics*, *Ocean Systems Engineering*, and *International Journal of Applied Mechanics.*

1 Introduction to Structural Vibration

1.1 WHAT IS VIBRATION?

Vibration may be regarded as any motion that repeats itself after an interval of time, or one may define vibrations as oscillations of a system about a position of equilibrium (Kelly 2007). Examples of vibratory motion include the swinging of a pendulum, the motion of a plucked guitar string, tidal motion, the chirping of a male cicada by rubbing its wings, the flapping of airplane wings in turbulence, the soothing motion of a massage chair, or the swaying of a slender tall building due to wind or an earthquake.

The key parameters in describing vibration are amplitude, period, and frequency. The amplitude of vibration is the maximum displacement of a vibrating particle or body from its position of equilibrium, and this is related to the applied energy. The period is the time taken for one complete cycle of the motion. The frequency is the number of cycles per unit time or the reciprocal of the period. The angular (or circular) frequency is the product of the frequency and 2π, and hence its unit is radians per unit time.

Vibrations may be classified as either *free vibration* or *forced vibration*. Free vibration takes place when a system oscillates under the action of forces inherent within the system itself—when externally imposed forces are absent. A system under free vibration will vibrate at one or more of its natural frequencies, which are dependent on the mass and stiffness distributions as well as the boundary conditions. In contrast, forced vibration occurs when an external periodic force is applied to the system.

When the effects of friction can be neglected, the vibrations are referred to as undamped. Realistically, all vibrations are damped to some degree. If a free vibration is only slightly damped, its amplitude gradually decreases until the motion comes to an end after a certain time. If the damping is sufficiently large, vibration is suppressed, and the system then quickly regains its original equilibrium position. A damped forced vibration is maintained so long as the periodic force that causes the vibration is applied. The amplitude of the vibration is affected by the magnitude of the damping forces.

From an energy viewpoint, vibration may be defined as a phenomenon that involves alternating interchange of potential energy and kinetic energy. If the system is damped, then some energy is dissipated in each cycle of the vibration, and the vibratory motion will ultimately come to an end. If a steady motion of vibration is to be maintained, then the energy dissipated due to damping has to be compensated by an external source.

1.2 BRIEF HISTORICAL REVIEW ON VIBRATION OF STRINGS, MEMBRANES, BEAMS, AND PLATES

According to Rao (1986, 2005), it is likely that the interest in vibration dates back to the time of the discovery of early musical instruments such as whistles, strings, or drums, which produce sound from vibration. Drawings of stringed instruments have been found on the walls of Egyptian tombs that were built around 3000 BC.

In the course of seeking why some notes sounded more pleasant than others, the Greek mathematician and philosopher Pythagoras (582–507 BC) conducted experiments on vibrating strings, and he observed that the pitch of the note (the frequency of the sound) was dependent on the tension and length of the string. Galileo (1638), the Italian physicist and astronomer, took measurements to establish a relationship between the length and frequency of vibration for a simple pendulum and for strings; he also observed the resonance of two connecting bodies. Marinus Mersenne (1636), a French mathematician and theologian, also studied the behavior of vibrating strings. English scientist Robert Hooke (1635–1703) and French mathematician and physicist Joseph Sauveur (1653–1716) performed further studies on the relationship between the pitch and frequency of a vibrating taut string. Sauveur is noted for introducing the terms *nodes* (stationary points), *loops*, *fundamental frequency*, and *harmonics*, and he is the first scientist to record the phenomenon of *beats*.

The breakthrough in formulating the governing equations for structural vibration problems may be attributed to Sir Isaac Newton (1687), who was the first to formulate the laws of classical mechanics, and to Gottfried Leibniz (1693) as well as Newton for creating calculus. Euler (1744) and Bernoulli (1751) discovered the differential equation governing the lateral vibration of prismatic bars and investigated its solution for the case of small deflections. Lagrange (1759) also made important contributions to the theory of vibrating strings. Euler (1766) derived the equations for the vibration of rectangular membranes under uniform tension as well as for the vibration of a ring. Poisson (1829) derived the governing equation for vibrating circular membranes and gave the solutions for the axisymmetric vibration mode. Pagani (1829) worked out the nonaxisymmetric vibration solution for circular membranes. Coulomb (1784) investigated the torsional oscillations of a metal cylinder suspended by a wire.

The German physicist Chladni observed nodal patterns on flat square plates at their resonant frequencies using sand spread evenly on the plate surface. The sand formed regular patterns as the sand accumulated along the nodal lines of zero vertical displacements upon induction of vibration. Figure 1.1 shows the patterns of square plates that were originally published in Chladni's book (Chladni 1802). In 1816, Sophie Germain successfully derived the differential equation for the vibration of plates by means of calculus of variations. However, she made a mistake in neglecting the strain energy due to the twisting of the plate mid-plane. The correct version of the governing differential equation, without its derivation, was found posthumously among Lagrange's notes in 1813. Thus, Lagrange has been credited as being the first to present the correct equation for thin plates. By using trigonometric series introduced by Fourier around that time, Navier (1823) was able to readily determine the exact vibration solutions for rectangular plates with simply supported

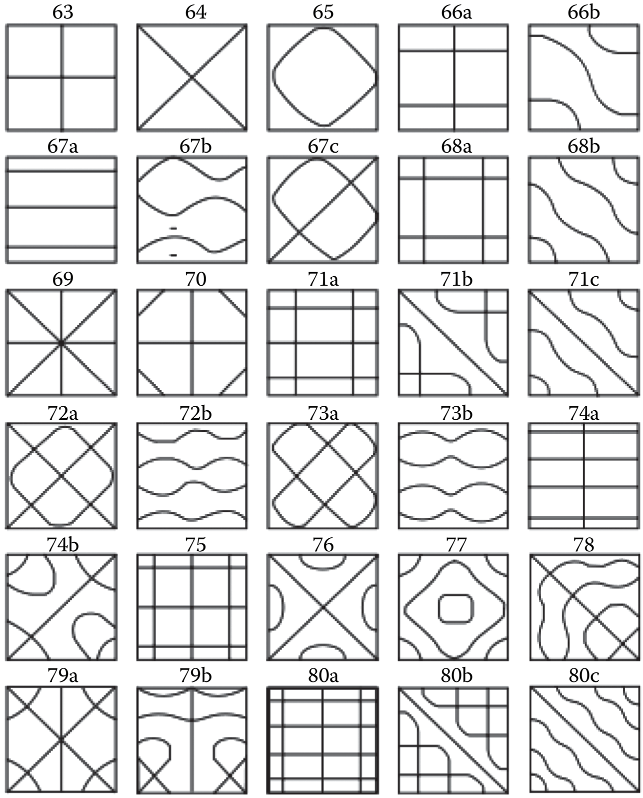

FIGURE 1.1 Chladni's original figures of vibrating square plates showing nodal lines. *Source*: http://en.wikipedia.org/wiki/File:Chladini.Diagrams.for.Quadratic.Plates.svg.

edges. Poisson (1829) extended Navier's work to circular plates. The extended plate theory that considered the combined bending and stretching actions of a plate has been attributed to Kirchhoff (1850). His other significant contribution is the application of a virtual displacement method for solving plate problems.

Lord Rayleigh (1877) presented a theory to explain the phenomenon of vibration that to this day is still used to determine the natural frequencies of vibrating structures. Based on the plate assumptions made by Kirchhoff (1850) and Rayleigh's theory, early researchers used analytical techniques to solve the vibration problems of plates. For example, Voigt (1893) and Carrington (1925) successfully derived the exact vibration frequency solutions for a simply supported rectangular plate and a fully clamped circular plate, respectively. Ritz (1909) was one of the early researchers to solve the problem of the freely vibrating plate, which does not have an exact solution. He demonstrated how to reduce the upper-bound frequencies by including more than a single trial (admissible) function and performing a minimization with respect to the unknown coefficients of these trial functions. The method became known as the Ritz method. Liew and Wang (1992, 1993) automated the Ritz method for analysis of arbitrarily shaped plates.

The theories of vibration of beams and plates were investigated further by Timoshenko (1921) and Mindlin (1951), and their theories allow for the effects of transverse shear deformation and rotary inertia. Other, more refined beam and plate theories that do away with the need for a shear correction factor were developed by Bickford (1982), Reddy (1984), and Reddy and Phan (1985), who employed higher-order polynomials in the expansion of the displacement components through the beam or plate thickness. Leissa (1969) produced an excellent monograph entitled "Vibration of Plates," which contains a wealth of vibration solutions for a wide range of plate shapes and boundary conditions. Originally published by NASA in 1969, Leissa's monograph was reprinted in 1993 by the Acoustical Society of America due to popular demand.

1.3 IMPORTANCE OF VIBRATION ANALYSIS IN STRUCTURAL DESIGN

When designing structures, the effect of vibration on them is a very important factor to consider. Obviously, structures used to support heavy centrifugal machines like motors and turbines are subjected to vibration. Vibration causes excessive wear of bearings, material cracking, fasteners to become loose, noise, and abrasion of insulation around electrical conductors, resulting in short circuiting (Wowk 1991). When cutting a metal, vibration can cause chatter, which affects the quality of the surface finish. Structural vibration may cause discomfort and even fear in the occupants working in the building, make it difficult to operate machinery, and cause malfunctioning of equipment.

The natural frequencies of a structure are very important to structural and mechanical engineers when designing for human comfort, structural serviceability and operational requirements, and against the occurrence of resonance. Resonance occurs when the natural frequency of the structure coincides with the excitation frequency. This resonance phenomenon has to be avoided so as to prevent excessive deformation, fatigue cracks, and even the collapse of the entire structure. For example, the spectacular collapse of the Tacoma Narrows suspension bridge (that spanned the Tacoma Narrows strait of Puget Sound between Tacoma and the Kitsap Peninsula in the U.S. state of Washington) in 1940 was a result of resonance caused by strong wind gusts. Therefore, structural engineers design their structures to have a fundamental natural frequency of vibration that satisfies a specific minimum frequency given in design codes. For instance, the American Association of State Highway and Transportation Officials (AASHTO) specifies the minimum frequency for a pedestrian bridge to be 3 Hz. For office buildings, it is recommended that the natural frequency of floor structures be kept to within 4 Hz, whereas for performance stages and dance floors, this minimum limit of natural frequency may be raised to 8.4 Hz (Technical Guidance Note 2012).

Given the undesirable and devastating effects that vibrations can have on machines and structures, vibration analysis and testing have become a standard procedure in the design of structures (Richardson and Ramsey 1981; McConnell and Varoto 2008). Vibration may be reduced by using the illustrative vibrating mechanical

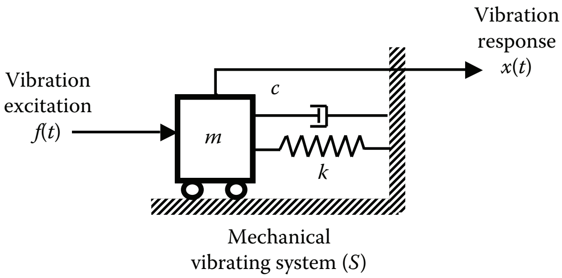

FIGURE 1.2 A vibrating mechanical system.

system shown in Figure 1.2, where the forcing excitations $f(t)$ to the mechanical system S cause the vibration response $x(t)$. The problem at hand is to suppress $x(t)$ to an acceptable level. The three general ways to do this are:

1. *Isolation*. Suppress the excitations of the vibration. This method deals with the forcing excitation $f(t)$
2. *Design modification*. Modify or redesign the mechanical system so that for the same levels of excitation, the resulting vibrations are acceptable. This method deals with the mechanical system S, which has a mass m, stiffness k, and damping coefficient c.
3. *Control*. Absorb or dissipate the vibrations using external devices, through implicit or explicit sensing and control. This method deals with the vibration response $x(t)$.

Within each category, there are several approaches for mitigating vibration. Actually, each of these approaches needs either redesign or modification. It is to be noted that the removal of faults (e.g., misalignments and malfunctions by repair or parts replacement) can also reduce vibrations. This approach may be included in any of the three categories listed here (De Silva 2007).

In order to understand isolation well, we need to know the concept of mechanical impedance (Wowk 1991). When vibrations travel through different materials and metal interfaces, they get reduced or attenuated. With the concept of impedance, we can insert materials into the force transmission path so as to reduce the amplitude of the vibration. Generally, any material with a lower stiffness than the adjacent material will function well to attenuate the force, and it works in both directions. Mechanical springs, air springs, cork, fiberglass, polymer, and rubber are the typical isolator materials. The performance of the isolator is a function of frequency.

On the other hand, vibration can also be useful in several industrial applications. For example, compactors, vibratory conveyors, hoppers, sieves, and washing machines take advantage of vibration to do the job. More interestingly, vibrations are found to be able to improve the efficiency of certain machining, casting, forging, and welding processes. Vibration is also used in nondestructive testing of materials and structures, in vibratory finishing processes, and in electronic circuits to filter out the unwanted frequencies (Rao 1986). It is also employed in shake tables to simulate earthquakes for testing structural designs against seismic action. Of course, most people enjoy the vibration of a massaging chair/device on their bodies.

1.4 SCOPE OF BOOK

In this book, we focus our attention on the free, harmonic, and flexural vibration of strings, membranes, beams, and plates. Damping is assumed to be small, and hence it is neglected. In each of the many structural vibration problems treated herein, we present the exact natural angular (or circular) frequencies and their accompanying mode shapes. Exact solutions are very important, as they clearly reveal the intrinsic features of the solutions and provide benchmarks to assess the validity, convergence, and accuracy of numerical solutions. Here, we define an exact solution as one that can be expressed in terms of a finite number of terms, and the proposed solution may contain elementary or common functions such as harmonic or Bessel functions. Special functions, such as hypergeometric functions, are excluded. Analytical solutions that are not exact, such as infinite series solutions and asymptotic solutions, are also excluded.

The governing differential equations of motion for the problems treated herein are obtained by using the method of elementary analysis, and the equations are solved for different boundary conditions. Analytical vibration solutions of structures with complicated geometries and boundary conditions are difficult or impossible to obtain. In such cases, numerical methods are required. However, for some cases of structural geometries and boundary conditions, it is possible to solve the differential equations exactly in a closed form. In this book, the authors present as many analytical vibration solutions as possible in one single volume for ready use by engineers, academicians, and researchers in structural dynamic analysis and design. This book addresses a variety of boundary conditions, restraints, and mass and stiffness distributions in the hope that the reader may better understand the effects of shape, restraints, and boundary conditions on vibration frequencies and mode shapes.

The numerous differential equations and their solutions presented in this book are also useful for academicians, especially when they wish to provide practical problems to the differential equations that they present to students of engineering science.

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2 Vibration of Strings

2.1 INTRODUCTION

Strings are basic structural elements that support only tension. They also approximate cables, with negligible bending, and chains, with numerous links. There seems to be no exact solution to the vibration of strings where the static shape deviates from a straight line. We assume that the tension, density, and deflection are continuous functions of position along the string, except perhaps at a single point in the interior span of the string. Thus, we limit our presentation to a string with at most two connected segments. For example, we include a string composed of two segments of different constant densities, but not multiple segments, since the solution of the latter can be extended similarly.

In the tables and figures of this chapter, we present the first five natural frequencies of vibration. The lowest one is the fundamental frequency, below which no natural vibration would occur.

2.2 ASSUMPTIONS AND GOVERNING EQUATIONS FOR STRINGS

A string is slender, i.e., its lateral dimensions are infinitesimal compared to the longitudinal length. It does not admit any bending moment, shear, or axial compression. Rotational inertia is negligible. We assume that there is a stable equilibrium straight state. The vibrations, mainly lateral, are small compared to the string length, which is finite.

One can derive the string equations by considering the dynamic balance on an elemental segment as shown in Figure 2.1, or if damping is absent, by the energy method. The governing equation of a string, derived in many texts (e.g., see Magrab 2004), is given by

$$\frac{\partial}{\partial x'}\left(T'(x')\frac{\partial w'}{\partial x'}\right)=\rho(x')\frac{\partial^2 w'}{\partial t'^2} \tag{2.1}$$

Here w' is the lateral deflection, x' is the distance from one end, $T'(x') > 0$ is the tension, $\rho(x') > 0$ is the density (mass per unit length), and t' is the time. The tension is governed by the static force balance, i.e.,

$$\frac{dT'}{dx'}+f'(x')=0 \tag{2.2}$$

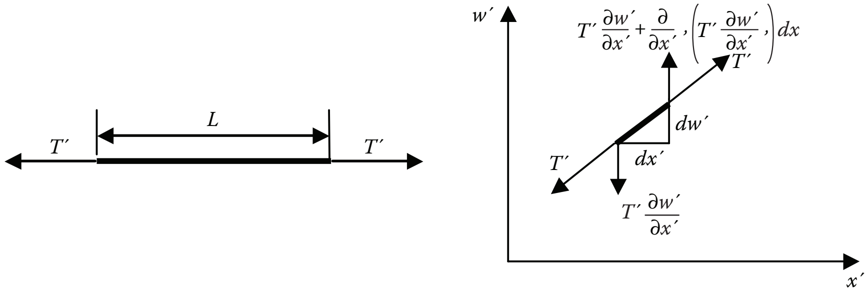


FIGURE 2.1 String under tension and an elemental segment of string.

where f' is the force per length acting along the string. Of particular interest is when the string is hanging (i.e., f' is constant).

By normalizing all lengths by the string length L, the tension by the maximum tension T_0, the density by the maximum density ρ_0, and the time by $L\sqrt{\rho_0/T_0}$ and dropping the primes, Equation (2.1) becomes

$$\frac{\partial}{\partial x}\left(T\frac{\partial w'}{\partial x}\right)=\rho\frac{\partial^2 w'}{\partial t^2} \tag{2.3}$$

For free vibrations, we can assume that

$$w'(x',t')=\bar{w}(x)e^{i\bar{\omega}t'} \tag{2.4}$$

where $i=\sqrt{-1}$ and $\bar{\omega}$ is the angular frequency of vibration. Let $w=\bar{w}/L, T=T'/T_0$, $t=t'/(L\sqrt{\rho_0/T_0})$, $\omega=\bar{\omega}L\sqrt{\rho_0/T_0}$, and recognizing that only the real part of w has significance, Equation (2.3) becomes

$$\frac{d}{dx}\left(T\frac{dw}{dx}\right)+\rho\omega^2 w=0 \tag{2.5}$$

2.3 BOUNDARY CONDITIONS

The boundary conditions at an end of a string include

- Fixed end, where

$$w=0 \tag{2.6}$$

- Sliding end, where there is no transverse resistance

$$\frac{dw}{dx}=0 \tag{2.7}$$

- Massed end, where, by transverse force balance,

$$T'\frac{\partial w'}{\partial x'} = \mp m\frac{\partial^2 w'}{\partial t'^2} \tag{2.8}$$

  or in a nondimensional form

$$T\frac{dw}{dx} \mp \alpha\omega^2 w = 0 \tag{2.9}$$

  where m is the point mass at the end, $\alpha = m/\rho_0 L$ is a mass ratio, the top sign is for an end with the normal in the x-direction, and the bottom sign otherwise.

- Elastically lateral supported end, where

$$T'\frac{\partial w'}{\partial x'} = \mp kw' \tag{2.10}$$

  or in a nondimensional form

$$T\frac{dw}{dx} = \mp\beta w \tag{2.11}$$

  where k is the spring constant and $\beta = kL/T_0$ is a normalized spring constant.

There are other boundary conditions, such as viscous dashpots, which are not as important. The aforementioned boundary conditions can be combined into a canonical form, i.e.,

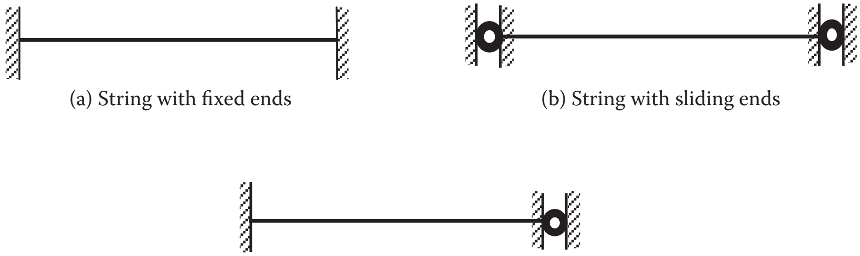
$$T\frac{dw}{dx} \mp (\alpha\omega^2 + \beta)w = 0 \tag{2.12}$$

For a fixed end, α or β is infinite, and for a sliding end, $\alpha = \beta = 0$.

2.4 CONSTANT PROPERTY STRING

In this case, the tension and density are constants. By setting $T = \rho = 1$, Equation (2.5) becomes

$$\frac{d^2 w}{dx^2} + \omega^2 w = 0 \tag{2.13}$$



(a) String with fixed ends

(b) String with sliding ends

(c) String with one end fixed and the other end sliding

FIGURE 2.2 Strings with different end conditions.

Let α_0, β_0 be the values at $x = 0$, and α_1, β_1 be the values at $x = 1$. The solution of Equation (2.13) is

$$w = C_1 \sin(\omega x) + C_2 \cos(\omega x) \tag{2.14}$$

In view of the boundary conditions in Equation (2.12), one obtains

$$\omega C_1 - (\alpha_0 \omega^2 - \beta_0) C_2 = 0 \tag{2.15}$$

$$\omega(\cos\omega\, C_1 - \sin\omega\, C_2) - (\alpha_1\omega^2 - \beta_1)(\sin\omega\, C_1 + \cos\omega\, C_2) = 0 \tag{2.16}$$

For nontrivial C_1, C_2, the exact characteristic equation for the frequency ω is

$$\omega[(\alpha_1\omega^2 - \beta_1)\cos\omega + \omega\sin\omega] - (\alpha_0\omega^2 - \beta_0)[\omega\cos\omega - (\alpha_1\omega^2 - \beta_1)\sin\omega] = 0 \tag{2.17}$$

If both ends are fixed, set α_0 or β_0, and α_1 or β_1 to infinity. Thus, we obtain $\sin\omega = 0$ or $\omega = n\pi$, where n is a positive integer. The fundamental frequency, or the frequency below which the string would not vibrate, is $\omega = \pi$. If both ends are sliding, set $\alpha_0 = \beta_0 = 0$ and $\alpha_1 = \beta_1 = 0$, and the frequencies are the same, i.e., $\omega = n\pi$. For one end fixed and one end sliding, we find $\cos\omega = 0$ or $\omega = (n - 1/2)\pi$. The frequencies for other combinations can be generated from Equation (2.17). Strings with different end conditions are shown in Figure 2.2. Mode shapes for strings with different end conditions are shown in Figures 2.3a, 2.3b, and 2.3c. Since the vibration amplitudes are arbitrary, they are made equal in the figures.

2.5 TWO-SEGMENT CONSTANT PROPERTY STRING

We consider a composite string composed of two connected constant-property segments. Let a subscript 1 denote the segment $0 \le x \le b$ and a subscript 2 denote the segment $b \le x \le 1$. At the joint, the string is continuous

$$w_1(b) = w_2(b) \tag{2.18}$$

Also there may be a point mass and a supporting spring at the joint. By carrying out a transverse force balance at $x = b$, one obtains

$$T_2 \frac{dw_2}{dx} - T_1 \frac{dw_1}{dx} + (\alpha\omega^2 - \beta)w = 0 \tag{2.19}$$

Three important cases will be illustrated. In each case we assume that the tension is the same, i.e., $T_1 = T_2 = 1$, and the ends are fixed.

2.5.1 Different Densities

Figure 2.4 shows a two-segment composite string. Segment 1 has the maximum density ($\rho_1 = 1$), whereas segment 2 has the smaller density ($\rho_2 < \rho_1$).

Let $\gamma = \rho_2/\rho_1 \leq 1$. The governing equations are

$$\frac{d^2 w_1}{dx^2} + \omega^2 w_1 = 0 \tag{2.20}$$

$$\frac{d^2 w_2}{dx^2} + \gamma\omega^2 w_2 = 0 \tag{2.21}$$

The boundary conditions are that the deflections are zero at the ends, i.e.,

$$w_1(0) = 0, \quad w_2(1) = 0 \tag{2.22}$$

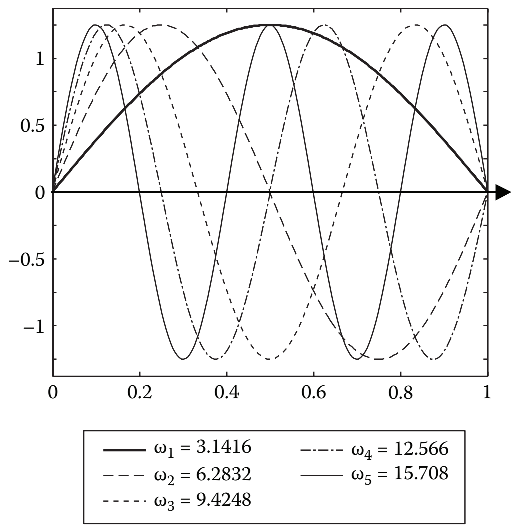


FIGURE 2.3 (a) Mode shapes for a string with fixed ends. (*continued*)

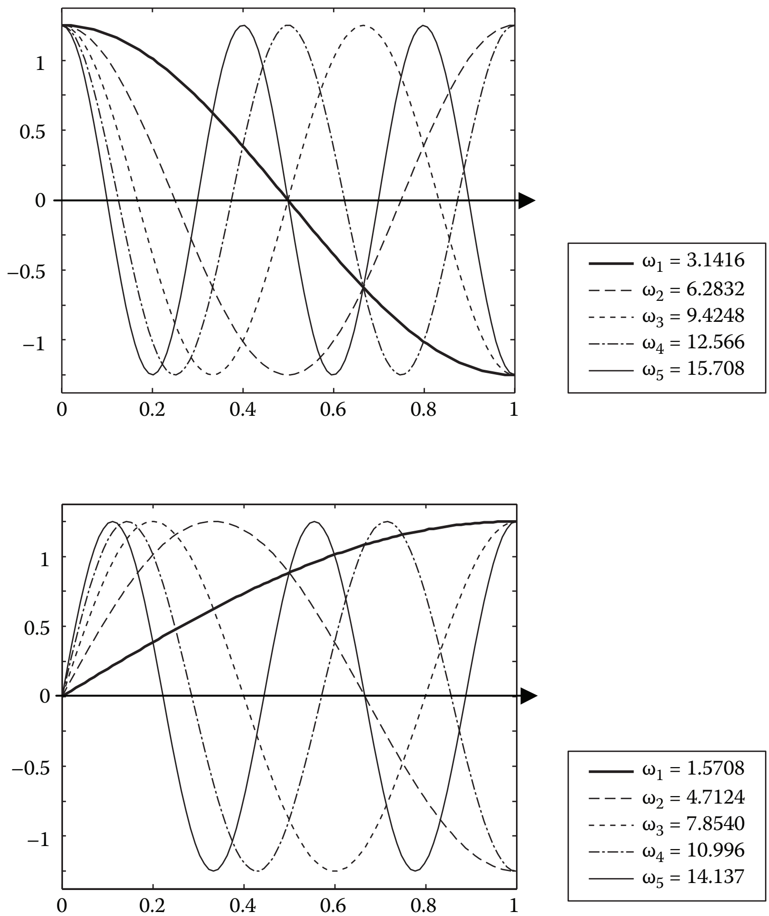


FIGURE 2.3 (b) Mode shapes for a string with sliding ends. (c) Mode shapes with one end fixed and one end sliding.

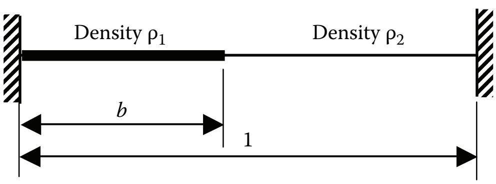


FIGURE 2.4 Composite string with two segments of different densities.

The solutions for the foregoing governing equations and boundary conditions are

$$w_1 = C_1 \sin(\omega x), \quad w_2 = C_2 \sin\left[\sqrt{\gamma}\omega(x-1)\right] \tag{2.23}$$

At the joint, we have from Equation (2.19)

$$\frac{dw_1}{dx}(b) = \frac{dw_2}{dx}(b) \tag{2.24}$$

Equations (2.18) and (2.24) yield the characteristic equation

$$\sqrt{\gamma}\sin(\omega b)\cos\left[\sqrt{\gamma}\omega(1-b)\right] + \cos(\omega b)\sin\left[\sqrt{\gamma}\omega(1-b)\right] = 0 \tag{2.25}$$

Equation (2.25) is equivalent to that found by Levinson (1976).

The first five frequencies for various γ and b are given in Table 2.1. Notice that the higher the average density, the lower is the frequency. Figure 2.5 shows sample mode shapes of a two-segment string with $\gamma = 0.5$, $b = 0.5$.

TABLE 2.1
Frequencies for Two-Segment String

b	ω	$\gamma = 0.1$	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.7$	$\gamma = 0.9$
0.1	ω_1	9.5592	5.6896	4.4281	3.7497	3.3103
	ω_2	16.112	11.063	8.7665	7.4690	6.6140
	ω_3	23.223	15.872	12.941	11.136	9.9067
	ω_4	32.944	20.714	16.994	14.754	13.189
	ω_5	42.260	26.121	21.109	18.358	16.464
0.3	ω_1	5.5976	4.7628	4.1072	3.6338	3.2841
	ω_2	12.929	8.9029	7.6873	7.0030	6.4982
	ω_3	17.060	14.192	12.906	10.651	9.7650
	ω_4	25.060	17.955	15.794	14.270	13.062
	ω_5	29.373	23.387	19.622	17.641	16.284
0.5	ω_1	3.9648	3.7728	3.5799	3.3945	3.2221
	ω_2	9.4712	8.4791	7.5415	6.8937	6.4531
	ω_3	15.065	12.024	10.840	10.191	9.6663
	ω_4	19.354	16.060	14.888	13.771	12.906
	ω_5	23.038	20.681	18.314	17.010	16.111
0.7	ω_1	3.3401	3.2984	3.2552	3.2106	3.1648
	ω_2	7.2563	7.0657	6.8509	6.6226	6.3939
	ω_3	11.455	11.029	10.506	10.002	9.5921
	ω_4	15.743	14.480	13.864	13.181	12.741
	ω_5	20.042	18.299	17.052	16.434	15.945
0.9	ω_1	3.1505	3.1486	3.1466	3.1446	3.1426
	ω_2	6.3462	6.3331	6.3196	6.3054	6.2908
	ω_3	9.6043	9.5694	9.5318	9.4913	9.4477
	ω_4	12.918	12.854	12.782	12.702	12.614
	ω_5	16.275	16.175	16.061	15.930	15.785