project Origami

ACTIVITIES FOR EXPLORING MATHEMATICS

SECOND EDITION

THOMAS HULL

CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

project Origami

ACTIVITIES FOR EXPLORING MATHEMATICS

SECOND EDITION

This page intentionally left blank

project Origami

ACTIVITIES FOR EXPLORING MATHEMATICS

SECOND EDITION

THOMAS HULL

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
AN A K PETERS BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20130109

International Standard Book Number-13: 978-1-4665-6809-9 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Dedicated to dr. sarah-marie belcastro
who brainstormed the title and concept of this book
and then supported it throughout

This page intentionally left blank

CONTENTS

PREFACE TO SECOND EDITION

The first edition of *Project Origami* was published in 2006, and since then I have received a lot of feedback. Every semester I get emails from people using the book in one way or another. Some are college professors or high-school teachers who tell me of an activity that they used that went well or tell me about an idea that they had or an approach that worked with their students. Some of the emails are from students themselves, asking for a pointer on a project they are attempting or asking for further resources to explore. Others still are from people who are fans of origami-math and who want to thank me for this book.

And, of course, I used this book plenty of times myself! I taught several courses on the mathematics of origami at Merrimack College and Western New England University, and whenever I teach college-level geometry, multivariable calculus, or graph theory, I draw from the activities in this book.

As any teacher knows, the act of teaching is not unidirectional, with information only passing from the teacher to the student. Rather, it is more like a feedback loop, with the teacher learning new things by watching the students learn and respond to the material. Therefore, it should come as no surprise that after years of these emails about *Project Origami* and my own teaching, new activities developed. Conversations with students and colleagues gave me ideas; sometimes a student would find an origami model themselves online or in a book and start asking mathematical questions about it. Before I knew it was happening, I discovered that I had material for half a dozen more origami-math activities. Once I realized this, I knew that producing a second edition of the book was inevitable.

This—new material generated by excitement and other people using this book—is the happy side of producing a second edition. The more embarrassing side is that in any book that contains a lot of information and gets extensively used, mistakes will be found. Many of the mistakes reported to me (or discovered by me) fit into the category of typos or regretful omissions and are easily fixed. Other mistakes, however, are of the mathematical variety. Despite the fact that the first edition manuscript went through extensive beta-testing by dozens of college and university professors (and their students) across the country, there were still mathematical errors that did not get caught.

The most egregious of these errors was in the Five Intersecting Tetrahedra activity. The solution given to that activity in the first edition was very close, but not 100% correct. This is corrected in the second edition, and in fact, a new solution is given that is much more simple than the previous one.

The process of preparing a second edition of *Project Origami* has given me the opportunity to re-read the whole book with a critical eye. I was pleasantly surprised that during the five years since the book first was published, some of my views on the simplest ways to present or teach this material have changed. Even the presentation of relatively straightforward results, like the matrix model for flat vertex folds, looked like it could be improved. Thus, nearly all of the activities from the first edition have been edited, with solutions and teaching tips improved here and there.

In my view, this second edition is a much better book that the first. The table of contents has been expanded from 22 activities to 30, over a hundred new pages have been added, and the further experiences of myself and dozen of other people (who are thanked in the acknowledgements section) have greatly improved many of the other activities. I hope you will agree that is worth it!

Thomas C. Hull
Western New England University
Springfield, MA

INTRODUCTION

Why did I write this book?

This book is the first of what I hope will be a variety of books on the mathematics of origami. It grew out of my life-long passion for the two subjects. I started learning origami at age eight, after my uncle gave me an origami instruction book. While many of the instructions in this translated-from-Japanese book were impossibly cryptic, I managed to figure out many of them and, for some reason, it stuck. Around the same time I was realizing that I was good at math, that the patterns found in addition and multiplication were easy—and fun—to memorize. I also distinctly remember noticing a link between origami and mathematics during those years. I had folded an animal, probably the classic flapping bird, and instead of putting it in my ever-growing box of folded models, I carefully unfolded it. The pattern of crease lines on the unfolded paper was intricate and lovely. Clearly, it seemed to me, there was some mathematics going on here. The pattern of lines must be following some geometric rules. But understanding these rules was well beyond my comprehension, or so I thought at the time.

My next visit to the crossroads of origami and mathematics occurred while I was in college. By that time I was well-versed in complex-level origami, having devoured the books of John Montroll, Robert Lang, Jun Maekawa, and Peter Engel. (See [Eng89], [Kas83], [Lang95], [Mon79].) I had been to a few origami conventions in New York City (hosted by the nonprofit organization now known as OrigamiUSA) and even invented a number of my own origami designs. I had also taken many math classes and was considering a career in the mathematical sciences. But then one thing happened that forced me to think about and explore the intersection of origami and math—I obtained a copy of the classic book *Origami for the Connoisseur* by Kunhiko Kasahara and Toshi Takahama [Kas87]. At first I thought this was just another complex-level origami book. In fact, I bought it because it contained instructions for John Montroll's infamous stegosaurus model (impeccably detailed and made from one uncut square of paper). Little did I know that this book also contained instructions for something that would grip my interest like a vice and result in dozens of hours of procrastination.

This book provided my first exposure to modular origami, whereby many small squares of paper are folded into identical "units" which are then locked together to form a variety of shapes. The units in *Origami for the Connoisseur* allowed one to make representations of all the Platonic solids: the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. Prior to this I had only a casual understanding of these objects, but after folding many, sometimes hundreds of units to make these and other polyhedral shapes, I became intimately familiar with them. Modular origami was, quite literally, my first tutor on the subject of polyhedral geometry.

In retrospect, it is easy for me to see what was happening, although at the time I only knew that I was having fun and making beautiful, geometric objects with which to decorate my dorm room. Origami was teaching me, giving me a context in which I had to explore and master properties of various polyhedral objects. How do I arrange the units around each vertex to form a cuboctahedron? How many units of each color would I need to make an interesting coloring of the icosidodecahedron?

Over the years afterward, during graduate school and as a professor, first at Merrimack College and then Western New England University, I continued to collect everything I could find about the mathematics of origami. Since many sources were hard to find or were merely hints at underlying patterns, I often had to do my own research to help put the pieces together. In the process I saw origami intersect a variety of mathematical topics, from the more obvious realm of geometry to the fields of algebra, number theory, and combinatorics. It seemed that the more I looked, the more branches of math origami overlapped.

Simultaneous to this gathering of origami-math material, I began giving lectures on the subject for college and high-school students and their teachers. From this the interest in origami as a mathematical education tool became very clear. Teachers would regularly ask me where they could find more information on how to use origami in their classes. Eventually a few books did emerge, like [Fra99], that offered ways to use modular origami to teach geometric concepts, but none of these were done at the college level or touch the variety of topics that origami can offer.

Thus came about this book. My goal was to compile many of the origami-math aspects that I had found and present them in a way that would be easy for college or advanced high-school teachers to use in their classes.

How to use this book

Thirty activities are included in this book that cover a variety of mathematical areas. The intent is for mathematics instructors to be able to find something of use no matter what college or, perhaps, high-school course is being taught.

Each activity begins with a list of courses in which it might fit. In the appendix you'll find a cross-reference that lists which activity might match a given course.

However, it is important to realize that many of these activities can be effectively used in a variety of courses at a variety of levels. The angle trisection activity, for example, has been very popular among high-school geometry teachers, yet it also makes for a very illustrative diversion in an upper-level Galois Theory course. The PHiZZ unit Buckyball activity can be a great extended project for "liberal arts" general-education math classes, but it also provides a hands-on way for students of graph theory to explore the connections between 3-edge-coloring cubic planar graphs and the Four Color Theorem, not to mention the opportunity to classify geodesic spheres in an upper-level geometry class.

In short, the key word in terms of using this book is *flexibility*. Each activity includes handouts that may be photocopied and used with your class as well as notes to the instructor on solutions, how the handouts can be used, suggestions on pedagogy, and further directions that can be taken.

But you, depending on your class, your time, and your interest in origami, can and are encouraged to find your own ways to use this material. Perhaps it would be better for you to present the Folding Equilateral Triangles in a Square and Can Origami Trisect an Angle? activities at the same time. Perhaps you'd prefer to use only part of a handout or add on investigative questions of your own. Perhaps these would fit your class better as homework or extra-credit assignments. Perhaps one of these activities could be the basis for a senior research project. Perhaps you could spend a whole year using these activities for your college's Math Club or your high school's Math Circle.

To provide readers with as much flexibility as possible, our publisher, A K Peters/CRC Press, is making all of the handouts in this book available online. Many professors who beta-tested these activities had access to PDF versions of the handouts and thus were able to modify them to taste. Some copied the graphics into separate documents so that they could write their own text and modify the questions. Others removed certain questions or combined several activities into one. Others chose to insert more explanation for their students. You are the one most familiar with your students, and thus we want to give you the ability to tailor the activities to your liking.

The online PDF versions of the handouts can be found on the CRC Press web site: go to http://www.crcpress.com/product/isbn9781466567917 and click on the "Downloads" tab.

I am especially interested to know what people do with these activities. If you modify or find interesting ways to utilize them, please feel free to email me and share your experiences: thull@wne.edu.

Discovery-based learning

The main pedagogical approach behind all the activities is one that is active and discovery-based (as opposed to, say, a lecture-based approach). There is a logical choice for this that deserves some explanation.

One of the main attractions of using origami to teach math is that it requires hands-on participation. There's no chance of someone hiding in the back of the room or falling asleep when everyone is trying to fold a hyperbolic paraboloid (see the Rigid Folds 1 activity). The fact that origami is, by definition, hands-on makes it a natural fit for active learning. One could even make the argument that while folding paper, especially when making geometric models, latent mathematical learning will always happen. There's no way a student can make a dodecahedron out of thirty PHiZZ units without an understanding of some fundamental properties of this object.

Therefore, when choosing to use origami as a vehicle for more organized mathematics instruction, an easy choice is to let the students *discover things for themselves*. This approach to teaching mathematics, where students are allowed to experiment and discover basic principles and theorems themselves, was pioneered by David Henderson in college-level geometry courses (see [Hen01]). The approach is based not only on exploration but also in students *learning how to ask the right questions* while exploring.

I tried to achieve a mixture of this in the handouts in this book. Some of them try to lead the student toward asking the proper questions that lead to theorems, like in the Haga's "Origamics" activities. Others, like the Exploring Flat Vertex Folds activity, is deliberately very open-ended. The specific purpose in such open-ended activities is for students to gain experience with asking questions and building conjectures.

I highly encourage instructors to not shy away from this approach. Too often professors feel that they need to instruct their students on the fine art of conjecture-building. But the best way to learn this process is to just do it. Some students behave as if they were just waiting to be asked to make conjectures; once you get them going, they can't stop! Others do have difficulty with open-ended assignments, but again, these difficulties arise from not knowing how to ask questions. Engaging such students in a Socratic dialog often helps a lot.

For example, a student who can't find any patterns in flat vertex folds might be asked, "Well, is there anything going on with the mountain and valley creases?" If that doesn't help, then a more specific question, "How many of them are there at your vertex?" will get things going. It's these questions that help students see that the piece of folded paper is their experimental laboratory. The math ceases to be an abstract entity, only existing in their mind. It becomes tangible, something they can hold in their hand and count or use to compute data from which patterns, conjectures, and theorems flow.

Nothing gives students a feeling of ownership of such discovery like the personal touch of their own names. Sure, the fact that the difference between the number of mountain and valley creases at a flat foldable vertex is always two is known as Maekawa's Theorem (see the Exploring Flat Vertex Folds activity). But it might as well be christened "Danielle's Conjecture" for a few classes as students discover and try to prove it.

However, it should be noted that a completely 100% discovery-based instruction method is not for everyone. Instructors who are more comfortable with lecture-based instruction can still use the handouts for, say, 20 minutes of in-class activity and then wrap up the main points and student observations via lecture. Still, it might be more interesting to see what the students have come up with and ask some of them to present their results to the whole class.

The value of the discovery-based approach should be clear, in that it provides students with the experience of being a mathematical researcher. If helping train your students for independent research or for a senior capstone experience is one of your goals, then by all means give this approach a try. In fact, if you think about the skills and experiences needed to become comfortable with mathematical inquiry, you just might end up totally changing the way you approach teaching your course. For example, it's very important for math researchers to understand that not succeeding is OK—that failure is a natural part of discovery. Thus, when exploring one of these origami activities in class, instructors should be prepared for their students to *not succeed* and realize that this is fine. This leads to the next topic.

Preparing yourself

The preparation required for these activities takes several forms.

First of all, if the activity has a strong folding component, like folding modular units or folding a crane, instructors need to practice folding these things themselves in advance. What's more, instructors need to think, as they fold, about how they would explain the folding process to a classroom of students or to individuals who are stuck. Teaching origami is quite a bit different from teaching math. It involves trying to communicate three-dimensional movements by "show and tell." The handout instructions for folding in these activities are meant to help, but some people have a very hard time translating two-dimensional instructions into three-dimensional movements of their hands and paper. *Always assume* that there will be students who need one-on-one help with the folding instructions.

If the technology is available, using a document camera (also known as a digital imager or Elmo) can be a big help. Document cameras allow one to place their hands and a piece of paper underneath a camera that will then project this image on an overhead screen. Using this, a whole class can see what your hands are doing, up-close, as you fold the paper. In my experience, this is by far the most efficient way of teaching a whole class to fold paper. It also works very well for showing how to lock modular units together. Such units are often small, and a good document camera will allow you to zoom-in on the details of putting the units together properly.

Note, however, that while it is important for instructors who, say, are using the Making Origami Buckyballs activity to become very familiar with the PHiZZ unit, understand its locking process well, and make a 30-unit dodecahedron of their own (and properly 3-color it), other longer projects can be left to the students to figure out. Instructors are not likely to have the time to make a 270-unit Buckyball or an 84-unit torus beforehand, although these projects are fun and make great office decorations. Students should be encouraged to attempt larger projects. The fact that you might not have done them yourself can give students an extra feeling of accomplishment over their achievements.

Aside from the paper folding itself, it goes without saying that all instructors will have to tailor these activities to their own classes. The chances of a successful experience with these activities will increase dramatically if you make sure that your goals and expectations of the activity are clearly focused. Is your main goal to reinforce student understanding of Euler's formula and its uses (as in the Buckyball activity)? Is it for your students to see hands-on applications of the algebra of $\mathbb{Z}_n$ and number theory (as done in the Folding Strips into Knots and Fujimoto Approximation activities)? Or do you see the main goal as being to introduce more active participation in class or for students to explore and discover mathematics on their own?

The answers to these questions will allow you to clarify how to use the activity in class—how much time to spend on the hands-on part versus group discussion, or whether to assign the folding instructions for homework beforehand, or whether to expect the students to come up with very many conjectures on their own in class. Of course, the first time any of us try a new activity, especially one with an active or discovery-based learning component, it needs to be thought of as an experiment. The second time you try using any of these activities will require much less preparation.

Where to find paper

The question of where to obtain paper is a bit complicated. It entirely depends on what you or your students will be folding. While paper is paper, it comes in many different types. Some projects and activities can be done with any type of paper, but often there are preferences that can make the students' and instructor's job easier. I'll break these preferences down into categories.

For PHiZZ units, Flat Vertex Fold, Haga's Origamics, Matrix Model, Butterfly Bomb activities

I recommend three-inch square memo cube paper, which can be easily bought (for about $3 for 500 sheets) from office supply stores and comes in a rainbow of colors. Look for it near the Post-it note section, but *make sure* you do not buy Post-it notes! (The sticky side gets in the way of folding and sliding modular units into one another.) The best memo cube paper is the type that comes in its own plastic container—this paper is more accurately square than the type that doesn't come in a box. Also, if you look carefully you can find *blank* memo cube paper. If you're unlucky all they'll have is paper that's blank on one side and has "while you were out" office messages printed on the other side. That works just as well, and your students may find it more humorous anyway.

Business cards

Once you get bitten by the business card modular origami bug (and yes, there are many other modular units to be made from business cards than those presented in this book), you'll be very interested in collecting large supplies of discarded cards. This can sometimes be very easy to do. Visit an office supply or printing store where they print business cards for customers and ask if they have any unwanted cards. Often such places will have boxes of cards with printer errors or that were never picked up and have been sitting around for months. If you make it known that you're interested in such unwanted business cards, they'll often save such prizes for you when they turn up.

In a pinch, you can buy blank cards, but ones with printing on them can be much more interesting. Along those lines, be on the lookout for colorful or nicely patterned business cards at restaurants. Pinching ten or so of these at a time can slowly build a good collection. You can also ask students to acquire business cards of their own beforehand and bring them to class.

Strips of paper (for folding knots)

It can be difficult to find rolls of thin paper. Ticker-tape paper is ideal, but you can also get rolls of accounting tape, which is the paper accountants use for those calculators that print out the calculations as they go. You can usually find rolls of such paper at office supply stores.

Actual origami paper

This is paper that is colored on one side and white on the other, and origamists often call it "kami" or "plain kami." It folds very well and is considered "special" origami paper. It is the paper you probably want students using if they are folding cranes (for the Folding and Coloring a Crane activity) or other traditional origami models. You can find it at any art supply store. It usually costs $5–$6 (US) for 100 squares, 6 inches per side, in a variety of colors. You can also order it on the web at OrigamiUSA (a national nonprofit organization—if you're an advocate of origami, or want to become one, you should become a member, since it gives you a magazine, access to members-only web content, the ability to attend origami conventions, and a 10% discount on buying things from them. See http://origamiusa.org).

Other options (and the Five Intersecting Tetrahedra)

The most basic paper you can use is photocopy paper. You can use up that pile of scrap 8 1/2 inch by 11 inch paper you have stacked in your office by cutting it into squares with a paper cutter. This makes great all-purpose, no frills paper to use in class. It's fine paper to use when folding cranes and is *very* good to use when making the Five Intersecting Tetrahedra model, since it is heavier than normal origami paper. Also, you can get it in a variety of colors from any office supply store or from the Print Center at your college or university.

In fact, a very good resource for square paper and business cards (and maybe even strips of paper) is your friendly Print Center on campus. While not everyone has a friendly Print Center at their school, it would be worth your time to find out if you do. Pay them a visit and tell them that you're doing origami in your classes. They'll probably be happy to cut paper to size for you or give you discarded business cards, or they might have long strips of paper handy.

Other sources

In each activity I've tried to provide references for the material as well as for places where more information can be learned.

Since interest in origami-math has been increasing, there are some books now available that are devoted to certain aspects of the subject. Also, there are a few books with chapters devoted to paper folding as well as some proceedings and other books that are useful. Since these sources might be very valuable, depending on your specific interests in origami-math, they deserve special mention.

Galois Theory by David Cox [Cox04]. This book is excellent anyway because David Cox is such a good writer. But Chapter 10 is devoted to geometric constructions, and Section 3 of this chapter is on origami. This is probably the best exposition of an algebraic, Galois Theory approach to origami geometric constructions available. Instructors interested in using the Folding a Parabola, Can Origami Trisect an Angle?, and Solving Cubic Equations activities in an advanced algebra class should consult this book.

Geometric Folding Algorithms: Linkages, Origami, Polyhedra by Erik Demaine and Joseph O'Rourke [Dem07]. This book is a must-read for anyone interested in the field of computational origami, where questions are asked about how feasible it is to find answers to thorny, and even not-so-thorny, folding problems. As the title indicates, the authors look at folding and unfolding linkages (which can be thought of as one-dimensional folding), paper, and polyhedra. Anyone who is confused about why this would be an active area of research in theoretical computer science should look at this book and see the applications to robotics, protein folding, and numerous other things. Much of the math presented in this book is in alignment with the math presented in *Project Origami*.

Geometric Origami by Robert Geretschläger [Ger08]. This book focuses on origami geometric constructions, giving a very axiomatic and synthetic approach. Fans of geometry and the Folding a Parabola and angle trisection activities will enjoy Geretschläger's book very much.

Origamics: Mathematical Explorations Through Paper Folding by Kazuo Haga [Haga08]. This book is an English translation of many of Haga's Japanese writings on using very simple, geometric folding problems to engage students in the process of mathematical discovery. Haga's approach to geometric origami is rather unique, and you can get a big taste of it in the Haga's "Origamics" activity. If you like that, then definitely buy this book.

Origami^3 [Hull02-2], *Origami*^{4} [Lang09], and *Origami*^{5} [Wang11]. These three books are the proceedings of the third, fourth, and fifth international meetings of Origami Science, Mathematics, and Education (OSME, for short). The first two such meetings took place in Italy (1989) and Japan (1994), but the proceedings for those meetings are out of print and very hard to find. The other proceedings are still in print and present excellent snapshots of the state of origami research in science, math, and education. While I am, as editor of one of the volumes, biased, I feel confident in saying that no matter what your taste in origami you'll find many of the articles in these books of great interest. Make sure your library has them if you would like your students to be able to see what current research in origami is like.

Origami Design Secrets: Mathematical Methods for an Ancient Art, Second Edition, by Robert Lang [Lang11]. Robert Lang is one of the pre-eminent creators of complex, artistic origami models, and this book is his *magnum opus*. It describes in detail Lang's *TreeMaker* algorithm as well as other origami design techniques. While none of the activities in *Project Origami* deal directly with origami model design (that is, trying to answer the question, "How do you fold an insect from a square without making any cuts?"), the techniques that modern origamists use follow from mathematical principles of origami (for example, things like Maekawa and Kawasaki's Theorems from the Exploring Flat Vertex Folds activity). Students who get bitten by the origami bug should devour this book. It's a great source for student projects in this area.

Mathematical Reflections: In a Room with Many Mirrors by Peter Hilton, Derek Holton, and Jean Pedersen [Hil97]. This book (in Springer's Undergraduate Texts in Mathematics series) has a 57-page chapter titled Paper-Folding and Number Theory. It collects much of the research done by Peter Hilton and Jean Pedersen on the number theory behind folding strips of paper into polygons and polyhedra. This is very related to the topics covered in my Fujimoto Approximation and Folding Strips into Knots activities, although Hilton et al. use a different approach and take the material in different directions. If these activities appeal to you, definitely explore this chapter.

Origami for the Connoisseur by Kunihiko Kasahara and Toshie Takahama [Kas87]. Of the many origami instructions books in print, this one is the most mathematical (and was mentioned earlier in this Introduction). It contains instructions for many geometric models, like polyhedra and spiral shells, both from single sheets of paper and modular. It also contains references to Maekawa and Kawasaki's Theorems as well as some of Haga's origamics activities. While several of the models are very complicated, requiring expert origami skills, others are surprisingly simple and elegant. This is a gem of a book.

Geometric Constructions by George E. Martin [Mar98]. The last chapter (14 pages) of this book is devoted to geometric constructions via paper folding. Martin's approach is purely geometric, as opposed to Cox's algebraic analysis, so this would appeal to teachers of geometry who want to learn more about origami geometry. Martin concentrates on only the most sophisticated of the single-fold origami operations—the one explored in the Solving Cubic Equations activity. This is all one needs, however, to perform constructions such as angle trisections and cube doublings. Martin also compares this to other construction methods, for instance, using a marked ruler.

Origami Polyhedra Design by John Montroll [Mon09]. John Montroll is an origami legend. He was one of the first people to achieve the level of complexity in origami (as seen in his 1979 book [Mon79]) that we today associate with complex-level origami. He also is very interested in folding polyhedral shapes from single, uncut squares, and in this book he provides instructions for many such folding projects as well as explains the math behind them. In this way he shows some great applications of trigonometry as well as planar and 3D geometry, making the math used very accessible for motivated high-school students. Teachers looking for interesting ways to show how trigonometry and geometry are used in origami would enjoy this book.

How to Fold It: The Mathematics of Linkages, Origami, and Polyhedra by Joseph O'Rourke [ORo11]. As the title suggests, this book is thematically similar to Demaine and O'Rourke's book [Dem07], but at the same time it is very different! O'Rourke's *How to Fold It* is much smaller and meant for high-school or college students to follow (whereas [Dem07] is more of a research monograph). The book contains many projects with clear explanations and would make a good companion text for *Project Origami*.

Fragments of Infinity by Ivars Peterson [Pet01]. This is a popular math book for a general audience and has a 22-page chapter on origami called Plane Folds. While not a math text, it does give a good overview of flat origami crease patterns, Maekawa's Theorem, Lang's *TreeMaker* algorithm, and origami tessellations. In particular, it includes some wonderful pictures of Chris Palmer's complex folded tessellations. If you found the Folding a Square Twist activity exciting, definitely check this out.

Geometric Exercises in Paper Folding by T. Sundra Row [Row66]. This book is a classic. T. Sundra Row was an Indian mathematics teacher who, in the late 1800s, wrote this book on the basic geometric constructions that can be performed by paper folding. It attracted the attention of Felix Klein, and after he referenced it in some of his publications, Western publishers began printing it world-wide. The latest printing was by Dover, and it should not be hard to find in most libraries. A careful reading of the book makes it unclear whether Row knew that origami could do things like trisect angles (no method for this is given in the book, but Row does discuss how paper folding relates to solving some types of cubic equations). Nonetheless, this is an excellent source of methods for folding a variety of polygons and shapes in paper. While written in the very formal style of over a hundred years ago, the construction methods are simple and could easily be adapted for modern geometry classes (for both college and high school).

ACKNOWLEDGMENTS

This book was made possible via a variety of support. First and foremost is the Paul E. Murray Fellowship that I received at Merrimack College which funded the creation of the book's first draft. Without the generosity of the Murray family, this project might never have gotten off the ground. In general both Merrimack College and the Hampshire College Summer Studies in Mathematics have been incredibly supportive by providing me with environments in which to study and teach the mathematics of paper folding freely and creatively.

I have been very lucky to receive the help and input from a large number of people in the creation of this book. Discussions and feedback from those in the origami and mathematics community have been invaluable: Roger Alperin, sarah-marie belcastro, Ethan Berkove, Vera Cherepinsky, David Cox, Erik and Marty Demaine, Koshiro Hatori, Miyuki Kawamura, David Kelly, Jason Ku, Robert Lang, Jeannine Mosely, James Tanton, Tamara Veenstra, and Carolyn Yackel.

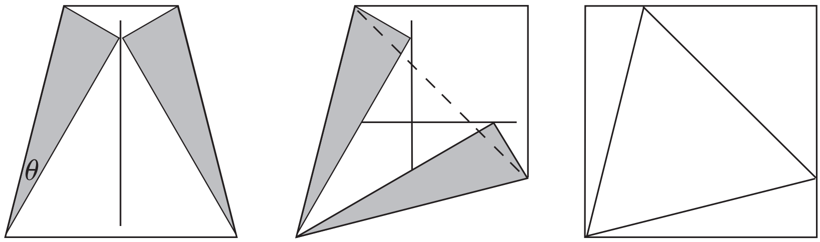
I was also fortunate to have the help of many Project NExT fellows who beta-tested these activities in their own mathematics classes during the spring and fall of 2005. Project NExT (which stands for New Experiences in Teaching) is a fellowship program of the Mathematical Association of America designed to help new math professors become better teachers and scholars without becoming lost or overwhelmed by the academic mathematics community. Their collective pedagogical wisdom and experience has directly shaped this book. Furthermore, through Project NExT word of this book spread into the greater mathematical community, where numerous other faculty and students in graduate school, college, and even high-school asked to be beta-testers. In particular, I need to thank Cristina Bacuta, Don Barkauskas, Mark Bollman, David Brenner, Kyle Calderhead, Scott Dillery, Melissa Giardina, Susan Goldstine, Aparna Higgins, Barbara Kaiser, Michael Lang, Chloe Mandell, Hope McIlwain, Blake Mellor, Andrew Miller, Cheryl Chute Miller, Donna Molinek, George Moss, Katarzyna Potocka, Jason Ribando, Liz Robertson, Cameron Sawyer, Amanda Serenevy, Brigitte Servatius (and her students Roger Burns, Onalie Sotak, and John Temple), Linda Van Niewaal, Kathryn Weld, Jennifer Wilson, and Yi Zhou.

Thanks must also go to all the students that I've had in my classes where origami-math has been taught, which has included the University of Rhode Island, Merrimack College, the Hampshire College Summer Studies in Mathematics, the University of Cincinnati, and Western New England University. Not many people realize, I think, that if you are a teacher who cares about what you do and thinks deeply about it, you learn just as much from your students as they learn from you. The students that I have learned from are too numerous to mention, but I would like to thank Hannah Alpert, Mike Borowczak, Michael Calderbank, Alessandra Fiorenza, Emily Gingras, Josh Greene, Monique Landry, Kevin Malarkey, Wing

Mui, Emily Peters, Gowri Ramachandran, Jan Siwanowicz, Ari Turner, Jeanna Volpe, Haobin Yu, the 1995–1996 graduate students in mathematics at the University of Rhode Island, and my Spring 2005 Combinatorial Geometry class at Merrimack College, who were unknowing guinea pigs for these activities as the first edition of book was being finished.

Also, many of these activities were tested on and shared with the middle- and high-school teachers who took my Origami in Mathematics and Education course as part of Western New England University's Master of Arts for Teachers in Mathematics (MAMT) program. These "students" would learn some origami-math in my class and then the next day use it in their own classes, giving me immediate feedback. In particular, ideas generated by Ann Farnham and Diane Glettenberg influenced some of the new activities in the second edition of *Project Origami*.

Activity 1
FOLDING EQUILATERAL TRIANGLES IN A SQUARE



For courses: precalculus, elementary algebra, trigonometry, geometry, calculus (optimization), modeling

---

Summary

Students are asked to find a way to fold an equilateral triangle from a square piece of paper. Then the challenge of finding the largest possible equilateral triangle that can be folded from a square is given. Of course, students need to prove that their conjectured triangle is the largest possible.

Content

The geometry component of this problem only requires the ability to work with $30°$-$60°$-$90°$ triangles. However, more creative geometrical insights can lead to more elegant solutions.

For a calculus class, this problem could actually be posed without any mention of origami: What is the largest equilateral triangle that can be inscribed in a square? But knowing that paper folders actually use this knowledge can provide extra motivation. This is a challenging modeling problem that can be completely done without resorting to derivatives, provided the students set up the model carefully, know trigonometry solidly, and do a proper graphical analysis. As an optimization problem, it breaks away from the mold that is typically encountered in calculus textbooks, thus forcing students to apply their knowledge to a brand-new, real-life situation.

Handouts

Three optional handouts are provided:

(1) Introduces the general problem of folding an equilateral triangle inside a square.

(2) Provides a few guided steps in setting up the optimization model.

(3) Leads students step-by-step through the optimization model.

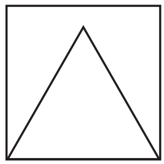
Time commitment

Handout 1 will require about 40 minutes of class time, including student exploration and presentation of their triangle-folding methods to the rest of the class.

Handout 2 or 3, if done in class, could take 50–60 minutes total, depending on how quick your students are at making mathematical models.

How to Fold an Equilateral Triangle

The goal of this activity is to fold an equilateral triangle from a square piece of paper.



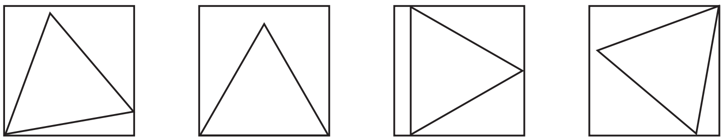
Question 1: First fold your square to produce a 30°-60°-90° triangle inside it. Hint: You want your folds to make the hypothenuse twice as long as one of the sides. Keep trying! Explain why your method works in the space below.

Question 2: Now use what you did in Question 1 to fold an equilateral triangle inside a square.

Follow-up: If the side length of your original square is 1, what is the length of a side of your equilateral triangle? Would it be possible to make the triangle's side length bigger?

What's the Biggest Equilateral Triangle in a Square?

If we are going to turn a square piece of paper into an equilateral triangle, we'd like to make the **biggest possible** triangle. In this activity your task is to make a mathematical model to find the equilateral triangle with the **maximum area** that we can fit inside a square. Follow the steps below to help set up the model.



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? Why?

Question 2: Assuming Question 1, draw a picture of what your triangle-in-the-square might look like, where the "common corner" of the triangle and square is in the lower left. Now you'll need to create your model by introducing some variables. What might they be? (Hint: One will be the angle between the bottom of the square and the bottom of the triangle. Call this one θ.)

Question 3: One of your variables will be your *parameter* that you'll change until you get the maximum area of the triangle. Pick one variable (and try to pick wisely—a bad choice may make the problem harder) and then come up with a formula for the area of the triangle in terms of your variable.

Question 4: With your formula in hand, use techniques you know to find the value of your variable that gives you the maximum area for the equilateral triangle. Be sure to pay attention to the proper range of your parameter.

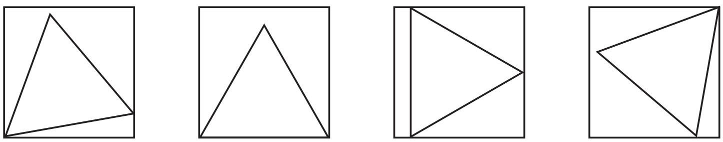
Question 5: So, what is your answer? What triangle gives the biggest area? Find a folding method that produces this triangle.

Follow-up: Your answer to Question 5 can also give a way to fold the largest *regular hexagon* inside a square piece of paper. Can you see how this would work?

What's the Biggest Equilateral Triangle in a Square?

In this activity your task is to find the biggest equilateral triangle that can fit inside a square of side length 1. (Note: An equilateral triangle is the triangle with all sides of equal length and all three angles measuring 60°.) The step-by-step procedure will help you find a mathematical model for this problem, and then to solve the optimization problem of finding the triangle's position and maximum area.

Here are some random examples:



Question 1: If such a triangle is maximal, then can we assume that one of its corners will coincide with a corner of the square? (Hint: The answer is yes. Explain why.)

Question 2: Assuming Step 1 above, draw a picture of what your triangle-in-the-square might look like, where the common corner of the two figures is in the lower left. (Hint: See one of the four examples above.) Now you'll need to create your model by labeling your picture with some variables. (Hint: Let θ be the angle between the bottom of the square and the bottom of the triangle. Let x be the side length of the triangle.)

Question 3: Come up with the formula for the area of the triangle in terms of one variable, x. Then, find an equation that relates your two variables, x and θ. Combine the two to get the formula for the area of the triangle in terms of only one variable, θ. (Hint: Your last formula will be $A = \frac{\sqrt{3}}{4} \sec^2 \theta$.)

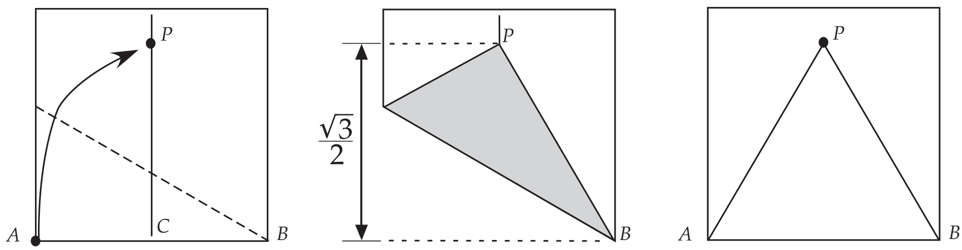
Question 4: What is the range of your variable θ? Explain. (Hint: The range should be $0^\circ \leq \theta \leq 15^\circ$.)

Question 5: Most important part: With your formula and the range for θ in hand, use techniques of optimization to find the value of θ that gives you the maximum area for the equilateral triangle. Also, find the value of this maximum area. (Hint: For simplicity, you may want to express all trigonometric functions in terms of sin and cos).

SOLUTION AND PEDAGOGY

Folding an equilateral triangle

There are a number of ways to fold an equilateral triangle in a square. All involve finding a way to produce a 60° angle. Your students might find new and creative ways to do this, but the most common way people discover is shown below. (We assume in these pictures that the side of the original square has length 1.)



The origami "move" here is to take one corner, A, and fold it to the center line (so the paper must have been creased in half first) *while at the same time* making sure the crease you make goes through corner B.[1] We let P be the image of point A under this fold. Then we have that ABP is an equilateral triangle. This can be seen in a number of different ways:

- Let C be the midpoint of AB. Then considering $\triangle BCP$, we have that BP has length 1 (since it is the image of AB) and BC has length $1/2$. The Pythagorean Theorem then tells us that CP has length $\sqrt{3}/2$, so $\triangle BCP$ is a 30°-60°-90° triangle. Then creasing AP gives us an equilateral triangle.
- Since BP is the image of AB under the folding, BP has length 1. We can then either say, "Now fold B to the center line in the same way," or "By symmetry," to get that AP has length 1 as well. Thus $\triangle ABP$ is an equilateral triangle.

In the solution pictured here, the length of the side of our triangle is the same as the side of the square. However, if we imagine rotating the triangle counterclockwise a little bit about the point A, we could then expand the sides some and still remain inside the square. So it *is* possible to make a bigger equilateral triangle inside the square.

Pedagogy. Many students will first try to construct a 30°-60°-90° triangle by trying to make the right angle be at a corner of the square. This is not the easiest thing to do, and suggesting that such students try folding the corner inside the square instead can get them over this mental block. Suggesting that they use the $1/2$ center line can also be offered.

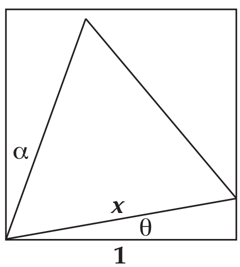
[1] This is a standard origami move: A point p_1 is folded onto a line l, but that is not enough to determine where exactly the crease should be made. So a second point, p_2, is needed, where we make sure the crease line goes through p_2 as well as making p_1 land on l. See the Folding a Parabola activity for more information.

Oftentimes students overhear ideas from other groups in class, or a good idea gets suggested from one group to another. That's fine, but everyone should write down a proof that their triangle is really 30°-60°-90° or equilateral. Groups should present their proofs to the class so that everyone can see that it can be done in more than one way. Writing up their proofs formally can be assigned individually for homework, if desired. (This should be easy after the group work, but writing things up "for real" is still a very valuable activity.)

Finding the maximal triangle

There are two versions of this handout: one that provides only a frame for the problem, leaving all the details to the students, and one that walks the students through the problem, step-by-step. The solutions are basically the same and presented here in tandem.

For the first question on the handout, the answer is yes. If no corner of the equilateral triangle is on a corner of the square, then the triangle must not be touching one side of the square (since the triangle has three corners and the square has four sides). Assume this is the left side. Then the three corners of the triangle must be touching the three other sides of the square, for otherwise we could make the triangle bigger. Then we can slide the triangle to the left until it touches this left side with one of the corners that touches either the top or bottom side as well. This puts a corner of the triangle on a corner of the square.

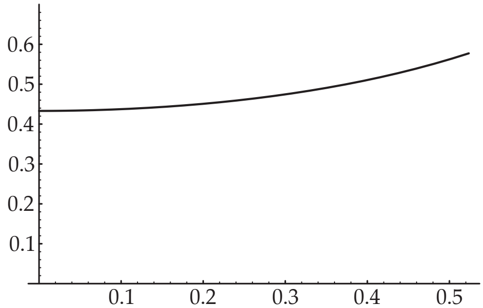


To set up the model, students will need a picture something like the above figure. The base of the triangle (length x) should extend from the bottom left corner to the right side of the square. Then we need to consider the range $0° \leq \theta \leq 15°$, for if $\theta > 15°$, then we'll have $\alpha \leq 15°$ and we'd be in a case symmetric to one with $\theta \geq 15°$. In other words, the symmetry of the square restricts the range of θ that we need to consider.

We need to find a formula for the area A of the equilateral triangle and then try to maximize this formula in terms of θ. (We want to do this in terms of θ, instead of x, because θ is the variable that tells us the position of the triangle in the square.) Since the base of the triangle is x, its height is $(\sqrt{3}/2)x$. So $A = (\sqrt{3}/4)x^2$, but we wanted it in terms of θ. Well, $\cos\theta = 1/x$, so $x = 1/\cos\theta = \sec\theta$. Thus we have

$$A = \frac{\sqrt{3}}{4}\sec^2\theta.$$

We could take the derivative of this and try to maximize it using calculus, but we don't really need to. Since $\cos\theta$ is a decreasing function on the interval $0 \leq \theta \leq \pi/12$ (we really should be working in radians, after all), we know that $\sec\theta$ is an increasing function on this interval. The same will be true of $\sec^2\theta$, so the maximum value of A will be on the right-most endpoint of the interval, $\theta = \pi/12$. Students can see this by graphing the function $A(\theta)$:



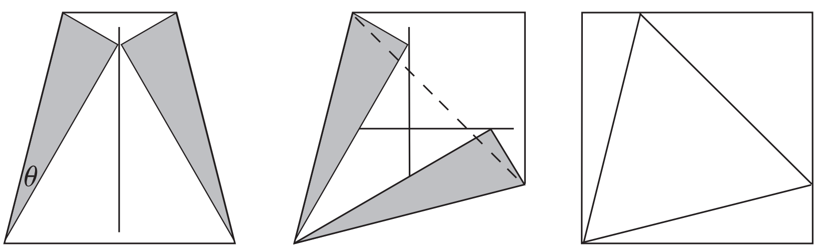
Thus the maximum area is achieved at $\theta = \pi/12 = 15°$. This results in a picture where one corner of the triangle is on a corner of the square and the triangle is symmetric about a diagonal of the square.

Students who do use derivatives to solve this would get

$$\frac{dA}{d\theta} = 2\frac{\sqrt{3}}{4}\sec^2\theta\tan\theta = \frac{\sqrt{3}\sin\theta}{2\cos^3\theta}.$$

Since $0 \leq \theta \leq 15°$, we know that $dA/d\theta = 0$ only when $\theta = 0$. This means that the area formula has a critical point at $\theta = 0$. But this is just an endpoint of our interval, so this means that the extreme values of the area A will happen at the endpoints $\theta = 0$ and $\theta = 15°$ (since there are no critical points in between). The question then is, which is a maximum and which is a minimum? We could take the second derivative of A and determine the concavity of the critical point $\theta = 0$, but taking such a derivative looks a little foreboding. Instead we could just check the value of A when $\theta = 0°$ and $\theta = 15°$. Fifteen degrees wins.

Students who do both of these handouts should be able to find a folding sequence for the maximal equilateral triangle. The pictures below serve as such a folding sequence as well as a "proof without words" that it works. (First note that $\theta = 15°$ in the left-most figure.) This folding sequence proof was developed by Emily Gingras, Merrimack College class of 2002.



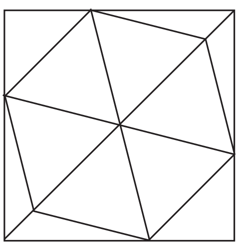
Pedagogy. Students familiar with the classic "fenced-in pen along a side of a barn" or "box folded out of a sheet of cardboard" calculus problems should see right away that our maximum equilateral triangle problem should be solvable using similar methods. However, the model for our problem is very different from those classic ones, and most students find it very challenging to set up the model properly. The hard and subtle part is making sure that you can parameterize the problem with a variable that tells you the triangle's position in the square. The best way to do this seems to be with an angle, and thus a formula for the triangle's area must be found in terms of this angle. In any case, this problem is at the right level of what calculus students learning optimization problems *should* be able to solve. But the value in this activity is for the students to sharpen their mathematical modeling skills, so the instructor should resist giving any more hints than those already given in the handout. Also, students should be encouraged to explore whatever avenue they choose to give a correct proof, be it a numerical, graphical, or analytical approach.

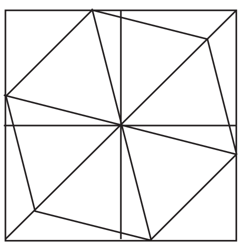
However, not all instructors will want to leave the details of such an activity entirely open. The second version of the optimization handout is for those who would like their students to see the proper procedure for such a problem and work out the details themselves. The format and pacing of this handout follows a suggestion by beta-tester Katarzyna Potocka of Ramapo College of New Jersey.

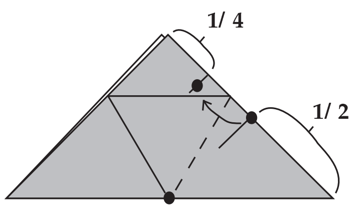
It can also be valuable to do this activity in a geometry course to emphasize the interconnections between mathematical disciplines. Typically, math major undergraduates in an upper-level geometry course will claim to have forgotten all of calculus, making this all the more worthy to do.

Follow-up activity

If you think about how a maximal regular hexagon would be inscribed in a square, as in the pictures below, and make horizontal and vertical half-way creases, you can see that one quarter of the square is exactly like the crease pattern for the maximal equilateral triangle. Therefore, the folding method for the triangle can be modified to give a maximal hexagon. The far right figure below abbreviates such a method.







Of course, these questions can be asked for folding any regular polygon inside a square, and while proving maximality gets more complicated, it's not beyond an undergraduate's means and can make good extended projects. The following figures show a way of proving the maximal hexagon case. Let θ be the angle it makes with the bottom edge of the square (whose side length is, again, 1) and let x be the length of a side of the hexagon. The hexagon is made up of six equilateral triangles, which makes it easy to compute the area of the hexagon: $A = 6 \times$ (area of one triangle) $= 6(x/2)(\sqrt{3}/2)x = (3\sqrt{3}/2)x^2$. But we want to maximize this with respect to θ.

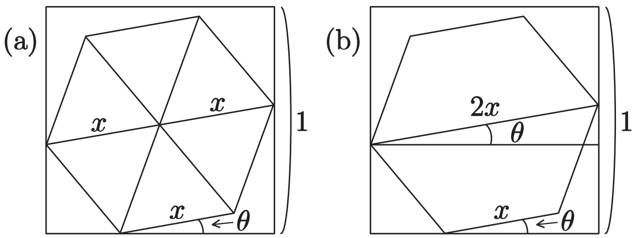


Figure (b) shows how we can do this. The diameter of the hexagon is $2x$, and if we assume that two opposite corners of the hexagon will be touching the left and right sides of the square, then we can form a right triangle from one of these corners (the left one, in the figure) with base of length 1 and hypothenuse one of the diagonals (length $2x$). Since the bottom of this triangle is parallel to the bottom of the square, and the hypothenuse is parallel to the bottom of the hexagon, we know that the base angle in this right triangle will be θ. Thus $\cos\theta = 1/2x$, or $x = (1/2)\sec\theta$. Thus the area of the hexagon is $A = (3\sqrt{3}/8)\sec^2\theta$.

To maximize this, we need to find the range of θ we need to consider. The symmetry of the hexagon shows us that $0^\circ \leq \theta \leq 15^\circ$ is all we need to consider. Like the triangle case, the largest endpoint of this interval, $\theta = 15^\circ$, gives the largest area. This will make one of the diagonals of the hexagon lie along a diagonal of the square.