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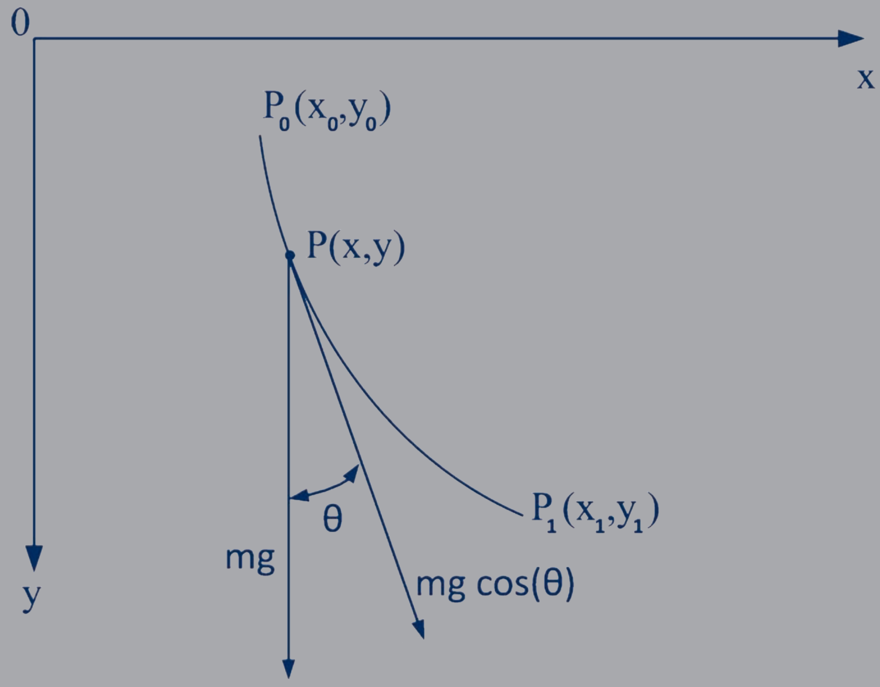
Nonlinear Optimal Control Theory



Leonard D. Berkovitz
Negash G. Medhin

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A CHAPMAN & HALL BOOK

Nonlinear Optimal Control Theory

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Leonard D. Berkovitz

Purdue University

Negash G. Medhin

North Carolina State University

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Foreword

This book provides a thorough introduction to optimal control theory for nonlinear systems. It is a sequel to Berkovitz's 1974 book entitled *Optimal Control Theory.* In optimal control theory, the Pontryagin principle, Bellman's dynamic programming method, and theorems about existence of optimal controls are central topics. Each of these topics is treated carefully. The book is enhanced by the inclusion of many examples, which are analyzed in detail using Pontryagin's principle. These examples are diverse. Some arise in such applications as flight mechanics, and chemical and electrical engineering. Other examples come from production planning models and the classical calculus of variations.

An important feature of the book is its systematic use of a relaxed control formulation of optimal control problems. The concept of relaxed control is an extension of L. C. Young's notion of generalized curves, and the related concept of Young measures. Young's pioneering work in the 1930s provided a kind of "generalized solution" to calculus of variations problems with nonconvex integrands. Such problems may have no solution among ordinary curves. A relaxed control, as defined in Chapter 3, assigns at each time a probability measure on the space of possible control actions. The approach to existence theorems taken in Chapters 4 and 5 is to prove first that optimal relaxed controls exist. Under certain Cesari-type convexity assumptions, optimal controls in the ordinary sense are then shown to exist. The Pontryagin maximum principle (Chapters 6 and 7) provides necessary conditions that a relaxed or ordinary control must satisfy. In the relaxed formulation, it turns out to be sufficient to consider discrete relaxed controls (see Section 6.3). This is a noteworthy feature of the authors' approach.

In the control system models considered in Chapters 2 through 8, the state evolves according to ordinary differential equations. These models neglect possible time delays in state and control actions. Chapters 10, 11, and 12 consider models that allow time delays. For "hereditary systems" as defined in Chapter 10, Pontryagin's principle takes the form in Theorem 10.3.1. Hereditary control problems are effectively infinite dimensional. As explained in Section 10.6, the true state is a function on a time interval $[-r, 0]$ where r represents the maximum time delay in the control system. Chapter 11 considers hereditary system models, with the additional feature that states are constrained by given bounds. For readers interested only in control systems without time delays, necessary conditions for optimality in bounded state problems are described in Section 11.6.

The dynamic programming method leads to first order nonlinear partial differential equations, which are called Hamilton-Jacobi-Bellman equations (or sometimes Bellman equations). Typically, the value function of an optimal control problem is not smooth. Hence, it satisfies the Hamilton-Jacobi-Bellman equation only in a suitable "generalized sense." The Crandall-Lions Theory of viscosity solutions provides one such notion of generalized solutions for Hamilton-Jacobi-Bellman equations. Work of A. I. Subbotin and co-authors provides another interesting concept of generalized solutions. Chapter 12 provides an introduction to Hamilton-Jacobi Theory. The results described there tie together in an elegant way the viscosity solution and Subbotin approaches. A crucial part of the analysis involves a lower Dini derivate necessary condition derived in Section 12.4.

The manuscript for this book was not quite in final form when Leonard Berkovitz passed away unexpectedly. He is remembered for his many original contributions to optimal control theory and differential games, as well as for his dedicated service to the mathematics profession and to the control community in particular. During his long career at Purdue University, he was a highly esteemed teacher and mentor for his PhD students. Colleagues warmly remember his wisdom and good humor. During his six years as Purdue Mathematics Department head, he was resolute in advocating the department's interests. An obituary article about Len Berkovitz, written by W. J. Browning and myself, appeared in the January/February 2010 issue of *SIAM News*.

Wendell Fleming

Preface

This book is an introduction to the mathematical theory of optimal control of processes governed by ordinary differential and certain types of differential equations with memory and integral equations. The book is intended for students, mathematicians, and those who apply the techniques of optimal control in their research. Our intention is to give a broad, yet relatively deep, concise and coherent introduction to the subject. We have dedicated an entire chapter to examples. We have dealt with the examples pointing out the mathematical issues that one needs to address.

The first six chapters can provide enough material for an introductory course in optimal control theory governed by differential equations. Chapters 3, 4, and 5 could be covered with more or less details in the mathematical issues depending on the mathematical background of the students. For students with background in functional analysis and measure theory, Chapter 7 can be added. Chapter 7 is a more mathematically rigorous version of the material in Chapter 6.

We have included material dealing with problems governed by integrodifferential and delay equations. We have given a unified treatment of bounded state problems governed by ordinary, integrodifferential, and delay systems. We have also added material dealing with the Hamilton-Jacobi Theory. This material sheds light on the mathematical details that accompany the material in Chapter 6.

The material in the text gives a sufficient and rigorous treatment of finite dimensional control problems. The reader should be equipped to deal with other types of control problems such as problems governed by stochastic differential equations and partial differential equations, and differential games.

I am very grateful to Mrs. Betty Gick of Purdue University and Mrs. Annette Rohrs of Georgia Institute of Technology for typing the early and final versions of the book. I am very grateful to Professor Wendell Fleming for reading the manuscript and making valuable suggestions and additions that improved and enhanced the quality of the book as well as avoided and removed errors. I also wish to thank Professor Boris Mordukovich for reading the manuscript and making valuable suggestions. All or parts of the material up to the first seven chapters have been used for optimal control theory courses in Purdue University and North Carolina State University.

This book is a sequel to the book *Optimal Control Theory* by Leonard

D. Berkovitz. I learned control theory from this book taught by him. We decided to write the current book in 1994 and we went through various versions.

L. D. Berkovitz was my teacher and a second father. He passed away on October 13, 2009 unexpectedly. He was caring, humble, and loved mathematics. He is missed greatly by all who were fortunate enough to have known him. This book was completed before his death.

Negash G. Medhin
North Carolina State University

Chapter 1

Examples of Control Problems

1.1 Introduction

Control theory is a mathematical study of how to influence the behavior of a dynamical system to achieve a desired goal. In optimal control, the goal is to maximize or minimize the numerical value of a specified quantity that is a function of the behavior of the system. Optimal control theory developed in the latter half of the 20th century in response to diverse applied problems. In this chapter we present examples of optimal control problems to illustrate the diversity of applications, to raise some of the mathematical issues involved, and to motivate the mathematical formulation in subsequent chapters. It should not be construed that this set of examples is complete, or that we chose the most significant problem in each area. Rather, we chose fairly simple problems in an effort to illustrate without excessive complication.

Mathematically, optimal control problems are variants of problems in the calculus of variations, which has a 300+ year history. Although optimal control theory developed without explicit reference to the calculus of variations, each impacted the other in various ways.

1.2 A Problem of Production Planning

The first problem, taken from economics, is a resource allocation problem; the Ramsey model of economic growth. Let $Q(t)$ denote the rate of production of a commodity, say steel, at time t. Let $I(t)$ denote the rate of investment of the commodity at time t to produce capital; that is, productive capacity. In the case of steel, investment can be thought of as using steel to build new steel mills, transport equipment, infrastructure, etc. Let $C(t)$ denote the rate of consumption of the commodity at time t. In the case of steel, consumption can be thought of as the production of consumer goods such as automobiles. We assume that all of the commodity produced at time t must be allocated to either investment or consumption. Then

$$Q(t) = I(t) + C(t) \qquad I(t) \geq 0 \quad C(t) \geq 0.$$

We assume that the rate of production is a known function F of the capital at time t. Thus, if $K(t)$ denotes the capital at time t, then

$$Q(t) = F(K(t)),$$

where F is a given function. The rate of change of capital is given by the capital accumulation equation

$$\frac{dK}{dt} = \alpha I(t) - \delta K(t) \quad K(0) = K_0, \; K(t) \geq 0,$$

where the positive constant α is the growth rate of capital and the positive constant δ is the depreciation rate of capital. Let $0 \leq u(t) \leq 1$ denote the fraction of production allocated to investment at time t. The number $u(t)$ is called the savings rate at time t. We can therefore write

$$\begin{aligned} I(t) &= u(t)Q(t) = u(t)F(K(t)) \\ C(t) &= (1 - u(t))Q(t) = (1 - u(t))F(K(t)), \end{aligned}$$

and

$$\begin{aligned} \frac{dK}{dt} &= \alpha u(t) F(K(t)) - \delta K(t) \\ K(t) &\geq 0 \quad K(0) = K_0. \end{aligned} \tag{1.2.1}$$

Let $T > 0$ be given and let a "social utility function" U, which depends on C, be given. At each time t, $U(C(t))$ is a measure of the satisfaction society receives from consuming the given commodity. Let

$$J = \int_0^T U(C(t)) e^{-\gamma t} \, dt,$$

where γ is a positive constant. Our objective is to maximize J, which is a measure of the total societal satisfaction over time. The discount factor $e^{-\gamma t}$ is a reflection of the phenomenon that the promise of future reward is usually less satisfactory than current reward.

We may rewrite the last integral as

$$J = \int_0^T U((1 - u(t))F(K(t)))e^{-\gamma t} dt. \tag{1.2.2}$$

Note that by virtue of (1.2.1), the choice of a function $u \colon [0, T] \to u(t)$, where u is subject to the constraint $0 \leq u(t) \leq 1$ determines the value of J. We have here an example of a *functional*; that is, an assignment of a real number to every function in a class of functions. If we relabel K as x, then the problem of maximizing J can be stated as follows:

Choose a savings program u over the time period $[0, T]$, that is, a function u defined on $[0, T]$, such that $0 \leq u(t) \leq 1$ and such that

$$J(u) = -\int_0^T U((1-u(t))F(\varphi(t)))e^{-\gamma t}dt \tag{1.2.3}$$

is minimized, where φ is a solution of the differential equation

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x \quad \varphi(0) = x_0,$$

and φ satisfies $\varphi(t) \geq 0$ for all t in $[0, T]$. The problem is sometimes stated as

Minimize:

$$J(u) = -\int_0^T U((1-u(t))F(x))e^{-\gamma t}dt$$

Subject to:

$$\frac{dx}{dt} = \alpha u(t)F(x) - \delta x, \quad x(0) = x_0, \quad x \geq 0, \quad 0 \leq u(t) \leq 1$$

1.3 Chemical Engineering

Let $x^1(t), \ldots, x^n(t)$ denote the concentrations at time t of n substances in a reactor in which n simultaneous chemical reactions are taking place. Let the rates of the reactions be governed by a system of differential equations

$$\frac{dx^i}{dt} = G^i\left(x^1, \ldots, x^n, \theta(t), p(t)\right) \qquad x^i(0) = x_0^i \quad i = 1, \ldots, n. \tag{1.3.1}$$

where $\theta(t)$ is the temperature in the reactor at time t and $p(t)$ is the pressure in the reactor at time t. We control the temperature and pressure at each instance of time, subject to the constraints

$$\begin{aligned} \theta_b &\leq \theta(t) \leq \theta_a \\ p_b &\leq p(t) \leq p_a \end{aligned} \tag{1.3.2}$$

where θ_a, θ_b, p_a, and p_b are constants. These represent the minimum and maximum attainable temperature and pressure.

We let the reaction proceed for a predetermined time T. The concentrations at this time are $x^1(T), \ldots, x^n(T)$. Associated with each product is an economic value, or price c^i, $i = 1, \ldots, n$. The price may be negative, as in the case of hazardous wastes that must be disposed of at some expense. The value of the end product is

$$V(p,\theta) = \sum_{i=1}^{n} c^i x^i(T). \tag{1.3.3}$$

Given a set of initial concentrations x_0^i, the value of the end product is completely determined by the choice of functions p and θ if the functions G^i have certain nice properties. Hence the notation $V(p,\theta)$. This is another example of a functional; in this case, we have an assignment of a real number to each pair of functions in a certain collection.

The problem here is to choose piecewise continuous functions p and θ on the interval $[0,T]$ so that (1.3.2) is satisfied and so that $V(p,\theta)$ is maximized.

A variant of the preceding problem is the following. Instead of allowing the reaction to proceed for a fixed time T, we stop the reaction when one of the reactants, say x^1, reaches a preassigned concentration x_f^1. Now the final time t_f is not fixed beforehand, but is the smallest positive root of the equation $x^1(t) = x_f^1$. The problem now is to maximize

$$V(p,\theta) = \sum_{i=2}^{n} c^i x^i(t_f) - k^2 t_f.$$

The term $k^2 t_f$ represents the cost of running the reactor.

Still another variant of the problem is to stop the reaction when several of the reactants reach preassigned concentrations, say $x^1 = x_f^1$, $x^2 = x_f^2, \ldots, x^j = x_f^j$. The value of the end product is now

$$\sum_{i=j+1}^{n} c^i x^i(t_f) - k^2 t_f.$$

We remark that in the last two variants of the problem there is another question that must be considered before one takes up the problem of maximization. Namely, can one achieve the desired final concentrations using pressure and temperature functions p and θ in the class of functions permitted?

1.4 Flight Mechanics

In this problem a rocket is taken to be a point of variable mass whose moments of inertia are neglected. The motion of the rocket is assumed to take place in a plane relative to a fixed frame. Let $y = (y^1, y^2)$ denote the position vector of the rocket and let $v = (v^1, v^2)$ denote the velocity vector of the rocket. Then

$$\frac{dy^i}{dt} = v^i \qquad y^i(0) = y_0^i \qquad i = 1,2, \tag{1.4.1}$$

where $y_0 = (y_0^1, y_0^2)$ denotes the initial position of the rocket.

Let $\beta(t)$ denote the rate at which the rocket burns fuel at time t and let $m(t)$ denote the mass of the rocket at time t. Thus,

$$\frac{dm}{dt} = -\beta. \tag{1.4.2}$$

If $a > 0$ denotes the mass of the vehicle, then $m(t) \geq a$.

Let $\omega(t)$ denote the angle that the thrust vector makes with the positive y^1-axis at time t. The burning rate and the thrust angle will be at our disposal subject to the constraints

$$0 \leq \beta_0 \leq \beta(t) \leq \beta_1 \qquad \omega_0 \leq \omega(t) \leq \omega_1, \tag{1.4.3}$$

where β_0, β_1, ω_0, and ω_1 are fixed.

To complete the equations of motion of the rocket we analyze the momentum transfer in rectilinear rocket motion. At time t a rocket of mass m and velocity v has momentum mv. During an interval of time δt let the rocket burn an amount of fuel $\delta\mu > 0$. At time $t + \delta t$ let the ejected combustion products have velocity v'; their mass is clearly $\delta\mu$. At time $t + \delta t$ let the velocity of the rocket be $v + \delta v$; its mass is clearly $m - \delta\mu$. Let us consider the system which at time t consisted of the rocket of mass m and velocity v. At time $t + \delta t$ this system consists of the rocket and the ejected combustion products. The change in momentum of the system in the time interval δt is therefore

$$(\delta\mu)v' + (m - \delta\mu)(v + \delta v) - mv.$$

If we divide the last expression by $\delta t > 0$ and then let $\delta t \to 0$, we obtain the rate of change of momentum of the system, which must equal the sum of the external forces acting upon the system. Hence, if F is the resultant external force per unit mass acting upon the system we have

$$Fm - (v' - v)\,\frac{d\mu}{dt} = m\,\frac{dv}{dt}.$$

If we assume that $(v' - v)$, the velocity of the combustion products relative to the rocket is a constant c, and if we use $d\mu/dt = \beta$, we get

$$F - c\beta/m = dv/dt.$$

If we apply the preceding analysis to each component of the planar motion we get the following equations, which together with (1.4.1), (1.4.2), and (1.4.3) govern the planar rocket motion

$$\begin{aligned} \frac{dv^1}{dt} &= F^1 - \frac{c\beta}{m}\cos\omega \\ \frac{dv^2}{dt} &= F^2 - \frac{c\beta}{m}\sin\omega \qquad v^i(0) = v_0^i, \quad i = 1, 2. \end{aligned} \tag{1.4.4}$$

Here, the components of the force F can be functions of y and v. This would be the case if the motion takes place in a non-constant gravitational field and if drag forces act on the rocket.

The control problems associated with the motion of the rocket are of the following type. The burning rate control β and the thrust direction control ω are to be chosen from the class of piecewise continuous functions (or some other appropriate class) in such a way that certain of the variables t, y, v, m attain specific terminal values. From among the controls that achieve these values, the control that maximizes (or minimizes) a given function of the remaining terminal values is to be determined. In other problems, an integral evaluated along the trajectory in the state space is to be extremized.

To be more specific, consider the "minimum fuel problem." It is required that the rocket go from a specified initial point y_0 to a specified terminal point y_f in such a way that the fuel consumed is minimized. This problem is important for the following reason. Since the total weight of rocket plus fuel plus payload that can be constructed and lifted is constrained by the state of the technology, it follows that the less fuel consumed, the larger the payload that can be carried by the rocket. From (1.4.2) we have

$$m_f = m_0 - \int_{t_0}^{t_f} \beta(t)dt,$$

where t_0 is the initial time, t_f is the terminal time (time at which y_f is reached), m_f is the final mass, and m_0 is the initial mass. The fuel consumed is therefore $m_0 - m_f$. Thus, the problem of minimizing the fuel consumed is the problem of minimizing

$$P(\beta, \omega) = \int_{t_0}^{t_f} \beta(t)dt \tag{1.4.5}$$

subject to (1.4.1) to (1.4.4). This problem is equivalent to the problem of maximizing m_f. In the minimum fuel problem the terminal velocity vector v_f will be unspecified if a "hard landing" is permitted; it will be specified if a "soft landing" is required. The terminal time t_f may or may not be specified.

Another example is the problem of rendezvous with a moving object whose position vector at time t is $z(t) = (z^1(t), z^2(t))$ and whose velocity vector at time t is $w(t) = (w^1(t), w^2(t))$, where z^1, z^2, w^1, and w^2 are continuous functions. Let us suppose that there exist thrust programs β and ω satisfying (1.4.3) and such that rendezvous can be effected. Mathematically this is expressed by the assumption that the solutions y, v of the equations of motion corresponding to the given choice of β and ω have the property that the equations

$$\begin{aligned} y(t) &= z(t) \\ v(t) &= w(t) \end{aligned} \tag{1.4.6}$$

have positive solutions. Such controls (β, ω) will be called admissible. Since for each admissible β and ω the corresponding solutions y and v are continuous, and since the functions z and w are continuous by hypothesis, it follows that for each admissible pair (β, ω) there is a smallest positive solution $t_f(\beta, \omega)$ for which (1.4.6) holds. The number $t_f(\beta, \omega)$ is the rendezvous time. Two problems are possible here. The first is to determine from among the admissible controls one that delivers the maximum payload; that is, to maximize $m_f = m_f(t_f(\beta, \omega))$. The second is to minimize the rendezvous time $t_f(\beta, \omega)$.

1.5 Electrical Engineering

Example 1.5.1. A control surface on an airplane is to be kept at some arbitrary position by means of a servo-mechanism. Outside disturbances such as wind gusts occur infrequently and are short with respect to the time constant of the servo-mechanism. A direct-current electric motor is used to apply a torque to bring the control surface to its desired position. Only the armature voltage v into the motor can be controlled. For simplicity we take the desired position to be the zero angle and we measure deviations in the angle θ from this desired position. Without the application of a torque the control surface would vibrate as a damped harmonic oscillator. Therefore, with a suitable normalization the differential equation for θ can be written as

$$\frac{d^2\theta}{dt^2} + a\,\frac{d\theta}{dt} + \omega^2\theta = u \qquad \theta(0) = \theta_0 \qquad \theta'(0) = \theta_0'. \tag{1.5.1}$$

Here u represents the restoring torque applied to the control surface, the term $a d\theta/dt$ represents the damping effect, and ω^2 is the spring constant. If no damping occurs, then $a = 0$. Since the source of voltage cannot deliver a voltage larger in absolute value than some value v_0, the restoring torque must be bounded in absolute value. Hence it follows that we must have

$$|u(t)| \le A, \tag{1.5.2}$$

where A is some positive constant.

If we set

$$x^1 = \theta \qquad x^2 = \frac{d\theta}{dt}$$

we can rewrite Eq. (1.5.1) as follows:

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 & x^1(0) &= \theta_0 \\ \frac{dx^2}{dt} &= -ax^2 - \omega^2 x^1 + u & x^2(0) &= \theta_0'. \end{aligned} \tag{1.5.3}$$

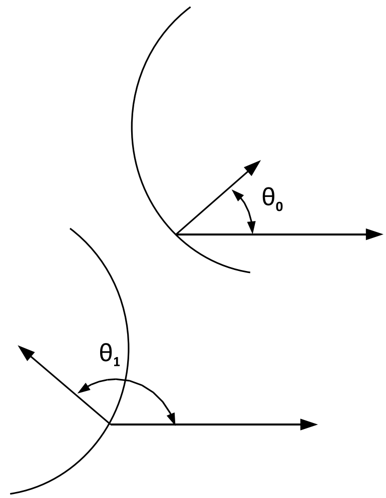

FIGURE 1.1 [From: G. Stephens Jones and Aaron Strauss, An example of optimal control, *SIAM Review*, Vol. 10, 25–55 (1968).]

The problem is the following. A short disturbance has resulted in a deviation $\theta = \theta_0$ from the desired position and a deviation $d\theta/dt = \theta_0'$ from rest. How should the voltage be applied over time so that the control surface is brought back to the set position $\theta = 0$, $d\theta/dt = 0$ in the shortest possible time? In terms of (1.5.3), the problem is to choose a function u from an appropriate class of functions, say piecewise continuous functions, such that u satisfies (1.5.2) at each instant of time and such that the solution (x^1, x^2) of (1.5.3) corresponding to u reaches the origin in (x^1, x^2)-space in minimum time.

Example 1.5.2. Figure 1.1 depicts an antenna free to rotate from any angular position θ_0 to any other angle θ_1. The equation of motion under an applied torque T is given by

$$I\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} = T \qquad \theta(0) = \theta_0 \quad \theta'(0) = \theta_0', \tag{1.5.4}$$

where β is a damping factor and I is the moment of inertia of the system about the vertical axis.

The objective here is to move from the position and velocity (θ_0, θ_0') at an initial time t_0 to the state and velocity $(\theta_1, 0)$ at some later time t_1 in a way that the following criteria are met.

(a) The transfer of position must take place within a reasonable (but not specified) period of time.

(b) The energy expended in making rotations must be kept within reasonable (but not specified) bounds in order to avoid excessive wear on components.

(c) The fuel or power expended in carrying out the transfer must be kept within reasonable (but not specified) limits.

Since the energy expended is proportional to $(d\theta/dt)^2$ and the fuel or power expended is proportional to the magnitude of the torque, a reasonable performance criterion would be

$$J = \int_{t_0}^{t_1} (\gamma_1 + \gamma_2 \left(\frac{d\theta}{dt}\right)^2 + \gamma_3 |T|) dt,$$

where $\gamma_1 > 0$, $\gamma_2 \geq 0$, $\gamma_3 \geq 0$, and t_1 is free.

The control torque T is constrained in magnitude by a quantity $k > 0$, that is, $|T| \leq k$, and $(d\theta/dt)$ is constrained in magnitude by 1, that is, $|d\theta/dt| \leq 1$.

If as in Example 1.5.1 we set

$$x^1 = \theta \qquad x^2 = \frac{d\theta}{dt},$$

we can write (1.5.4) as the system

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 & \qquad x^1(0) &= \theta_0 \\ \frac{dx^2}{dt} &= -\frac{\beta}{I}\, x^1 + \frac{T}{I} & \qquad x^2(0) &= \theta_0'. \end{aligned} \tag{1.5.5}$$

The problem then is to choose a torque program (function) T that minimizes

$$J(T) = \int_{t_0}^{t_1} (\gamma_1 + \gamma_2 (x^2)^2 + \gamma_3 |T|) dt$$

subject to (1.5.5), the terminal conditions $x^1(t_1) = \theta_1$, $x^2(t_1) = 0$, t_1 free and the constraints

$$|T(t)| \leq k \qquad |x^2(t)| \leq 1.$$

This example differs from the preceding examples in that we have a constraint $|x^2(t)| \leq 1$ on the state as well as a constraint on the control.

1.6 The Brachistochrone Problem

We now present a problem from the calculus of variations; the brachistochrone problem, posed by John Bernoulli in 1696. This problem can be regarded as the starting point of the theory of the calculus of variations. Galileo

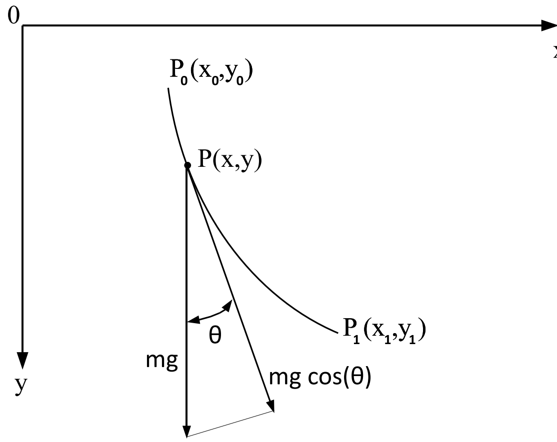

FIGURE 1.2

also seems to have considered this problem in 1630 and 1638, but was not as explicit in his formulation.

Two points P_0 and P_1 that do not lie on the same vertical line are given in a vertical plane with P_0 higher than P_1. A particle, or point mass, acted upon solely by gravity is to move along a curve C joining P_0 and P_1. Furthermore, at P_0 the particle is to have an initial speed v_0 along the curve C. The problem is to choose the curve C so that the time required for the particle to go from P_0 to P_1 is a minimum.

To formulate the problem analytically, we set up a coordinate system in the plane as shown in Fig. 1.2.

Let P_0 have coordinates (x_0, y_0) with $y_0 > 0$, let P_1 have coordinates (x_1, y_1) with $y_1 > 0$, and let C have $y = y(x)$ as its equation. At time t, let $(x(t), y(t))$ denote the coordinates of the particle as it moves along the curve C, let $v(t)$ denote the speed, and let $s(t)$ denote the distance traveled. We shall determine the time required to traverse C from P_0 to P_1.

From the principle of conservation of energy, we have that

$$\frac{1}{2}m(v^2 - v_0^2) = mg(y - y_0). \tag{1.6.1}$$

Also,

$$v = \frac{ds}{dt} = \frac{ds}{dx}\,\frac{dx}{dt} = [1 + (y')^2]^{1/2}\,\frac{dx}{dt}. \tag{1.6.2}$$

Hence, using (1.6.1) and (1.6.2), we get that

$$dt = \frac{[1 + (y')^2]^{1/2}}{v}\,dx = \left[\frac{1 + (y')^2}{2g(y - \alpha)}\right]^{1/2} dx,$$

where

$$\alpha = y_0 - v_0^2/2g.$$

Thus, the time of traverse T along C is given by

$$T = \frac{1}{(2g)^{1/2}} \int_{x_0}^{x_1} \left[\frac{1 + (y')^2}{y - \alpha} \right]^{1/2} dx.$$

The problem of finding a curve C that minimizes the time of traverse is that of finding a function $y = y(x)$ that minimizes the integral

$$\int_{x_0}^{x_1} \left[\frac{1 + (y')^2}{y - \alpha} \right]^{1/2} dx. \tag{1.6.3}$$

Note that if $v_0 = 0$, then the integral is improper.

We put this problem in a format similar to the previous ones as follows. Change the notation for the independent variable from x to t. Then set

$$y' = u \qquad y(t_0) = y_0. \tag{1.6.4}$$

A continuous function u will be called admissible if it is defined on $[t_0, t_1]$, if the solution of (1.6.4) corresponding to u satisfies $y(t_1) = y_1$, if $y(t) > y_0$ on $[t_0, t_1]$, and if the mapping $t \to [(1 + u^2(t))/(y(t) - \alpha)]^{1/2}$ is integrable on $[t_0, t_1]$. Our problem is to determine the admissible function u that minimizes

$$J(u) = \int_{t_0}^{t_1} \left(\frac{1 + u^2}{y - \alpha} \right)^{1/2} dt \tag{1.6.5}$$

in the class of all admissible u.

The brachistochrone problem can be formulated as a control problem in a different fashion. By (1.6.1) and (1.6.2), the speed of the particle along the curve C is given by $(2g(y - \alpha))^{1/2}$. Hence, if θ is the angle that the tangent to C makes with the positive x-axis, then

$$\begin{aligned} \frac{dx}{dt} &= (2g(y - \alpha))^{1/2} \cos\theta \\ \frac{dy}{dt} &= (2g(y - \alpha))^{1/2} \sin\theta. \end{aligned}$$

Let $u = \cos\theta$. Then the equations of motion become

$$\begin{aligned} \frac{dx}{dt} &= (2g(y - \alpha))^{1/2} u & x(t_0) &= x_0 \\ \frac{dy}{dt} &= (2g(y - \alpha))^{1/2} (1 - u^2)^{1/2} & y(t_0) &= y_0. \end{aligned} \tag{1.6.6}$$

The problem is to choose a control u satisfying $|u| \le 1$ such that the point (x, y), which at initial time t_0 is at (x_0, y_0), reaches the prescribed point (x_1, y_1) in minimum time. If t_1 is the time at which P_1 is reached, then this is equivalent to minimizing $t_1 - t_0$. This in turn is equivalent to minimizing

$$\int_{t_0}^{t_1} dt \tag{1.6.7}$$

subject to (1.6.6), the terminal condition (x_1, y_1), and the constraint $|u(t)| \leq 1$.

The brachistochrone problem can be modified in the following fashion. One can replace the fixed point P_1 by a curve Γ_1 defined by $y = y_1(x)$ and seek the curve C joining P_0 to Γ_1 along which the mass particle must travel if it is to go from P_0 to Γ_1 in minimum time. We can also replace P_0 by a curve Γ_0 where Γ_0 is at positive distance from Γ_1 and ask for the curve C joining Γ_0 and Γ_1 along which the particle must travel in order to minimize the time of transit.

1.7 An Optimal Harvesting Problem

We present here a population model of McKendric type with crowding effect. The reformulation of the control problem coincides with the reformulation by Gurtin and Murphy [40], [68]. The age-dependent population model is given by

$$\begin{aligned}\frac{\partial p(r,t)}{\partial t} + \frac{\partial p(r,t)}{\partial t} &= -\mu(r)p(r,t) - f(N(t))p(r,t) - u(t)p(r,t) \qquad (1.7.1)\\ p(r,0) &= p_0(r)\\ p(0,t) &= \beta \int_0^\infty k(r)p(r,t)dr, \quad k(r) = \widetilde{k}(r)h(r)\\ N(t) &= \int_0^\infty p(r,t)dr\end{aligned}$$

where $p(r,t)$ denotes the age density distribution at time t and age r, $\mu(r)$ is the mortality rate, $k(r)$ is the female sex ratio at age r, $h(r)$ is the fertility pattern, and β is the specific fertility rate of females. The function $f(N(\cdot))$ indicates decline in the population due to environmental factors such as crowding. The function $u(\cdot) \geq 0$ is the control or harvesting strategy.

We consider the problem of maximizing the harvest

$$J(u) = \int_0^T u(t)N(t)dt \qquad (1.7.2)$$

where $0 \leq u(\cdot) \leq M$ is piecewise continuous and (1.7.1) is satisfied. The upper bound M on $u(\cdot)$ is the maximum effort.

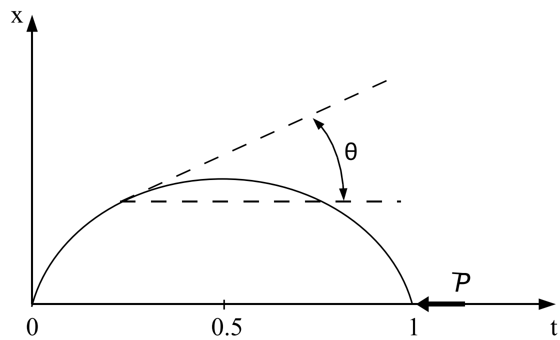

FIGURE 1.3 [From: H. Maurer and H. D. Mittelmann, *Optimal Control Applications and Methods*, 12, 19–31 (1991).]

1.8 Vibration of a Nonlinear Beam

Consider the classical nonlinear Euler beam [56] with deflection limited by an obstacle parallel to the plane of the beam. The beam is axially compressed by a force P, which acts as a branching parameter α.

We assume that the energy of a beam that is compressed by a force P is given by

$$I_\alpha = \frac{1}{2}\int_0^1 \dot{\theta}^2 dt + \alpha \int_0^1 \cos\theta(t) dt.$$

Here $\alpha = P/EJ$, where EJ is the bending stiffness, t denotes the arc length, $\theta(t)$ is the angle between the tangential direction of the beam at t and the reference line (see Fig. 1.3), and the length of the beam is $\ell = 1$.

For the deflection of the beam away from the reference line we have

$$\dot{x} = \sin\theta, \qquad \dot{\theta} = \frac{\ddot{x}}{\sqrt{1-\dot{x}^2}}.$$

Hence, the energy can also be written as

$$I_\alpha = \frac{1}{2}\int_0^1 \frac{\ddot{x}^2}{1-\dot{x}^2}\, dt + \alpha \int_0^1 \sqrt{1-\dot{x}^2}\, dt.$$

We assume that $|\dot{x}(t)| < 1$, that is, $-\pi/2 < \theta(t) < \pi/2$ holds on $[0, 1]$.

The variational problem for the simply supported beam consists of minimizing the energy subject to the boundary conditions

$$x(0) = x(1) = 0$$

and the state constraints

$$-d \le x(t) \le d, \qquad 0 \le t \le 1, \qquad d > 0.$$

In the case of a clamped beam, one replaces the boundary conditions by

$$x(0) = 0, \qquad \theta(0) = 0, \qquad x(1) = \theta(1) = 0.$$

Chapter 2

Formulation of Control Problems

2.1 Introduction

In this chapter we discuss the mathematical structures of the examples in the previous chapter.

We first discuss problems whose dynamics are given by ordinary differential equations. We motivate and give precise mathematical formulations and equivalent mathematical formulations of apparently different problems. We then point out the relationship between optimal control problems and the calculus of variations. Last, we present various formulations of hereditary problems. These problems are also called delay or lag problems.

2.2 Formulation of Problems Governed by Ordinary Differential Equations

Many of the examples in the preceding chapter have the following form. The state of a system at time t is described by a point or vector

$$x(t) = (x^1(t), \dots, x^n(t))$$

in n-dimensional euclidean space, $n \geq 1$. Initially, at time t_0, the state of the system is

$$x(t_0) = x_0 = (x_0^1, \dots, x_0^n).$$

More generally, we can require that at the initial time t_0 the initial state x_0 is such that the point (t_0, x_0) belongs to some pre-assigned set $\mathcal{T}_0$ in (t, x)-space. The state of the system varies with time according to the system of differential equations

$$\frac{dx^i}{dt} = f^i(t, x, z) \qquad x^i(t_0) = x_0^i \qquad i = 1, \dots, n, \tag{2.2.1}$$

where $z = (z^1, \dots, z^m)$ is a vector in real euclidean space $\mathbb{R}^m$ and the functions f^i are real valued continuous functions of the variables (t, x, z).

By the "system varying according to (2.2.1)" we mean the following. A function u with values in m-dimensional euclidean space is chosen from some prescribed class of functions. In this section we shall take this class to be a subclass $\mathcal{C}$ of the class of piecewise continuous functions. When the substitution $z = u(t)$ is made in the right-hand side of (2.2.1), we obtain a system of ordinary differential equations:

$$\frac{dx^i}{dt} = f^i(t, x, u(t)) = F^i_u(t, x) \qquad i = 1, \ldots, n. \tag{2.2.2}$$

The subscript u on the F^i_u emphasizes that the right-hand side of (2.2.2) depends on the choice of function u. For each u in $\mathcal{C}$ it is assumed that there exists a point (t_0, x_0) in $\mathcal{T}_0$ and a function $\phi = (\phi^1, \ldots, \phi^n)$ defined on an interval $[t_0, t_2]$ with values in $\mathbb{R}^n$ such that (2.2.2) is satisfied. That is, we require that for every t in $[t_0, t_2]$

$$\phi'^i(t) = \frac{d\phi^i}{dt} = f^i(t, \phi(t), u(t)) \qquad \phi^i(t_0) = x^i_0 \quad i = 1, \ldots, n.$$

At points of discontinuity of u this equation is interpreted as holding for the one-sided limits. The function ϕ describes the evolution of the system with time and will sometimes be called a *trajectory*.

The function u is further required to be such that at some time t_1, where $t_0 < t_1$, the point $(t_1, \phi(t_1))$ belongs to a pre-assigned set $\mathcal{T}_1$ and for $t_0 \leq t < t_1$ the points $(t, \phi(t))$ do not belong to $\mathcal{T}_1$. The set $\mathcal{T}_1$ is called the terminal set for the problem. Examples of terminal sets, taken from Chapter 1, are given in the next paragraph.

In the production planning problem, $\mathcal{T}_1$ is the line $t = T$ in the (t, x) plane. In the first version of the chemical engineering problem, the set $\mathcal{T}_1$ is the hyperplane $t = T$; that is, those points in (t, x)-space with $x = (x^1, \ldots, x^n)$ free and t fixed at T. In the last version of the chemical engineering problem, $\mathcal{T}_1$ is the set of points in (t, x)-space whose coordinates x^i are fixed at x^i_f for $i = 1, \ldots, j$ and whose remaining coordinates are free. In some problems it is required that the solution hit a moving target set $G(t)$. That is, at each time t of some interval $[\tau_0, \tau_1]$ there is a set $G(t)$ of points in x-space, and it is required that the solution ϕ hit $G(t)$ at some time t. Stated analytically, we require the existence of a point t_1 in $[\tau_0, \tau_1]$ such that $\phi(t_1)$ belongs to $G(t_1)$. An example of this type of problem is the rendezvous problem in Section 1.4. The set $\mathcal{T}_1$ in the moving target set problem is the set of all points $(t, x) = (t, z(t), w(t), m)$ with $\tau_0 \leq t \leq \tau_1$ and $m > 0$.

The discussion in the preceding paragraphs is sometimes summarized in less precise but somewhat more graphic language by the statement that the functions u are required to transfer the system from an initial state x_0 at time t_0 to a terminal state x_1 at time t_1, where $(t_0, x_0) \in \mathcal{T}_0$ and $(t_1, x_1) \in \mathcal{T}_1$. Note that to a given u in $\mathcal{C}$ there will generally correspond more than one trajectory ϕ. This results from different choices of initial points (t_0, x_0) in $\mathcal{T}_0$

or from non-uniqueness of solutions of (2.2.2) if no assumptions are made to guarantee the uniqueness of solutions of (2.2.2) with given initial data (t_0, x_0).

It is often further required that a function u in $\mathcal{C}$ and a corresponding solution ϕ satisfy a system of inequality constraints

$$R^i(t, \phi(t), u(t)) \geq 0 \qquad i = 1, 2, \ldots, r, \tag{2.2.3}$$

for all $t_0 \leq t \leq t_1$, where the functions $R^1, \ldots, R^r$ are given functions of (t, x, z). For example, in the production planning problem discussed in Section 1.2 the constraints can be written as $R^i \geq 0$, $i = 1, 2, 3$, where $R^1(t, x, z) = x$, $R^2(t, x, z) = z$, and $R^3(t, x, z) = 1 - z$. In Example 1.5.1, the constraints can be written as $R^i \geq 0$, $i = 1, 2$, where $R^1(t, x, z) = z + A$ and $R^2(t, x, z) = A - z$.

In the examples of Chapter 1, the control u is to be chosen so that certain functionals are minimized or maximized. These functionals have the following form. Let f^0 be a real valued continuous function of (t, x, z), let g_0 be a real valued function defined on $\mathcal{T}_0$, and let g_1 be a real valued function defined on $\mathcal{T}_1$. For each u in $\mathcal{C}$ and each corresponding solution ϕ of (2.2.2), define a *cost* or *payoff* or *performance index* as follows:

$$J(\phi, u) = g_0(t_0, \phi(t_0)) + g_1(t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(s, \phi(s), u(s))ds.$$

If the function J is to be minimized, then a u^* in $\mathcal{C}$ and a corresponding solution ϕ^* of (2.2.2) are to be found such that $J(\phi^*, u^*) \leq J(\phi, u)$ for all u in $\mathcal{C}$ and corresponding ϕ. In other problems, the functional J is to be maximized. Examples of J taken from Chapter 1 are given in the next paragraph.

In the examples of Chapter 1, the set $\mathcal{T}_0$ is always a point (t_0, x_0). The differential equations in the examples, except in Section 1.3, are such that the solutions are unique. In Section 1.3 let us assume that the functions G^i are such that the solutions are unique. Thus, in these examples the choice of u completely determines the function ϕ. In the economics example, $J(\phi, u)$ is the total cost $J(u)$ given by (1.2.3). The function f^0 is given by $-U((1-z)F(x))e^{-\gamma t}$ and the functions g_0 and g_1 are identically zero. In the first chemical engineering example of Section 1.3, $J(\phi, u) = V(p, \theta)$, where $V(p, \theta)$ is given by (1.3.3). The functions f^0 and g_0 are identically zero. In the minimum fuel problem of Section 1.4, $J(\phi, u) = P(\beta, \omega)$, where P is given by (1.4.5). Here $f^0 = \beta$ and g_0 and g_1 are identically zero. An equivalent formulation is obtained if one takes $J(\phi, u) = -m_f$. Now $f^0 = 0$, $g_0 = 0$, and $g_1 = -m_f$.

We conclude this section with a discussion of two generalizations that will appear in the mathematical formulation to be given in the next section. The first deals with the initial and terminal data. The initial set $\mathcal{T}_0$ and the terminal set $\mathcal{T}_1$ determine a set $\mathcal{B}$ of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{2n+2}$ as follows:

$$\mathcal{B} = \{(t_0, x_0, t_1, x_1) : (t_0, x_0) \in \mathcal{T}_0, \quad (t_1, x_1) \in \mathcal{T}_1\}. \tag{2.2.4}$$

Thus, a simple generalization of the requirement that $(t_0, \phi(t_0)) \in \mathcal{T}_0$ and

$(t_1, \phi(t_1)) \in \mathcal{T}_1$ is the following. Let there be given a set $\mathcal{B}$ of points in $\mathbb{R}^{2n+2}$. It is required of a trajectory ϕ that $(t_0, \phi(t_0), t_1, \phi(t_1))$ belong to $\mathcal{B}$. That is, we now permit possible relationships between initial and terminal data. We shall show later that in some sense this situation is really no more general than the situation in which the initial and terminal data are assumed to be unrelated.

The second generalization deals with the description of the constraints on u. For each (t, x), a system of inequalities $R^i(t, x, z) \geq 0$, $i = 1, \ldots, r$ determines a set $U(t, x)$ in the m-dimensional z-space; namely

$$U(t, x) = \{z : R^i(t, x, z) \geq 0, i = 1, \ldots, r\}.$$

The requirement that a function u and a corresponding trajectory satisfy constraints of the form (2.2.3) can therefore be written as follows:

$$u(t) \in U(t, \phi(t)) \qquad t_0 \leq t \leq t_1.$$

Thus, the constraint (2.2.3) is a special case of the following more general constraint condition.

Let Ω be a function that assigns to each point (t, x) of some suitable subset of $\mathbb{R}^{n+1}$ a subset of the z-space $\mathbb{R}^m$. Thus,

$$\Omega : (t, x) \to \Omega(t, x),$$

where $\Omega(t, x)$ is a subset of $\mathbb{R}^m$. The constraint (2.2.3) is replaced by the more general constraint

$$u(t) \in \Omega(t, \phi(t)).$$

2.3 Mathematical Formulation

The formulation will involve the Lebesgue integral. This is essential in the study of solutions to the problem. The reader who wishes to keep the formulation on a more elementary level can replace "measurable controls" by "piecewise continuous controls," replace "absolutely continuous functions" by "piecewise $C^{(1)}$ functions," and interpret the solution of Eq. (2.3.1) as we interpreted the solution of Eq. (2.2.2).

We establish some notation and terminology. Let t denote a real number, which will sometimes be called time. Let x denote a vector in real euclidean space $\mathbb{R}^n$, $n \geq 1$; thus, $x = (x^1, \ldots, x^n)$. The vector x will be called the *state variable.* We shall use superscripts to denote components of vectors and we shall use subscripts to distinguish among vectors. Let z denote a vector in euclidean m-space $\mathbb{R}^m$, $m \geq 1$; thus, $z = (z^1, \ldots, z^m)$. The vector z will be called the *control variable.* Let $\mathcal{R}$ be a region of (t, x)-space and let $\mathcal{U}$ be a region of z-space, whereby a region we mean an open connected set. Let $\mathcal{G} = \mathcal{R} \times \mathcal{U}$, the cartesian product of $\mathcal{R}$ and $\mathcal{U}$. Let $f^0, f^1, \ldots, f^n$ be real valued functions defined on $\mathcal{G}$. We shall write

$$f = (f^1, \ldots, f^n) \qquad \widehat{f} = (f^0, f^1, \ldots, f^n).$$

Let $\mathcal{B}$ be a set of points

$$(t_0, x_0, t_1, x_1) = (t_0, x_0^1, \ldots, x_0^n, t_1, x_1^1, \ldots, x_1^n)$$

in $\mathbb{R}^{2n+2}$ such that (t_i, x_i), $i = 0, 1$ are in $\mathcal{R}$ and $t_1 \geq t_0 + \delta$, for some fixed $\delta > 0$. The set $\mathcal{B}$ will be said to define the *end conditions* for the problem.

Let Ω be a mapping that assigns to each point (t, x) in $\mathcal{R}$ a subset $\Omega(t, x)$ of the region $\mathcal{U}$ in z-space. The mapping Ω will be said to define the *control constraints*. If $\mathcal{U}(t, x) = \mathcal{U}$ for all (t, x) in $\mathcal{R}$, then we say that there are no control constraints.

Henceforth we shall usually use vector-matrix notation. The system of differential equations (2.2.2) will be written simply as

$$\frac{dx}{dt} = f(t, x, u(t)),$$

where we follow the usual convention in the theory of differential equations and take dx/dt and $f(t, x, u(t))$ to be column vectors. We shall not distinguish between a vector and its transpose if it is clear whether a vector is a row vector or a column vector or if it is immaterial whether the vector is a row vector or a column vector. The inner product of two vectors u and v will be written as $\langle u, v \rangle$. We shall use the symbol $|x|$ to denote the ordinary euclidean norm of a vector. Thus,

$$|x| = \left(\sum_{i=1}^{n} |x^i|^2 \right)^{1/2} = \langle x, x \rangle^{1/2}.$$

If A and B are matrices, then we write their product as AB.

If $f = (f^1, \ldots, f^n)$ is a vector valued function from a set Δ in some euclidean space to the euclidean space $\mathbb{R}^n$ such that each of the real value functions $f^1, \ldots, f^n$ is continuous (or $C^{(k)}$, or measurable, etc.) then we shall say that f is continuous (or $C^{(k)}$, or measurable, etc.) on the set Δ. Similarly, if a matrix A has entries that are continuous functions (or $C^{(k)}$, or measurable functions, etc.) defined on a set Δ in some euclidean space, then we shall say that A is continuous (or $C^{(k)}$, or measurable, etc.) on Δ.

Definition 2.3.1. A *control* is measurable function u defined on an interval $[t_0, t_1]$ with range in $\mathfrak{U}$.

Definition 2.3.2. A *trajectory* corresponding to a control u is an absolutely continuous function ϕ defined on $[t_0, t_1]$ with range in $\mathbb{R}^n$ such that:

(i) $(t, \phi(t)) \in \mathcal{R}$ for all t in $[t_0, t_1]$

(ii) ϕ is a solution of the system of differential equations

$$\frac{dx}{dt} = f(t, x, u(t)); \tag{2.3.1}$$

that is,

$$\phi'(t) = f(t, \phi(t), u(t)) \text{ a.e. on } [t_0, t_1].$$

The point $(t_0, \phi(t_0))$ will be called the *initial point* of the trajectory and the point $(t_1, \phi(t_1))$ will be called the *terminal point* of the trajectory. The point $(t_0, \phi(t_0), t_1, \phi(t_1))$ will be called the *end point* of the trajectory.

Note that since ϕ is absolutely continuous, it is the integral of its derivative. Hence (ii) contains the statement that the function $t \to f(t, \phi(t), u(t))$ is Lebesgue integrable on $[t_0, t_1]$.

The system of differential equations (2.3.1) will be called the *state equations.*

We emphasize the following about our notation. We are using the letter z to denote a point of $\mathcal{U}$; we are using the letter u to denote a function with range in $\mathcal{U}$.

Definition 2.3.3. A control u is said to be an *admissible control* if there exists a trajectory ϕ corresponding to u such that

(i) $t \to f^0(t, \phi(t), u(t))$ is in $L_1[t_0, t_1]$.

(ii) $u(t) \in \Omega(t, \phi(t))$ a.e. on $[t_0, t_1]$.

(iii) $(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{B}$.

A trajectory corresponding to an admissible control as in Definition 2.3.3 will be called an *admissible trajectory.*

Definition 2.3.4. A pair of functions (ϕ, u) such that u is an admissible control and ϕ is an admissible trajectory corresponding to u will be called an *admissible pair.*

Note that to a given admissible control there may correspond more than one admissible trajectory as a result of different choices of permissible end points. Also, even if we fix the endpoint, there may be several trajectories corresponding to a given control because we do not require uniqueness of solutions of (2.3.1) for given initial conditions.

We now state the control problem.

Problem 2.3.1. Let $\mathcal{A}$ denote the set of all admissible pairs (ϕ, u) and let $\mathcal{A}$ be non-empty. Let

$$J(\phi, u) = g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt, \tag{2.3.2}$$

where (ϕ, u) is an admissible pair and g is a given real valued function defined on $\mathcal{B}$. Let $\mathcal{A}_1$ be a non-empty subset of $\mathcal{A}$. Find a pair (ϕ^*, u^*) in $\mathcal{A}_1$ that minimizes (2.3.2) in the class $\mathcal{A}_1$. That is, find an element (ϕ^*, u^*) in $\mathcal{A}_1$ such that

$$J(\phi^*, u^*) \leq J(\phi, u) \quad \text{for all} \quad (\phi, u) \text{ in } \mathcal{A}_1.$$

The precise formulation of Problem 2.3.1 is rather lengthy. Therefore, the following statement, which gives the essential data of the problem, is often used to mean that we are considering Problem 2.3.1.

Minimize (2.3.2) *in the class* $\mathcal{A}_1$ *subject to the state equation* (2.3.1)*, the end conditions* $\mathcal{B}$*, and the control constraints* Ω*.*

We have stated Problem 2.3.1 as a minimization problem. In some applications it is required that the functional J be maximized. There is, however, no need to consider maximum problems separately because the problem of maximizing J is equivalent to the problem of minimizing $-J$. Hence we shall confine our attention to minimum problems.

Definition 2.3.5. A pair (ϕ^*, u^*) that solves Problem 2.3.1 is called an *optimal pair*. The trajectory ϕ^* is called an *optimal trajectory* and the control u^* is called an *optimal control*.

The first term on the right in (2.3.2) is the function g evaluated at the end points of an admissible trajectory. Thus, it assigns a real number to every admissible trajectory and so is a functional G_1 defined on the admissible trajectories. The functional G_1 is defined by the formula

$$G_1(\phi) = g(t_0, \phi(t_0), t_1, \phi(t_1)).$$

Other examples of functionals defined on admissible trajectories are

$$G_2(\phi) = \max\{|\phi(t)| : t_0 \leq t \leq t_1\}$$

and

$$G_3(\phi) = \max\{|\phi(t) - h(t)| : t_0 \leq t \leq t_1\},$$

where h is a given continuous function defined on an interval I containing all the intervals $[t_0, t_1]$ of definition of admissible trajectories. The functionals G_2 and G_3 arise in problems in which in addition to minimizing (2.3.2) it is also desired to keep the state of the system close to some preassigned state.

The preceding discussion justifies the consideration of the following generalization of Problem 2.3.1.

Problem 2.3.2. Let everything be as in Problem 2.3.1, except that (2.3.2) is replaced by

$$\widehat{J}(\phi, u) = G(\phi) + \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt, \tag{2.3.3}$$

where G is a functional defined on the admissible trajectories. Find a pair (ϕ^*, u^*) in $\mathcal{A}_1$ that minimizes (2.3.3) in the class $\mathcal{A}_1$.

2.4 Equivalent Formulations

Certain special cases of Problem 2.3.1 are actually equivalent to Problem 2.3.1 in the sense that Problem 2.3.1 can be formally transformed into the special case in question. This information is useful in certain investigations where it is more convenient to study one of the special cases than to study Problem 2.3.1. The reader is warned that in making the transformation to the special case some of the properties of the original problem, such as linearity, continuity, convexity, etc. may be altered. In any particular investigation one must check that the pertinent hypotheses made for the original problem are valid for the transformed problem.

Special cases of Problem 2.3.1 are obtained by taking $f^0 = 0$ or $g = 0$. In keeping with the terminology for related problems in the calculus of variations, we shall call a problem in which $f^0 = 0$ a Mayer problem and we shall call a problem in which $g = 0$ a Lagrange problem. Problem 2.3.1 of Section 2.3 is sometimes called a Bolza problem, also as in the calculus of variations. We shall show that the Mayer formulation and the Lagrange formulation are as general as the Bolza formulation by showing that Problem 2.3.1 can be written either as a Mayer problem or as a Lagrange problem.

We formulate Problem 2.3.1 as a Mayer problem in a higher dimensional euclidean space. Let $\widehat{x} = (x^0, x) = (x^0, x^1, \ldots, x^n)$. Let $\widehat{\mathcal{R}} = \mathbb{R}^1 \times \mathcal{R}$ and let $\widehat{\mathcal{G}} = \widehat{\mathcal{R}} \times \mathcal{U}$. The functions $f^0, f^1, \ldots, f^n$ are defined on $\widehat{\mathcal{G}}$ since they are defined on $\mathcal{G}$ and they are independent of x^0. Let the mapping $\widehat{\Omega}$ be defined on $\widehat{\mathcal{R}}$ by the equation $\widehat{\Omega}(t, \widehat{x}) = \Omega(t, x)$. Let

$$\widehat{\mathcal{B}} = \{(t_0, \widehat{x}_0, t_1, \widehat{x}_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B}, \quad x_0^0 = 0\}.$$

Let (ϕ, u) be an admissible pair for Problem 2.3.1. Let $\widehat{\phi} = (\phi^0, \phi)$, where ϕ^0 is an absolutely continuous function such that

$$\phi^{0'}(t) = f^0(t, \phi(t), u(t)) \qquad \phi^0(t_0) = 0$$

for almost every t in $[t_0, t_1]$. By virtue of (i) of Definition 2.3.3 such a function ϕ^0 exists and is given by

$$\phi^0(t) = \int_{t_0}^{t} f^0(s, \phi(s), u(s))ds.$$

Then $(\widehat{\phi}, u)$ is an admissible pair for a problem in which $\mathcal{R}, \mathcal{G}, \Omega, \mathcal{B}$, replaced by $\widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}, \widehat{\mathcal{B}}$, respectively, and in which the system of state equations (2.3.1) is replaced by

$$\begin{aligned} \frac{dx^0}{dt} &= f^0(t, x, u(t)) \\ \frac{dx}{dt} &= f(t, x, u, (t)). \end{aligned} \tag{2.4.1}$$

If we set $\widehat{f} = (f^0, f)$, then Eq. (2.4.1) can be written as

$$\frac{d\widehat{x}}{dt} = \widehat{f}(t, x, u(t)).$$

Conversely, to every admissible pair $(\widehat{\phi}, u)$ for a problem involving $\widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}, \widehat{\mathcal{B}}$ and (2.4.1) there corresponds the admissible pair (ϕ, u) for Problem 2.3.1, where ϕ consists of the last n-components of $\widehat{\phi}$. Let

$$\widehat{g}(t_0, \widehat{x}_0, t_1, \widehat{x}_1) = g(t_0, x_0, t_1, x_1) + x_1^0$$

and let

$$\widehat{J}(\widehat{\phi}, u) = \widehat{g}(t_0, \widehat{\phi}(t_0), t_1, \widehat{\phi}(t_1)).$$

Then $\widehat{J}(\widehat{\phi}, u) = J(\phi, u)$, where $\widehat{\phi} = (\phi^0, \phi)$. Hence the Mayer problem of minimizing $\widehat{J}$ subject to state equations (2.4.1), control constraints $\widehat{\Omega}$, and end conditions $\widehat{\mathcal{B}}$ is equivalent to Problem 2.3.1.

We now show that Problem 2.3.1 can be formulated as a Lagrange problem. Let $\widehat{x}, \widehat{\mathcal{R}}, \widehat{\mathcal{G}}, \widehat{\Omega}$ be as in the previous paragraph. Let

$$\widehat{\mathcal{B}} = \{(t_0, \widehat{x}_0, t_1, \widehat{x}_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B}, \quad x_0^0 = g(t_0, x_0, t_1, x_1)/(t_1 - t_0)\}. \tag{2.4.2}$$

(Recall that for all points in $\mathcal{B}$ we have $t_1 > t_0$.) Let (ϕ, u) be an admissible pair for Problem 2.3.1 and let $\widehat{\phi} = (\phi^0, \phi)$ where $\phi^0(t) \equiv g(t_0, x_0, t_1, x_1)/(t_1 - t_0)$. Then $(\widehat{\phi}, u)$ is an admissible pair for a problem in which $\mathcal{R}, \mathcal{G}, \Omega, \mathcal{B}$ are replaced by roofed quantities with $\widehat{\mathcal{B}}$ as in (2.4.2) and with state equations

$$\begin{aligned} \frac{dx^0}{dt} &= 0 \\ \frac{dx}{dt} &= f(t, x, u(t)). \end{aligned} \tag{2.4.3}$$

Conversely, to every admissible pair $(\widehat{\phi}, u)$ for the problem with roofed quantities there corresponds the admissible pair (ϕ, u) for Problem 2.3.1, where ϕ consists of the last n components of $\widehat{\phi}$. If we replace f^0 of Problem 2.3.1 by $f^0 + x^0$ and let

$$\widehat{J}(\widehat{\phi}, u) = \int_{t_0}^{t_1} (f^0(t, \phi(t), u(t)) + \phi^0(t))dt \tag{2.4.4}$$

then $\widehat{J}(\widehat{\phi}, u) = J(\phi, u)$. Hence the Lagrange problem of minimizing (2.4.4) subject to state equations (2.4.3), control constraints $\widehat{\Omega}$, and end conditions $\widehat{\mathcal{B}}$ is equivalent to Problem 2.3.1.

In Problem 2.3.1 the initial time t_0 and the terminal time t_1 need not be fixed. We now show that Problem 2.3.1 can be written as a problem with fixed initial time and fixed terminal time. We do so by changing the time parameter to s via the equation

$$t = t_0 + s(t_1 - t_0) \qquad 0 \le s \le 1$$

and by introducing new state variables as follows.

Let w be a scalar and consider the problem with state variables (t, x, w), where x is an n-vector and t is a scalar. Let s denote the time variable. Let the state equations be

$$\begin{aligned} \frac{dt}{ds} &= w \qquad\qquad \frac{dw}{ds} = 0 \\ \frac{dx}{ds} &= f(t, x, \bar{u}(s))w \end{aligned} \tag{2.4.5}$$

where $\bar{u}$ is the control and f is as in Problem 2.3.1. Let

$$\begin{aligned} \bar{\mathcal{B}} = \{&(s_0, t_0, x_0, w_0, s_1, t_1, x_1, w_1) : s_0 = 0, s_1 = 1, \\ &(t_0, x_0, t_1, x_1) \in \mathcal{B}, \quad w_0 = t_1 - t_0\}. \end{aligned} \tag{2.4.6}$$

Note that the initial and terminal times are now fixed. Let $\bar{\Omega}(s, t, x, w) = \Omega(t, x)$. Let $\bar{\phi} = (\tau, \xi, \omega)$ be a solution of (2.4.5) corresponding to a control $\bar{u}$, where the Greek-Latin correspondence between (τ, ξ, ω) and (t, x, w) indicates the correspondence between components of $\bar{\phi}$ and the system (2.4.5). Let

$$\bar{J}(\bar{\phi}, \bar{u}) = g(\tau(0), \xi(0), \tau(1), \xi(1)) + \int_0^1 f^0(\tau(s), \xi(s), \bar{u}(s))\omega(s)ds. \tag{2.4.7}$$

Consider the fixed end-time problem of minimizing (2.4.7) subject to the state equations (2.4.5), the control constraints $\bar{\Omega}$, and the end conditions $\bar{\mathcal{B}}$.

Since $t_1 - t_0 > 0$, it follows that for any solution of (2.4.5) satisfying (2.4.6) we have $\omega(s) = t_1 - t_0$, a positive constant, for $0 \le s \le 1$. Let (ϕ, u) be an admissible pair for Problem 2.3.1. It is readily verified that if

$$\begin{aligned} \tau(s) &= t_0 + s(t_1 - t_0) \qquad & \xi(s) &= \phi(t_0 + s(t_1 - t_0)) \\ \bar{u}(s) &= u(t_0 + s(t_1 - t_0)) \qquad & \omega(s) &= t_1 - t_0, \end{aligned}$$

then $(\bar{\phi}, \bar{u}) = (\tau, \xi, \omega, \bar{u})$ is an admissible pair for the fixed end-time problem and $\bar{J}(\bar{\phi}, \bar{u}) = J(\phi, u)$. Conversely, let $(\bar{\phi}, \bar{u})$ be an admissible pair for the fixed end-time problem. If we set

$$\phi(t) = \xi\left(\frac{t - t_0}{t_1 - t_0}\right) \qquad u(t) = \bar{u}\left(t - t_0 t_1 - t_0\right), \qquad t_0 \le t \le t_1,$$

then since $\tau(s) = t_0 + s(t_1 - t_0)$, we have $t = \tau(s)$ for $0 \le s \le 1$. It is readily verified that (ϕ, u) is admissible for Problem 2.3.1 and that $J(\phi, u) = \bar{J}(\bar{\phi}, \bar{u})$. Hence Problem 2.3.1 is equivalent to a fixed end-time problem.

The following observation will be useful in the sequel. Since for any admissible solution of the fixed time problem we have $\omega(s) = t_1 - t_0 > 0$, we can take the set $\bar{\mathcal{R}}$ for the fixed end-time problem to be $[0,1] \times \mathcal{R} \times \mathbb{R}^+$, where $\mathbb{R}^+ = \{w : w > 0\}$.

A special case of the end conditions occurs if the initial and terminal data are separated. In this event, a set $\mathcal{T}_0$ of points (t_0, x_0) in $\mathbb{R}^{n+1}$ and a set $\mathcal{T}_1$ of points (t_1, x_1) in $\mathbb{R}^{n+1}$ are given and an admissible trajectory is required to satisfy the conditions

$$(t_i, \phi(t_i)) \in \mathcal{T}_i, \qquad i = 0, 1. \tag{2.4.8}$$

The set $\mathcal{B}$ in this case is given by (2.2.4). We shall show that the apparently more general requirement (iii) of Definition 2.3.3 can be reduced to the form (2.4.8) by embedding the problem in a space of higher dimension as follows.

Let $y = (y^1, \ldots, y^n)$ and let y^0 be a scalar. Let $\widehat{y} = (y^0, y)$. Let the sets $\mathcal{R}$ and $\mathcal{G}$ of Problem 2.3.1 be replaced by sets $\widetilde{\mathcal{R}} = \mathcal{R} \times \mathbb{R}^{n+1}$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{R}} \times \mathcal{U}$. Then the vector function $\widehat{f} = (f^0, f)$ is defined on $\widetilde{\mathcal{G}}$ since it is independent of $\widehat{y}$. Let $\widetilde{\Omega}(t, x, \widehat{y}) = \Omega(t, x)$. Let the state equations be

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, u(t)) \\ \frac{d\widehat{y}}{dt} &= 0. \end{aligned} \tag{2.4.9}$$

Let

$$\begin{aligned} \widetilde{\mathcal{T}}_0 &= \{(t_0, x_0, y_0^0, y_0) : (t_0, x_0, y_0^0, y_0) \in \mathcal{B}\} \\ \widetilde{\mathcal{T}}_1 &= \{(t_1, x_1, y_1^0, y_1) : y_1^0 = t_1,\ y_1^i = x_1^i, i = 1, \ldots, n\}. \end{aligned}$$

Replace condition (iii) of Definition 2.3.2 by the condition

$$(t_i, \widetilde{\phi}(t_i)) \in \widetilde{\mathcal{T}}_i \qquad i = 0, 1, \tag{2.4.10}$$

where $\widetilde{\phi}$ is a solution of (2.4.9). Then it is easily seen that a function u is an admissible control for Problem 2.3.1 if and only if it is an admissible control for the system (2.4.9) subject to control constraints $\widetilde{\Omega}$ and end-condition (2.4.10). Moreover, the admissible trajectories $\widetilde{\phi}$ are of the form $\widetilde{\phi} = (\phi, t_1, x_1)$. Hence if we take the cost functional to be $\widetilde{J}$, where

$$\widetilde{J}(\widetilde{\phi}, u) = J(\phi, u),$$

then Problem 2.3.1 is equivalent to a problem with end conditions of the form (2.4.8).

2.5 Isoperimetric Problems and Parameter Optimization

In some control problems, in addition to the usual constraints there exists constraints of the form

$$\begin{aligned} \int_{t_0}^{t_1} h^i(t, \phi(t), u(t))dt &\le c^i \qquad i = 1, \ldots, q \\ \int_{t_0}^{t_1} h^i(t, \phi(t), u(t))dt &= c^i \qquad i = q+1, \ldots, p, \end{aligned} \tag{2.5.1}$$

where the functions h^i are defined on $\mathcal{G}$ and the constants c^i are prescribed. Constraints of the form (2.5.1) are called isoperimetric constraints. A problem with isoperimetric constraints can be reduced to a problem without isoperimetric constraints as follows.

Introduce additional state variables $x^{n+1}, \ldots, x^{n+p}$ and let $\widetilde{x} = (x, \bar{x})$, where $\bar{x} = (x^{n+1}, \cdots, x^{n+p})$. Let the state equations be

$$\begin{aligned} \frac{dx^i}{dt} &= f^i(t, x, u(t)) \qquad\qquad i = 1, \ldots, n \\ \frac{dx^{n+i}}{dt} &= h^i(t, x, u(t)), \qquad\qquad i = 1, \ldots, p \end{aligned} \tag{2.5.2}$$

or

$$\frac{d\widetilde{x}}{dt} = \widetilde{f}(t, x, u, (t)),$$

where $\widetilde{f} = (f, h)$. Let the control constraints be given by the mapping $\widetilde{\Omega}$ defined by the equation $\widetilde{\Omega}(t, \widetilde{x}) = \Omega(t, x)$. Let the end conditions be given by the set $\widetilde{\mathcal{B}}$ consisting of all points $(t_0, \widetilde{x}_0, t_1, \widetilde{x}_1)$ such that: (i) $(t_0, x_0, t_1, x_1) \in \mathcal{B}$; (ii) $x_0^i = 0$, $i = n+1, \ldots, n+p$; (iii) $x_1^i \le c^i$, $i = n+1, \ldots, n+q$; and (iv) $x_1^i = c^i$, $i = n+q+1, \ldots, n+p$. For the system with state variable $\widetilde{x}$, let $\mathcal{R}$ be replaced by $\widetilde{\mathcal{R}} = \mathcal{R} \times \mathbb{R}^p$ and let $\mathcal{G}$ be replaced by $\widetilde{\mathcal{G}} = \widetilde{\mathcal{R}} \times \mathcal{U}$.

Let (ϕ, u) be an admissible pair for Problem 2.3.1 such that the constraints (2.5.1) are satisfied. Let $\widetilde{\phi} = (\phi, \bar{\phi})$, where

$$\bar{\phi}(t) = \int_0^t h(s, \phi(s), u(s))ds \qquad \bar{\phi}(0) = 0.$$

Then $(\widetilde{\phi}, u)$ is an admissible pair for the system with state variable $\widetilde{x}$. Conversely, if $(\widetilde{\phi}, u)$ is admissible for the $\widetilde{x}$ system then (ϕ, u), where ϕ consists of the first n components of $\widetilde{\phi}$, is admissible for Problem 2.3.1 and satisfies the isoperimetric constraints. Hence by taking the cost functional for the problem in $\widetilde{x}$-space to be $\widetilde{J}$, where $\widetilde{J}(\widetilde{\phi}, u) = J(\phi, u)$, we can write the problem with constraints (2.5.1) as an equivalent problem in the format of Problem 2.3.1.

In Problem 2.3.1, the functions $f^0, f^1, \ldots, f^n$ defining the cost functional and the system of differential equations (2.3.1) are regarded as being fixed. In some applications these functions are dependent upon a parameter vector $w = (w^1, \ldots, w^k)$, which is at our disposal. For example, in the rocket problem of Section 1.4 we may be able to vary the effective exhaust velocity over some range $c_0 \leq c \leq c_1$ by proper design changes. The system differential equations (2.3.1) will now read

$$\frac{dx}{dt} = f(t, x, w, u(t)) \qquad w \in W,$$

where W is some preassigned set in $\mathbb{R}^k$. For a given choice of control u a corresponding trajectory ϕ will in general now depend on the choice of parameter value w. Hence, so will the value $J(\phi, u, w)$ of the cost functional. The problem now is to choose a parameter value w^* in W for which there exists an admissible pair (ϕ^*, u^*) such that $J(\phi^*, u^*, w^*) \leq J(\phi, u, w)$ for all w in W and corresponding admissible pairs (ϕ, u).

The problem just posed can be reformulated in the format of Problem 2.3.1 in $(n + k + 1)$-dimensional space as follows. Introduce new state variables $w = (w^1, \ldots, w^k)$ and consider the system

$$\begin{aligned} \frac{dx^i}{dt} &= f^i(t, x, w, u(t)) \qquad && i = 1, \ldots, n \\ \frac{dw^i}{dt} &= 0 && i = 1, \ldots, k. \end{aligned} \tag{2.5.3}$$

Let $\widetilde{x} = (x, w)$, let $\widetilde{\mathcal{R}} = \mathcal{R} \times \mathbb{R}^k$, let $\widetilde{\mathcal{G}} = \widetilde{\mathcal{R}} \times \mathcal{U}$, and let $\widetilde{\Omega}(t, x, w) = \Omega(t, x)$. Let the end conditions be given by

$$\widetilde{\mathcal{B}} = \{(t_0, x_0, w_0, t_1, x_1, w_1) : (t_0, x_0, t_1, x_1) \in \mathcal{B},\ w_0 \in W\}.$$

Let $\widetilde{J}(\widetilde{\phi}, u) = J(\phi, w, u)$. It is readily verified that the problem of minimizing J subject to (2.5.3), the control constraints $\widetilde{\Omega}$, and end conditions $\widetilde{\mathcal{B}}$ is equivalent to the problem involving the optimization of parameters.

2.6 Relationship with the Calculus of Variations

The brachistochrone problem in Section 1.6 is an example of the simple problem in the calculus of variations, which can be stated as follows. Let t be a scalar, let x be a vector in $\mathbb{R}^n$, and let x' be a vector in $\mathbb{R}^n$. Let $\mathcal{G}$ be a region in (t, x, x')-space. Let f^0 be a real valued function defined on $\mathcal{G}$. Let $\mathcal{B}$ be a given set of points (t_0, x_0, t_1, x_1) in $\mathbb{R}^{2n+2}$ and let g be a real valued function defined in $\mathcal{B}$. An admissible trajectory is defined to be an absolutely continuous function ϕ defined on an interval $[t_0, t_1]$ such that: