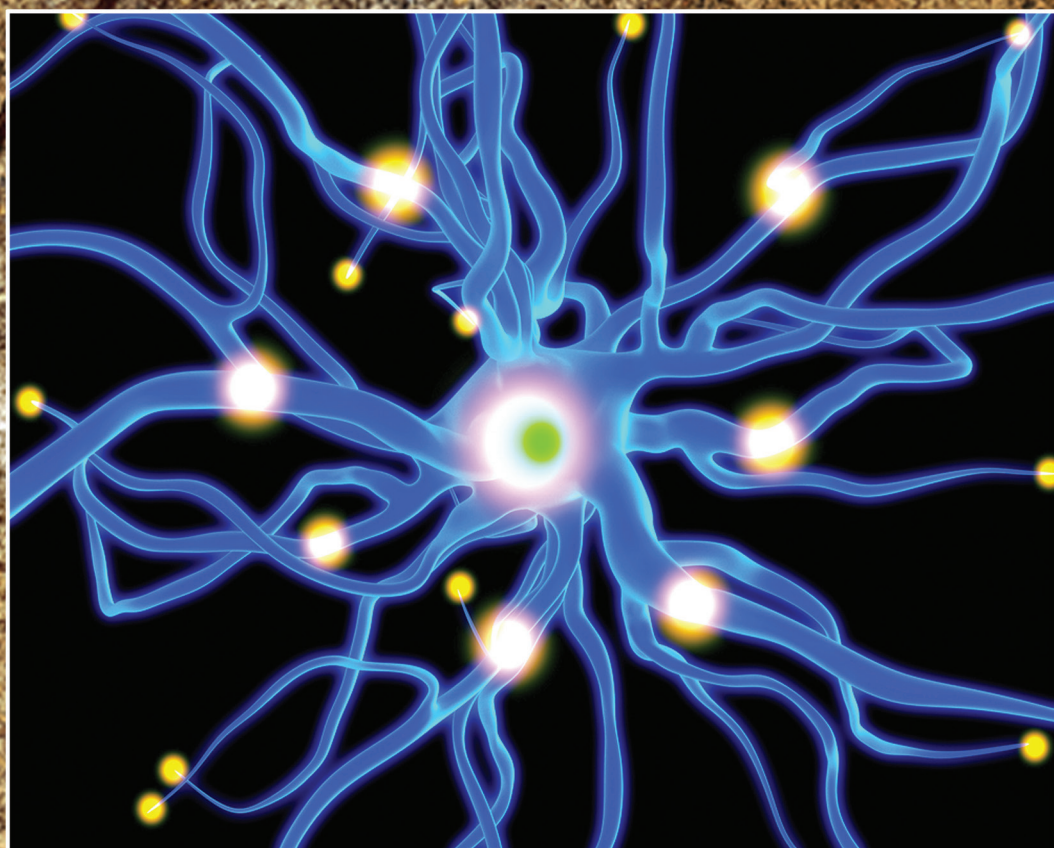


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Differential Equations with MATLAB[®]

Exploration, Applications, and Theory



Mark A. McKibben
Micah D. Webster



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Exploration, Applications, and Theory

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CRC Press

Taylor & Francis Group
Boca Raton London New York

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CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

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Version Date: 20140724

International Standard Book Number-13: 978-1-4665-5708-6 (eBook - PDF)

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To our mentors...

Sergiu Aizicovici, Sam “Cal” Calavitta, Patrick Guidotti, and David Keck.

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Preface

The mathematical modeling of complex phenomena that evolve over time relies heavily on the analysis of a variety of systems of ordinary and partial differential equations. Such models are developed in very disparate areas of study, ranging from the physical and biological sciences, engineering, and population ecology to economics and financial markets. The fact that the use of differential equations is so widespread ensures that students, scholars, and professionals across disciplines, at one time or another, will encounter a differential equation that describes a scenario of particular interest to them. Yet, a traditional course in differential equations focuses on basic techniques of solving certain types of equations and analyzing some very basic mathematical models. Even more innovative differential equations courses only scratch the surface of analyzing differential equations from a qualitative viewpoint. Such first courses simply do not provide the tools necessary to analyze differential equations and their utility in mathematical modeling is not imparted onto the student.

The mathematical community has responded to this apparent gap by developing mathematical modeling courses and second courses in differential equations. There are vast differences in the approaches and topic choice used in these courses. Many such books tend to focus primarily on discrete dynamical systems, while others provide a vast collection of topics (including discrete dynamical systems, optimization methods, probabilistic models, and graph theory) that cover different types of models. The few books that fall into this latter category are thorough, but tend to be encyclopedic compendia of ideas designed for those already familiar with the subject. They would be daunting to a beginning-level student who had only completed a course in multivariable calculus.

This book is an attempt to fill the apparent gap in the existing literature on books related to mathematical modeling and differential equations, and is unique in many ways. Our primary goal is to instill in the reader a sense of intuition and practical and theoretical “know-how” as it pertains to mathematical models involving ordinary and partial differential equations. We provide a bigger, unifying picture inherent to the study and analysis of more than 20 distinct models spanning disciplines, and make this material accessible to a student who has only completed a course in multivariable calculus by conveying the thought process of building the surrounding theory from the ground up, and actively involving them in this mathematical enterprise throughout the text. Pedagogically, the goals of the project are achieved by engaging the reader by posing questions of all types (from verifying details and illustrating theorems with examples to posing (and proving) conjectures of actual results and analyzing broad strokes that occur within the development of the theory itself, and applying it to specific models) throughout the development of the material. Heuristic commentary and motivation are also included to provide an intuitive flow through the chapters. As such, the exposition in the text, at times, may lack the “polished style” of a mathematical monograph, and the language used will be colloquial English rather than the standard mathematical language. But, this style has the benefit of encouraging the reader to not simply passively read the text, but rather work through it, which is essential to obtaining a meaningful grasp of the material.

The book is divided into two main parts. The first is devoted to the study of systems of linear ordinary differential equations. We introduce and develop, to various degrees, math-

ematical models from ten disparate fields of study, including pharmacokinetics, chemistry, classical mechanics, neural networks, physiology, and electrical circuits. These models play crucial roles throughout the chapter. For one, their analysis motivates different aspects of the theory being developed. We use interactive activities extensively to guide the readers through different processes and sequences of observations that lead to the development of theoretical results. Once the theory is established, the models then serve a secondary role of illustrating the applicability of the abstract theory. The theoretical results in Part I are developed in three distinct cases, each one of which builds upon the previous. These cases are the homogeneous case, the nonhomogeneous case (with a time-dependent forcing term), and the semilinear case (where the forcing term depends on time and state). The topics of existence and uniqueness, continuous dependence on parameters and initial conditions, long-term behavior, and convergence schemes permeate all theoretical chapters.

Part II mimics Part I in structure almost identically, but this time focuses on linear partial differential equations. The guiding principle of the approach in Part II is to express the partial differential equations that arise in the mathematical modeling of phenomena in ten different fields of study, including heat conduction, wave propagation, fluid flow through fissured rocks, pattern formation, and financial mathematics, as abstract version of the linear ordinary differential equations that were studied in Part I. Once this connection is made, the theoretical results from Part I can be subsumed as a special case of those in Part II.

The structure of this book is likely to be somewhat different than others through which you have worked. For one, the text is NOT simply intended to be read passively. Rather, it is chock full of questions posed to the reader! A passive reading of this text will prove to be ineffective, whereas carefully working through it will result in a solid understanding of the material. Everything from computations to underlying assumptions must be carefully examined in order to fully understand the material. An integral part of this text is the collective of questions posed throughout. We consciously indicate when the discussion is intentionally terse by including parenthetical questions and instructions, such as (**Why?**), (**Do so!**), and (**Explain how.**). Sometimes, we even bring the discussion to a screeching halt with a **STOP!** It is imperative that you attempt to answer each of these questions to ensure you develop a solid background as you work through the text.

While the primary goal of the text is to develop an overarching mathematical theory, we are doing so under the assumption that your prior exposure to theoretical mathematics is minimal. We intend to build your intuition and involve you in the enterprise of building the theory from the ground up. That said, we need to arm you with the essential tools of analysis, but in a manner that balances intuition, practicality, and mathematical rigor. This is accomplished with a preliminary chapter in each main part of the book which provides the essential tools to be used in that part, along with some hands-on **EXPLORE!** activities. Certainly, the completion of these portions of the text is by no means a substitute for a formal study of analysis. But, they do provide you with just enough knowledge to proceed through the next step of the development of the theory.

Applying the theory is, to many readers, arguably the most important and satisfying part of the journey. Our dual emphasis on the analysis of mathematical models is not typical in textbooks. We feel that this connection to concrete applications helps build intuition, provides motivation for why someone would want to develop such a theory, and post fortiori arms having developed the theory by examining the plentiful collection of applications. There are **APPLY IT!** activities sprinkled throughout the text, asking you to reexamine models previously mentioned in light of newly-formed theoretical results.

Finally, the glue that binds all aspects of our journey is our extensive use of MATLAB® Graphical User Interfaces (GUIs). As you will see, this feature of the text will enable you to discover patterns and make conjectures that otherwise would be solely reachable using

real analysis and, for all practical purposes, be out of reach. Although not a goal of this text, the numerical algorithms used in GUIs to construct solutions to differential equations mimic the theoretical approach outlined in the text. For more details, an interested reader is directed to [44, 35, 8, 23, 29, 28].

It is our hope that this book will be useful not only in the classroom, but also as a book that anyone with an interest can pick up and develop a sense of mastery of the subject. We welcome you and hope that you find your journey through this text as enjoyable and fulfilling as it was for us to write it.

The MATLAB GUIs for use with this textbook can be downloaded from the following webpage. http://www.wcupa.edu/ACADEMICS/SCH_CAS.MAT/mmckibben/DEwithMATLAB.asp

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Acknowledgments

The two-plus years that it took to co-write this text was one of the most rewarding, fulfilling, and dare we say, fun, experiences in each of our professional careers. Various stages of writing this book and developing the MATLAB GUIs were truly energizing and uplifting, while others required us to plumb the very depths of our patience and perseverance; it was all we could do to restrain each other from hurling the computer off the desk during momentary lapses of sanity. That said, this book is truly a labor of love, and we hope that the joy we experienced writing it (evident perhaps most outwardly by our occasional playfulness in the writing) and our love of the subject are infectious to our readers.

There are many people without whom the project would likely not have come to fruition. Our respective spouses, Jodi and Noelle, have been indelible sources of encouragement, energy, and support throughout this project.

Mark's mother, Pat, has been supportive of all his academic endeavors for decades and Micah's young son, Zane, taught him to focus, adapt, and to live in the moment. We both would like to thank Dr. Richard Craster and Dr. Roberto Sassi for their support and permission to alter some of their MATLAB codes for use in several Part II GUIs.

Our respective mentors, Sergui Aizicovici, Sam Calavitta, Patrick Guidotti, and David Keck were instrumental in the development of our academic and professional careers. Their unbridled enthusiasm and love of mathematics continue to be a significant driving force.

Cynthia Young, Dean of the College of Arts and Sciences at UCF; Patrick Guidotti, Professor of Mathematics at UCI; and Geoff Cox, Assistant Professor of Mathematics at VMI all gave valuable feedback on initial stages of the proposal.

Our students have been very helpful throughout the process. At Goucher, we have Alexa Gaines, who beta tested Part I of the text, including all the exercises, **EXPLORE!** Projects, and GUIs. And, the MA 347 class we co-taught effectively alpha-tested early versions of Part I of the text. More recently two MAT 343 classes at West Chester University beta-tested certain EXPLORE! Projects and GUIs from Part I.

From West Chester University of Pennsylvania, graduate students Gavin Hobbs and Andrew Stump put forth a concerted effort beta-testing the MATLAB GUIs and writing up careful solutions of the corresponding exercises. This is tedious work that would not

have been completed were it not for their assistance. These solutions will be included on the webpage for the book.

We would also like to thank Goucher College for supporting this endeavor through a Junior Faculty Development Grant and a Faculty Development Grant.

We would like to thank the entire Taylor and Francis team. To our editors Bob Stern and Bob Ross, thank you for patiently guiding us through the publication process from beginning to end. To our project coordinator Jessica Vakili, who handled our production, stylistic, and marketing questions in a very helpful and timely manner. And to Pawan Sharma, who professionally produced various diagrams in the text. And finally, to Shashi Kumar, for his efforts in typesetting the final manuscript.

And, last but not least, we would like to thank you, the reader, for embarking on this journey with us through an amazing field of mathematics. We hope your study of it is as fulfilling as ours has been thus far.

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Part I

Ordinary Differential Equations

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Chapter 1

Welcome!

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Welcome to the rich world that is differential equations. What prompted you to embark on this journey? Are you hunting for answers to questions about the Universe? Are you, as the result of allergies, contemplating taking two tablets instead of one to alleviate symptoms, but seek mathematical affirmation of the effectiveness of this decision? Have you wondered why it is that while you are standing atop the Empire State Building - swaying back and forth, back and forth, ... - the building doesn't simply concede defeat and snap in half? Are you curious as to how quickly the avian flu virus can spread throughout the world, infecting unsuspecting passerbyes? Or, how about those airplane wings - why don't they simply break apart in the middle of flight?

Whatever your reason for opening this book, your study of this material can shed some light on the answers to such questions. Many interesting, often surprising and sometimes downright unbelievable adventures await you. Again, welcome, hang on, and enjoy the ride.

1.1 Introduction

What are differential equations? No succinct definition truly captures the essence of this sprawling area of study. Researchers study phenomena occurring in nature guided by the goal of predicting the behavior of a solution of a system of equations governed by a set of discipline-specific laws. Relying on experimental observation at the onset of an investigation can be costly, subject to error, and can prove to be futile if the imposed assumptions turn out to be incorrect or infeasible. An often more pragmatic first step is to formulate a mathematical model of the situation and study its properties. Assuming success in this regard, a natural second step is to test the validity of the model using actual experimentation. The equations and/or assumptions are reexamined and tweaked, the abstract theory is modified, and then the system is again tested experimentally.

The phenomena with which we shall concern ourselves in Part I can all be characterized by quantities that change with time and can be formally described using a system of ordinary differential equations. The birth of differential equations as an area of study goes back to 1687 when they first appeared in Sir Isaac Newton's book *Philosophiae Naturalis Principia Mathematica*. The equations he posed were consequences of his observation that one should study forces to which a system responds in order to describe the laws of nature. Since

then, many researchers have made significant contributions, creating a massive continually-growing literature.

Many situations in physics, chemistry, economics, etc. that you have likely encountered are described by differential equations. Among the common examples of these phenomena are bacteria and population growth and decay, elementary financial models, and predator-prey systems. As you will discover along your journey through this text, differential equations play a crucial role in the modeling and analysis of numerous complicated scenarios.

1.2 This Book Is a Field Guide. What Does That Mean For YOU?

A perusal of the Table of Contents will reveal that your journey through differential equations will be split into two main parts. The first part is devoted to a discussion of certain classes of linear *ordinary* differential equations. Various theoretical concepts necessary to understand mathematical models whose mathematical description falls into this category will be introduced and developed, and subsequently addressed and generalized several times as the complexity of the models increases. All told, eleven distinct models spanning disciplines will be investigated in Part I, and an encompassing abstract theory will be developed.

Part II focuses on linear partial differential equations. Here, we will study another ten distinct mathematical models, and develop an encompassing abstract theory analogous to the one formulated in Part I. The theoretical results established in Part I can effectively be interpreted as special cases of the theory developed here.

The structure of this book may be somewhat different than others through which you have worked. For one, the text is NOT simply intended to be read passively. Rather, it is chock full of questions posed to you. If this journey were envisioned as a trek through a densely-vegetated rain forest, then our approach is akin to arming each and every reader with a machete intended to be used to help forge the path to an elusive Mayan ruins that legend suggests is home to riches beyond belief. Hyperbole aside, YOU are asked to work through many of the details. A passive reading of this text will prove to be ineffective, whereas carefully working through it will result in a solid understanding of the material. Everything from computations to assessing the validity of underlying assumptions must be carefully examined in order to fully understand the material. We consciously indicate when the discussion is intentionally terse by including parenthetical questions and instructions, such as (**Why?**), (**Do so!**), and (**Explain how.**). Sometimes, we even bring the discussion to a screeching halt with a **STOP!** It is imperative that you not leave any stone unturned and attempt to answer each of these questions. Failing to do so could start as a seemingly minor drop of confusion and then, snowball into a thunderous sea of misunderstanding as you proceed through the book.

While the primary goal of the text is to develop an overarching mathematical *theory*, we are doing so under the assumption that your prior exposure to theoretical mathematics is minimal. We intend to build your intuition and involve you in the enterprise of building the theory from the ground up. That said, we need to arm you with the essential tools of analysis, but in a manner that balances intuition, practicality, and mathematical rigor. This is accomplished with a preliminary chapter in both main parts of the book which provides the essential tools to be used in that part, along with the inclusion of hands-on **EXPLORE!** activities. The purpose of the **EXPLORE!** activities is just that, to explore.

We provide questions to guide you and point you in some direction. All we ask is that you walk, or even run, in some direction, even if you do not know the final destination. Please, do not feel that you should know the *correct* answer in the traditional math book sense. The correct answer to an **EXPLORE!** question is more of a thought process followed by articulating what you have considered and your conclusion. Certainly, the completion of these portions of the text is by no means a substitute for a formal study of analysis. But, they do provide you with just enough knowledge to proceed through the next step of the development of the theory.

Applying the theory is, to many readers, arguably analogous to finding the lost treasure as the culmination of a long, tiresome journey. Our dual emphasis on the analysis of mathematical models is not typical in textbooks. We feel that this connection to concrete applications helps build intuition, provides motivation for why such a theory should be developed, and *post fortiori* affirms having developed the theory by examining the plentiful collection of applications. There are **APPLY IT!** activities sprinkled throughout the text, asking you to reexamine models in light of new theoretical results.

Finally, the glue that binds all aspects of our journey is our extensive use of MATLAB Graphical User Interfaces (GUIs). As you will see, this feature of the text enables you to discover patterns and make conjectures that otherwise would be solely reachable using real analysis making them intractable.

1.3 Mired in Jargon—A Quick Language Lesson!

Superficially, a *differential equation* (DE) is an equation involving derivatives. If an equation contains derivatives of functions dependent only on a single variable, then it is called an *ordinary differential equation* (ODE), while an equation containing partial derivatives (which arise when the functions of interest depend on more than one variable) is appropriately called a *partial differential equation* (PDE). Some examples of such equations are

$$u'(x) = f(x, u(x)) \quad (1.1)$$

$$\frac{d^2u(x)}{dx^2} + 3\frac{du(x)}{dx} = \sin x \quad (1.2)$$

$$\frac{\partial^2u(t, x)}{\partial t^2} + c^2\frac{\partial^2u(t, x)}{\partial x^2} = 0. \quad (1.3)$$

Equations (1.1) and (1.2) are ODEs while (1.3) is a PDE. The variables upon which the functions depend, namely x (and also t in the case of (1.3)) are called *independent variables*. Often, the independent variables in an equation are clearly understood and in such cases we suppress the dependence on them and write a cleaner, more succinct equation. For instance, the following equations are often written in place of (1.1) - (1.3), respectively:

$$u' = f(x, u) \quad (1.4)$$

$$\frac{d^2u}{dx^2} + 3\frac{du}{dx} = \sin x \quad (1.5)$$

$$\frac{\partial^2u}{\partial t^2} + c^2\frac{\partial^2u}{\partial x^2} = 0. \quad (1.6)$$

The *order* of a differential equation is the order of the highest derivative appearing in the equation. The order of (1.1) is 1, while the order of both (1.2) and (1.3) is 2.

With the basic terminology in place, we loosely define what is meant by solving a differential equation. The notion is intuitive enough. Indeed, when asked to “solve (1.1),” you are to determine a real-valued *function* $u = u(x)$ that *satisfies* the equation (1.1). Unlike algebraic equations, the unknown quantity of interest is an actual function, rather than a real-number. Defining what precisely is meant by saying “a function satisfies a differential equation” can be very tricky, and we will need to pay more careful attention to various nuances as we proceed through the text. Momentarily, though, it is reasonable to think of this obtaining a true statement as the result of plugging the function into the equation. Consider the following example.

Example 1.3.1. Verify that the function $u(x) = e^{-x}$ is a solution of the ODE $u' = -u$.

Solution: Observe that $u'(x) = -e^{-x}$ and that $-u(x) = -e^{-x}$, for any real number x . Substituting these into the given ODE results in a true statement, for any real number x . So, $u(x)$ is a solution to the given ODE.

We make two remarks regarding Example 1.3.1. One, let C be an arbitrary real number and consider the function u_C defined by $u_C(x) = Ce^{-x}$. Clearly, u_C is also a solution of the given ODE. (**Why?**) The presence of the arbitrary constant C prompts us to refer to this as a *general solution* of the DE. This demonstrates that without additional information, a DE can have more than one solution; in fact, it can have infinitely many. Two, the solution of the DE is a function and as such, comes equipped with its own domain which indicates when the DE is solvable. A complete understanding of the behavior of a solution of a DE requires that we know its domain. It is customary to include this domain when describing the solution of a DE.

Exercise 1.3.1. Verify that for any real constant C , $u_C(t) = Ce^{-3t}$ is a general solution of the ODE $u''(t) + 2u'(t) - 3u(t) = 0$. What is the domain of the solution?

Exercise 1.3.2. Verify that $z(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x)$ is a general solution of the Cauchy-Euler ODE $x^2 z''(x) + xz'(x) + z(x) = 0$, where C_1 and C_2 are arbitrary real constants. What is the domain of the solution?

1.4 Introducing MATLAB

MATLAB[®] is a software package produced by Mathworks, Inc. (mathworks.com) and is available on a systems ranging from personal computers to supercomputers. The name MATLAB stems from “Matrix Laboratory,” which alludes to the software package’s foundation in linear algebra. MATLAB has evolved into an interactive development tool for scientific and engineering programs, or in fact any setting for which numeric computation is required.

This textbook takes advantage of MATLAB’s Graphical User Interface (GUI) environment to minimize the use of MATLAB syntax. To obtain all the GUIs for the textbook go to http://www.wcupa.edu/_academics/sch_cas.mat/mmcKibben/book3.asp and download the files to your computer. Place the files in a folder and record the name of the folder for later use. In order to execute the text’s GUIs one only needs to identify the *current folder* and the *command window* in MATLAB.

If necessary, change the current folder to the one that contains this text’s GUI files. This can be done by manually typing the location of the folder or by using the *browse for folder* button (the button with three dots located in the top right of screen in Figure 1.1).

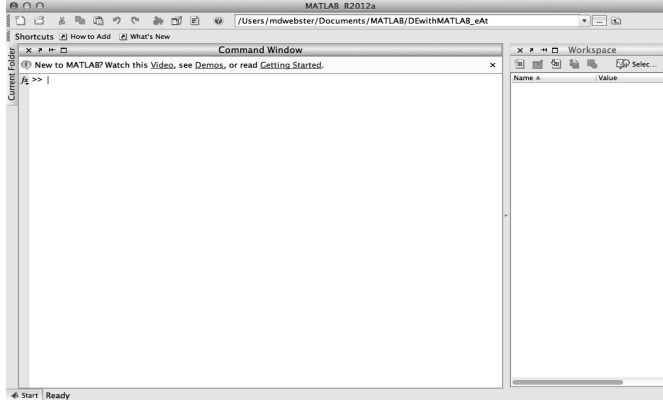


FIGURE 1.1: MATLAB Command Window

For example, in Figure 1.1 the current folder is titled *DEwithMATLAB_eAt*, so in order for the GUIs to work all the files need to be placed in that folder. When prompted by an exercise to use a specific GUI you will be directed to the command window. The command window is the large panel in the middle of the screen in Figure 1.1. A defining feature of the command window is the prompt `>>`. To open a GUI, type the name as provided in the exercise instructions after the `>>` prompt and hit return. For example,

```
>> MATLAB_Elementary_Population_Growth
```

will open the MATLAB GUI of the same name and output an interface for which one can visualize solutions, learn new concepts, and answer questions posed in exercises, MATLAB-Exercises, **EXPLORE!**s, and **APPLY IT!**s. The vast majority of GUIs that accompany this text are self-contained in the sense that all the information you need can be found within the panel of the interface. However, in a few cases additional information can be found in the command window. In these situations the GUI's *Instructions* and *Help* button will direct you to the command window and explain how to retrieve any additional information.

The GUIs in this book have been tested under MATLAB release, version 7.14.0.739 (R2012a). For a detailed description of MATLAB, its features, and syntax refer to the MATLAB manual along with the Mathworks website,

<http://www.mathworks.com/help/matlab/index.html>.

1.5 A First Look at Some Elementary Mathematical Models

A differential equation has an entire family of solutions associated with it until additional restrictions are imposed on the behavior of the solution. A natural question is, “How do we isolate a particular solution of a DE?” To explore this issue, we shall examine some familiar elementary models.

Model I.1. Freely-Falling Bodies

The height, at any time $t \geq 0$, of a moving body falling freely along a linear vertical path due to gravity (neglecting air resistance) is governed by the following mathematical model :

$$\begin{cases} \underbrace{\frac{d^2y}{dt^2}(t)}_{\text{Acceleration}} = \underbrace{-g}_{\text{Gravity}}, \\ y(0) = s_0 \text{ (Initial Height)}, \\ y'(0) = v_0 \text{ (Initial Velocity)}, \end{cases} \quad (1.7)$$

where t denotes time and $y = y(t)$ the vertical position of the body above the ground.

Our goal is to determine the actual function y satisfying all parts of (1.7). In addition to the ODE, we have imposed two extra conditions (called *initial conditions*) that will enable us to isolate a particular solution of the differential equation that we seek and, as such, further enable us to provide a more precise description of the quantity of interest. Whenever we consider a DE, together with conditions of this type, we refer to the entire problem as an *initial-value problem* (IVP) .

Let us momentarily disregard the initial conditions and attempt to solve the ODE. Upon inspection, it seems reasonable that solving this differential equation for y simply requires us to consecutively integrate both sides with respect to t twice. (**Why is this justified?**) Doing so yields

$$\begin{aligned} \frac{d^2y}{dt^2}(t) = -g &\implies \int \frac{d^2y}{dt^2}(t)dt = \int -g dt \\ &\implies \frac{dy}{dt}(t) + C_1^* = -gt + C_1^{**}, \end{aligned}$$

where C_1^* and C_1^{**} are arbitrary real constants (called *parameters*). We can combine these constants into a single arbitrary constant C_1 (**Why?**) to obtain the cleaner equation

$$\frac{dy}{dt}(t) = -gt + C_1.$$

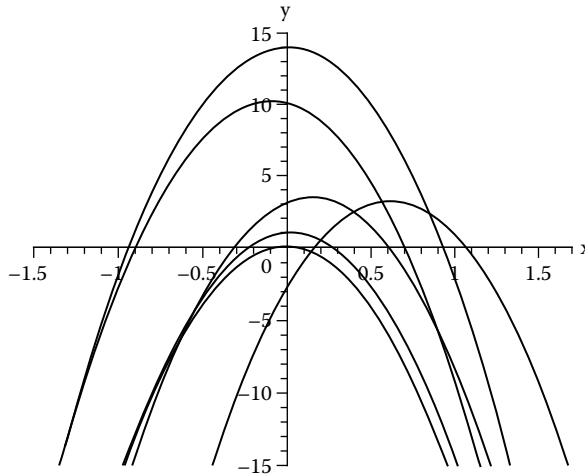
This equation is meaningful for all $t \geq 0$ and so, we can integrate once again to obtain

$$y(t) = -\frac{g}{2}t^2 + C_1t + C_2, \quad (1.8)$$

where C_1 and C_2 are arbitrary real constants. Equation (1.8) is the general solution of IVP (1.7) and is sometimes referred to as a *two-parameter family of solutions* of (1.7). If we stop here, we have infinitely many solutions to the differential equation portion of IVP (1.7). However, demanding that this solution satisfy the two initial conditions enables us to isolate a unique function that satisfies all three conditions. Using these two conditions enables us to uniquely determine C_1 and C_2 as follows:

$$\begin{aligned} y(0) = s_0 &\implies s_0 = 0 + 0 + C_2 = C_2 \\ y'(0) = v_0 &\implies v_0 = 0 + C_1 = C_1 \end{aligned}$$

Hence, the *particular solution* of (1.7) is given by $y(t) = -\frac{g}{2}t^2 + v_0t + s_0$, which is the familiar formula developed in an elementary physics course.



Model 1.2 Elementary Population Growth

A very elementary model of population growth that ignores limitation of resources, among other factors, can be formed under the assumption that the substance/population grows simply at a rate that is proportional to its size. This immediately yields the IVP

$$\begin{cases} \frac{dx(t)}{dt} = kx(t), & t > t_0, \\ x(t_0) = x_0, \end{cases} \quad (1.9)$$

where k is the constant of proportionality, $x(t)$ is the size of the population at time t , $\frac{dx}{dt}$ is the rate of growth or decay, and x_0 is the initial size of the population.

It is easy to see that the solution of (1.9) is $x(t) = e^{kt}x_0$. If $k > 0$, then the population is growing exponentially, while if $k < 0$, it decreases exponentially (**Why?**). Such a model is useful when determining the half-life of a radioactive substance.

Exercise 1.5.1. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at any time. After 3 hours, it is observed that there are 400 bacteria present. After 10 hours, there are 2,000 bacteria present. What was the initial number of bacteria present?

MATLAB-Exercise 1.5.1. We will use **MATLAB** to investigate the population model given by (1.9), the population model posed in Exercise 1.5.1. Open **MATLAB** and in the command line, type:

MATLAB_Elementary_Population_Growth

Use the GUI to answer the following questions. Make certain to click on the ‘Description of System’ button for a description of the GUI and instructions on how to use the GUI. A typical screenshot can be found in Figure 1.2.

- i.) Use your answers in Exercise 1.5.1 as parameters for the GUI to find the population after 5 hours.
- ii.) Suppose there were measurement errors caused by misaligned lab equipment and, in actuality, there were 405 bacteria after 3 hours and 2100 bacteria after 10 hours. Compute the new initial population and proportionality constant.

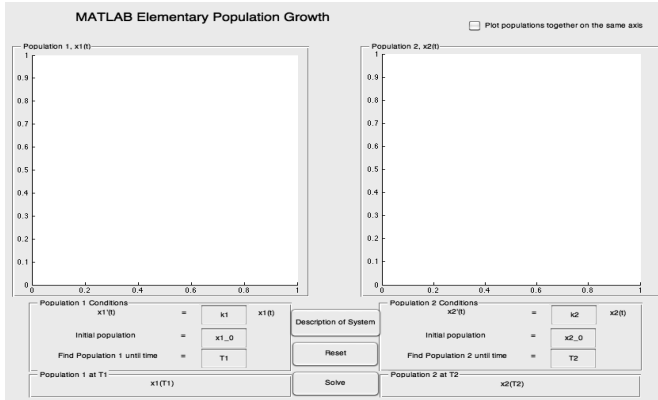


FIGURE 1.2: MATLAB Elementary Population Growth GUI

- iii.) Let us refer to the situation described in Exercise 1.5.1 as *Population 1*, and the one in ii) as *Population 2*. Use the GUI to plot both populations on the same axis over a 5 hour time span. Comment on the shapes of the graphs. For example, do they look the same? Are they getting further apart? Closer together?

■

Model I.3 Newton's Law of Heating and Cooling

Did you ever wonder why it seems that within a very short time of taking an ice-cold soda out of the refrigerator to drink outside on a scorching summer day, the soda becomes warm or, why does a Thanksgiving turkey cool so quickly once it is taken out of an oven and set onto the dining room table? These phenomena can be modeled using *Newton's Law of Heating and Cooling*, which says that the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium (called the *ambient temperature*).

Let $T(t)$ represent the temperature of the body at time t , T_m the constant temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the body cools. Then, Newton's Law of Heating and Cooling can be expressed mathematically as the following IVP:

$$\begin{cases} \frac{dT(t)}{dt} = k(T(t) - T_m), & t > t_0, \\ T(t_0) = T_0, \end{cases} \quad (1.10)$$

where k is the constant of proportionality and T_0 is the temperature of the body at time $t = t_0$. If the body is cooling, then it makes sense that $T(t) \geq T_m$. Likewise, if the body is warming, the reverse inequality is true. (**Why?**) The solution of IVP (1.10) is given by

$$T(t) = T_m + (T_0 - T_m)e^{k(t-t_0)}. \quad (1.11)$$

STOP! Verify this.

Exercise 1.5.2. A thermometer is taken from inside a house to the outdoors, where the air temperature is 5 degrees Fahrenheit. After 1 minute the thermometer reads 55 degrees Fahrenheit and after 5 minutes the reading is 30 degrees Fahrenheit. What was the initial temperature of the room?

MATLAB-Exercise 1.5.2. We will use **MATLAB** to investigate the temperature model given by (1.11), the situation described in Exercise 1.5.2. Open **MATLAB** and in the command line, type:

MATLAB_Heating_Cooling

Use the GUI to answer the following questions. Make certain to click on the ‘Description of System’ button for a description of the GUI and instructions on how to use the GUI. A typical screenshot can be found in Figure 1.3.

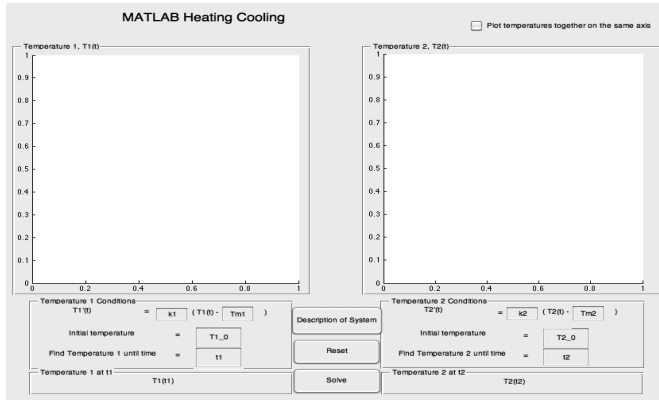


FIGURE 1.3: MATLAB Heating Cooling GUI

- i.) Use your answers in Exercise 1.5.2 as parameters for the GUI to find the temperature reading on the thermometer after 8 minutes.
- ii.) All measurement devices, including thermometers, are imperfect, meaning they do not necessarily output the actual measurement, but rather a value within some tolerance of the true value. Repeat Exercise 1.5.2 assuming the thermometer was reporting temperatures 1% warmer than the true values and that the outdoor temperature is truly 5 degrees Fahrenheit.
- iii.) Let us refer to the situation described in Exercise 1.5.2 as *Temperature 1* and the one in ii) as *Temperature 2*. Use the GUI to plot both temperatures on the same set of axes over an eight minute time span. Comment on the shapes of the graphs. For example, do they look the same? Are they getting further apart? Closer together?
- iv.) Make a conjecture for solutions to (1.11) when the parameters are a “little off.”



A Basic Philosophy of Mathematical Modeling

Before studying more complicated models, we provide a crude outline of the underlying philosophy used when forming so-called mathematical models. Below is a list of key steps to keep in mind when embarking on a journey whose goals are to form, analyze, solve, and refine mathematical models :

1. Clearly state all assumptions.

This step constitutes the foundation of the modeling process. You must describe all

relationships that exist among the quantities being studied, as accurately as possible. Try to avoid hidden assumptions, and realize that the quality of the assumptions you impose determines the validity of the model.

2. Clearly identify variables and parameters.

The most frequently used independent variable is time, though other commonly occurring ones include position and angle. The dependent variables (that is, the quantities of interest that change with respect to change in the independent variable) are much wider in scope. Parameters include other measurable quantities that must be incorporated into the equations of the model, such as properties of material whose temperature is being studied, rate of decay, and density of a population.

3. Formulate equations relating the quantities identified in Step 2.

This step requires the use of underlying laws (geometric, physical, and economic) that govern the setting in which the model arises. The process of formulating these equations becomes more natural as you gain experience.

4. Study the equations formulated in Step 3 in an effort to understand the behavior of the quantities of interest.

The actual solution of an ODE or system of ODEs is often not attainable. But, there are other ways to extract information about the behavior of the phenomena that do not require an actual solution.

5. Test the model using actual data.

When working in the sciences, for instance, one formulates a model and then performs a few “test cases” in which the outcome is known a priori in order to test the accuracy of the model. We will not actually do this in the present text, as our main goal is the analysis of existing models.

6. Modify assumptions and formulate an enhanced, improved model.

No model is perfect. There is no single model that accounts for every possible variable that arises. And, even if such a model did exist, the complexity of the equations would render extracting useful information about the phenomena impossible. However, a typical approach is to start simple and move forward taking small steps towards a sufficiently realistic model.

We will examine a large collection of models in this text, but will rarely focus on going back to first principles when formulating them. Doing so thoroughly would require extensive knowledge of each area. Rather, our approach can be better described as a *macro-analysis* in that we establish a single framework in which to study more global features of broad collections of models.

Chapter 2

A Basic Analysis Toolbox

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If you have taken an elementary differential equations course, you are familiar with various techniques used to determine solutions of differential equations, as well as possibly some methods used to describe characteristics of the solutions corresponding to different initial conditions. Pushing beyond such a rudimentary analysis with an eye toward performing a more rigorous investigation inevitably requires some mathematical sophistication. Contrary to the opinion that mathematics is all about performing calculations to solve problems, discovery of patterns, forming cogent arguments, and performing careful analysis is at the heart of the discipline. But, how does one *do mathematics* carefully and properly?

The purpose of this chapter is to provide you with a succinct, hands-on working knowledge of some concepts in elementary analysis focusing on notation, main definitions and

results, and the techniques used throughout the text. Occasionally, other topics will be introduced when needed.

2.1 Some Basic Mathematical Shorthand

Symbolism is used heftily in mathematical exposition. Careful usage of some basic notation can streamline the verbiage. Some of the common symbols used are as follows.

Let P and Q be statements. (If the statement P changes depending on the value of some parameter x , we denote this dependence by writing $P(x)$.)

- 1.) The statement “not P ,” is called the *negation* of P .
- 2.) The statement “If P , then Q ” is called an *implication*, and is denoted by “ $P \implies Q$ ” (read “ P implies Q ”). Here, P is called the *hypothesis* and Q is the *conclusion*.
- 3.) The statement “ P if, and only if, Q ” is denoted by “ P iff Q ” or “ $P \iff Q$.” Precisely, this means “($P \implies Q$) and ($Q \implies P$).”
- 4.) The statement “ $Q \implies P$ ” is the *converse* of “ $P \implies Q$.”
- 5.) The statement “not $Q \implies$ not P ” is the *contrapositive* of “ $P \implies Q$.” These two statements are logically equivalent.
- 6.) The symbol “ \exists ” is an existential quantifier and is read as “there exists” or “there is at least one.”
- 7.) The symbol “ \forall ” is a universal quantifier and is read as “for every” or “for any.”

Exercise 2.1.1. Let P , Q , R , and S be statements.

- i.) Form the negation of “ $\exists x$ such that $P(x)$ holds.”
- ii.) Form the negation of “ $\forall x$, $P(x)$ holds.”

Example 2.1.1. Form the negation of “ P and (Q and R).”

This sort of formal exercise in logic arises when determining if a mathematical result is applicable to a given situation. For instance, if there are 3 distinct hypothesis, P , Q and R , that all must be satisfied in order to apply a theorem, all it would take to render the theorem inapplicable is for at least one of the hypotheses to not hold. That is, the negation of “ P and (Q and R)” is “not P or not Q or not R .”

Remark 2.1.1. Implication is a transitive relation in the sense that

$$((P \implies Q) \text{ and } (Q \implies R)) \implies (P \implies R).$$

For instance, a sequence of algebraic manipulations used to solve an equation is technically such a string of implications from which we conclude that the values of the variable obtained in the last step are the solutions of the original equation. Mathematical arguments are comprised of strings of implications, albeit of a somewhat more sophisticated nature.

A *theorem* is a statement about mathematical objects. It has a special form in that it consists of a *hypothesis* and a *conclusion*. A *mathematical proof* of a theorem is a logical argument which guarantees the conclusion holds when the hypotheses are satisfied. You have encountered many theorems in your study of the calculus, some of which are easily stated and others whose structure is more complicated because the hypotheses involve multiple

conditions.

STOP! State the following calculus theorems as precisely as you can. Then, check your responses using a calculus text.

1. Product Rule for Derivatives
2. Mean Value Theorem for Derivatives
3. Fundamental Theorem of Calculus

Undoubtedly, while “how to use” each of the above theorems likely came to mind, formally and precisely stating them often poses a challenge to a beginner. It is not uncommon to overlook stating the conditions that must be satisfied in order to employ the rule suggested by each theorem, yet these conditions are crucial in the sense that the rules do not necessarily apply unless they are satisfied. For example, you cannot apply the Fundamental Theorem of Calculus to compute

$$\int_{-2}^1 \frac{1}{x} dx$$

and you cannot apply the Mean Value Theorem to $f(x) = |x|$ on $[-2, 3]$. (**Why?**) As you work through the text, be mindful about the conditions being imposed, what they mean, and the ramifications of imposing them.

2.2 Set Algebra

Informally, a *set* can be thought of as a collection of objects (e.g., real numbers, vectors, matrices, functions, other sets, etc.); the contents of a set are referred to as its *elements*. We usually label sets using upper case letters and their elements by lower case letters. Two sets that arise often and for whom specific notation will be reserved are:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{R} = \text{the set of all real numbers}$$

If P is a certain property and A is the set of all objects having property P , we write $A = \{x : x \text{ has } P\}$ or $A = \{x | x \text{ has } P\}$. A set with no elements is *empty*, denoted by \emptyset .

If A is not empty and a is an element of A , we denote this fact by “ $a \in A$.” If a is not an element of A , a fact denoted by “ $a \notin A$,” then where is it located? This prompts us to prescribe a universal set \mathcal{U} that contains all possible objects of interest in our discussion. The following definition provides an algebra of sets.

Definition 2.2.1. Let A and B be sets.

- i.) A is a *subset* of B , written $A \subset B$, whenever $x \in A \implies x \in B$.
- ii.) A *equals* B , written $A = B$, whenever $(A \subset B)$ and $(B \subset A)$.
- iii.) The *complement of A relative to B* , written $B \setminus A$, is the set $\{x | x \in B \text{ and } x \notin A\}$. The complement relative to \mathcal{U} is denoted by \bar{A} .
- iv.) The *union of A and B* is the set $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
- v.) The *intersection of A and B* is the set $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
- vi.) The *Cartesian product of A and B* is the set $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$. More generally, if A_1, \dots, A_n are sets, the Cartesian product of A_1, \dots, A_n is the set $A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) : a_1 \in A_1, \dots, a_n \in A_n\}$. If $A_1 = \dots = A_n$, then denoting this common set by A , we denote the Cartesian product by simply A^n .

Verifying that two sets are equal requires that we prove two implications. Use this fact when appropriate to complete the following exercises:

Exercise 2.2.1. Let A, B , and C be sets. Illustrate each of the following pictorially and prove each of them:

- i.) $A \subset B$ iff $\widetilde{B} \subset \widetilde{A}$.
- ii.) $A = (A \cap B) \cup (A \setminus B)$
- iii.) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- iv.) $\widetilde{(A \cap B)} = \widetilde{A} \cup \widetilde{B}$ and $\widetilde{(A \cup B)} = \widetilde{A} \cap \widetilde{B}$

Exercise 2.2.2. Explain how you would show two sets A and B are NOT equal.

2.3 Functions

The functions typically encountered in a calculus course are defined by explicit formulas that assign elements in one set to outputs in another. We will need to think more broadly and carefully about functions in this text and so, we introduce a more formal definition of a function below.

Definition 2.3.1. Let A and B be sets.

- i.) A function from A to B is a subset $f \subset A \times B$ satisfying
 - a) $\forall x \in A, \exists y \in B$ such that $(x, y) \in f$,
 - b) $(x, y_1) \in f$ and $(x, y_2) \in f \implies y_1 = y_2$. We say f is B -valued, denoted by $f : A \rightarrow B$.
- ii.) The set A is called the *domain* of f , denoted $\text{dom}(f)$.
- iii.) The *range* of f , denoted by $\text{rng}(f)$, is given by $\text{rng}(f) = \{f(x) | x \in A\}$.

Remark 2.3.1. Notation: When defining a function using an explicit formula, say $y = f(x)$, the notation $x \mapsto f(x)$ is often used to denote the function. Also, we indicate the general dependence on a variable using a dot, say $f(\cdot)$. If the function depends on two independent variables, we distinguish between them by using a different number of dots for each, say $f(\cdot, \cdot)$. Also, the term *mapping* is used synonymously with the term *function*. An immediate consequence of Definition 2.3.1 is that $\text{rng}(f) \subset B$.

Exercise 2.3.1. Precisely define what it means for two functions f and g to be equal.

We sometimes wish to apply functions in succession in the following sense.

Definition 2.3.2. Suppose that $f : \text{dom}(f) \rightarrow A$ and $g : \text{dom}(g) \rightarrow B$ with $\text{rng}(g) \subset \text{dom}(f)$. The *composition of f with g* , denoted $f \circ g$, is the function $f \circ g : \text{dom}(g) \rightarrow A$ defined by $(f \circ g)(x) = f(g(x))$.

Exercise 2.3.2. Show that, in general, $f \circ g \neq g \circ f$.

Definition 2.3.3. $f : A \rightarrow B$ is called

- i.) *one-to-one* if $f(x_1) = f(x_2) \implies x_1 = x_2, \forall x_1, x_2 \in A$;
- ii.) *onto* whenever $\text{rng}(f) = B$.

Exercise 2.3.3. Let $f : \text{dom}(f) \rightarrow A$ and $g : \text{dom}(g) \rightarrow B$ be such that $f \circ g$ is defined. Prove the following:

- i.) If f and g are onto, then $f \circ g$ is onto.
- ii.) If f and g are one-to-one, then $f \circ g$ is one-to-one.

Functions that are one-to-one and onto are invertible in the following sense.

Definition 2.3.4. Let $f : \text{dom}(f) \rightarrow \text{rng}(f)$. The function $g : \text{rng}(f) \rightarrow \text{dom}(f)$ such that

$$\begin{aligned} g(f(x)) &= x, \text{ for every } x \in \text{dom}(f), \\ f(g(y)) &= y, \text{ for every } y \in \text{rng}(f), \end{aligned}$$

is called the *inverse of f* and is denoted by f^{-1} . A function f for which such an inverse function exists is said to be *invertible*.

At times, we need to compute the functional values for all members of a subset of the domain, or perhaps determine the subset of the domain whose collection of functional values is a prescribed subset of the range. These notions are made precise below.

Definition 2.3.5. Let $f : A \rightarrow B$.

- i.) For $X \subset A$, the *image of X under f* is the set $f(X) = \{f(x) | x \in X\}$.
- ii.) For $Y \subset B$, the *pre-image of Y under f* is the set

$$f^{-1}(Y) = \{x \in A | \exists y \in Y \text{ such that } y = f(x)\}.$$

We often consider functions whose domains and ranges are subsets of \mathbb{R} . For such functions, the notion of monotonicity is often a useful characterization.

Definition 2.3.6. Let $f : \text{dom}(f) \subset \mathbb{R} \rightarrow \mathbb{R}$ and suppose that $\emptyset \neq S \subset \text{dom}(f)$. We say that f is

- i.) *nondecreasing on S* whenever $x_1, x_2 \in S$ with $x_1 < x_2 \implies f(x_1) \leq f(x_2)$;
- ii.) *nonincreasing on S* whenever $x_1, x_2 \in S$ with $x_1 < x_2 \implies f(x_1) \geq f(x_2)$.

Remark 2.3.2. When the inequality is strict, we replace “nondecreasing” by “increasing” and “nonincreasing” by “decreasing.”

The arithmetic operations of real-valued functions are defined in the natural way.

Definition 2.3.7. (Arithmetic of Functions)

Suppose that $f : \text{dom}(f) \subset \mathbb{R} \rightarrow \mathbb{R}$ and $g : \text{dom}(g) \subset \mathbb{R} \rightarrow \mathbb{R}$. The functions $f + g$, $f - g$, and $f \cdot g$ are defined as follows:

- i.) $f + g : \text{dom}(f) \cap \text{dom}(g) \subset \mathbb{R} \rightarrow \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x)$, for all $x \in \text{dom}(f) \cap \text{dom}(g)$,
- ii.) $f - g : \text{dom}(f) \cap \text{dom}(g) \subset \mathbb{R} \rightarrow \mathbb{R}$ is defined by $(f - g)(x) = f(x) - g(x)$, for all $x \in \text{dom}(f) \cap \text{dom}(g)$,
- iii.) $f \cdot g : \text{dom}(f) \cap \text{dom}(g) \subset \mathbb{R} \rightarrow \mathbb{R}$ is defined by $(f \cdot g)(x) = f(x) \cdot g(x)$, for all $x \in \text{dom}(f) \cap \text{dom}(g)$.

Exercise 2.3.4. Suppose that $f : \text{dom}(f) \subset \mathbb{R} \rightarrow \mathbb{R}$ and $g : \text{dom}(g) \subset \mathbb{R} \rightarrow \mathbb{R}$ are nondecreasing (resp. nonincreasing) functions on their domains.

- i.) Which of the functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$, if any, are nondecreasing (resp. nonincreasing) on their domains? Explain.
- ii.) Assuming that $f \circ g$ is defined, must it be nondecreasing (resp. nonincreasing) on its domain? Explain.

2.4 The Space $(\mathbb{R}, |\cdot|)$

2.4.1 Order Properties

The basic arithmetic and order features of the real number system are likely familiar. For our purposes, we shall begin with a set \mathbb{R} equipped with two operations, addition and multiplication, satisfying these algebraic properties:

1. addition and multiplication are both commutative and associative;
2. multiplication distributes over addition;
3. adding zero to any real number yields the same real number;
4. multiplying a real number by one yields the same real number;
5. every real number has a unique additive inverse; and
6. every nonzero real number has a unique multiplicative inverse.

Moreover, \mathbb{R} equipped with the natural “ $<$ ” ordering is an ordered field and obeys the following properties:

Proposition 2.4.1. (Order Features of \mathbb{R})

For all $x, y, z \in \mathbb{R}$, the following are true:

- i.) Exactly one of the relationships $x = y$, $x < y$, or $y < x$ holds;
- ii.) $x < y \implies x + z < y + z$;
- iii.) $(x < y)$ and $(y < z) \implies x < z$;
- iv.) $(x < y)$ and $(c > 0) \implies cx < cy$;
- v.) $(x < y)$ and $(c < 0) \implies cx > cy$;
- vi.) $(0 < x < y)$ and $(0 < w < z) \implies 0 < xw < yz$.

The following is an immediate consequence of these properties and is often the underlying principle used when verifying an inequality.

Proposition 2.4.2. If $x, y \in \mathbb{R}$ such that $x < y + \epsilon$, $\forall \epsilon > 0$, then $x \leq y$.

Proof. Suppose not; that is, $y < x$. Observe that for $\epsilon = \frac{x-y}{2} > 0$, $y + \epsilon = \frac{x+y}{2} < x$. (**Why?**) This is a contradiction. Hence, it must be the case that $x \leq y$. \square

Remark 2.4.1. The above argument is a very simple example of a proof by contradiction. The strategy is to assume that the conclusion is false and then use this additional hypothesis to obtain a false statement or a contradiction of another hypothesis in the claim.

2.4.2 Absolute Value

The above is a heuristic description of the familiar algebraic structure of \mathbb{R} . When equipped with a distance-measuring artifice, deeper properties of \mathbb{R} can be defined and studied. This is done with the help of the absolute value function.

Definition 2.4.1. For any $x \in \mathbb{R}$, the *absolute value* of x , denoted $|x|$, is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This can be viewed as a measurement of distance between real numbers within the context of a number line. For instance, the solution set of the equation “ $|x - 2| = 3$ ” is the set of real numbers x that are “3 units away from 2,” namely $\{-1, 5\}$.

Exercise 2.4.1. Determine the solution set for the following equations.

i.) $|x - 3| = 0$

ii.) $|x + 6| = 2$

Proposition 2.4.3. The following properties hold for all $x, y, z \in \mathbb{R}$ and $a \geq 0$:

- i.) $-|x| = \min\{-x, x\} \leq x \leq \max\{-x, x\} = |x|$;
- ii.) $|x| \geq 0, \forall x \in \mathbb{R}$;
- iii.) $|x| = 0$ iff $x = 0$;
- iv.) $\sqrt{x^2} = |x|$;
- v.) $|xy| = |x||y|$;
- vi.) $|x| \leq a$ iff $-a \leq x \leq a$;
- vii.) $|x + y| \leq |x| + |y|$;
- viii.) $|x - y| \leq |x - z| + |z - y|$;
- ix.) $||x| - |y|| \leq |x - y|$;
- x.) $|x - y| < \epsilon, \forall \epsilon > 0 \implies x = y$.

Remark 2.4.2. The properties listed in Prop. 2.4.3 will be used heftily throughout the text.

- 1.) (vii) is called the *triangle inequality*. Its main utility lies in providing an estimate on a sum in terms of its parts.
- 2.) (viii) is an immediate consequence of (vii) obtained by “putting in and taking out” z and then applying (vii). Indeed observe

$$\begin{aligned} |x - y| &= |(x - z) + (z - y)| \\ &\leq |x - z| + |z - y| \end{aligned}$$

This is useful when trying to form an upper bound on $|x - y|$ in terms of quantities different from just $|x|$ and $|y|$.

- 3.) (x) is a useful trick to use when showing two quantities x and y are equal. In words, if the absolute value of the difference of two real numbers is smaller than every positive number, then the two numbers must be equal.

The following inequalities are sometimes useful when establishing estimates.

Proposition 2.4.4. Let $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathbb{R}$.

- i.) (Cauchy-Schwarz) $\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$,
- ii.) (Minkowski) $\left(\sum_{i=1}^n (x_i + y_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$,
- iii.) $|\sum_{i=1}^n x_i|^M \leq \left(\sum_{i=1}^n |x_i|\right)^M \leq n^{M-1} \sum_{i=1}^n |x_i|^M, \forall M \in \mathbb{N}$.