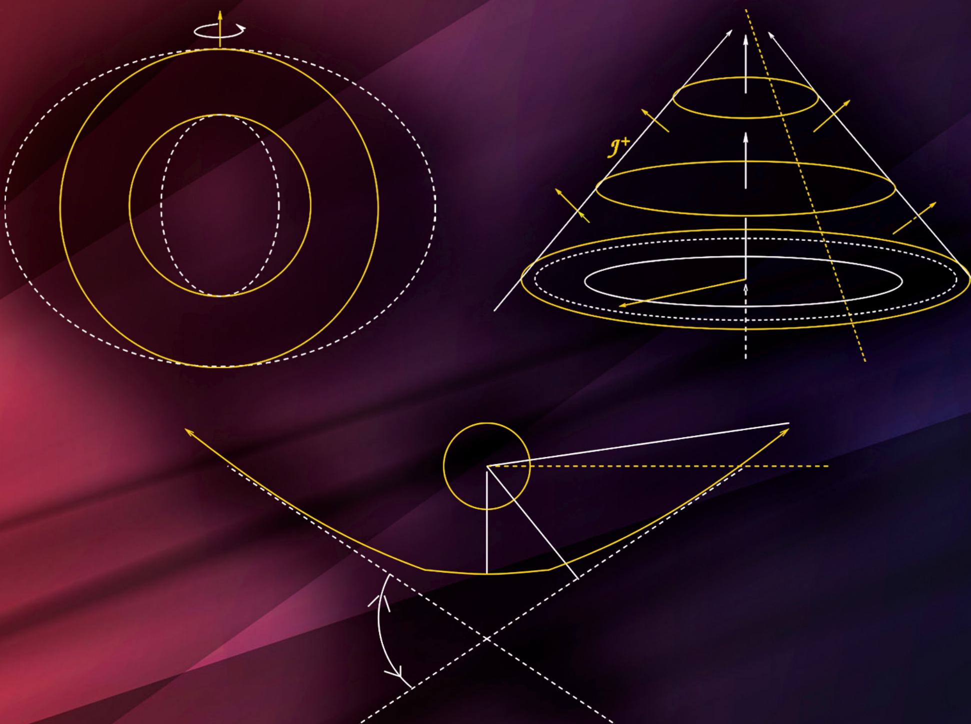
General Relativity

Basics and Beyond



Ghanashyam Date

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Preface

It fills me with a sense of joy and humility to present this book on the eve of the centenary year of the publication of Albert Einstein's General Theory of Relativity. When general relativity arrived, it had an aura of mystery due to its sophisticated view of space, time and gravitational phenomena. From the early phase when primary elaboration of the theory was mathematical in nature, it has evolved into a phase where it is being confronted by increasingly sophisticated experiments that have been successful so far. Students are often attracted to the theory and want to know what yet can be done with it. The book is envisaged as an attempt to familiarize students and prospective researchers with the basic features of the theory and offer a perspective on its more advance features.

There are many excellent textbooks from the classics by Misner–Thorne–Wheeler, Weinberg and Wald to the more recent ones by Sean Carroll, James Hartle and Thanu Padmanabhan, with differing styles and emphasis and there are excellent review articles on frontline topics. The idea here is to combine the 'textbook' and 'the review'. Thus, I have tried to adopt the pedagogical style of a textbook while avoiding an emphasis on detailed treatments, and at the same time, tried to present the essential ideas and just enough background material needed for students to appreciate the issues and current research.

There was also a conscious effort to emphasize the physical ideas and motivations, contrasting the mathematical idealizations which are important in appreciating the scope and limitations of the theory. Consequently, requisite mathematical background of differential geometry is summarized in the last chapter while the main text emphasizes the physical aspects.

The first five chapters usually form the core of an introductory course on General Relativity (GR) and constitute the "Basics" part of the book. The first chapter traces Einstein's arguments and informally motivates the mathematical model for space-time. In the second chapter, we first discuss the basic physical quantities related to space-time measurements and their relation to a metric in an arbitrary coordinate system. This is followed by examples of space-times corresponding to different types of gravitational fields. Some of these are revisited subsequently for further elaboration. Chapter 3 discusses adaptation of dynamics in a Riemannian geometry framework while the next chapter presents the Einstein equation together with its elementary properties. The fifth chapter discusses different phenomena either predicted by GR or influenced by GR. This also contains the classic tests of general relativity.

The "Beyond" part of the book, takes a look at some of the more sophisticated features of GR. Chapter 6 discusses the physical requirements of a well-defined deterministic framework for non-gravitational dynamics and the constraints it puts on the global structure of space-times. Surprisingly, the singular features seen in physically motivated examples turn out to have more general presence. The structure of the physically acceptable space-times is such that if certain conditions—such as complete gravitational collapse or an everywhere expanding universe—are realized in nature, then space-time will necessarily have regions where GR will cease to be applicable.

Not all physical situations are as grim. There are physical bodies of finite extent and it becomes necessary to look at the space-time geometry far away from them. This is especially relevant in the context of energy being carried away in the form of gravitational waves. Chapter 7 discusses the characterization of the appropriate asymptotic space-times.

In the next three chapters, we revisit black holes, gravitational waves and cosmological space-times. Apart from considering the general definition of black holes, we examine and discuss their quasi-local generalization in terms of the trapping, isolated and dynamical horizons. In the second look at gravitational waves, we trace the issues that were involved in settling the 'reality' of gravitational waves and briefly discuss the basic features of the challenge involved in their direct detection. The cosmological space-times are discussed primarily to get a glimpse of the possible nature of the space-like singularities.

Chapter 11 discusses the evolutionary interpretation for the class of globally hyperbolic space-times and reviews the initial value formulation. This forms a basis for numerical relativity presented in the next chapter. The Hamiltonian formulation paves a way for canonical quantization of gravity. While the book is focused on classical general relativity, introductory summaries of the main approaches to a quantum theory of gravity are included in Chapter 13. An alternative view of emergent gravity is also briefly mentioned.

There were many topics I wanted to include in this book, but could not. These are listed in the fourteenth chapter together with some concluding remarks. The Epilogue contains a summary of the requisite differential geometry and some of the results used in the main text.

There are many people to whom I owe a debt of gratitude. My understanding and appreciation of GR have been shaped by many influences over several years which are hard to demarcate. I must mention Naresh Dadhich and thank him for the numerous discussions and his generous encouragement. Within the context of this book, I would like to acknowledge critical feedback from my former teacher, Arvind Kumar on an earlier draft of Chapter 2 and my former student Alok Laddha for his comments on Chapter 7. I would also like to thank Thanu Padmanabhan for his help on the emergent gravity view and Sudipta Sarkar for a discussion on Jacobson's work. I must not forget the

students of my institute who had taken my courses on GR and those from places other than India who took short-term courses on various occasions under the SERC Schools in Theoretical High Energy Physics (India). The book has grown out of various lecture notes. I thank all of these students. I thank my friend and colleague, Gautam Menon, for help proofreading and for his helpful suggestions. There are times of meeting deadlines where responsibilities get shuffled and prioritized. This cannot be done without support from the family. I thank Nisha, Aditya, and my parents for it.

Ghanashyam Date

Part I

The Basics

Chapter 1

From Newton to Einstein: Synthesis of General Relativity

1.1 Space, Time, Observers

We all have an intuitive sense of what space is and what time is. Space is something in which 'bodies move' and time is something that sequences these movements. To make these notions quantitative we need to adopt a procedure to assign numbers to 'locations' and put time stamps on events. It is in terms of these assignments or *coordinates* that we make the space time explicit and it is this explicit model that is used in physics. All the tourist maps we use and the scheduling we struggle to achieve are based on precisely such 'made explicit' space and time. There is *no unique way to assign coordinates and time stamps*. Herein enters an *observer* (= adopted procedure).

With such a procedure at hand, it is possible to formulate the phenomenon of motion of bodies in terms of *kinematics* - description of motion and *dynamics* - laws of motion. The key point to note is that there is always an observer implicit directly in kinematics and indirectly in dynamics.

Einstein now observes several examples of relationships between classes of observers and the phenomena being described. Consider the problem of determining the distance between two points say by laying down meter sticks. The answer will evidently depend on how the meter sticks are laid. Drawing on the experience of measuring distances along short straight lines and using the procedure of assigning the *Cartesian coordinates* an observer can determine the distance between two points with Cartesian coordinates (x^1, y^1, z^1) and (x^2, y^2, z^2) to be given by

$$\text{Distance}^2 \;=\; (x^2 - x^1)^2 + (y^2 - y^1)^2 + (z^2 - z^1)^2 \ .$$

Now the interesting observation is that all observers assigning Cartesian coordinates will *verify* that the distance between two given points is numerically the same (assuming the same units are used!). Hence, as far as the problem of determining distance between points is concerned, *any* of this class of observers will do fine. *Mathematically*, the coordinates assigned by any two observers are related by the *transformation law*:

$$(x')^i = \sum_{j=1}^{3} A^i{}_j x^j + B^i, \quad \text{where } A^i{}_j \text{ is a 3-by-3 orthogonal matrix.}$$

These leave the Cartesian nature of coordinates unchanged as well as the expression for distance invariant. For $B^i = 0$, Einstein calls this *relativity of orientation.*

The next example he considers is the phenomenon of motion of particles, governed by Newton's laws formulated in the so-called inertial frames. The class of observers whose descriptions are equivalent are those who are in uniform relative motion, possibly differing in the orientation of the axes of the Cartesian frames and possibly with difference in the 'zero' of their clocks. This is of course Galilean relativity. What is left invariant is the $mass \times acceleration$.

When phenomenon of motion is extended to include electromagnetic field and the motion of charges under their influence, a contradiction arises. Analysis of the famous moving magnet and conductor problem in the magnet's rest frame and the conductor's rest frame presents two alternatives. *Either* have *Galilean transformations* among the electric and magnetic fields so as to get the same force in both the frames *or,* allow a *new transformation law for the force* so as to be consistent with the Lorentz transformations which leave the Maxwell's equation invariant. Which one of these is 'correct'?

On the one hand, confirmation of constancy of speed of light puts Lorentz transformations on a firmer ground and on the other hand Galilean transformations contain an unwarranted assumption of observer independence of simultaneity. Einstein chooses Lorentz transformations and we have the *theory of special theory.* What two observers in uniform relative motion must agree on is the *same value of the speed of light in vacuum.*

This affects the kinematics in a profound manner. We will discuss the derivations a little later but let us note at this stage that length of a stick measured by a moving observer is a little *less* than that measured by an observer at rest with respect to the stick. Likewise when an observer compares the successive ticks of a moving clock with a stationary clock, the moving clock always ticks *slower.* These consequences of the demand of invariance of the speed of light go by the names *length contraction* and *time dilation* respectively.

The new kinematics does not leave invariant the other Newtonian law, namely the law of gravitational force. Once again we face a similar dilemma as before: Do we limit the applicability of the new kinematics or do we modify the law of gravitational force?

There is a peculiarity with the law of gravitation. The 'charge' that enters in the force law, the *gravitational mass*, happens to be numerically equal to the measure of the inertia of a body, its *inertial mass.* This makes different bodies of varied compositions, weights fall to the ground with the same acceleration.

There is no 'reason' for such conceptually widely different quantities to be numerically equal, except perhaps it is a clue to the *nature* of *gravitational interaction.*

All bodies fall at the same rate also means that an observer does so too and therefore, relative to the observer, the bodies continue to maintain their state of uniform motion. In the absence of any force of any other origin, this just means that the freely falling observer is the *Newtonian inertial observer*! The clue of equality of the two masses provides us with a *definition of inertial frames* as precisely those in which gravitational field *cannot* be detected. Furthermore, an observer who detects gravitational field, is *accelerated* relative to an inertial frame. Thus we can trade-off a gravitational field, for an observer accelerated relative to an inertial observer. Since relatively accelerated observers are involved, Lorentzian kinematics is not immediately applicable. Rotating platforms provide a convenient 'laboratory' for a thought experiment.

Imagine determining the circumference and the radius of a rotating platform. The measuring sticks tangential to the circumference will undergo Lorentz contraction while those along the radial direction will not be contracted. Thus the ratio of the circumference to radius of the *rotating* platform, obtained by taking the ratio of the number of measuring sticks along the circumference and the number along the radius, will be *greater than* 2π [1] while that of a non-rotating platform will be 2π. Hence, the *geometry* on a rotating platform will be *non-Euclidean.* But by equivalence principle, acceleration is equivalent to a gravitational field (locally) and therefore one must infer that gravity affects the geometry. This gravitational field is of course inferred by the observer who is co-rotating with the platform. We will return to the rotating platform later again.

Thus the *response* (motion) of bodies to a gravitational field is *independent* of their masses and the gravitational field also changes the geometry of space. Since a gravitational field is produced by masses, the spatial geometry is also influenced by the masses. Thus, *geometry of space is changeable.* This is quite a novel inference! Does *space-time* geometry also change with distribution of masses?

This could be so if clocks tick at different rates in a gravitational field. Consider an observer stationed at a height of h from the ground and another observer freely falling. The freely falling observer will have a speed $v = gt$ relative to the stationary observer after a time t and will have a fallen through a distance of $s = \frac{1}{2}gt^2$. As per Lorentzian kinematics, the rate of freely falling clock will be,

$$\Delta\tau_{\text{falling}} = \Delta\tau_{\text{fixed}}\sqrt{1 - g^2t^2} = \Delta\tau_{\text{fixed}}\sqrt{1 - 2gs} = \Delta\tau_{\text{fixed}}\sqrt{1 - 2\Delta\Phi_{\text{grav}}}$$

The final expression is depends only on the gravitational potential difference between the stationary observer and instantaneous position of the freely falling observer.

It is clear from this argument that the gravitational potential affects the rates of clocks and since gravitational potential changes with the distribution of masses, so does the clock rates and hence the *space-time geometry* too is affected by distribution of masses.

Thus, replacing gravitational field by an accelerated observer and the Lorentzian kinematics leads us to a space-time geometry which is affected by presence of gravitational field which in turn depends on distribution of masses. One puzzle still remains. If gravitational field can be 'gotten rid off' as in a freely falling lift, is gravity 'fictitious'? It can't be. After all Earth *is* freely falling in the gravitational field of the Sun and real tides - which are effects of Newtonian gravity - do exist! So, while metrical property within a freely falling lift will be that in the absence of gravitational field, something else must remain encoded in the geometry that will account for the tides.

From the examples of two-dimensional surfaces, we know that the non-Euclidean geometries have non-zero *curvature*. This is most easily seen on the surface of a sphere. Consider a triangle made up of sides which are portions of great circles on the sphere. If a triangle is 'large', with two points on the equator and the third one the north pole (say), then the sum of angles is *greater* than 180^0 degrees. Now bring the two equatorial points closer to the pole. Note that the generic latitude is *not* a great circle (the longitudes always are). So the small triangle will look more and more 'distorted', but the sum of its angles will get closer and closer to 180^0. In short, non-zero curvature is detectable as deviation from Euclidean geometry, only for larger triangles. The same is true for tidal forces in Newtonian gravity. The differential forces on two extremes of a body are larger when the separation of the two extremes is larger. Thus we see a parallel between the effects of curvature in geometry and the tidal forces of gravity.

At a qualitative level then, we see that effects of gravitational field can be mimicked by a space-time geometry which has curvature which in turn must depend on the distribution of masses since Newtonian gravitational potential does. The observed equality of gravitational mass and inertial mass, combined with Lorentzian kinematics leads to replacing gravitational interaction as revealing a space-time geometry which is curved in general and is changeable. Space-time is a dynamical entity. In the process, the principle of relativity also gets extended to *all* observers regardless of their state of motion. As Einstein says [1]: *"Theory of relativity is intimately connected with a theory of space and time ..."* In the subsequent chapters we will formalize and make these arguments precise and quantitative.

1.2 General Relativity and Space-Time Arenas

We will proceed somewhat informally and heuristically to arrive at the mathematical model for space-time. The precise details are given in chapter 14.

We have already alluded to the assignment of coordinates (and time stamps) as the a defining character of an observer. We are quite familiar with assignment of *Cartesian coordinates* on a plane: choose a point (origin) and a pair of orthogonal directions at that point (we use a protractor to determine orthogonality) call them the x-axis and the y-axis; go 'x'-units along the x-axis and then 'y'-units along the y-direction and assign the coordinates (x, y) to the point reached, 'P'. Repeat for other points. For the same choice of origin and the axes, we may reverse the order of traversal from origin to the same point i.e. first go 'y-'-units along the y-axis and then 'x'-units in the direction of the x-axis. From experience, we know that we will reach the same point and assign the same coordinates to it. A different observer may choose the same origin but a different pair of axes, can still reach the same point, 'P', but now with different values for its coordinates. Another observer may even choose a different origin. Nevertheless, each observer is able to follow this procedure for arbitrary values of (x, y) and thus label the points on a plane in an unambiguous and on-to-one manner. We even know how to relate the coordinates assigned by different observers, namely, $x' = O_{1,1}x + O_{1,2}y + C_1,\ y' = O_{2,1}x + O_{2,2}y + C_2$, where, the matrix O_{ij} is an orthogonal matrix, $O^T O = \mathbb{1}$. We can see readily that if we follow the same procedure on the surface of a sphere, then even for the same choice of an origin and the same pair of axes, the point reached depends on the order of traversal! Secondly, the relation between the coordinates assigned by two observers not a simple linear one as before. We also recognize this as a feature of the 'curved' nature of the sphere. We can attempt a similar exercise on the surface of a saddle and discover the same features. Clearly, the plane surface is rather an exception in the class of two-dimensional surfaces and therefore the ambiguities in the procedure for assigning coordinates is quite generic. we may have to be content with (i) *any arbitrary procedure of assigning coordinates* - but in a one-to-one manner and (ii) allow arbitrary (invertible) relations among different coordinates.

Of course labeling the points is only a first step an observer has to undertake. An observer has to observe and describe phenomena in terms of the reference system of coordinates chosen. How can different observers be sure that they are describing the same phenomena and compare notes to evolve a consensus on the laws of nature? Is it possible at all? Let us keep in mind the surface of the Earth as a concrete example. We know that temperatures at various locations have their specific values, irrespective of the labeling of the locations. Likewise, the wind patterns or ocean currents are described by a field of arrows, again independent of the labeling of the locations. Therefore there exist quantities on the sphere which have an existence independent of the labelling of the location. However, when we want to describe the variations of these quantities with the locations in quantitative terms, each observer can only do so using his or her reference system. Clearly, for the same quantity, we will have multiple descriptions in terms of multiple coordinates systems. Using the relations among the coordinates, we can transform one description into another one. Consistency requires that the quantities describing the phenomena must transform in a *specific manner* reflecting the fact that the quantities *exist independent of the coordinates*. For example, if a point P has two sets of coordinates (x, y) and (x', y') and the temperature in the vicinity is described by two functions $T(x, y)$ and $T'(x', y')$, then we must have, $T'(x', y') = T(x, y)$ at P. Similarly, if we have two descriptions of wind velocities as $(dx/dt, dy/dt)$ and $(dx'/dt, dy'/dt)$, then we must have the relations,

$$\frac{dx'}{dt} = \frac{\partial x'}{\partial x}\frac{dx}{dt} + \frac{\partial x'}{\partial y}\frac{dy}{dt}, \quad \frac{dy'}{dt} = \frac{\partial y'}{\partial x}\frac{dx}{dt} + \frac{\partial y'}{\partial y}\frac{dy}{dt}.$$

We have only used the chain rule of differentiation and the assumption that the relation among different coordinates is not only invertible but also differentiable. In a similar manner, we can see that the *gradients* of the temperature distribution must be related as,

$$\frac{\partial T'}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial T}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial T}{\partial y}, \quad \frac{\partial T'}{\partial y'} = \frac{\partial x}{\partial y'}\frac{\partial T}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial T}{\partial y}$$

Here we have also used the fact that $T'(x', y') = T(x, y)$ in applying the chain rule. If we use a more compact notation of denoting the coordinates as x^i, the coordinate relations as $x'^i(x^j)$ then we can write the equations as,

$$T'(x') = T(x), \quad \frac{dx'^i}{dt} = \frac{\partial x'^i}{\partial x_j}\frac{dx^j}{dt}, \quad \frac{\partial T'}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial T}{\partial x^j}$$

We have also introduced the Einstein summation convention, namely, repeated indices in an expression imply summation over the values of the indices. What we see is the beginning of *tensors*—sets of quantities that transform in a specific manner which imply that they represent entities that exist independent of assignments of coordinates. The temperature is a *scalar*, the velocities are *contravariant tensor of rank 1* and the gradients are *covariant tensor of rank 1*. Generalizations to multi-index quantities and details are given in the chapter 14.

This is similar to the case of special relativity's *4-tensor notation*, except that the implicit transformations are not the *Lorentz transformations* but the *general coordinate transformations*. This also means that the partial derivatives are evaluated at the same point where two sets of quantities are related and that these vary from point-to-point, the transformations being *non-linear* in general. Hence elementary algebraic operations such as addition, multiplications of tensors can only be defined pointwise.

One of the first casualty of allowing arbitrary assignment of coordinates is that the relation between coordinate differences and physically measured length is more remote. From the example of Cartesian coordinates on a plane, we know that the distance between two points, measured by using meter sticks (say) is related to the coordinate differences by the square root of the sum of their squares. If we were to use the polar coordinates, (r, θ), then the expression is, $(\Delta s)^2 = (\Delta r)^2 + r^2(\Delta\theta)^2 := \sum_{ij} g_{ij}(r,\theta)(\Delta x)^i(\Delta x)^j$. For points on a sphere, the coordinate differences have to be sufficiently small (ideally infinitesimal) to match with the length obtained by putting small measuring sticks along the surface of the sphere. The matching would be necessarily approximate as no finite length measuring stick can be confined to the curved surface. Even after restricting to small enough coordinate differences, we need to ensure that the measured length, Δs^2, is numerically the same if computed using differences from a different coordinate system. This can possibly be true, if the coefficients g_{ij} in the second coordinate system are different in just the right manner: $\sum_{ij} g'_{ij}\Delta x'^i \Delta x'^j = \sum_{ij} g_{ij}\Delta x^i \Delta x^j$. In the light of the discussion of tensors, this demand just means that (a) Δx^i transform as the contravariant rank 1 tensor and (b) g_{ij} transform as covariant rank two tensor. This will make the distance an invariant (coordinate independent) quantity. Since $\Delta x^i \approx (dx^i/dt)\Delta t$ and the velocity is a tensor while Δt is manifestly independent of the coordinates, the (a) above is satisfied. The requirement of Δx^i being sufficiently small comes about because for sufficiently small Δt, there is a unique 'straight' path along which we may lay the measuring sticks. The point to note is that we must have an quantity such as g_{ij} so that measured lengths can be computed using coordinate differences. This quantity is called a *metric tensor* while the expression $(\Delta s)^2$ is called the *line element.* There are infinitely many possible choices for a metric tensor.

An observer cannot be satisfied by just making observations at one point. We will want to set up differential equations, ordinary and partial, to make theoretical predictions. So we need to define derivatives of tensors which should also be tensors. Differentiation involves comparing values at neighboring points and tensors forbid such comparisons. Suppose we are given a tensor field, $A^i(q)$, for points q in the vicinity of a point p. If we consider the derivatives, $\frac{\partial A^i(x)}{\partial x^j}$, in two different coordinate systems, we see immediately that the derivatives do *not* transform as a tensor, due to an offending term containing double derivatives of the form $\frac{\partial^2 x'^i}{\partial x^j \partial x^k}$. For linear transformations such as Lorentz transformations, we don't encounter this, but for general coordinates, we cannot escape it. To construct a tensorial derivative, called a *covariant derivative*, we need to introduce a quantity $\Gamma^i{}_{jk}$ with appropriate transformations and define: $\nabla_j A^i := \frac{\partial A^i}{\partial x^j} + \Gamma^i{}_{jk}A^k$, with $\Gamma'^i{}_{jk}(x') := \frac{\partial x'^i}{\partial x^l}\frac{\partial x^m}{\partial x'^j}\frac{\partial x^n}{\partial x'^k}\Gamma^l{}_{mn}(x) + \frac{\partial x'^i}{\partial x^m}\frac{\partial^2 x^m}{\partial x'^j \partial x'^k}$. This quantity is called an *affine connection.* Notice that a *choice of* Γ is constrained only by the transformation rule and there are infinitely many choices possible. For every choice we can define covariant derivatives for all tensor fields (see chapter 14). Now, unlike the usual coordinate derivatives, the covariant derivatives *do not commute* i.e. $\nabla_i \nabla_j A^k - \nabla_j \nabla_i A^k := R^k{}_{lij}(\Gamma)\ A^l \neq \mathbf{0}$. The 4 index quantity $R^k{}_{lij}$ is manifestly a tensor (since the left-hand side is) and depends only on Γ and its first derivatives. This is the famous *Riemann Curvature Tensor*. We are thus naturally lead to a framework involving tensors, an arbitrarily chosen tensor - the metric tensor g_{ij} and an arbitrarily chosen affine connection - $\Gamma^i{}_{jk}$, with the associated Riemann curvature tensor. It turns out that the arbitrariness in the choice of the connection can be completely removed by demanding that $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ and $\nabla_k g_{ij} = 0\ \forall i, j, k$. The connection so restricted is called the Riemann–Christoffel connection which is dependent on the metric tensor and the corresponding Riemann tensor is also determined by the metric. We now have a model for a space-time: It is a collection of 'events', made explicit by arbitrarily assigned coordinates, an arbitrarily chosen metric with a non-vanishing Riemann tensor in general. All determinable physical quantities of interest being tensors of appropriate ranks satisfying differential equations involving covariant derivatives. This model is nothing but a Riemannian manifold, defined more precisely in chapter 14.

To familiarize ourselves, we will discuss several examples of Riemannian manifolds in the next chapter.

Chapter 2

Examples of Space-Times

We will take a specification of a space-time as a set of coordinates x^μ with a non-singular metric $g_{\mu\nu}(x)$ with Lorentzian signature, given as *an infinitesimal invariant interval*, also known as *line element*, and study some of its properties[1]. Specifically, we consider,

Minkowski (No gravity)	$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$
Rindler (Uniform)	$\Delta s^2 = -g_0^2 z^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2, \quad z > 0$
Rotating Disk (Centrifugal)	$\Delta s^2 = -f(\rho)\Delta t^2 + 2h(\rho)\Delta t \Delta\phi + g(\rho)\Delta\phi^2 + \Delta\rho^2 + \Delta z^2$ $f(\rho) := e^{-\omega^2\rho^2} - \rho^2\omega^2 e^{+\rho^2\omega^2}$, $h(\rho) := -\omega g(\rho)$, $\quad g(\rho) := \rho^2 e^{+\rho^2\omega^2}$
Schwarzschild (Spherical)	$\Delta s^2 = -\left(1 - \frac{2GM}{r}\right)\Delta t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\Delta r^2 + r^2\Delta\Omega^2$
FRW (Cosmological)	$\Delta s^2 = -\Delta t^2 + a^2(t)\left\{\frac{\Delta r^2}{1-\kappa r^2} + r^2\Delta\Omega^2\right\}$ where, $\Delta\Omega^2 := \left(\Delta\theta^2 + \sin^2\theta\Delta\phi^2\right)$
Plane wave (Undulating)	$\Delta s^2 = (\eta_{\mu\nu} + h_{\mu\nu})\Delta x^\mu \Delta x^\nu$, where, $h_{\mu\nu}(x) = \epsilon_{\mu\nu}(k)e^{ik\cdot x} + \bar{\epsilon}(k)_{\mu\nu}e^{-ik\cdot x}$ and $\eta_{\mu\nu} =$ diag (-1, 1, 1, 1) .

In order to appreciate interpretation of physical consequences of the space-time model, we will focus on familiar quantities such as *physical lengths, elapsed times measured by clocks, local speed ('speedometer reading'), local acceleration ('acceleration due to gravity')*. We take as given, a line element

[1] Our notation: *space-time* coordinates are indexed by Greek letters, $\mu, \nu, \ldots$ taking values $0, 1, 2, 3$; *space* coordinates are indexed by Roman letters, $i, j, \ldots$ taking values $1, 2, 3$. The invariant interval $\Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu$. The metric signature is $- + ++$ and speed of light $c = 1$.

and its *physical interpretation.* We view the given coordinate system as a 'map' *on* the space-time with the metric coefficients as giving the rule to link coordinate intervals with physical quantities.

On a pseudo–Riemannian manifold, an infinitesimal *invariant interval* can be positive (*space-like*), negative (*time-like*) or null (*light-like*). Time-like intervals are given by *elapsed time on a physical clock* while space-like intervals are the *lengths measured by a physical measuring stick.* This could be deduced from the principle of equivalence applied to a freely falling lift [2], but we will simply take it as part of the interpretational scheme. We can classify smooth curves in the manifold into time-like, space-like and light-like according as the nature of infinitesimal intervals along the curves. Motion of small bodies ('point particles') in space is represented by time-like curves (or world lines) while propagation of light, in the geometrical optics approximation, is represented by light-like curves in the space-time manifold.

Elapsed Times: Any small physical clock is represented by a time-like curve. Define the clock's *coordinate velocity,* $V^i := \frac{\Delta x^i}{\Delta t}$. Consider two events on the clock's world-line defined by two consecutive 'ticks' of the clock. Let the coordinate intervals for these two events be $(\Delta t, \Delta x^i := V^i \Delta t)$. The corresponding *invariant interval* is given by,

$$\Delta\tau^2_{\vec{V}} := -\Delta s^2 = -g_{00}\Delta t^2\left(1 + \frac{2g_{0i}V^i}{g_{00}} + \frac{g_{ij}V^iV^j}{g_{00}}\right) \qquad (2.1)$$

By definition, the invariant interval is the elapsed time measured by this clock. Notice that for a clock 'at rest' ($V^i = 0$), $\Delta\tau^2 > 0$ implies that $g_{00} < 0$ and then for $V^i \neq 0$, the expression in parenthesis must be positive.

For a clock at rest, $V^i = 0$, we get $\Delta\tau_{\vec{0}} = \sqrt{-g_{00}}\Delta t$ and this provides the *interpretation of the coordinate interval: it is the elapsed time as measured by a clock at rest, divided by* $\sqrt{-g_{00}}$. It has a dependence on the *location* of the clock, through the metric coefficient. It follows,

$$\Delta\tau_{\vec{V}} = \Delta\tau_{\vec{0}}\left[1 + \frac{2g_{0i}V^i}{g_{00}} + \frac{g_{ij}V^iV^j}{g_{00}}\right]^{\frac{1}{2}} \qquad (2.2)$$

For the Minkowski line element, $g_{00} = -1, g_{0i} = 0, g_{ij} = \delta_{ij}$, and we infer the *special relativistic time dilation* by noting that $\Delta\tau_{\vec{V}}$ is the time measured by the moving clock while $\Delta\tau_{\vec{0}}$ is the time measured by the stationary clock.

This appears to be 'opposite' to the usual special relativistic time dilation. It is not. The two events whose invariant interval is given by $\Delta\tau_{\vec{V}}$ are defined by the two consecutive ticks of the *moving clock.* This would usually be denoted by '$\Delta\tau_0$' ('proper time'). The *same* interval as measured by a clock at rest, would usually be denoted by Δt and we have denoted it by $\Delta\tau_{\vec{0}}$. Thus the time dilation derived above is the same one as obtained in special relativity when the metric is Minkowskian.

Next, consider two clocks A and B, both with coordinate velocities zero. Choose two events A and B on their respective world-lines such that the coordinates are (t, x^i_A) and (t, x^i_B) respectively. Consider two subsequent points

A' and B' on their world-lines with the *same* time coordinate $t' = t + \Delta t$. The spatial coordinates will remain the same since the coordinate velocities are zero. The elapsed physical times are then related as,

$$\Delta\tau_A = \sqrt{\frac{g_{00}|_A}{g_{00}|_B}}\ \Delta\tau_B \tag{2.3}$$

Notice that for Minkowski line element (and the FRW line element), $g_{00} = -1$ at both locations and hence the two elapsed times are the same. This ratio gives the *gravitational time dilation.* Taking the invariant time intervals to define inverses of frequencies, we get the prediction that frequencies undergo a change in a gravitational field. This was indeed first measured and verified by Pound and Rebka in 1959 [3–5]. The quantitative estimate is obtained using the Schwarzschild line element. The choice of pairs of events with the same coordinate interval, can be achieved in practice by clock A sending consecutive pulses to clock B. The coordinate time intervals at both clocks will be the same when the coordinate velocities are the same *and* the metric is assumed to be almost time independent over the flight time interval. This is of course realized in the near Earth space-time. We discuss the general case of frequency shifts in section 3.2.

Physical Lengths: Similar considerations apply to spatial invariant intervals ($\Delta s^2 > 0$) as well, in particular physical length intervals are also 'observer dependent'. To see this, recall an argument in the context of *special relativity.*

Imagine two events A and B defined by a car crossing two ends of a road. The coordinates assigned by a road observer will be $(0, \vec{0}), (\Delta t, L_{road}\vec{n})$. The coordinates assigned by the car observer will be $(0, \vec{0}), (\Delta t', 0)$. Let the speed of the car relative to road be β_{car} so that $L_{road} = \beta_{car}\Delta t$. Let the speed of the road relative to the car be β_{road} (in the opposite direction of course) so that $L_{car} = \beta_{road}\Delta t'$. Since the two events are the same, the invariant interval must be the same i.e.

$$\begin{aligned} -\Delta t^2 + L_{road}^2 &= -\Delta t'^2 \\ L_{road}^2(1 - \beta_{car}^{-2}) &= -L_{car}^2\beta_{road}^{-2} \\ \therefore L_{car} &= L_{road}\sqrt{1-\beta_{car}^2}\left(\frac{\beta_{road}}{\beta_{car}}\right) \end{aligned} \tag{2.4}$$

The usual length contraction formula results when we assert that $\beta_{road} = \beta_{car}$ i.e. speed of road measured by car observer is the same as the speed of car measured by the road observer. Had we insisted on the lengths L_{car}, L_{road}, of the road as measured by the two observer are same, we would have got the two speeds to be different, with the β_{road} being not bounded by 1. Clearly, we should interpret the above equation (2.4) as implying that the physical lengths measured by two observers *can* be different while the two speeds are the same: $\beta_{road} = \beta_{car} < 1$. Notice that this identification makes the *velocity truly relative.*

The *necessity* of length contraction can also be seen in the explanation of the observation of muons at the ground level after traversing the atmosphere even though, naively, the rest-frame-life-time of 2.2 μsec would not be sufficient to travel through the thickness of the atmosphere. The ground observer explains this by invoking time dilation to stretch the half life while the muon-rest-frame observer gets the simplest explanation by invoking length contraction to squeeze the thickness of the atmosphere.

We have discussed length contraction using the Minkowski metric. Its generalization to general space-times is given below by a different argument using the definition of local speed.

Local Speed: Imagine a spaceship going from a location A to another one B. The duration of the journey can be measured by an on-board clock and we can ask for an average speed for the journey. How is this to be determined in terms of the arbitrary local coordinates (and the metric coefficients)? For the everyday experience of going in a car the speed shown by speedometer denotes the physical distance traversed in a time shown by a clock, either on board or on the ground. The natural definition of speed would thus be the ratio of a physical distance to a proper time. The problem is to identify, for a given time-like curve, the *spatial* distance covered in some physical time - we *need a definition* of splitting the space-time into space and time.

Recall that space-time coordinates are just labels and it is only in conjunction with metric coefficients that physical meanings are ascribed. Thus to properly identify a split as space and time, we have to specify a form of metric as well, apart from simply labelling $t := x^0$. This is achieved by taking a form for the metric as,

$$\Delta s^2 \;=\; -N^2\Delta t^2 + \bar{g}_{ij}(\Delta x^i + N^i\Delta t)(\Delta x^j + N^j\Delta t) \quad \text{where,} \tag{2.5}$$

$\bar{g}_{ij}$ is positive definite with inverse $\bar{g}^{ij}$. As matrices,

$$g_{\mu\nu} \;=\; \begin{pmatrix} -N^2 + \bar{g}_{ij}N^iN^j & \bar{g}_{ij}N^i \\ \bar{g}_{ij}N^j & \bar{g}_{ij} \end{pmatrix} \leftrightarrow$$

$$g^{\mu\nu} \;=\; \begin{pmatrix} -N^{-2} & N^{-2}N^j \\ N^{-2}N^i & \bar{g}^{ij} - N^{-2}N^iN^j \end{pmatrix} \tag{2.6}$$

Such a form can always be taken locally and serves to identify time-like directions.

And now, *for a given coordinate system*, we define 'space' to be the 't = constant' hypersurface. We had already deduced $g_{00} < 0$ just below eq. (2.1) and g^{00} is manifestly negative. The *unit* (time-like) normal to such a hypersurface is given by $n_\mu = \frac{1}{\sqrt{-g^{00}}}(1,0,0,0)$ and the corresponding $n^\mu = g^{\mu 0}/\sqrt{-g^{00}}$. Define the associated projector, $P^\mu_{\ \nu} := \delta^\mu_{\ \nu} + n^\mu n_\nu$ which projects any vector to a *space-like vector*. Consider two points on the world line of the spaceship, with coordinate interval: $\Delta x^\mu := v^\mu \Delta\tau$, $v \cdot v := g_{\mu\nu}v^\mu v^\nu = -1$.