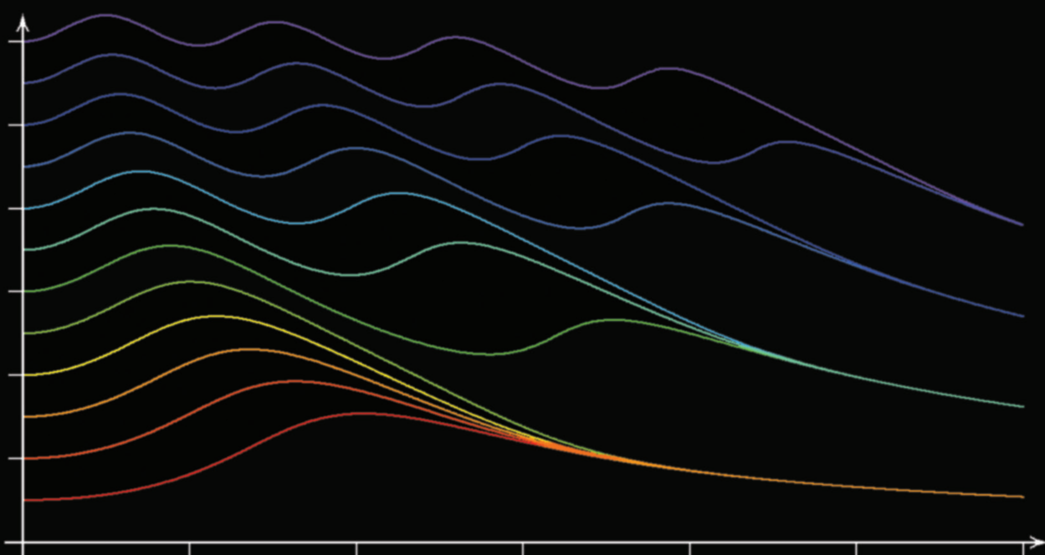
Asymptotic Analysis and Perturbation Theory



William Paulsen

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William Paulsen

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Preface

The goal of this textbook is to present the topics of asymptotic analysis and perturbation theory to a level obtainable to students who have only completed the standard calculus sequence. Even though the most common application of asymptotics is in analyzing differential equations, students need not have prior knowledge of differential equations for this text. Rather, the book begins by immediately introducing the asymptotic notation, and applying this new tool to problems that the students will already be familiar with: limits, inverse functions, and integrals. In fact, only the simplest differential equations, such as first order linear or separable equations, will the students need to learn how to solve exactly. Hence, there is very little overlap between this text and a standard differential equation textbook.

The text follows the traditional organization, with plenty of exercises at the end of each section, and the answers to the odd numbered problems in the back. However, it also includes an abundance of computerized graphs and tables that will illustrate how well the asymptotic approximations approach the actual solutions. These graphs and charts enhance the student's learning of the material, giving them visual evidence that these approximation methods can be applied to the many types of problems that the student will encounter in his or her field.

This book will benefit instructors in that it will allow them to offer a course in Applied Mathematics that does not require a differential equations prerequisite. It will benefit students by bringing this difficult subject material to an easy to comprehend level. The book will benefit the mathematics department by making a course which is attractive to both majors and non-majors alike. The fields of engineering, physics, and even computer science utilize the study of asymptotic analysis and perturbation theory.

Although the emphasis of this book is problem-solving, there are some proofs scattered throughout the book. The purpose of these proofs is to give the students a justification for the methods that they will be using. Just as there are some proofs in a freshman level calculus book which are not as rigorous as the corresponding proofs in an advanced calculus text, these proofs are more informal, and often will refer the students to other sources for the details. These proofs enrich the students understanding of the material.

Another focus of this textbook is flexibility. Knowing that the readership will be extremely diverse, the aim was to include material that would be beneficial to both beginning students and researchers. Also, the book was designed to be completely self-contained, requiring only a calculus sequence background. There is a section giving the necessary background material for complex variables, since this knowledge tends to be lacking in the undergraduate curriculum. References to differential equations is deferred until chapter 4, where the small amount of background is covered, with minimal duplication of a standard differential equations course. Since the goal is to only *approximate* the solutions to such equations, it is not necessary for the students to know how to solve differential equations exactly, except for first order linear or separable equations. Hence, an undergraduate course can easily be designed using this text.

There is also more than enough material needed for a semester course. Professors may choose to skip chapter 3, (or even chapter 4, if differential equations is a prerequisite,) in order to reach the latter chapters. On the other hand, the first 6 chapters will make a good undergraduate course on asymptotics. There are a myriad of possibilities between these two extremes.

Finally, there are plenty of homework problems of various levels of difficulty. Most sections have between 20 to 30 problems, giving professors enough choices for assignments. Also, the answers to the odd numbered problems appear in the back of the book.

Acknowledgments

I want to begin by thanking my mentor, Carl Bender, for lighting the spark that got me excited about asymptotics and perturbation theory. He inspired me in the direction of my research, which eventually gave me the motivation for writing this book.

I also would like to express my appreciation to my wife Cynthia and my son Trevor for putting up with me during this past two years, since this project ended up taking much more of my time than I first realized. They have been very patient with me and are looking forward to me finally being done.

About the Author

William Paulsen is a Professor of Mathematics at Arkansas State University. He has taught asymptotics in a dual level undergraduate/graduate level course since 1994. He received his B.S. (summa cum laude), M.S., and Ph.D. degrees in Mathematics at Washington University in St. Louis. He was on the winning team for the 45th William Lowell Putnam Mathematical Competition.

Dr. Paulsen has authored over 15 papers in applied mathematics. Most of these papers make use of *Mathematica*®, including one which proves that Penrose tiles can be 3-colored, thus resolving a 30-year old open problem posed by John H. Conway. He has authored an abstract algebra textbook, "Abstract Algebra: an Interactive Approach," also published by CRC press.

Dr. Paulsen has also programmed several new games and puzzles in Javascript and C++. One of these puzzles, Duelling Dimensions, was syndicated through Knight Features. Other puzzles and games are available on the Internet.

Dr. Paulsen lives in Harrisburg, Arkansas with his wife Cynthia, his son Trevor, and two pugs and a dachshund.

Symbol Description

$\sim$	behaves similar to	1
$\ll, \gg$	much less than, much greater than	3
$O(g(x))$	is of order $g(x)$	6
$o(g(x))$	is less than order $g(x)$	7
$S(x)$	Stieltjes integral function	16
$\sinh(x)$	hyperbolic sine function	18
$\sinh^{-1}(x)$	inverse hyperbolic sine	18
$\cosh(x)$	hyperbolic cosine function	18
$\mathrm{sech}(x)$	hyperbolic secant function	19
$\tanh(x)$	hyperbolic tangent function	20
$\coth x$	hyperbolic cotangent function	297
$\mathrm{csch}(x)$	hyperbolic cosecant function	313
$W(x)$	Lambert W function	26
$\mp$	minus or plus sign (vs. $\pm$)	32
$E_1(x)$	exponential integral function	41
γ	Euler-Mascheroni constant	43
$\mathrm{Si}(x)$	sine integral	43
$\mathrm{Ci}(x)$	cosine integral	51
$\mathrm{erf}(x)$	error function	48
$\mathrm{erfc}(x)$	complementary error function	482
$\Gamma(x)$	gamma function	59
$\displaystyle\prod_{n=1}^{\infty}$	infinite product	59
$\zeta(x)$	Riemann zeta function	77
$\tilde{z}$	point on a different sheet of Riemann surface	78
$\displaystyle\int_C f(z)\,dz$	complex contour integral	77
$J_v(x)$	Bessel function of the first kind	96
$\mathrm{Ai}(x)$	Airy function of the first kind	109
$S^m(A_n)$	iterated Shanks transformations	112
$S_m(A_n)$	generalized Shanks transformation	115
$R_m(A_n)$	generalized Richardson's extrapolation	120
$\delta(t)$	Dirac delta function	139

$\mathop{\mathrm{K}}\limits_{n=0}^{\infty}$	infinite continued fraction	146
$\diagup\!\!($	divide by the quantity	146
$P_M^N(x)$	N, M-Padé approximate	155
$W(y_1, y_2)$	Wronskian of two (or more) functions	168
$\sum\limits_n$	sum over all integers n	191
$\mathrm{Bi}(x)$	Airy function of the second kind	210
$D_v(x)$	parabolic cylinder function	212
$Y_v(x)$	Bessel function of the second kind	224
δ_i	Kronecker delta function	228
Δa_n	difference operator on a sequence	254
$\Delta^k a_n$	iterated difference operator	254
$\Delta^{-1} a_n$	indefinite anti-difference operator	256
$\binom{n}{i}$	binomial coefficients	254
$\psi(x)$	digamma function	269
B_n	Bernoulli numbers	289
$B_n(x)$	Bernoulli polynomials	289
$\mathrm{Cthi}(x)$	hyperbolic cotangent integral function	298
$n^{[-k]}$	$\Gamma(n)/\Gamma(n+k)$	305
ϵ	small parameter	317
δ	small scaled parameter	321
$y_{\mathrm{out}}(x)$	outer solution	339
$y_{\mathrm{in}}(t)$	inner solution	339
$y_{\mathrm{comp}}(x)$	uniformly valid composite approximation	345
$y_{\mathrm{out},n}(x)$	outer solution expanded to order ϵ^n	356
$y_{\mathrm{in},n}(t)$	inner solution expanded to order ϵ^n	356
y_{match}	solution in the overlapping region	357
$y_{\mathrm{comp},n}$	n^{th} order composite approximation	357
y_{exact}	exact solution	357
$(f(x))_{\mathrm{out},n}$	convert to outer variable, expanded to order ϵ^n	369
$(f(x))_{\mathrm{in},n}$	convert to inner variable, expanded to order ϵ^n	369
$y_{\mathrm{comp},m,n}$	mixed order composite approximation	369
$y_{\mathrm{in,left}}$	left hand inner solution	382
$y_{\mathrm{in,right}}$	right hand inner solution	382
$y_{\mathrm{leftmatch}}$	left hand matching function	382
$y_{\mathrm{rightmatch}}$	right hand matching function	382
$y_{\mathrm{WKBJ},n}$	n^{th} order WKBJ approximation	394
$\max\limits_x$	maximum over all possible x	409

y_{right}	right hand WKBJ approximation	419
y_{left}	left hand WKBJ approximation	419
$G_0(x)$	modified parabolic cylindrical equation	429
$\hbar$	Planck's constant	436
$y_{\text{strain},n}$	strained coordinate approximation of order n	448
$y_{\text{multi},n}$	multi-variable unstrained approximation of order n	464
$y_{\text{two-var},n}$	two variable expansion of order n	468
$\text{Ein}(x)$	entire exponential integral function	483
$\text{Cin}(x)$	entire cosine integral function	483
$\text{Shi}(x)$	hyperbolic sine integral	483
$\text{Chi}(x)$	hyperbolic cosine integral	483
$\text{Chin}(x)$	entire hyperbolic cosine integral	483
$\text{Thi}(x)$	hyperbolic tangent integral	483
$I_v(x)$	modified Bessel function of the first kind	488
$K_v(x)$	modified Bessel function of the second kind	488
$\text{He}_n(x)$	Hermite polynomial of order n	491

Chapter 1

Introduction to Asymptotics

Asymptotics has been called the "calculus of approximations." It provides a powerful tool for approximating the solutions to wide classes of problems, including limits, integrals, differential equations, and difference equations. Although the basic definitions are easy to understand, it requires skill to use asymptotics effectively and accurately. The goal of this chapter is to teach the necessary skills for a basic understanding of asymptotics. In later chapters we will apply the techniques to more difficult problems that cannot be solved any other way. In the process, we will learn the properties of some very important functions in applied mathematics.

1.1 Basic Definitions

Since the foundation of standard calculus is the concept of a limit, it is not surprising that the "calculus of approximations" will also hinge on limits. Usually we will consider a finite limit $\lim\limits_{x\to a} f(x)$, but we can also have infinite limits, so a can be ∞ or $-\infty$.

1.1.1 Definition of $\sim$ and $\ll$

We begin with the two fundamental definitions of asymptotics.

DEFINITION 1.1 Given two functions, $f(x)$ and $g(x)$, we say that $f(x)$ is *similar* to $g(x)$ as x approaches a, written

$$f(x) \sim g(x) \quad \text{as} \quad x \to a,$$

if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

For example, $\sin x \sim x$ as $x \to 0$, since $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$. Also, $x + 1 \sim x$ as $x \to \infty$, since $\lim\limits_{x\to\infty} \frac{x+1}{x} = \lim\limits_{x\to\infty} 1 + \frac{1}{x} = 1$.

PROPOSITION 1.1

The relation $\sim$ as $x \to a$ is an *equivalence relation* on the set of all functions that are non-zero near a. That is, $\sim$ obeys the *reflexive property*:

$$f(x) \sim f(x) \text{ as } x \to a, \tag{1.1}$$

the *symmetric property*:

$$\text{if } f(x) \sim g(x) \text{ as } x \to a, \text{ then } g(x) \sim f(x) \text{ as } x \to a, \tag{1.2}$$

and the *transitive property*:

$$\text{if } f(x) \sim g(x) \text{ and } g(x) \sim h(x) \text{ as } x \to a, \text{ then } f(x) \sim h(x) \text{ as } x \to a. \tag{1.3}$$

PROOF: Since $f(x)$ is non-zero near a, we have

$$\lim_{x\to a} \frac{f(x)}{f(x)} = 1,$$

so $f(x) \sim f(x)$ as $x \to a$. To prove the symmetric property, note that

$$\lim_{x\to a} \frac{g(x)}{f(x)} = \frac{1}{\lim\limits_{x\to a} (f(x)/g(x))} = 1.$$

Finally, the transitive property follows from the fact that

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} \frac{g(x)}{h(x)} = 1.$$

□

Note that the reflective property only applies to functions that are non-zero near a. Unfortunately, we *cannot* say that $0 \sim 0$. We will stress this point by highlighting the following statement.

$$\boxed{\text{There is no function } f(x) \text{ such that } f(x) \sim 0 \text{ as } x \to a.} \tag{1.4}$$

Otherwise, we would have

$$\lim_{x\to a} \frac{0}{f(x)} = 1,$$

which is impossible. This marks a stark difference between asymptotics and standard limit notations. We can say that

$$\sin(x) \sim 1 \text{ as } x \to \pi/2,$$

but we *cannot* say that

$$\sin(x) \sim 0 \text{ as } x \to \pi.$$

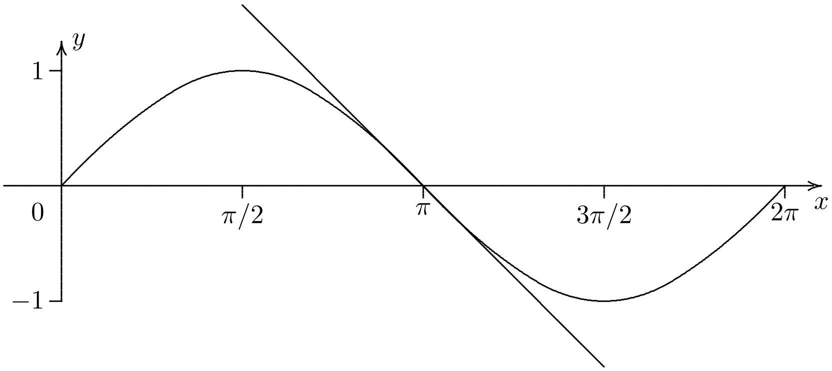

FIGURE 1.1: The two graphs reveal that $\sin(x) \sim \pi - x$ as $x \to \pi$. Note that the ratio of the two functions approaches 1 as x approaches π.

In fact, there is a linear function that is similar to $\sin(x)$ as $x \to \pi$, namely, the tangent to the curve at that point

$$\sin(x) \sim \pi - x \text{ as } x \to \pi.$$

Figure 1.1 gives a visualization of this asymptotic relationship.

The second main notation is also defined in terms of limits:

DEFINITION 1.2 We say that $f(x)$ is *much less than* $g(x)$ as x approaches a, written

$$f(x) \ll g(x) \qquad \text{as} \qquad x \to a,$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

We can think of this as "$f(x)$ is a drop in the bucket compared to $g(x)$, when x is close enough to a." Likewise, if a is ∞, we can say that "$f(x)$ is a drop in the bucket compared to $g(x)$, for sufficiently large x." We can similarly define $f(x) \gg g(x)$ as $x \to a$.

For example, $x^2 \ll x$ as $x \to 0$, since $\lim\limits_{x \to 0} x^2/x = 0$. However, $x^2 \gg x$ as $x \to \infty$, since $\lim\limits_{x \to \infty} x/x^2 = 0$.

The $\ll$ notation also has a special property:

PROPOSITION 1.2

If $f(x) \ll g(x)$ and $g(x) \ll h(x)$ as $x \to a$, then $f(x) \ll h(x)$ as $x \to a$. This property is called the *partial ordering property* of $\ll$ as $x \to a$.

PROOF: If $f(x) \ll g(x)$ and $g(x) \ll h(x)$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \lim_{x\to a} \frac{g(x)}{h(x)} = 0 \cdot 0 = 0.$$ ∎

The two fundamental notations of asymptotics are in fact related.

PROPOSITION 1.3

If $f(x) \sim g(x)$ as $x \to a$, then the relative error between the functions is going to zero, that is,

$$f(x) - g(x) \ll g(x) \qquad \text{as} \qquad x \to a.$$

Likewise, if $h(x) \ll f(x)$ as $x \to a$, then adding (or subtracting) $h(x)$ to $f(x)$ will produce a function similar to $f(x)$. That is,

$$f(x) \pm h(x) \sim f(x) \qquad \text{as} \qquad x \to a.$$

PROOF: If $f(x) \sim g(x)$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x) - g(x)}{g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} - \frac{g(x)}{g(x)} = 1 - 1 = 0.$$

So $f(x) - g(x) \ll g(x)$ as $x \to a$. On the other hand, if $h(x) \ll f(x)$ as $x \to a$, then

$$\lim_{x\to a} \frac{f(x) \pm h(x)}{f(x)} = \lim_{x\to a} \frac{f(x)}{f(x)} \pm \frac{h(x)}{f(x)} = 1 \pm 0 = 1.$$

So $f(x) \pm h(x) \sim f(x)$ as $x \to a$. ∎

Comparing algebraic functions such as polynomials is particularly easy. If $a > b$ then as $x \to \infty$, $x^a \gg x^b$. However, if we consider the limit as $x \to 0$, this reverses the direction: $x^a \ll x^b$. Thus $x^3 + 3x^2 - 2x \sim x^3$ as $x \to \infty$ since $3x^2 - 2x \ll x^3$. In fact, any polynomial is similar to its highest order term as $x \to \infty$. However, as $x \to 0$, $x^3 + 3x^2 - 2x \sim -2x$, the lowest order term.

1.1.2 Hierarchy of Functions

It will be important to understand how fast different functions grow, particularly as $x \to \infty$. Given two functions, we could ask whether one function grows more rapidly than another. This gives us a type of hierarchy to the different functions. We have already seen that given two polynomials of different degrees, the one with the larger degree will be much greater than the other as $x \to \infty$. However, functions which increase exponentially will grow faster than any polynomial.

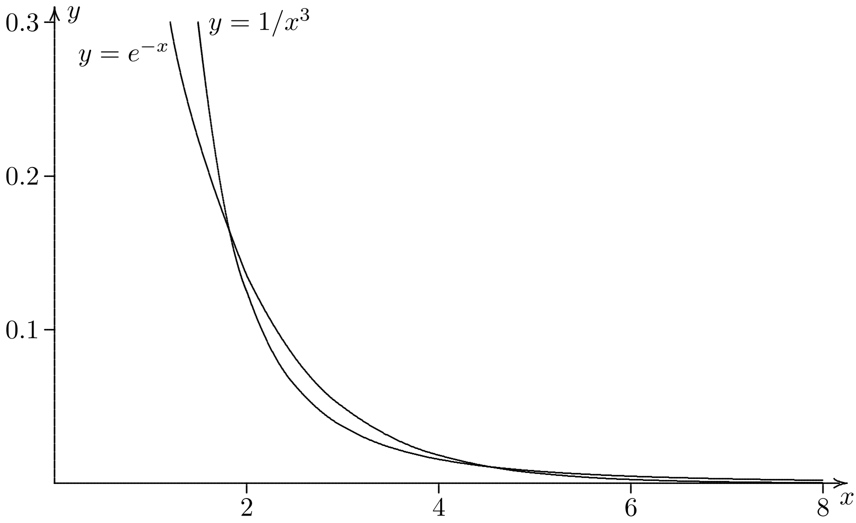

FIGURE 1.2: The graphs of $y = e^{-x}$ and $y = x^{-3}$. Although both converge to 0 as $x \to \infty$, the e^{-x} approaches zero faster as x increases, showing that $e^{-x} \ll x^{-3}$ as $x \to \infty$.

PROPOSITION 1.4
If $a > 0$ and b is a real constant, then $e^{ax} \gg x^b$ as $x \to \infty$. Also, $e^{-ax} \ll x^b$ as $x \to \infty$.

PROOF: Pick a positive integer n bigger than b. We can apply L'Hôpital's rule n times to the limit

$$\lim_{x\to\infty} \frac{x^n}{e^{ax}} = \lim_{x\to\infty} \frac{nx^{n-1}}{ae^{ax}} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{a^2 e^{ax}} = \cdots = \lim_{x\to\infty} \frac{n!}{a^n e^{ax}} = 0.$$

Thus, $e^{ax} \gg x^n$ as $x \to \infty$, and since $x^n \gg x^b$, $e^{ax} \gg x^b$ as $x \to \infty$. Also note that

$$\lim_{x\to\infty} \frac{e^{-ax}}{x^b} = \lim_{x\to\infty} \frac{x^{-b}}{e^{ax}} = 0,$$

so $e^{-ax} \ll x^b$ as $x \to \infty$. ∎

Figure 1.2 is a graphical illustration that $e^{-x} \ll x^{-3}$ as $x \to \infty$. Note that this does *not* mean that $e^{-x} < x^{-3}$ for all x. In fact, the curves cross each other in two places. Only what happens for large values of x counts towards deciding which function is much smaller as $x \to \infty$.

Finally, we can compare two exponential functions by determining which has the larger exponent.

$$\boxed{\text{If } a > b, \text{ then } e^{ax} \gg e^{bx} \text{ as } x \to \infty.} \tag{1.5}$$

The proof is left as an exercise. See problem 31.

To compare two exponential functions with different bases, we can convert all of the bases to e. Thus, to compare e^{2x} and 8^x we observe that $8^x = (e^{\ln 8})^x = e^{(\ln 8)x}$. Since $\ln 8 \approx 2.079$, we have $e^{2x} \ll 8^x$ as $x \to \infty$.

We can determine how the logarithm function fits into the ranking by using L'Hôpital's Rule. It is easy to see that $\ln(x) \gg 1$ as $x \to \infty$, and we can observe that, for any $a > 0$, $\lim\limits_{x\to\infty} \frac{\ln(x)}{x^a} = 0$. (See problem 32.) Therefore,

$$\boxed{\text{If } a > 0, \text{ then } 1 \ll \ln x \ll x^a \text{ as } x \to \infty.} \tag{1.6}$$

Thus, as $x \to \infty$, $x^{1/1000} \gg \ln(x) \gg 1 = x^0$. The logarithm function squeezes in between the tight gap between x^0 and x^ϵ as $x \to \infty$, where ϵ is an extremely small positive number. This fact is helpful for computing limits involving logarithms. For example, $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$, since $\ln x \ll \sqrt{x}$ as $x \to \infty$.

When $x \to 0$, $\ln(x)$ approaches $-\infty$, and we can use the property that $\ln(x) = -\ln(1/x)$ to derive the following result.

$$\boxed{\text{If } a > 0, \text{ then } 1 \ll \ln x \ll x^{-a} \text{ as } x \to 0.} \tag{1.7}$$

See problem 33. For example, $\lim_{x\to 0} \sqrt[3]{x}\ln(x) = 0$, since $\ln x \ll x^{-1/3}$ as $x \to 0$.

1.1.3 Big O and Little o Notation

Two more useful notations that are sometimes used are referred to as the "big O" and "little o" notation.

DEFINITION 1.3 We say that a function $f(x)$ is *of order* $g(x)$ as x approaches a, denoted by $f(x) = O(g(x))$ as $x \to a$, if the ratio $f(x)/g(x)$ is *bounded* for x near a. If a is finite, we can say this by saying that there are M and ϵ such that

$$|f(x)| \le M|g(x)| \text{ whenever } 0 < |x - a| < \epsilon. \tag{1.8}$$

Likewise, we say that $f(x) = O(g(x))$ as $x \to \infty$ if there are M and N such that

$$|f(x)| \le M|g(x)| \text{ whenever } x > N. \tag{1.9}$$

It is clear that if $f(x) \sim g(x)$ or $f(x) \ll g(x)$ as $x \to a$, then $f(x) = O(g(x))$ as $x \to a$. In fact, if $f(x) \sim kg(x)$ for some constant k, then $f(x) = O(g(x))$ as $x \to a$. However, the big O notation is useful when there is no clear asymptotic behavior of $f(x)$.

For example, $\sin x = O(1)$ as $x \to \infty$, since we can pick $M = 1$ and $N = 0$. Then of course, $|\sin x| \leq 1$ for all $x > 0$. Note in this example that there is *no* function $f(x)$ for which $f(x) \sim \sin(x)$ as $x \to \infty$, because $\sin x$ is not non-zero near ∞. The ratio $f(x)/\sin(x)$ would be undefined whenever x is a multiple of π, so technically, the limit as $x \to \infty$ does not exist. We will later see in subsection 2.4.4 how we can asymptotically analyze periodic and near-periodic functions.

The big O notation is often used with series to show the order of the first term left out. For example, the familiar Maclaurin series for $\cos(x)$ can be written

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \qquad \text{as } x \to 0.$$

The $O(x^6)$ in this equation replaces some function that is of order x^6 as $x \to 0$. In other words, the function

$$\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}$$

must be of order x^6. In fact,

$$\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} \sim \frac{x^6}{720} \qquad \text{as } x \to 0,$$

as indicated by the next term in the Maclaurin series.

The little o notation is similar, except that the function must be strictly smaller than the function inside the o.

DEFINITION 1.4 We say that a function $f(x)$ is *less than order* $g(x)$ as x approaches a, denoted by $f(x) = o(g(x))$ as $x \to a$, if the ratio $f(x)/g(x)$ approaches 0 as x approaches a.

To say that $f(x) = o(g(x))$ as $x \to a$ is equivalent to saying that $f(x) \ll g(x)$, but the little o notation can also be used for series to indicate the accuracy of the series. For example, one can write

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \qquad \text{as } x \to 0,$$

which emphasizes that there is no x^5 term.

Problems for §1.1

For problems **1** through **12**: State whether the following statements are true or false.

1 $x^2 - 2 \sim 2$ as $x \to 2$
2 $x^2 - 4 \sim 0$ as $x \to 2$
3 $x^2 \sim x$ as $x \to 0$
4 $x^2 + x \sim x$ as $x \to 0$
5 $x^2 + x \sim x^2$ as $x \to \infty$
6 $x^2 + x \sim 2x$ as $x \to 1$
7 $x^2 \ll x$ as $x \to 0$
8 $\frac{x}{1000} \ll x$ as $x \to \infty$
9 $x \ll -2$ as $x \to 0$
10 $\sqrt{x+1} \sim \sqrt{x}$ as $x \to \infty$
11 $e^{x+1} \sim e^x$ as $x \to \infty$
12 $\sin(x+1) \sim \sin(x)$ as $x \to \infty$

For problems **13** through **20**: Find the polynomial of lowest degree that is similar to the following functions as $x \to a$.

13 x^2 $a = 2$
14 $\sin x$ $a = \pi/2$
15 $\cos x$ $a = \pi/2$
16 $\sin x$ $a = 0$
17 $2x^3 - 3x^2 + 1$ $a = 1$
18 $\sin x - 1$ $a = \pi/2$
19 $1 + \cos x$ $a = \pi$
20 0 $a = 0$

For problems **21** through **26**: Find the polynomial $p(x)$ of lowest degree so that the equation is a true statement.

21 $e^x = p(x) + O(x^3)$ as $x \to 0$
22 $e^x = p(x) + o(x^3)$ as $x \to 0$
23 $\sin(x) = p(x) + o(x^5)$ as $x \to 0$
24 $\sin(x) = p(x) + O(x^5)$ as $x \to 0$
25 $\cos(x) = p(x) + o(x^5)$ as $x \to 0$
26 $\sqrt{x} = p(x) + O((x-1)^3)$ as $x \to 1$

27 Is there a function $f(x)$ for which no other function can be much greater than $f(x)$ as $x \to \infty$? Why or why not?

28 Is there a function $f(x)$ for which no other function can be much smaller than $f(x)$ as $x \to \infty$? Why or why not?

29 Is it possible for a function to be $O(0)$ as $x \to \infty$? Why or why not?

30 Is it possible for a function to be $o(0)$ as $x \to \infty$? Why or why not?

31 Prove equation 1.5. That is, show that if $a > b$, then $e^{ax} \gg e^{bx}$ as $x \to \infty$.

32 Prove equation 1.6. That is, use L'Hôpital's rule to show that $\ln x \ll x^a$ as $x \to \infty$, where a is a positive constant.

33 Prove equation 1.7. That is, use L'Hôpital's rule to show that $\ln x \ll x^{-a}$ as $x \to 0$, where a is a positive constant.

1.2 Limits via Asymptotics

One of the basic applications of asymptotics is as an alternative to L'Hôpital's rule for finding limits. The basic principal is to replace parts of a limit, such as the numerator or denominator, with another function that is similar as $x \to a$. This usually will not affect the limit, and we will cover the exceptions as we encounter them.

$$\boxed{\text{If } f(x) \sim g(x) \text{ and } h(x) \sim k(x) \text{ as } x \to a \text{, then } \lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{g(x)}{k(x)}} \tag{1.10}$$

provided either of these limits exist. The reasoning is simple:

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{k(x)} \cdot \frac{k(x)}{h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} \cdot \frac{k(x)}{h(x)} \cdot \lim_{x\to\infty} \frac{g(x)}{k(x)} = \lim_{x\to\infty} \frac{g(x)}{k(x)}.$$

Example 1.1
Find

$$\lim_{x\to\infty} \frac{x^3 + 3x^2 - 2x + 1}{3x^3 + 2x^2 - 5x}.$$

SOLUTION: Since $x^3 + 3x^2 - 2x + 1 \sim x^3$ and $3x^3 + 2x^2 - 5x \sim 3x^3$ we have

$$\lim_{x\to\infty} \frac{x^3 + 3x^2 - 2x + 1}{3x^3 + 2x^2 - 5x} = \lim_{x\to\infty} \frac{x^3}{3x^3} = 1/3.$$

Note that L'Hôpital's Rule would have to be applied 3 times to solve this problem. ∎

We can also replace a function inside a square root or other radical with a similar function as $x \to a$, without affecting the limit.

$$\boxed{\text{If } f(x) \sim g(x) \text{ as } x \to a \text{, and } c \text{ is real, then } [f(x)]^c \sim [g(x)]^c \text{ as } x \to a.} \tag{1.11}$$

This is easy to verify with limits.

$$\lim_{x\to\infty} \frac{[f(x)]^c}{[g(x)]^c} = \lim_{x\to\infty} \left(\frac{f(x)}{g(x)}\right)^c = \left(\lim_{x\to\infty} \frac{f(x)}{g(x)}\right)^c = 1^c = 1.$$

Example 1.2
Find

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 3x - 2}}{3x + 1}.$$

SOLUTION: The plan is to replace the complicated function $4x^2 + 3x - 2$ inside the square root with the simpler function $4x^2$. Since $4x^2 + 3x - 2 \sim 4x^2$ as $x \to \infty$, this substitution will not affect the limit. So

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 3x - 2}}{3x + 1} = \lim_{x\to\infty} \frac{\sqrt{4x^2}}{3x} = \lim_{x\to\infty} \frac{2x}{3x} = 2/3.$$

□

We can also substitute one factor of the numerator or denominator with a similar function, without affecting the limit. See problem 25.

Example 1.3
Find

$$\lim_{x\to\infty} \frac{(2x+3)\sqrt{3e^x + \cos x}}{(e^{x/2} + x^{100})\sqrt{x^2+4}}.$$

SOLUTION: Note that $|\cos x| < 1$ for all x, so $\cos x \ll e^x$ as $x \to \infty$. Also, $x^{100} \ll e^{x/2}$ as $x \to \infty$, so we have

$$\lim_{x\to\infty} \frac{(2x+3)\sqrt{3e^x + \cos x}}{(e^{x/2} + x^{100})\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{2x\sqrt{3e^x}}{e^{x/2}\sqrt{x^2}} = 2\sqrt{3}.$$

□

Can we substitute a similar function inside of a logarithm? That is,

$$\text{if } g(x) \ll f(x) \text{ as } x \to a, \text{ is } \ln(f(x) + g(x)) \sim \ln(f(x))?$$

The answer is usually yes, but not always. As long as $f(x)$ approaches 0 or ∞ as $x \to a$, then the substitution will produce a similar function. However, if $f(x)$ approaches 1, there is a complication. To see this, consider the limit

$$\lim_{x\to 0} \frac{\ln(1+x)}{x}.$$

It is certainly true that $x \ll 1$ as $x \to 0$, but we cannot say that $\ln(1+x) \sim \ln(1)$, because $\ln(1) = 0$, and we already established that *no* function is similar to 0.

So what do we do in this situation? By looking at the Maclaurin series for $\ln(1+x)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + O(x^6), \tag{1.12}$$

we see that $\ln(1+x) \sim x$ as $x \to 0$. From this result, we can establish the following:

$$\boxed{\text{If } g(x) \ll 1 \text{ as } x \to a, \text{ then } \ln(1+g(x)) \sim g(x) \text{ as } x \to a.} \tag{1.13}$$

Note that this is the exceptional case, not the rule, for dealing with logarithms. If $g(x) \ll f(x)$ as $x \to a$, and $f(x)$ is approaching 0 or ∞, or even some constant other than 1, then $\ln(f(x) + g(x)) \sim \ln(f(x))$ as $x \to a$. See problem 27.

Example 1.4
Find the limit

$$\lim_{x\to\infty} x \ln\left(\frac{x+3}{x+1}\right).$$

SOLUTION: Since $x+3 \sim x$ and $x+1 \sim x$ as $x \to \infty$, we see that the expression within the logarithm function is approaching 1. So we must first decompose the improper rational function, and rewrite the limit as

$$\lim_{x\to\infty} x \ln\left(1 + \frac{2}{x+1}\right).$$

Now, $2/(x+1)$ is approaching 0 as $x \to \infty$, so we can use equation 1.13:

$$x \ln\left(1 + \frac{2}{x+1}\right) \sim x\frac{2}{x+1} \sim \frac{2x}{x} = 2.$$

∎

Example 1.5
Find the limit

$$\lim_{x\to\infty} \frac{\ln(3x^5+2x^2+10)}{\ln(5x^7+3x^4+10)}.$$

SOLUTION: Since the argument of the logarithms are going to ∞, we can substitute $3x^5+2x^2+10 \sim 3x^5$ and $5x^7+3x^4+10 \sim 5x^7$ as $x \to \infty$ without affecting the limit. Thus, as $x \to \infty$,

$$\frac{\ln(3x^5+2x^2+10)}{\ln(5x^7+3x^4+10)} \sim \frac{\ln(3x^5)}{\ln(5x^7)} = \frac{\ln 3 + 5\ln x}{\ln 5 + 7\ln x}.$$

Since $\ln x \gg 1$ as $x \to \infty$, this simplifies further:

$$\frac{\ln 3 + 5\ln x}{\ln 5 + 7\ln x} \sim \frac{5\ln x}{7\ln x} = \frac{5}{7}.$$

∎

Is it possible to substitute a similar function in an exponent without affecting the limit? Note that even though $x+1 \sim x$,

$$\lim_{x\to\infty} \frac{e^{x+1}}{e^x} = \lim_{x\to\infty} \frac{e^x \cdot e}{e^x} = e \neq 1.$$

Hence, the answer is a resounding *no*.

$$\boxed{\text{Just because } f(x) \sim g(x) \text{ as } x \to a \text{ does } \textit{not} \text{ mean that } e^{f(x)} \sim e^{g(x)}.} \tag{1.14}$$

Even though we cannot make simplifications withing the exponent, we can instead use the properties of exponentials to simplify a limit involving complicated powers.

Example 1.6
Find the limit

$$\lim_{x\to\infty} \frac{(4x)^{x-2}x^3}{x^{x+1}2^{2x+3}}.$$

SOLUTION: Expanding the exponentials, we obtain

$$\frac{(4x)^{x-2}x^3}{x^{x+1}2^{2x+3}} = \frac{(4x)^x(4x)^{-2}x^3}{x^x x^1 2^{2x}2^3} = \frac{4^x x^x 4^{-2}x^{-2}x^3}{x^x x(2^2)^x 8} = \frac{4^{-2}}{8} = \frac{1}{128}.$$ ∎

There is also an issue as to whether we can substitute the *base* of a complicated exponential function with a similar function without affecting the limit. For example, in the limit

$$\lim_{x\to 0}(1+x)^{1/x}, \tag{1.15}$$

can we replace the $1+x$ with 1, since these are similar as $x\to 0$? Calculus students should recognize this as the indeterminate 1^∞ form, so the answer is no.

$$\boxed{f(x)\sim g(x) \text{ does } \textit{not} \text{ imply that } f(x)^{h(x)}\sim g(x)^{h(x)} \text{ if } h(x)\to\infty.} \tag{1.16}$$

The situation where the exponent is going to ∞ is best handled by taking the logarithm of both sides of an equation, so that the properties of logarithms can be used. In the limit of equation 1.15, we set $y = \lim_{x\to 0}(1+x)^{1/x}$, so that $\ln y = \lim_{x\to 0}\ln(1+x)/x$. Using equation 1.13 quickly produces $\ln y = 1$, so $y = e$ is the original limit.

Example 1.7
Find the limit

$$\lim_{x\to\infty} \frac{(x+3)^{x+1}x^{x-1}}{(x+1)^{2x}}.$$

SOLUTION: We can first simplify this expression:

$$\frac{(x+3)^{x+1}x^{x-1}}{(x+1)^{2x}} = \frac{(x+3)^x(x+3)x^x x^{-1}}{((x+1)^2)^x} = \frac{x+3}{x}\left(\frac{x^2+3x}{x^2+2x+1}\right)^x.$$

The limit of the first factor is clearly one, so we will set y equal to the second factor. Then

$$\ln y \sim x\ln\left(\frac{x^2+3x}{x^2+2x+1}\right) = x\ln\left(1+\frac{x-1}{x^2+2x+1}\right) \sim x\frac{x-1}{x^2+2x+1} \sim 1.$$

Since $\ln y = 1$, the original limit is $e^1 = e$. ∎

Problems for §1.2

For problems **1** through **12**: Find the following limits, using asymptotics. Note that most of these cannot be done with L'Hôpital's rule alone.

1 $\lim\limits_{x\to\infty} \frac{3x^3-5x^2+4x-6}{2x^3+7x^2-4x+5}$

2 $\lim\limits_{x\to\infty} \frac{4x^5+3x^2+5}{5x^5-8x^4+3x}$

3 $\lim\limits_{x\to\infty} \frac{e^x+x^{100}}{e^x+\cos x}$

4 $\lim\limits_{x\to\infty} \frac{2^{x-1}+3^{x-1}+x^3}{x^5-3^{x+1}}$

5 $\lim\limits_{x\to\infty} \frac{\sqrt{3x^2-3x+4}}{2x-1}$

6 $\lim\limits_{x\to\infty} \frac{3x^2-4}{\sqrt{x^4+3x-1}}$

7 $\lim\limits_{x\to\infty} \frac{\sqrt{e^{2x}+x^4}}{e^x+2^x}$

8 $\lim\limits_{x\to\infty} \frac{2^x-x^5}{\sqrt{4^x+e^x+x^2}}$

9 $\lim\limits_{x\to\infty} \frac{(e^x+\cos x)\sqrt{4x^2-5x+3}}{(3x+2)\sqrt{e^{2x}+7^x}}$

10 $\lim\limits_{x\to\infty} \frac{(x^2+\sin x)\sqrt{e^{4x}+54^x}}{(e^{2x}+x^{10})\sqrt{5x^4+3x-3}}$

11 $\lim\limits_{x\to\infty} \frac{(3x+3)^{20}(x-2)^{10}}{(2x+5)^{30}}$

12 $\lim\limits_{x\to\infty} \frac{(5x^2+4x-3)^6(4x^2+3x-2)^8}{(3x+5)^{18}(x-5)^{10}}$

For problems **13** through **24**: Find the following limits involving logarithms. Some of these require using equation 1.13.

13 $\lim\limits_{x\to\infty} \frac{\ln(x^3-3x^2+4)}{\ln(x^2-4x+6)}$

14 $\lim\limits_{x\to\infty} \frac{\ln(5x^2-4x+3)}{\ln(3x^5+2x^3+4x-1)}$

15 $\lim\limits_{x\to\infty} \frac{\ln(e^{2x}+2^x+x^9)}{\ln(e^{3x}+(\ln x)^3+x)}$

16 $\lim\limits_{x\to\infty} \frac{\ln(4x^2e^{3x}+x^3e^{2x})}{\ln(5x^5e^{2x}+3xe^{4x})}$

17 $\lim\limits_{x\to\infty} \frac{\ln(8x^3e^{3x}+4x^7 20^x)}{\ln(7x^5e^{2x}+2x8^x)}$

18 $\lim\limits_{x\to\infty} e^x\ln(1+e^{-x})$

19 $\lim\limits_{x\to 0} \frac{\ln(1+x)}{\ln(1+2x)}$

20 $\lim\limits_{x\to 2} \frac{\ln(x^2+2x-7)}{\ln(x^2-x-1)}$

21 $\lim\limits_{x\to\infty} \frac{(x+1)^{2x}}{(x^2+3x)^x}$

22 $\lim\limits_{x\to\infty} \frac{(x^2-5x)^x}{(x+3)^{2x}}$

23 $\lim\limits_{x\to\infty} \frac{(x^2+3x+2)^x}{(x^2+5x-6)^x}$

24 $\lim\limits_{x\to\infty} \frac{(x^3+4x^2+x\ln x)^{2x}}{(x^2+3x+5)^{3x}}$

25 Show that if $f(x)\sim g(x)$ as $x\to a$, and $h(x)$ is a non-zero function, then $f(x)h(x)\sim g(x)h(x)$ as $x\to a$.

26 If $f(x)\sim g(x)$ as $x\to a$, can we always say that $f(x)+h(x)\sim g(x)+h(x)$ as $x\to a$? Why or why not?

27 Show that if $f(x)\sim g(x)$ with $f(x)$ either approaches 0 or ∞ as $x\to a$, then $\ln(f(x))\sim\ln(g(x))$ as $x\to a$. You can assume that both $f(x)$ and $g(x)$ have a derivative.

Hint: In these two cases, $\frac{\ln(f(x))}{\ln(g(x))}$ is of the form $\frac{\infty}{\infty}$, so we can use L'Hôpital's rule.

1.3 Asymptotic Series

Knowing the asymptotic behavior of a function gives us an idea of what is happening with the function as $x\to a$. However, we can get a much sharper picture of the behavior with an *asymptotic series.*

DEFINITION 1.5 We say that

$$f(x) \sim g_1(x) + g_2(x) + g_3(x) + \cdots \text{ as } x \to a, \tag{1.17}$$

or

$$f(x) \sim \sum_{n=1}^{\infty} g_n(x) \text{ as } x \to a, \tag{1.18}$$

provided each term of the series is asymptotic as $x \to a$ to the function created by subtracting the previous terms of the series from $f(x)$. That is,

$$\lim_{x\to a} \frac{f(x)}{g_1(x)} = 1, \qquad \lim_{x\to a} \frac{f(x) - g_1(x)}{g_2(x)} = 1, \qquad \lim_{x\to a} \frac{f(x) - g_1(x) - g_2(x)}{g_3(x)} = 1,$$

$$\lim_{x\to a} \frac{f(x) - g_1(x) - g_2(x) - g_3(x)}{g_4(x)} = 1, \text{ etc.} \tag{1.19}$$

So an asymptotic series gives us an infinite number of asymptotic relations, each giving a sharper picture to the behavior of the function $f(x)$ as $x \to a$.

Example 1.8

Find an asymptotic series for $\cos(x)$ as $x \to 0$.

SOLUTION: Since $\cos(0) = 1$, the first order approximation is $\cos(x) \sim 1$ as $x \to 0$. If we peel away this approximation, what is the behavior of $\cos(x) - 1$ near 0? From the Taylor series,

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots,$$

we see that $\cos(x) - 1 \sim -x^2/2$, so this is the second term in the asymptotic series. In a sense, the asymptotic series keeps peeling away approximations from the function like an onion, except that onions don't have an infinite number of layers. It is clear that each time we subtract a term of the Taylor series, the next term of the series will describe the behavior. Thus,

$$\cos(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ as } x \to 0. \tag{1.20}$$

$\square$

In this case, the Taylor series is the same as the asymptotic series. In fact, if a non-truncating Taylor series centered at a converges to a function $f(x)$, then the asymptotic series of $f(x)$ as $x \to a$ will be the same as the Taylor series. But there are some important differences between Taylor series and asymptotic series.

First of all, each term in a Taylor series must be a polynomial, in particular, one of the form $c_n(x-a)^n$. Asymptotic series, on the other hand, have no such

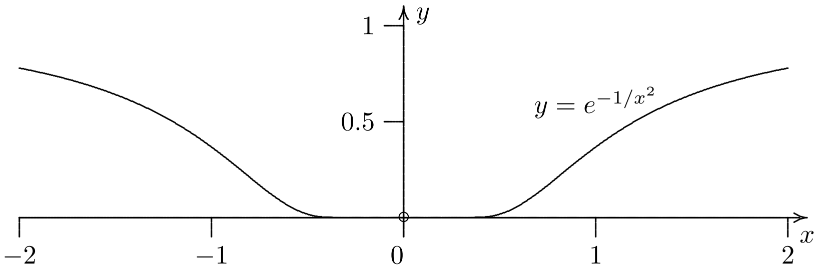

FIGURE 1.3: The graph of e^{-1/x^2}. This function approaches 0 as $x \to 0$ faster than any power of x, so this function will be *subdominant* to any Maclaurin series.

restriction. Often the terms involve fractional powers, exponential functions, or even logarithms. The only restriction is that $g_{n+1}(x) \ll g_n(x)$ as $x \to a$ for all terms in the series.

But more importantly, an asymptotic series is a *relative* property of a function, whereas a convergent Taylor series is an *absolute* property. In order to prove that a given series is the asymptotic series of $f(x)$, one must consider both $f(x)$ and the terms of the series. On the other hand, one can determine whether or not a Taylor series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges or not without knowing the function that it converges to. Hence, the convergence is an absolute property intrinsic to the coefficients c_n.

Let us clarify this distinction. Suppose we are given a series of functions with $g_{n+1}(x) \ll g_n(x)$, and ask what function is asymptotic to that series. The answer is that there are infinitely many functions that have that series as its asymptotic series! For example, $\cos(x) + e^{-1/x^2}$ has the same asymptotic series as $\cos(x)$.

$$\cos(x) + e^{-1/x^2} \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ as } x \to 0.$$

What is happening here? Note that because e^x grows faster than any polynomial as $x \to \infty$, $e^{-1/x^2} \ll x^n$ as $x \to 0$ for *all* n. Figure 1.3 shows the graph of this function. Hence e^{-1/x^2} will be smaller as $x \to 0$ than *all* of the terms of the cosine series, so no matter how many terms of the cosine series are subtracted from $\cos(x)$, the next largest factor will be the next term in the cosine series.

Any function which is smaller than all of the terms of an asymptotic series is said to be *subdominant* to the series. Because of subdominance, we cannot have a unique function associated with an asymptotic series.

Not every function has an asymptotic series as $x \to a$. Consider, for example, the hyperbolic function $\cosh(x)$ as $x \to \infty$. Since

$$\cosh(x) = \frac{e^x + e^{-x}}{2},$$

the first order approximation as $x \to \infty$ is $e^x/2$. If we subtract off this first term, we get $e^{-x}/2$, the second order term. But when this term is subtracted, we get 0, and by equation 1.4 this cannot be asymptotic to any function. Hence, $\cosh(x)$ does not have an asymptotic series as $x \to \infty$. So unlike Taylor series, an asymptotic series must contain an infinite number of non-zero terms.

Another important difference between Taylor series and asymptotic series is that Taylor series must *converge* if they are to be useful. For example, consider the power series

$$\sum_{n=0}^{\infty} (-1)^n n! x^n = 1 - x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + \cdots.$$

We can use the ratio test to see if this converges, that is, if $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$. But

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)! x^{n+1}}{n! x^n} = (n+1)x.$$

Unless $x = 0$, this will go to ∞ as $n \to \infty$, so the series converges only for $x = 0$. As a Taylor series, this is not very useful.

However, there *is* a function for which this is the asymptotic series! In chapter 2, we will determine that the function

$$S(x) = \int_0^{\infty} \frac{e^{-t}}{1 + xt}\, dt, \tag{1.21}$$

which is called the *Stieltjes integral function* has the asymptotic series

$$\int_0^{\infty} \frac{e^{-t}}{1 + xt}\, dt \sim \sum_{n=0}^{\infty} (-1)^n n! x^n \text{ as } x \to 0^+.$$

See example 2.7. The reason for the one sided limit is that $S(x)$ is undefined for negative x. (The integrand is undefined at the point $t = -1/x$.) In spite of the fact that the series diverges, the asymptotic series precisely describes the behavior of $S(x)$ near $x = 0$, namely, the ratios

$$\frac{S(x)}{1}, \qquad \frac{S(x) - 1}{-x}, \qquad \frac{S(x) - 1 + x}{2x^2}, \qquad \frac{S(x) - 1 + x - 2x^2}{-6x^3}, \qquad \text{etc.,}$$

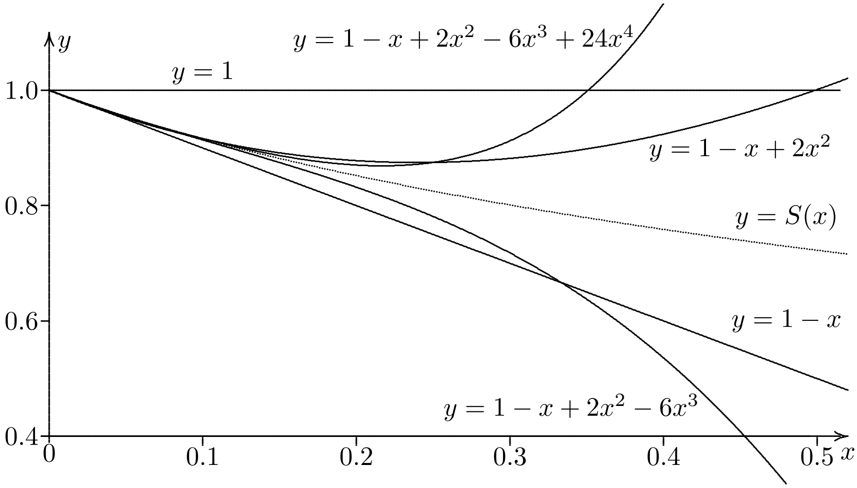

FIGURE 1.4: The graph shows the successive asymptotic approximations to the Stieltjes integral function $S(x)$. Note that the more terms of the series included, the better the approximation near $x = 0$, yet the approximation pulls away from $S(x)$ sooner.

all approach 1 as $x \to 0^+$. See figure 1.4 for a graphical illustration.

These limits illustrate the key difference between Taylor series and asymptotic series. For a Taylor series, if we pick a value x close to a, then the more terms we add gets us closer and closer to the function. For an asymptotic series, we first pick a number of terms, and we have an accurate approximation to the function, getting more accurate as x approaches a. Hence, asymptotic series can be used to compute complicated limits, regardless of whether the series converges or diverges.

For limits in which there is cancellation in the first order approximation, we can replace a function with not just a similar function, but with the first several terms of its asymptotic series.

Example 1.9
Find the limit

$$\lim_{x\to 0^+} \frac{S(x)}{x^2 - x^3} - \frac{1}{x^2}$$

where $S(x)$ is the Stieltjes integral function.

SOLUTION: If we consider just the first order approximation, we find that $S(x)/(x^2 - x^3) \sim 1/x^2$, causing cancellation to occur. The solution is to keep more terms of the asymptotic series. Although it is possible to find the asymptotic series for $S(x)/(x^2 - x^3)$ via long division, it is easier to rewrite the original limit in terms of a single fraction.

$$\frac{S(x)}{x^2 - x^3} - \frac{1}{x^2} = \frac{x^2 S(x) - x^2 + x^3}{x^4 - x^5}.$$

There is still cancellation in the numerator as $x \to 0$, but if we keep three terms of the asymptotic series for $S(x)$,

$$x^2 \left(1 - x + 2x^2 + O(x^3)\right) - x^2 + x^3 = 2x^4 + O(x^5).$$

Thus,

$$\frac{S(x)}{x^2 - x^3} - \frac{1}{x^2} = \frac{2x^4 + O(x^5)}{x^4 + O(x^5)} = 2 + O(x).$$

Thus, the limit as $x \to 0^+$ is 2. □

Unfortunately, it is impossible to know ahead of time how many terms of the asymptotic series must be used to compute the limit. The only advise is to try a reasonable number of terms, and if all of these cancel out, try again with more terms.

Example 1.10

Find the limit

$$\lim_{x \to 0} \frac{\sin(x)\sin^{-1}(x) - \sinh x \sinh^{-1}(x)}{x^2(\cos(x) - \cosh(x) + \sec(x) - \operatorname{sech}(x))}.$$

SOLUTION: We will need to use equation 1.20, along with the following:

$$\sin(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \sim x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \text{ as } x \to 0. \tag{1.22}$$

$$\sinh(x) \sim \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \sim x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots \text{ as } x \to 0. \tag{1.23}$$

$$\sin^{-1}(x) \sim \sum_{n=0}^{\infty} \frac{(2n)! x^{2n+1}}{2^{2n}(n!)^2(2n+1)} \sim x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots \text{ as } x \to 0. \tag{1.24}$$

$$\sinh^{-1}(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! x^{2n+1}}{2^{2n}(n!)^2(2n+1)} \sim x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + \cdots \text{ as } x \to 0. \tag{1.25}$$

$$\cosh(x) \sim \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \sim 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots \text{ as } x \to 0. \tag{1.26}$$

$$\sec(x) \sim 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{1385x^8}{8!} + \frac{50521x^{10}}{10!} + \cdots \text{ as } x \to 0. \tag{1.27}$$

$$\operatorname{sech}(x) \sim 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \frac{1385x^8}{8!} - \frac{50521x^{10}}{10!} + \cdots \text{ as } x \to 0. \quad (1.28)$$

Since the denominator does not involve any products, let us begin there, keeping three terms in each of the series.

$$\begin{aligned} x^2\Bigg[&\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right) - \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right) \\ &+ \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6)\right) - \left(1 - \frac{x^2}{2} + \frac{5x^4}{24} + O(x^6)\right)\Bigg] \\ &= O(x^8). \end{aligned}$$

We discover that all terms canceled, so we will need to keep more terms.

$$\begin{aligned} x^2\Bigg[&\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)\right) - \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + O(x^8)\right) \\ &+ \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + O(x^8)\right) - \left(1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + O(x^8)\right)\Bigg] \\ &= \frac{x^8}{6} + O(x^{10}). \end{aligned}$$

Now we can proceed with the numerator, using the distributive law to perform the product of two series. By doing the denominator first, we learn that we must keep terms of order x^8. The numerator of the limit can now be computed to be

$$\begin{aligned} &\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + O(x^9)\right)\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + O(x^9)\right) \\ &\quad - \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + O(x^9)\right)\left(x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112} + +O(x^9)\right) \\ &\quad = \left(x^2 + \frac{x^6}{18} + \frac{x^8}{30} + O(x^{10})\right) - \left(x^2 + \frac{x^6}{18} - \frac{x^8}{30} + O(x^{10})\right) \\ &= \frac{x^8}{15} + O(x^{10}). \end{aligned}$$

Putting the pieces together, we get

$$\frac{\sin(x)\sin^{-1}(x) - \sinh x \sinh^{-1}(x)}{x^2(\cos(x) - \cosh(x) + \sec(x) - \operatorname{sech}(x))} \sim \frac{x^8/15 + O(x^{10})}{x^8/6 + O(x^{10})} = \frac{2}{5} + O(x^2). \qquad \square$$

This example reviewed many of the classical Maclaurin series, but there are a few more that may be used in future exercises, so let us cover these here. All of these series are for $x \to 0$.

Exponential series, converges for all x:

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad (1.29)$$

Logarithmic series, converges for $|x| < 1$:

$$\ln(1+x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1.30}$$

Binomial series, converges for $|x| < 1$:

$$\begin{aligned}(1+x)^k \sim 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \\ + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots\end{aligned} \tag{1.31}$$

$$\frac{1}{\sqrt{1-4x}} \sim \sum_{n=0}^{\infty} \frac{(2n)!x^n}{(n!)^2} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots \tag{1.32}$$

Inverse tangent series:

$$\tan^{-1} x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{1.33}$$

Inverse hyperbolic tangent series:

$$\tanh^{-1} x \sim \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \tag{1.34}$$

Tangent series:

$$\tan(x) \sim x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + \cdots \tag{1.35}$$

Hyperbolic tangent series:

$$\tanh(x) \sim x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \frac{1382x^{11}}{155925} + \cdots \tag{1.36}$$

Some limits require plugging one asymptotic series into another asymptotic series.

Example 1.11
Find the limit

$$\lim_{x\to\infty} \sqrt{x^4 + 4x^3 + 7x^2} - x^2 - 2x.$$

SOLUTION: Again, a first order approximation yields $\sqrt{x^4} - x^2 = 0$, which cannot be the asymptotic approximation of this function. So we must use an asymptotic series.

The asymptotic series for the square root function is found by plugging $k = 1/2$ into the binomial series.

$$\sqrt{1+\epsilon} \sim 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{5}{128}\epsilon^4 + \cdots \text{ as } \epsilon \to 0. \tag{1.37}$$

To utilize this series, we rewrite $\sqrt{x^4 + 4x^3 + 7x^2}$ as $x^2\sqrt{1 + 4/x + 7/x^2}$, and as $x \to \infty$, $4/x + 7/x^2 \to 0$. So we replace ϵ with $4/x + 7/x^2$ in the series.

$$\sqrt{1 + \frac{4}{x} + \frac{7}{x^2}} = 1 + \frac{1}{2}\left(\frac{4}{x} + \frac{7}{x^2}\right) - \frac{1}{8}\left(\frac{4}{x} + \frac{7}{x^2}\right)^2 + \frac{1}{16}\left(\frac{4}{x} + \frac{7}{x^2}\right)^3 + O(1/x^4).$$

In expanding, we only have to keep terms up to order $1/x^3$. Thus,

$$\sqrt{1 + \frac{4}{x} + \frac{7}{x^2}} = 1 + \frac{2}{x} + \frac{3}{2x^2} - \frac{3}{x^3} + O(1/x^4).$$

Thus,

$$x^2\sqrt{1 + \frac{4}{x} + \frac{7}{x^2}} - x^2 - 2x = \frac{3}{2} - \frac{3}{x} + O(1/x^2).$$

So the limit is $3/2$. $\square$

Problems for §1.3

For problems **1** through **14**: By replacing functions with a few terms of their asymptotic series, find the following limits.

1 $\displaystyle\lim_{x\to 0} \frac{e^x - \sqrt{2x+1}}{x^2}$

2 $\displaystyle\lim_{x\to 0} \frac{\cos(x) - \sqrt{1-x^2}}{x^4}$

3 $\displaystyle\lim_{x\to \infty} \sqrt{x^2 + x} - x$

4 $\displaystyle\lim_{x\to \infty} \sqrt{2x^2 + 3x} - \sqrt{2x^2 + x}$

5 $\displaystyle\lim_{x\to 0} \frac{e^x}{x^2 + x^3} - \frac{1}{x^2}$

6 $\displaystyle\lim_{x\to 0} \frac{\cosh(x)}{2x^4 + x^6} - \frac{1}{2x^4}$

7 $\displaystyle\lim_{x\to 0} \frac{e^x \ln(x+1) + \ln(1-x)}{\cos(x)\cosh(x) - 1}$

8 $\displaystyle\lim_{x\to 0} \frac{\tan(x) - \sin(x)\cosh(x)}{x^5}$

9 $\displaystyle\lim_{x\to 0} \frac{\sin(x)\sin^{-1}(x) - x^2}{\tan(x)\tan^{-1}(x) - x^2}$

10 $\displaystyle\lim_{x\to \infty} \sqrt{x^4 + 2x^3} - x^2 - x$

11 $\displaystyle\lim_{x\to \infty} \sqrt[3]{x^3 + 2x^2} - x$

12 $\displaystyle\lim_{x\to 0} \frac{1}{\sin^2 x} - \frac{1}{\sinh^2 x}$

13 $\displaystyle\lim_{x\to 0} \frac{1}{\tanh^2 x} - \frac{1}{\tan^2 x}$

14 $\displaystyle\lim_{x\to 0} \frac{e^{-x^2}\cosh(x) - \cos(x)}{\sin(x)\sinh(x) - x^2}$

15 Find the limit

$$\lim_{x\to 0} \frac{S(x)e^x - 1}{x^2},$$

where $S(x)$ is the Stieltjes integral function, defined by equation 1.21.

16 Note that in figure 1.4, the curves $y = 1$ and $y = 1 - x + 2x^2$ cross at $x = 1/2$, the curves $y = 1-x$ and $y = 1-x+2x^2-6x^3$ cross at $x = 1/3$, and the curves $y = 1 - x + 2x^2$ and $1 - x + 2x^2 - 6x^3 + 24x^4$ cross at $x = 1/4$. Show that the pattern continues. That is, show that the n^{th} degree polynomial approximation and the $(n-2)^{\text{nd}}$degree polynomial approximation cross at $x = 1/n$.

For problems **17** through **28**: Find the first three (non-zero) terms of the asymptotic series as $x \to a$ for the following functions.

17 $\ln(x^2 + x^3)$ as $x \to \infty$

18 $\ln(e^x + 1)$ as $x \to \infty$

19 $\sqrt{x^4 + 2x^3 + 4x^2}$ as $x \to \infty$

20 $\sqrt{x^4 + 2x^3 + 4x^2}$ as $x \to 0$

21 $\sin(x)\sin^{-1}(x)$ as $x \to 0$

22 $\sin(x + x^3)$ as $x \to 0$

23 $e^{(x+x^2)}$ as $x \to 0$

24 $\ln(\sin(x))$ as $x \to 0$

25 $\sqrt[3]{x^3 + 4x^2 + 3x}$ as $x \to \infty$

26 $1/\ln(1 + x)$ as $x \to 0$

27 $\csc(x)$ as $x \to 0$

28 $\sinh(x)/\sin(x)$ as $x \to 0$

1.4 Inverse Functions

Although we have used asymptotics to calculate limits, we still have not applied this tool for what it is mainly designed to do: approximate functions that cannot be calculated any other way. One application of asymptotics comes from finding inverses of tricky functions. Recall that an inverse of a function $f(x)$ is the function $f^{-1}(x)$ such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$, at least for part of the domain. For a function like $f(x) = x^3 + x$, it is difficult to find a formula for $f^{-1}(x)$. Yet we can find the asymptotic series for $f^{-1}(x)$ as $x \to \infty$.

Example 1.12

Find the first three terms for the inverse of the function $f(x) = x^3 + x$ as $x \to \infty$.

SOLUTION: Since $x^3 + x \sim x^3$ as $x \to \infty$, it is natural to assume that the inverse function will be similar to $\sqrt[3]{x}$ as $x \to \infty$. But what will be the next term in the series? The plan is to peel off this first term, writing $f^{-1}(x) = \sqrt[3]{x} + g(x)$, and find the asymptotic approximation for $g(x)$. Since we know that $f(f^{-1}(x)) = x$, we have

$$(\sqrt[3]{x} + g(x))^3 + \sqrt[3]{x} + g(x) = x.$$

We can expand this out asymptotically, utilizing the fact that $g(x) \ll \sqrt[3]{x}$. As a general rule, we do not need to keep terms that involve the square (or higher power) of the unknown function.

$$x + 3g(x)\sqrt[3]{x^2} + O\left(g(x)^2\sqrt[3]{x}\right) + \sqrt[3]{x} + g(x) = x.$$

The x's cancel out, and $g(x)$ is small compared to $g(x)\sqrt[3]{x^2}$ as $x \to \infty$. By throwing out terms that are known to be smaller than a *non-canceling* term as $x \to \infty$, we get

$$3g(x)\sqrt[3]{x^2} \sim -\sqrt[3]{x},$$

which tells us that $g(x) \sim -x^{-1/3}/3$. So we now have two terms of the asymptotic series:

$$f^{-1}(x) \sim \sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}} \text{ as } x \to \infty.$$

To find the next term in the series, we repeat the process, assuming that $f^{-1}(x) = \sqrt[3]{x} - x^{-1/3}/3 + h(x)$. Since $f(f^{-1}(x)) = x$,

$$\left(\sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}} + h(x)\right)^3 + \sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}} + h(x) = x.$$

Cubing a trinomial is a bit tricky, but any term involving $h(x)^2$ will almost certainly be small. So we can first rewrite this as

$$\left(\sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}}\right)^3 + 3\left(\sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}}\right)^2 h(x) + O\left(h(x)^2\sqrt[3]{x}\right) + \sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}} + h(x) = x.$$

It is clear that the largest term involving $h(x)$ is $3\sqrt[3]{x^2}h(x)$, which does not cancel with any other terms. But we will have to expand the cube, to give us

$$\left(x - \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x}} - \frac{1}{27x}\right) + 3\sqrt[3]{x^2}h(x) + \sqrt[3]{x} - \frac{1}{3\sqrt[3]{x}} \sim x.$$

Since the x's canceled before, we expect them to cancel again. But this time, the $\sqrt[3]{x}$ and $x^{-1/3}/3$ also cancel, giving us

$$3\sqrt[3]{x^2}h(x) \sim \frac{1}{27x}.$$

So $h(x) \sim x^{-5/3}/81$. Thus, we have

$$f^{-1}(x) \sim x^{1/3} - \frac{1}{3\sqrt[3]{x}} + \frac{1}{81\sqrt[3]{x^5}} \text{ as } x \to \infty.$$

Figure 1.5 shows how each successive term gives a better approximation to the inverse function. Although the approximations are designed to be excellent for large values of x, even at $x = 1$ there is only a 0.5% error. □

Let us recap the steps that were used in this last example, since the same steps will be used for a variety of different types of problems.

1) Determine the first term of the asymptotic series. Many times, this can be done via simple approximations, but for more complicated problems

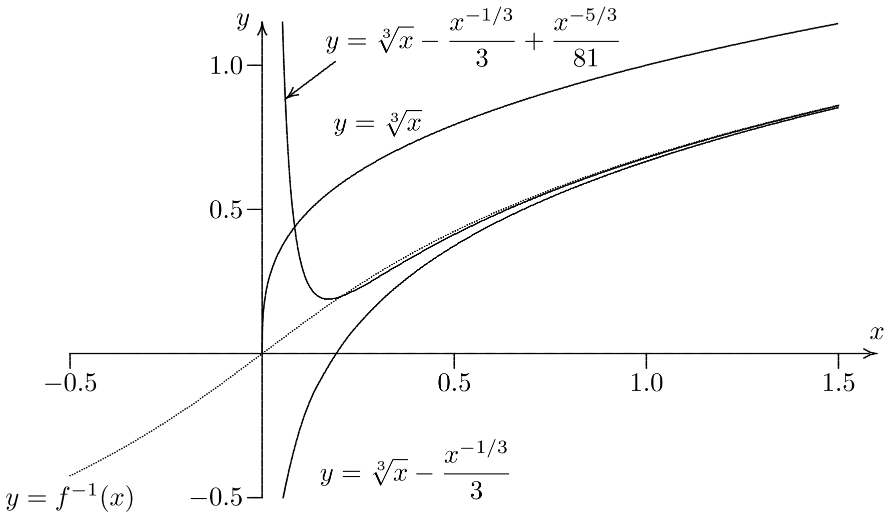

FIGURE 1.5: Comparing the inverse function of $x + x^3$ with the first three asymptotic approximations as $x \to \infty$. Note that the final approximation is indistinguishable from the inverse function for $x > 1$.

it may require more sophisticated methods such as a trial and error method called dominant balance. This first term is called the *leading behavior* of the solution.

2) Add an unknown function to the series so far. We can assume that this function is smaller than the previous term, which will help in later steps.

3) Plug this series into the equation that the function must satisfy.

4) Carefully expand this equation asymptotically, canceling terms whenever possible. Note that usually terms involving the unknown function squared need not be considered.

5) After the terms have canceled, we want to throw out any terms that are asymptotically smaller than a term that did not cancel.

6) The remaining terms should give an equation for the unknown function that is now easy to solve. This gives us the next term in the series.

7) Repeat steps 2-6 to get more terms in the series.

One can see that each term in the series is progressively harder to obtain. Nonetheless, it usually only takes a few terms of an asymptotic series to achieve incredible accuracy in the approximation.

Example 1.13

Analyze the inverse of the function $f(x) = xe^x$ as $x \to \infty$.

SOLUTION: Although there is only one term in $f(x)$, it is clear that the dominant factor that controls the behavior is the exponential function, so it is natural to assume that the inverse function will behave like a logarithm. So let us try $f^{-1}(x) \sim \ln(x)$. If we successfully find the next order term, this will verify that our guess is correct.

If we set $f^{-1}(x) = \ln(x) + g(x)$, then since $f^{-1}(f(x)) = x$, we have

$$\ln(xe^x) + g(xe^x) = x.$$

Using the properties of logarithms, we can simplify this to the exact equation $\ln(x)+g(xe^x) = 0$. Again, since the dominant factor of xe^x is the exponential, we get $g(e^x) \sim -\ln(x)$, so $g(x) \sim -\ln(\ln(x))$. Indeed, this is smaller than $\ln(x)$, so we have confirmed that the first term was correct.

Since we have an exact equation for $g(x)$, we can use this as a shortcut for finding the next term. We can let $g(x) = -\ln(\ln(x)) + h(x)$, to produce the equation

$$\ln(x) - \ln(\ln(xe^x)) + h(xe^x) = 0.$$

Now

$$\ln(\ln(xe^x)) = \ln(x + \ln(x)) = \ln\left(x\left(1 + \frac{\ln(x)}{x}\right)\right) = \ln(x) + \ln\left(1 + \frac{\ln(x)}{x}\right),$$

so the $\ln(x)$ cancels to produce

$$h(xe^x) = \ln\left(1 + \frac{\ln(x)}{x}\right). \tag{1.38}$$

Since $\ln(x)/x \ll 1$ and $\ln(1+\epsilon) \sim \epsilon$, we see that $h(xe^x) \sim \ln(x)/x$, or $h(x) \sim \ln(\ln(x))/\ln(x)$.

This is proceeding well enough to brave yet another term. Substituting $h(x) = \ln(\ln(x))/\ln(x) + k(x)$ into equation 1.38 and expanding the inner logarithms produces

$$\frac{\ln(x + \ln(x))}{x + \ln(x)} + k(xe^x) = \ln\left(1 + \frac{\ln(x)}{x}\right). \tag{1.39}$$

Expanding the right hand side asymptotically is easy using equation 1.12, but the first term has to first be rewritten as

$$\frac{\ln(x + \ln(x))}{x + \ln(x)} = \frac{1}{x}\left(\ln(x) + \ln\left(1 + \frac{\ln(x)}{x}\right)\right)\left(1 + \frac{\ln(x)}{x}\right)^{-1}.$$

Then equation 1.39 can be approximated asymptotically to give us

$$\begin{aligned}&\frac{1}{x}\left(\ln(x) + \frac{\ln(x)}{x} + O\left(\frac{\ln(x)^2}{x^2}\right)\right)\left(1 - \frac{\ln(x)}{x} + O\left(\frac{\ln(x)^2}{x^2}\right)\right) + k(xe^x)\\ &= \frac{\ln(x)}{x} - \frac{\ln(x)^2}{2x^2} + O\left(\frac{\ln(x)^3}{x^3}\right).\end{aligned}$$

Expanding the product, we find that the $\ln(x)/x$ terms will cancel, and we have

$$k(xe^x) = \frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{x^2} + O\left(\frac{\ln(x)^3}{x^3}\right).$$

Thus, we get the first two terms for $k(x)$,

$$\ln(\ln(x))^2/2(\ln(x))^2 - \ln(\ln(x))/(\ln(x))^2.$$

Putting all of the terms together, we get

$$f^{-1}(x) \sim \ln(x) - \ln(\ln(x)) + \frac{\ln(\ln(x))}{\ln(x)} + \frac{\ln(\ln(x))^2}{2\ln(x)^2} - \frac{\ln(\ln(x))}{\ln(x)^2}. \tag{1.40}$$

$\square$

The inverse function of $y = xe^x$ is called the Lambert W function. Its asymptotic series as $x \to \infty$ is more complicated than the ones we have seen before, since the logarithm terms complicate matters a bit. In fact, later terms will all have a power of $\ln(\ln(x))$ in the numerator, and a power of $\ln(x)$ in the denominator. Yet the simple function $1/x$ goes to zero faster than all of these terms. So how well does the asymptotic series approximate $W(x)$? Figure 1.6 shows that in spite of the fact that the individual terms to go 0 very slowly, the first few terms give a fairly accurate approximation to the function. There is less than 0.4% error for $x > 10$.

1.4.1 Reversion of Series

For a function in which $f(x) \sim cx$ as $x \to 0$, it is fairly clear that the inverse function will have $f^{-1}(x) \sim x/c$ as $x \to 0$. In fact, if we have a power series expansion for such a function, we can compute the power series expansion for its inverse.

Often the powers of x will appear in an arithmetic progression. For example, an odd function will have only odd powers of x in its Maclaurin series. In general, suppose that

$$f(x) \sim a_0 x + a_1 x^{b+1} + a_2 x^{2b+1} + \cdots = \sum_{n=0}^{\infty} a_n x^{bn+1}.$$

Then the inverse of $y = f(x)$ is given by

$$f^{-1}(y) = A_0 y + A_1 y^{b+1} + A_2 y^{2b+1} + \cdots = \sum_{n=0}^{\infty} A_n y^{bn+1},$$

where

$$A_0 = \frac{1}{a_0}, \quad A_1 = \frac{-a_1}{a_0^{b+2}}, \quad A_2 = \frac{1}{a_0^{2b+3}}((b+1)a_1^2 - a_0 a_2),$$

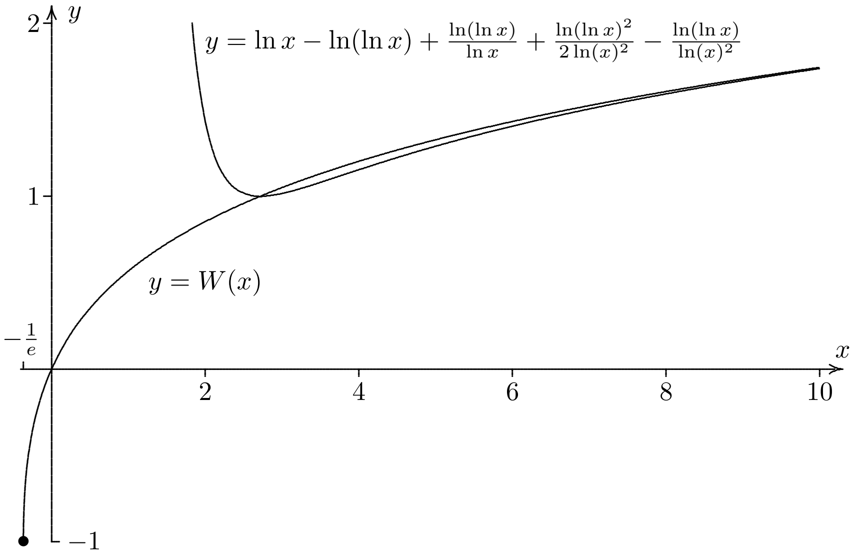

FIGURE 1.6: Comparing the Lambert W function (the inverse of $y = xe^x$) with the asymptotic approximation as $x \to \infty$. $W(x)$ is defined for $x \geq -1/e$, but the approximation is fairly good for $x \geq e$.

$$A_3 = \frac{1}{a_0^{3b+4}} \left(-\frac{(3b+3)(3b+2)}{6} a_1^3 + (3b+2)a_0 a_1 a_2 - a_0^2 a_3 \right), \tag{1.41}$$

$$A_4 = \frac{1}{a_0^{4b+5}} \left(\frac{(4b+4)(4b+3)(4b+2)}{24} a_1^4 - \frac{(4b+3)(4b+2)}{2} a_0 a_1^2 a_2 \right.$$
$$\left. + \frac{(4b+2)}{2} a_0^2 a_2^2 + (4b+2) a_0^2 a_1 a_3 - a_0^3 a_4 \right),$$

$$A_5 = \frac{1}{a_0^{5b+6}} \left(-\frac{(5b+5)(5b+4)(5b+3)(5b+2)}{120} a_1^5 \right.$$
$$+ \frac{(5b+4)(5b+3)(5b+2)}{6} a_0 a_1^3 a_2 - \frac{(5b+3)(5b+2)}{2} (a_0^2 a_1 a_2^2 + a_0^2 a_1^2 a_3)$$
$$\left. + (5b+2)(a_0^3 a_2 a_3 + a_0^3 a_1 a_4) - a_0^4 a_5 \right).$$

For most functions, we use $b = 1$, but occasionally we may want to take the inverse of an odd function, in which case we can use $b = 2$. The pattern for these coefficients can be given by

$$\boxed{A_n = \frac{1}{a_0^{bn+n+1}} \sum (-1)^{n+p_0} \frac{(bn+n-p_0)!}{(bn+1)!p_1!p_2!\cdots p_n!} a_0^{p_0} a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},} \tag{1.42}$$

where the sum is taken over all combinations of non-negative integers p_0, p_1, $\ldots p_n$ such that both $p_0 + p_1 + p_2 + \cdots p_n = n$, and $p_1 + 2p_2 + \cdots np_n = n$. See problem 19. If the original series has a non-zero radius of convergence, then the series for the inverse function will also have a non-zero radius of convergence, but it may be difficult to determine exactly what that radius is.

Example 1.14

We have already seen Lambert's W function, which is the inverse of $y = xe^x$. Although we have seen the asymptotic series for when $x \to \infty$, what is the behavior as $x \to 0$?

SOLUTION: We have

$$xe^x \sim x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots \quad \text{as } x \to 0.$$

Thus, we can plug in $b = 1$ and $a_n = 1/n!$ into equations 1.41 to get $A_0 = 1$, $A_1 = -1$, $A_2 = 3/2$, $A_3 = -8/3$, $A_4 = 125/24$, and $A_5 = -54/5$. Thus,

$$W(x) \sim x - x^2 + \frac{3x^3}{2} - \frac{8x^4}{3} + \frac{125x^5}{24} - \frac{54x^6}{5} + \cdots \quad \text{as } x \to 0.$$

In fact, there turns out to be a nice pattern:

$$W(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-1} x^n}{n!} \quad \text{as } x \to 0.$$

□

Example 1.15

Find the Maclaurin series for $\tan(x)$.

SOLUTION: We can use the fact that the series for its inverse has a simple pattern.

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Because this is an odd function, we can use $b = 2$, and replace

$$a_n = \frac{(-1)^n}{(2n+1)}$$

into equations 1.41 to get $A_0 = 1$, $A_1 = 1/3$, $A_2 = 2/15$, $A_3 = 17/315$, $A_4 = 62/2835$, and $A_5 = 1382/155925$. This gives use the first 6 terms of the tangent series:

$$\tan(x) \sim x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + \cdots \quad \text{as } x \to 0.$$

□

Problems for §1.4

For problems **1** through **12**: Find the first three (non-zero) terms of the asymptotic series as $x \to \infty$ for the inverse of the following functions.

1 $x^2 + 5x + 3$
2 $x^3 + 2x$
3 $x^3 + x + 1$
4 $x^3 + x^2$
5 $x^4 + x$
6 $x^4 + x^3$
7 $x^2 e^x$
8 $x + \ln(x)$
9 $e^x + \ln(x)$
10 $x \ln(x)$
11 $e^x + x$
12 $e^x \ln(x)$

For problems **13** through **18**: Many inverse functions can be expressed in terms of the Lambert W function. Express the inverse of the following functions in terms of $W(x)$.

13 $x + \ln(x)$
14 $x \ln(x)$
15 $x + e^x$
16 $\sqrt{x} e^x$
17 $x^2 + \ln(x)$
18 $x^2 \ln(x)$

19 Use equation 1.42 to find the formula for A_6 when $b = 1$

Hint: A *partition of* n is a set of positive integers that add up to n. For example, $1 + 1 + 1 + 3$ is a partition of 6. Rearrangement of the integers are not considered as different partitions. For example, there are seven partitions of 5: $1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 2$, $1 + 2 + 2$, $1 + 1 + 3$, $2 + 3$, $1 + 4$, and 5. Each partition of n can be padded with 0's to produce a set of n integers adding to n. For example, $1 + 1 + 1 + 3$ becomes $0 + 0 + 1 + 1 + 1 + 3$. Every partition of n becomes one term in equation 1.42. For each partition padded with 0's, we let p_i be the number of i's in the partition. For example, the partition $0 + 0 + 1 + 1 + 1 + 3$ produces $p_0 = 2$, $p_1 = 3$, $p_3 = 1$, and $p_2 = p_4 = p_5 = p_6 = 0$.

20 Find all of the partitions of 7. See problem 19 for an explanation of the partitions of n.

For problems **21** through **26**: Use equations 1.41 to find the first four non-zero terms in the Maclaurin series for the inverse of the function.

21 $x \cos x$
22 $x \cosh x$
23 $x + \ln(x + 1)$
24 $1/\sqrt{1 - 4x} - 1 - x$
25 $x + x^4 + x^7$
26 $x e^{x^3}$

27 Use equation 1.42 to find a formula for the n^{th} term in the Maclaurin series of the inverse of $f(x) = x + x^3$.

Hint: Using $b = 2$, the only partition of n that produces a non-zero term is the one where $p_1 = n$. See problem 19 for an explanation of the partitions of n.

28 The equations 1.41 can be used for a divergent series as well, producing an asymptotic series for the inverse function. Find the first six (non-zero) terms for the inverse of the function $S(x) - 1$, where $S(x)$ is the Stieltjes integral function defined by equation 1.21. Show that this series is divergent for all positive x.

1.5 Dominant Balance

In finding the asymptotic series for the solution of an equation, we must first determine its leading behavior. Sometimes this is easy, but usually this is a non-trivial problem. In these situations, a strategy that is very effective is the method of dominant balance.

The principle behind the dominant balance is quite simple. If there are three or more terms in an equation, usually two of the terms will be asymptotically larger than the others, that is, they will dominate the other terms. Also, these terms will balance each other, so we can form an asymptotic equation with only two terms. Such equations are usually very easy to solve.

The problem, of course, is determining *which* two terms are the ones that are dominant. This can only be determined by trial and error. In each case, we can test to see if the other terms are indeed small compared to the ones that we assumed were dominant.

Example 1.16

Find the behavior of the function defined implicitly by $x^2 + xy - y^3 = 0$ as $x \to \infty$.

SOLUTION: Since there are three non-zero terms, there are three choices for a pair of terms. If we assume y^3 is small as $x \to \infty$, then we have $x^2 \sim -xy$. This quickly leads to $y \sim -x$ as $x \to \infty$. But then $y^3 \sim -x^3$, which is *not* small compared to a term that we kept, x^2, as $x \to \infty$.

Suppose instead that we assume $x^2 \ll xy$. Then $xy \sim y^3$, producing $y \sim \pm\sqrt{x}$. But then $xy \sim \pm x^{3/2}$, so x^2 is larger as $x \to \infty$. So this possibility is ruled out.

The final case to try is to assume that xy is the smallest term. Then $x^2 \sim y^3$, which tells us that $y \sim x^{2/3}$. To check to see if this is consistent, we need to check that $xy \ll x^2$. Indeed, $xy \sim x^{5/3}$ which is smaller than x^2 as $x \to \infty$.

At this point, we have shown that $y \sim x^{2/3}$ is *consistent*, but this alone is not proof that the leading behavior is indeed $x^{2/3}$. In order to show that this is the correct leading behavior, we must find the next term in the series, and show that it is smaller than $x^{2/3}$.