ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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AN A K PETERS BOOK

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
AN A K PETERS BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper
Version Date: 20130208

International Standard Book Number-13: 978-1-4665-1500-0 (Hardback)

Library of Congress Cataloging-in-Publication Data

Henner, Victor.
Ordinary and partial differential equations / Victor Henner, Tatyana Belozerova, Mikhail Khenner.
pages cm
"An A K Peters book."
Includes bibliographical references and index.
ISBN 978-1-4665-1500-0 (hardcover : acid-free paper)
1. Differential equations--Textbooks. 2. Differential equations, Partial--Textbooks. I. Belozerova, Tatyana. II. Khenner, Mikhail. III. Title.

QA371.H46 2013
515'.352--dc23 2012038568

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Preface

This book presents ordinary differential equations (ODE) and partial differential equations (PDE) under one cover. All topics that form the core of a modern undergraduate and the beginner's graduate course in differential equations are presented at full length. We especially strived for clarity of presenting concepts and the simplicity and transparency of the text. At the same time, we tried to keep all rigor of mathematics (but without the emphasis on proofs—some simpler theorems are proved, but for more sophisticated ones we discuss only the key steps of the proofs). In our best judgment, a balanced presentation has been achieved, which is as informative as possible at this level, and introduces and practices all necessary problem-solving skills, yet is concise and friendly to a reader. A part of the philosophy of the book is "teaching-by-examples" and thus we provide numerous carefully chosen examples that guide step-by-step learning of concepts and techniques. The level of presentation and the book structure allows its use in engineering, physics, and mathematics departments.

The primary motivation for writing this textbook is that, to our knowledge, there has not been published a comprehensive textbook that covers both ODE and PDE. A professor who teaches ODE using this book can use the PDE sections to complement the main ODE course. Professors teaching PDE very often face the situation when students, despite having an ODE prerequisite, do not remember the techniques for solving ODE and thus can't do well in the PDE course. A professor can choose the key ODE sections, quickly review them in the course of, say, three or four lectures, and then seamlessly turn to the main subject, i.e., PDE.

The ODE part of the book contains topics that can be omitted (fully or partially) from the basic undergraduate course, such as the integral equations, the Laplace transforms, and the boundary value problems. For the undergraduate PDE course the most technical sections (for instance, where the nonhomogeneous boundary conditions are discussed) can be omitted from lectures and instead studied with the accompanying software. At least Sections 1 through 7 from Chapter 8, Fourier Series, should be covered prior to teaching PDEs. For class time savings, a few of these sections can be studied using the software.

The software is a very special component of the textbook. Our software covers both fields. The ODE part of the software is fairly straightforward—i.e., the software allows readers to compare their analytical solution and the results of a computation. For PDE the software also demonstrates the sequence of all the steps needed to solve the problem. Thus it leads a user in the process of solving the problem, rather than informs of the result of solving the problem. This feature is completely or partially absent from all software that we have seen and tested. After the software solution of the problem, a deeper investigation is offered, such as the study of the dependence of the solution on the parameters, the accuracy of the solution, the speed of a series convergence, and related questions. Thus the software is a platform for learning and investigating all textbook topics, an inherent part of the learning experience rather than an interesting auxiliary. The software enables lectures, recitations, and homeworks to be research-like, i.e., to be naturally investigative, thus hopefully increasing the student rate of success. It allows students to study a *limitless* number of problems (a known drawback of a typical PDE course is that, due to time constraints, students are limited to practicing solutions of a small number of simple problems using the "pen and paper" method).

The software is very intuitive and simple to use, and it does not require students to learn a (quasi)programming language as do the computer algebra systems (CAS), such as Mathematica and Maple. Most CAS require a significant time investment in learning commands, conventions, and other features, and often the undergraduate students are very reluctant, especially if they have reasons to think that they will not use CAS in the future; furthermore, the instructors are often not willing to spend valuable classroom time teaching the basics of using CAS. Besides, where using CAS to solve an ODE is a matter of typing in one command (*dsolve* in Maple or *DSolve* in Mathematica), which students usually can learn how to do with a minor effort, solving a PDE in CAS is more complicated and puts the burden on the instructor to create a worksheet with several commands, often as many as ten or fifteen, where students just change the arguments to enable the solution of an assigned problem. Creation of a collection of such worksheets that covers all sections of the textbook is only possible when the instructor teaches the course multiple times.

The software and tutorials contain a few topics, such as the classical orthogonal polynomials, generalized Legendre functions, and others, which are not included in the book to avoid its overload with content that is presently rarely taught in PDE courses (at least in the U.S. academic system). These topics with the help of the software can be assigned for an independent study, essay, etc.

The software tutorials for different chapters are placed in the appendices.

Finally, we would like to suggest the book *Mathematical Methods in Physics* [1] as a more complete and advanced PDE textbook. That book is written in the same style and uses the previous version of the software.

ACKNOWLEDGMENTS

We wish to thank Prof. Harley Flanders (University of Michigan) for advice and support.

We are also very grateful to Prof. Kyle Forinash (Indiana University Southeast) for permission to use in the current book several large fragments of our jointly written book *Mathematical Methods in Physics* [1].

I

Ordinary Differential Equations, Boundary Value Problems, Fourier Series, and the Introduction to Integral Equations

CHAPTER 1

First-Order Differential Equations

1.1 GENERAL CONSIDERATIONS

A *differential equation (DE)* is an equation that contains derivatives of an unknown function of one or more variables. When an unknown function depends on a single variable, an equation involves ordinary derivatives and is called an *ordinary differential equation (ODE).* When an unknown function depends on two or more variables, partial derivatives of an unknown function emerge and an equation is called a *partial differential equation (PDE*). In Part I of the book we study ODEs.

The *order* of a DE is the order of a highest derivative in the equation.

A *solution* of a differential equation is the function that when substituted for the unknown function in this equation makes it the identity. Usually a solution of a differential equation is sought in some domain D of an independent variable and an unknown function. The process of the solution ultimately boils down to integration, and therefore the solution is often called an *integral* of the DE.

For example, $y'(x) = e^x + x^2$ is the first-order ODE. To find the unknown function $y(x)$ we use *Leibnitz notation* $y'(x) = dy/dx$, write the equation as $dy = (e^x + x^2)dx$, and then integrate both sides. This gives $y(x) = e^x + x^3/3 + C$, where C is an arbitrary integration constant.

A solution that contains an arbitrary integration constant is called the *general solution* (or the *general integral*) of a DE. Such solution can be denoted as $y(x,C)$. For different C values the function $y = y(x,C)$ gives different curves in the (x,y)-plane. These curves are called the *integral curves*. All these curves are described by the same equation. In order to single out some particular curve (one that corresponds to a certain C value), a value of $y(x)$ can be specified at some $x = x_0$—which gives a point $P(x_0, y(x_0))$. For instance, let in the above example $y(0) = 1$. Setting $x_0 = 0$, $y_0 = 1$ in the solution $y(x) = e^x + x^3/3 + C$, gives $1 = 1 + 0 + C$. Thus, $C = 0$, and the solution is $y(x) = e^x + x^3/3$. Solutions obtained in such a way are called the *particular* solutions. We also say that the condition in the form $y(x_0) = y_0$ is the *initial condition*. For our example, the initial condition used to obtain the particular solution $y(x)=e^x+x^3/3$ is $y(0)=1$.

This example illustrates *an initial value problem (IVP)* for an ODE, also called the *Cauchy problem*. It shows that in order to obtain some concrete solution we need to specify initial conditions for the DE. The terminology initial conditions comes from mechanics, where the independent variable x represents time and is customarily symbolized as t.

Because the DE is first-order in our example, only one initial condition is necessary and sufficient in order to determine an arbitrary constant C. General solutions of higher-order ODEs contain more than one arbitrary constants and thus more than one initial conditions are needed. We will see later that the number of arbitrary constants in the general solution of an ODE equals the order n of the equation and the initial value problem for an nth-order ODE requires n initial conditions.

In Chapter 4 we will discuss problems of different type: the boundary value problems (BVP) for ordinary differential equations.

We can now generalize the example above and present the solution process for the first-order equation of a simple type:

$$y'=f(x).$$

After writing this equation as $dy=f(x)dx$, the general solution $y(x,C)$ is obtained by integration:

$$y(x)=\int f(x)dx+C.$$

(Note that the solution is pronounced successful even if $f(x)$ cannot be integrated down to the elementary functions.) If the initial condition is assigned, constant C is determined by plugging $y(x_0)=y_0$ in the general solution. Finally, the particular solution is obtained by replacing C in the general solution by its numerical value. This solution can be also written without an integration constant:

$$y=y_0+\int_{x_0}^{x} f(x)dx.$$

This form of the particular solution explicitly includes the initial condition $y(x_0)=y_0$. For instance, the particular solution of the example above can be written as

$$y=1+\int_0^x (e^x+x^2)\,dx.$$

After this elementary discussion, consider a general nth-order ODE:

$$F(x,y,y'\ldots y^{(n)})=0, \tag{1.1}$$

where $F(x,y,y'\ldots y^{(n)})$ is some function.

Often nth-order ODE is given in the form resolved for the highest-order derivative:

$$y^{(n)} = f(x, y, y' \dots y^{(n-1)}). \tag{1.2}$$

In either case (1.1) or (1.2), the general solution

$$y = y(x, C_1, \dots, C_n) \tag{1.3}$$

contains n arbitrary constants.

1.2 FIRST-ORDER EQUATIONS: EXISTENCE AND UNIQUENESS OF SOLUTION

First-order equations have either the form

$$F(x, y, y') = 0, \tag{1.4}$$

or the form

$$y' = f(x, y). \tag{1.5}$$

The general solution contains one arbitrary constant: $y = y(x, C)$.

Often the general solution is not *explicit*, $\boldsymbol{y = y(x,C)}$, but an *implicit* one:

$$\Phi(x, y, C) = 0.$$

A particular solution emerges when C takes on a certain numerical value.

Now consider the inverse problem: the determination of a differential equation that has a given solution $\Phi(x,y,C) = 0$. Here one has to differentiate the function Φ, considering y as a function of x (thus using the chain rule), and then eliminate C with the help of equation $\Phi(x,y,C) = 0$.

For example, let the general solution of some equation be $y = Cx^3$ and we wish to find the equation. Differentiation of the general solution gives $y' = 3Cx^2$. Substitution into this expression $C = y/x^3$ from the general solution, results in the differential equation $y' = 3y/x$.

In order to find the particular solution, a single initial condition

$$y(x_0) = y_0 \tag{1.6}$$

is needed. The problem (1.4), (1.6) (or (1.5), (1.6)) is the IVP (or Cauchy problem) for the first-order ODE. To be specific, we will assume in the forthcoming the IVP (1.5), (1.6).

The question that we now need to ask is this: Under what conditions on a function $f(x,y)$ does there exist a unique solution of the IVP? The following theorem provides the answer.

Picard's Theorem (Existence and Uniqueness of a Solution): Let a function f be continuous in a rectangle D: $x_0 - ° \le x \le x_0 + °$, $y_0 - ° \le y \le y_0 + °$ that contains the point (x_0, y_0). Also let the partial derivative $\partial f/\partial y$ exists and be bounded in D. Then the solution of IVP (1.5), (1.6) exists and is unique.

Proof of Picard's Theorem can be found in many books.

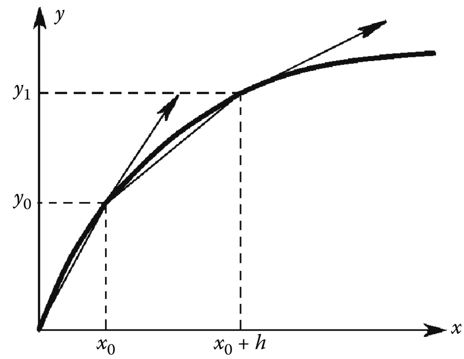


FIGURE 1.1 Illustration of Euler's method.

Conditions of Picard's Theorem can be illustrated by the simple method of the numerical integration of the ODE $y' = f(x, y)$ on the interval $[x_0, b]$. Let $[x_0, b]$ be divided into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, b]$ of equal length h, where $h = (b - x_0)/n$. The quantity h is the step size of the computation (Figure 1.1). The idea of the method is to approximate the integral curve by the set of the straight-line segments on the intervals $[x_i, x_{i+1}]$, such that each segment is tangent to the solution curve at one of the endpoints of the subintervals.

First, consider the interval $[x_0, x_0 + h]$ and find the value y_1 from equation $y' = f(x, y)$, as follows. Suppose h is small, then the derivative $y' \circ \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{h}$, and thus $y_1 = y_0 + y'h$.

Also, if h is small and the function $f(x,y)$ is continuous and changes slowly, then y' can be replaced by $f(x_0, y_0)$ on this interval. Then,

$$y_1 = y_0 + f(x_0, y_0)h.$$

Next, let repeat this construction on the interval $[x_0 + h, x_0 + 2h]$ (which is $[x_1, x_2]$). Taking

$$y' \circ \frac{y_2 - y_1}{h} \circ f(x_1, y_1),$$

we find $y_2 = y_1 + hf(x_1, y_1)$. Then, repeating the construction for other subintervals, we finally arrive to the *Euler's formula*:

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, \ldots, n-1. \tag{1.7}$$

(Euler's formula can be applied also when $b < x_0$. In this case the step size $h < 0$.) As $|h|$ decreases, the line segments become shorter and they trace the integral curve better – thus the accuracy of the approximate solution increases as $h \to 0$. The integral curve through the point x_0, y_0 represents the particular solution.

Another important question arises: What will happen with the solution if the initial conditions slightly change? Will the solution change also slightly? This question has not only theoretical significance but also a big practical meaning: how an error in initial conditions (which often are obtained from the experimental data, or from calculations performed with some limited precision) can affect the solution of the Cauchy problem. The answer to this question is this: It can be shown that if the conditions of the Picard's theorem are satisfied, the solution continuously depends on the initial conditions. It also can be shown that if the equation contains some parameter λ,

$$y' = f(x, y, ^\circ), \tag{1.8}$$

the solution continuously depends on λ if function f is a continuous function of λ. In Chapter 5 we will study the problem of *stability of solution*, which we only touched here.

In the following problems construct differential equations describing given families of integral curves.

Problems	Answers
1. $y=(x-C)^2$	$y'=2\sqrt{y}$
2. $x^3+Cy^2=5y$	$y'=\dfrac{3x^2y}{2x^3-5y}$
3. $y^3+Cx=x^2$	$y'=\dfrac{x^2+y^3}{3xy^2}$
4. $y=e^{Cx+5}$	$y'=\dfrac{y(\ln y-5)}{x}$
5. $y=Ce^x-3x+2$	$y'=y+3x-5$
6. $\ln y = Cx + y$	$y'=\dfrac{y(\ln y-y)}{x(1-y)}$
7. $y = C\cos x + \sin x$	$y'\cos x + y\sin x = 1$
8. $e^{-y}=x+Cx^2$	$y'=e^y-2/x$
9. $y^2+Cx=x^3$	$y'=\dfrac{2x^3+y^2}{2xy}$
10. $x = Cy^2 + y^3$	$y'=\dfrac{y}{2x+y^3}$

1.3 INTEGRAL CURVES, ISOCLINES

A differential equation $\frac{dy}{dx}=f(x,y)$ can be seen as a formula that provides a connection between the Cartesian coordinates of a point, (x,y), and the slope of the integral curve, $\frac{dy}{dx}$, at this point. To visualize this slope, we can actually draw at a point (x,y) a short line segment (called the *slope mark*) that has the slope $f(x,y)$. Repeating this for some other points in the Cartesian plane gives the *direction field*. (As a rule, tens or even hundreds of points are required for the construction of a quality direction field.) Thus, from a geometrical perspective, to solve a differential equation $y'=f(x,y)$ means to find curves that are tangent to the direction field at each point. As we pointed out in Section 1.2, these solution curves are called the integral curves.

When an analytical solution is unavailable, the construction of the direction field often can be made easier by first drawing lines of a constant slope, $y' = k$. Such lines are called *isoclines*. Since $y' = f(x, y)$, the equation of an isocline is $f(x,y) = k$. This means that an isocline is a *level curve* of the function $f(x,y)$.

Example 1.1

Consider equation $\frac{dy}{dx} = \frac{y}{x}$. We notice that at any point (x,y) the slope of the integral curve, y/x, is also the slope of straight lines leaving the origin (0,0) (for the right half-plane), and entering the origin (for the left half-plane). The direction field is shown in Figure 1.2 with short arrows. (Since the function $f(x,y) = y/x$ is not defined for $x = 0$, the equation does not have a solution on the y-axis, but the direction field can be plotted everywhere in the plane. Note that along the y-axis the slope is infinite.) The isoclines are given by equation $y/x = k$. Thus, in this example the isoclines are the straight lines coinciding with the direction field. The rectangle is a chosen domain for the graphical representation of the solution.

Example 1.2

Consider equation $\frac{dy}{dx} = -\frac{x}{y}$. Isoclines equation $-x/y = k$ gives straight lines shown in Figure 1.3. Along each isocline the value of y' does not change. This means that all slope marks (black arrows) along a particular isocline are parallel. Connecting slope marks on neighboring isoclines we can plot integral curves, which are obviously the circles centered at the origin: $x^2 + y^2 = C^2$. For the particular solution, starting from the point (x_0, y_0), we obtain the value of C (the circle radius): $C = \sqrt{x_0^2 + y_0^2}$. Function $f(x,y) = -x/y$ is not defined at $y = 0$, therefore the direction field can be plotted everywhere in the plane (x,y) except the x-axis. Thus we have to consider the solution in the form $y = \sqrt{C - x^2}$ in the upper half-plane and $y = -\sqrt{C - x^2}$ in the bottom half-plane.

In these figures, as well as in Figure 1.4, the isoclines and direction fields are plotted by the program **ODE 1st order**. The description of the accompanying software is in Section 1.9.

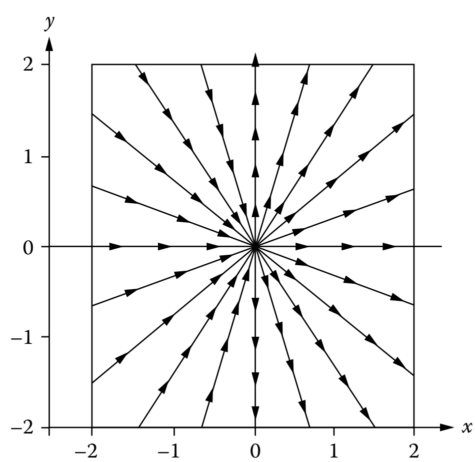


FIGURE 1.2 Direction field and isoclines for equation $y' = y/x$ (Example 1.1).

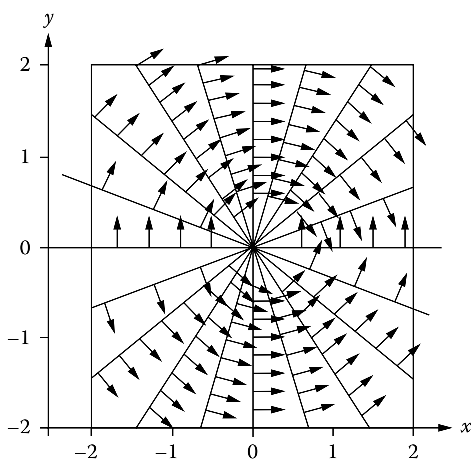


FIGURE 1.3 Direction field and isoclines for equation $y' = -x/y$ (Example 1.2).

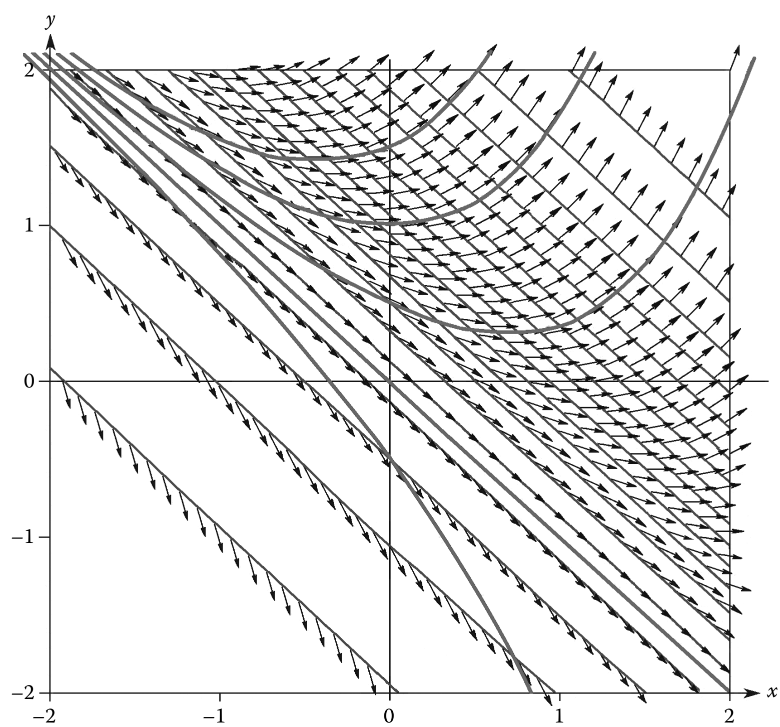


FIGURE 1.4 Direction field, isoclines, and five integral curves for equation $y' = x + y - 1$ (Example 1.3).

Example 1.3

Sketch several solution curves for equation $y' = x + y - 1$ using isoclines.

The isoclines equation is $x + y - 1 = k$, or $y = -x + (k + 1)$. First, we plot several isoclines for few values of k, and also plot the direction marks (that have slope k) along each isocline. Then, starting at a particular initial point (the initial condition), we connect the neighboring slope marks by a smooth curve in such a way that the curve's slope at each point is given by these marks. If we use reasonably large number of isoclines, then this process results in a qualitative plot of the integral curve. Another initial point gives another integral curve, etc. This approach can be useful in some situations, such as when the analytical solution is not known and the integral curves are relatively simple. Figure 1.4 shows the sketch of the isoclines and the direction field, as well as the five integral curves plotted for the initial points (0, –0.5), (0,0), (0,0.5), (0,1.0), (0,1.5).

In many problems, especially those of a physical or a geometrical nature, the variables x and y are indistinguishable in the dependent/independent sense (i.e., y may be considered independent and x dependent, or vice versa). Thus if such problem is described by a differential equation

$$\frac{dy}{dx} = f(x,y), \tag{1.9}$$

that has a solution $y = y(x)$, then it is natural to consider also the equation

$$\frac{dx}{dy} = \frac{1}{f(x,y)}, \tag{1.10}$$

that has the solution $x = x(y)$. From (1.9) and (1.10) it is clear that these solutions are equivalent, and thus their graphs (the integral curves) coincide.

In the situations without obvious dependent and independent variables and when one of equations (1.9) or (1.10) does not make sense at a certain point or points (thus the right-hand side is undefined), it is natural to replace the equation at such points by its counterpart. For instance, the right-hand side of equation $\frac{dy}{dx} = \frac{y}{x}$ is undefined at $x = 0$. The solution of equation, considering $x \neq 0$, is $y = Cx$. When this equation is replaced by $\frac{dx}{dy} = \frac{x}{y}$, one finds that the latter equation has the trivial solution $x = 0$. Thus, if variables x and y are equivalent, the original equation $\frac{dy}{dx} = \frac{y}{x}$ has the solution $x = 0$ in addition to the solution $y = Cx$.

Problems

Plot isoclines and the direction field for the given differential equation in the domain $[a,b;c,d]$. With the help of isoclines plot (approximately) two particular solutions of this equation passing through the points *a)* $y_1 = y(x_1)$; *b)* $y_2 = y(x_2)$ (with help of program **ODE 1st order**).

1. $y' = y - x^2$, $[-2,2;-2,2]$, a) $y(-1) = 1$; b) $y(0) = -1$
2. $y' = 2x - e^y$, $[-2,2;-2,2]$, a) $y(-1) = 1$; b) $y(0) = 0.5$
3. $y' = \dfrac{5x}{x^2 + y^2}$, $[1,5;1,5]$, a) $y(1) = 2$; b) $y(4) = 3$

4. $y' = x + 1 - 2y$,	[–1,1;–1,1],	a) $y(-0.5) = -1$;	b) $y(0) = 0.5$
5. $y' = \dfrac{y - 3x}{x + 3y}$,	[1,3;1,3],	a) $y(1.5) = 2.5$;	b) $y(2) = 1.5$
6. $y' = 2x^2 - y$,	[–2,2;–2,2],	a) $y(-1) = 1$;	b) $y(0) = -1.5$
7. $y' = \sin x \sin y$,	[–3,3;–3,3],	a) $y(-2) = -2$;	b) $y(2) = 1$
8. $y' = x^2 + \sin y$,	[–2,2;–2,2],	a) $y(-1) = -1$;	b) $y(1) = 1$
9. $y' = x(x - y)^2$,	[–1,1;–1,1],	a) $y(-1) = -0.5$;	b) $y(-0.5) = 0$
10. $y' = 3\cos^2 x + y$,	[–2,2;–2,2],	a) $y(-2) = 0$;	b) $y(2) = 1$

There is no general solution method applicable to all differential equations, and it is possible to solve analytically only the first-order equations of certain types. In the next sections we introduce these types and the solution methods.

1.4 SEPARABLE EQUATIONS

As we already pointed out, the first-order equation

$$\frac{dy}{dx} = f(x, y) \tag{1.11}$$

can be written in the equivalent form

$$dy = f(x,y)dx. \tag{1.12}$$

If function $f(x,y)$ can be represented as a fraction, $f_2(x)/f_1(y)$, then equation (1.12) takes the form

$$f_1(y)dy = f_2(x)dx. \tag{1.13}$$

In equation (1.13) variables x and y are separated to the right and to the left of the equality sign. Equations that allow transformation from (1.11) to (1.13) are called *separable,* and the process of transformation is called *separation of variables.* Next, by integrating both sides of (1.13), one obtains

$$\int f_1(y)dy = \int f_2(x)dx + C. \tag{1.14}$$

Expression (1.14) defines the general solution of the differential equation either explicitly, as a function $y(x) + C$, or implicitly, as a function $\Phi(x, y, C) = 0$. As we have noticed, even if the integrals in (1.14) cannot be evaluated in elementary functions, we pronounce that the solution of a separable equation has been determined.

The particular solution can be obtained from (1.14) by substituting in it the initial condition $y(x_0) = y_0$, which allows to determine the value of C. The particular solution can be written also as

$$\int_{y_0}^{y} f_1(y)dy = \int_{x_0}^{x} f_2(x)dx, \tag{1.15}$$

where no arbitrary constant is present. Obviously, formula (1.15) gives the equality 0 = 0 at the point (x_0, y_0); thus this particular solution includes the initial condition $y(x_0) = y_0$.

Often, in order to transform (1.12) into (1.13), it is necessary to divide both sides of (1.12) by a certain function of x,y. Zero(s) of this function may be the solution(s) of (1.11), which are lost when the division is performed. Thus we must take a note of these solutions before division.

Example 1.4

Consider again the differential equation from Example 1.2, $\frac{dy}{dx} = -\frac{x}{y}$. Multiplication of the equation by ydx gives $ydy = -xdx$. In the latter equation variables are separated. Next, integrate: $\int ydy = -\int xdx + C$. Evaluation of the integrals gives the general solution $x^2 + y^2 = C^2$. Let the initial condition be $y(3) = 4$. Substituting the initial condition in the general solution gives $C = 5$, thus the particular solution is the circle $x^2 + y^2 = 25$.

It is instructive to solve this exercise using the program **ODE 1st order**. The program offers a simple and convenient interface to the numerical solution procedures of first-order IVPs given in the form $y' = f(x, y)$, $y(x_0) = y_0$. (The description of **ODE 1st order** is in Section 1.9.) Using same interface one can plot a graph of a particular solution corresponding to the initial condition $y(x_0) = y_0$, that has been determined by the reader analytically. Numerical solution of IVP may encounter difficulties, if function $f(x,y)$ does not satisfy conditions of Picard's Theorem at some points. Thus in **ODE 1st order** , the region D for the integration of the differential equation must be chosen so that it does not contain such points. For instance, in Example 1.4 the domain D must include the point (3,4) (the initial condition), but it should not include any point with the zero y-coordinate. Since the particular solution is the circle of radius 5 centered at the origin, the interval of x-values for the numerical solution has to be $-5 < x < 5$. For the initial condition $y(3) = 4$ the numerical solution gives the upper semi-circle $y = +\sqrt{25 - x^2}$, and for the initial condition $y(3) = -4$ the numerical solution gives the bottom semi-circle $y = -\sqrt{25 - x^2}$.

Example 1.5

Next, we consider differential equation from Example 1.3: $y' = \frac{y}{x}$. Dividing both sides of equation by y and multiplying by dx, the equation becomes $\frac{dy}{y} = \frac{dx}{x}$. Dividing by y, we may have lost the solution $y = 0$. Indeed, plugging $y = 0$ into both sides of the original equation $y' = \frac{y}{x}$, we see that equation becomes the equality $0 = 0$. Thus $y = 0$ indeed is the solution. To find other solutions, we integrate $\frac{dy}{y} = \frac{dx}{x}$, and obtain: $\ln|y| = \ln|x| + C$. Exponentiation gives $|y(x)| = e^{\ln|x|+C} = e^{\ln|x|}e^C = C|x|$, where we used C again to denote arbitrary *positive* constant e^C. Now obviously, $y(x) = \pm Cx$. The trivial solution $y = 0$ can be included in the solution $y(x) = Cx$ by allowing $C = 0$. Thus finally, the general solution is $y = Cx$, where C is *arbitrary constant* (can be either positive, or negative, or zero).

To find particular solutions, any initial condition $y(x_0) = y_0$ is allowed, except x_0 cannot be zero. This is because the conditions of Picard's theorem are violated when $x_0 = 0$: the function $f(x,y) = y/x$ is unbounded. When $x_0 \neq 0$, the particular solution is $y = y_0 x / x_0$.

Numerical solutions are always conducted in a concrete domain D. In this example the line $x = 0$ (the y-axis) is the line of discontinuity. Therefore, for the initial condition taken at $x_0 > 0$, we can solve the equation in the domain $0 < x \le a$ (with some positive a), if $x_0 < 0$—in the domain $a \le x < 0$ (with some negative a).

Example 1.6

Consider differential equation $\frac{dy}{dx} = -\frac{y}{x}$. (Note that this differs from Example 1.5 only in that the right-hand side is multiplied by −1). Functions $f(x,y) = -y/x$ and $f_y(x,y) = -1/x$ are continuous at $x \neq 0$; thus equation satisfies the conditions of Picard's Theorem in the entire (x,y) plane, except again at the y-axis. Separating the variables we have $dy/y = -dx/x$ and after the integration obtain $\ln|y| = \ln C - \ln|x|$ (here we choose the arbitrary constant in the form $\ln C$, $C > 0$), which gives $|y| = C/|x|$. From there $y = \pm C/x$ and the general solution can be finally written as $y = C/x$, where C is an arbitrary constant ($C = 0$ is also allowed because $y = 0$ is a trivial solution of the given equation). This solution holds in the domains $x < 0$ and $x > 0$. The integral curves are the hyperbolas shown in Figure 1.5. An initial condition $y(x_0) = y_0$ taken at the point (x_0, y_0) in each of these

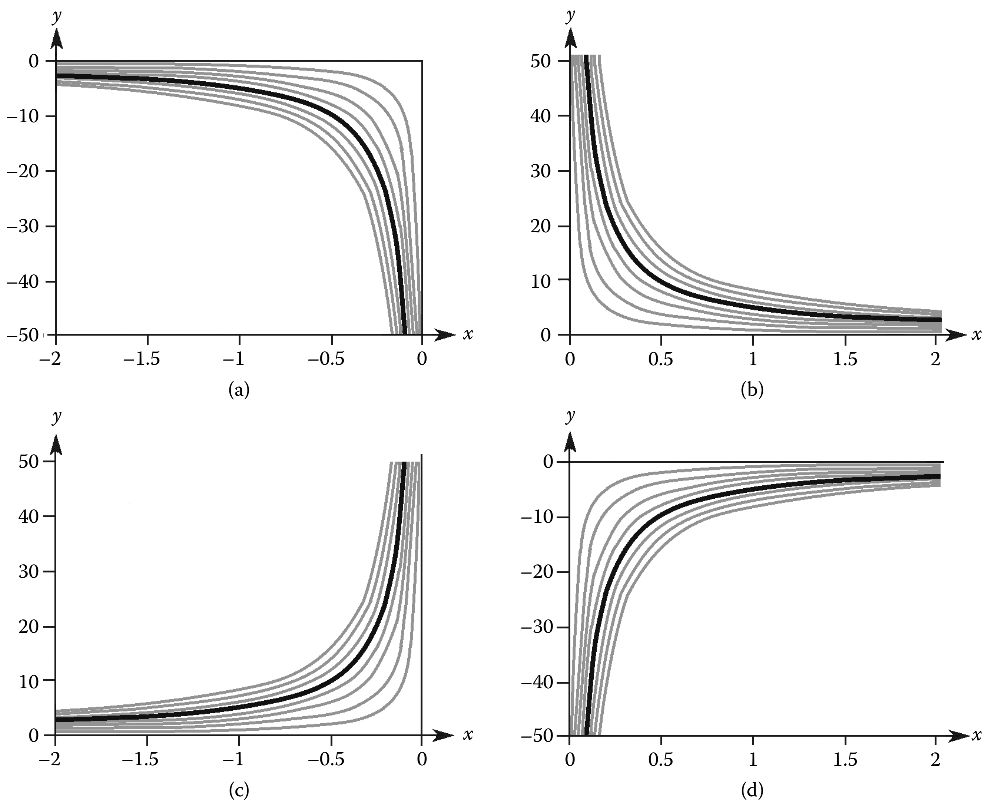


FIGURE 1.5 Integral curves for the equation $\frac{dy}{dx} = -\frac{y}{x}$ (Example 1.6). General solution is $y = C/x$. In (*a*) and (*b*) the curves for $C = 1, 2, \ldots, 9$ are shown; the particular solution $y = 5/x$ is a thick line. In (*c*) and (*d*) the curves for $C = -9, -8, \ldots, -1$ are shown; the particular solution $y = -5/x$ is a thick line.

four quadrants determines only one integral curve corresponding to the particular solution of the differential equation, as shown in Figure 1.5.

Example 1.7

Solve equation

$$y' = ky,$$

where k is a constant. For $k<0$ ($k>0$) this equation describes unlimited decay (growth) (see Section 1.8). Division by y and multiplication by dx gives $\frac{dy}{y} = kdx$. Integrating, $\ln|y| = kx + C$. Next, exponentiation gives $|y| = Ce^{kx}$, $C > 0$, or equivalently $y = Ce^{kx}$, where C is arbitrary and non-zero. By allowing $C = 0$, this formula again includes the trivial solution $y = 0$ which has been lost when the equation was divided by y. Thus $y(x) = Ce^{kx}$ is the general solution, where C is arbitrary constant.

Since the right-hand side of the equation $y' = ky$ satisfies the conditions of Picard's theorem everywhere in the plane, there is no limitations on the initial condition (any x_0 and y_0 can be used). Substituting the initial condition $y(x_0) = y_0$ in the general solution, we obtain $y_0 = Ce^{kx_0}$, thus $C = y_0e^{-kx_0}$. Substituting this C in the general solution, the particular solution (the solution of the IVP) takes the form $y = y_0e^{k(x-x_0)}$.

Example 1.8

Solve equation

$$x(1+y^2)dx = y(1+x^2)dy.$$

Division of both sides by $(1+y^2)(1+x^2)$ separates variables. This product obviously is never zero, thus there is no loss of solutions. Integrating,

$$\int \frac{xdx}{d^2+1} = \int \frac{ydy}{y^2+1} + C,$$

and then $\ln(1+x^2) = \ln(1+y^2) + C$. Note that the absolute value signs are not necessary since $1+x^2$ and $1+y^2$ are positive. Thus the general solution is

$$x^2 + 1 = C(y^2 + 1),$$

where $C > 0$.

Let the initial condition be $y(1) = 1$. Substitution in the general solution gives $C = 1$. Thus the particular solution is $x^2 = y^2$. Writing it in the form $y = \pm x$, it becomes clear that only the function $y = x$ satisfies the initial condition (see Figure 1.6).

Next, let try the initial condition at the origin: $y(0) = 0$. From the general solution we still get $C = 1$. Now both solutions $y = \pm x$ match the initial condition. This seems to

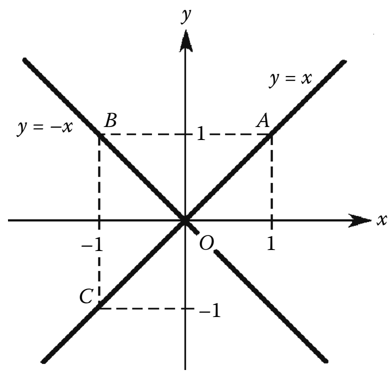


FIGURE 1.6 Example 1.8: the particular solution corresponding to the initial condition $y(1) = 1$.

contradict the uniqueness property stated in Picard's theorem. But from the equation in the form $y' = f(x, y)$ with

$$f(x,y) = \frac{x(1+y^2)}{y(1+x^2)}$$

it is clear that at $y = 0$ the function $f(x,y)$ has infinite discontinuity—thus the condition of Picard's theorem is violated and there is no guarantee of the uniqueness of the particular solution near the origin.

In practice this means that in the case of a nonunique solution, when an integral curve is traced, while passing the line of discontinuity one can slip from this curve to an adjacent one. For instance, let solve our example equation on the interval [–1,1] with the initial condition $y(1) = 1$, i.e., start at point A in Figure 1.6. Moving along the solution $y = x$ toward the origin, at the destination point (0,0) we have two possibilities for the following motion: continue along the line $y = x$ towards the final point C(–1,–1), or along the line $y = -x$ towards the final point B(–1,1). This demonstrates that when solution is not unique and one of the solution curves is traced, we may slip to an adjacent curve. Problems of this kind are especially serious for numerical solutions of differential equations, when there is a danger to miss a discontinuity point and not find several branches of solution. The choice of the proper integral curve in the case of a nonunique solution can be based on the real situation described by the differential equation. If, for instance, it is known that for $x < 0$ the values of a function y, describing a particular process, are positive, then on the interval [–1,1] the line AOB should be taken for the solution; if it is known that for $x < 0$ the values of a function y are negative, then the line AOC should be taken. Often the choice can be made based on the knowledge of the asymptotes at $x \to \infty$ or $x \to -\infty$ of the function described by a differential equation. The example discussed above demonstrates the importance of the analysis of the properties of a differential equation before starting the solution.

Problems	Answers
1. $2y'y = 1$	$y^2 = x + C$
2. $(x+1)^3 dy - (y-2)^2 dx = 0$	$\dfrac{1}{y-2} = \dfrac{1}{2(x+1)^2} + C, \quad y = 2$
3. $e^{x+y} - 3e^{-x-2y} y° = 0$	$e^{-2x} = -2e^{-3y} + C$
4. $y'(\sqrt{x} + \sqrt{xy}) = y$	$2\sqrt{y} + \ln y - 2\sqrt{x} = C$
	$y^2 = 5 + Ce^{1/x}$
5. $y'x^2 + y^2 = 0$	$y = -x$
6. $y'y + x = 1$	$(x-1)^2 + y^2 = C$
7. $y'x^2 y^2 - y + 1 = 0$	$\dfrac{y^2}{2} + y + \ln\lvert y-1\rvert = -\dfrac{1}{x} + C$
8. $(x^2+1)y^3 dx = (y^2-1)x^3 dy$	$1 + \ln\lvert x/y\rvert = (x^{-2} + y^{-2})/2 + C$
9. $xydx + (x+1)dy = 0$	$y = C(x+1)e^{-x}$
10. $x\dfrac{dx}{dt} + t = 1$	$x^2 = -t^2 + 2t + C$
11. $\sqrt{1+y^2}\,dx = xydy$	$x = Ce^{\sqrt{1+y^2}}, \ C > 0,$
12. $(x^2-1)y° + 2xy^2 = 0$	$y(\ln\lvert x^2-1\rvert + C) = 1; \ y = 0$
13. $y' = -\dfrac{xy}{x+2}$	$y = Ce^{-x}(x+2)^2; \ y = 0$
14. $\sqrt{y^2+5}dx = 2xydy$	$\ln\lvert x\rvert = C + \sqrt{y^2+5}; \ x = 0$
15. $2x^2 yy' + y^2 = 5$	$y^2 = 5 + Ce^{1/x}$
16. $y' - 2xy^2 = 3xy$	$y = \dfrac{3}{Ce^{-3x^2/2} - 2}; \ y = 0$
17. $xy' + y = y^2$	$y = \dfrac{1}{1-Cx}, \ y = 0$
18. $2x\sqrt{1+y^2}\,dx + ydy = 0$	$\sqrt{1+y^2 =} x^2 + C, \quad y = \pm 1$
19. $y' = \dfrac{\sqrt{y}}{x}$	$2\sqrt{y} = \ln\lvert x\rvert + C; \ y = 0$
20. $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$	arcsinx + arcsiny = C

In all these problems you can choose the initial condition and find the particular solution. Next, run the program **ODE 1st order** and compare the particular solution to the output. Using the program you can obtain the numerical solution of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, and also plot the integral curve for the analytical solution. Keep in mind that numerical solution may run into a difficulty if the function $f(x,y)$ is discontinuous. Thus in the program, the domain of integration of the differential equation, D, must be chosen such that it does not contain points of discontinuity of $f(x,y)$. This guarantees solution uniqueness and trouble-free numerical integration.

To summarize, separable equations can be always solved analytically. It is important that some types of equations can be transformed into separable equations using the change of a variable. Two such types of equations are the equations $y' = f(ax+by)$ and $y' = f(y/x)$.

1. Consider equation

$$y' = f(ax+by), \tag{1.16}$$

where a, b are constants.

Let

$$z = ax + by \tag{1.17}$$

be the new variable. Differentiating z with respect to x, we obtain $\frac{dz}{dx} = a + b\frac{dy}{dx}$. Notice that $\frac{dy}{dx} = f(z)$ (eq. (1.16)), thus

$$\frac{dz}{dx} = a + bf(z). \tag{1.18}$$

This is a separable equation:

$$\frac{dz}{a+bf(z)} = dx.$$

Integration gives:

$$x = \int \frac{dz}{a+bf(z)} + C.$$

After the integration at the right-hand side is completed, replacing the variable z by $ax + by$ gives the general solution of (1.16).

Example 1.9 Solve $\frac{dy}{dx} = 2x - 3y$.

Using new variable $z = 2x - 3y$, we find $z' = 2 - 3y' = 2 - 3z$. Separating variables, $\frac{dz}{2-3z} = dx$. Integration gives

$$x + C = -\frac{1}{3}\ln|2-3z|, \quad \text{or} \quad \ln|2-3z| = -3x + C.$$

Then,

$$|2-3z| = e^{-3x+C} = Ce^{-3x}, \quad \text{where } C > 0.$$

This can be written as $2-3z = Ce^{-3x}$ with C either positive or negative. Substituting $z = 2x-3y$ gives $2-6x+9y = Ce^{-3x}$. Thus the general solution is

$$y = Ce^{-3x} + 2x/3 - 2/9.$$

It is easy to check that for $C = 0$ the function $y = 2x/3 - 2/9$ also satisfies the given equation; thus C in the general solution is an arbitrary constant.

Example 1.10 Solve $y' = \frac{2}{x-y} - 3$.

Let $z = x-y$ be the new variable. Differentiation gives $z' = 1-y'$. Since $y' = \frac{2}{z} - 3$, the equation for the function z is $1-z° = \frac{2}{z} - 3$.

Separation of variables gives $dx = \frac{zdz}{4z-2}$. The right-hand side can be algebraically transformed: $\frac{zdz}{4z-2} = \frac{1}{4}\frac{4z-2+2}{4z-2}dz = \frac{1}{4}\left(1+\frac{2}{4z-2}\right)dz$. Integration of the latter equation gives $x + C = \frac{1}{4}z + \frac{1}{8}\ln|4z-2|$. Multiplying by 8, and letting $8C = C$ since C is arbitrary, finally gives the general solution in the implicit form:

$$8x + C = 2(x-y) + \ln|4x-4y-2|.$$

Problems	Answers
1. $(2x+y+1)dx-(4x+2y-3)dy = 0$	$2x+y-1 = Ce^{2y-x}$
2. $x-y-1-(y-x+2)y° = 0$	$-2x+2y+3 = Ce^{-2(x+y)}$
3. $y' = (y+x+3)(y+x+1)$	$-\dfrac{1}{x+y+2} = x+C$
4. $y' = \sqrt{2x-y+3}+2$	$2\sqrt{2x-y+3}+x+C = 0$
5. $y' = (y-4x+3)^2$	$\dfrac{1}{4}\ln\left\|\dfrac{y-4x+1}{y-4x+5}\right\| = x+C$
6. $y'-y = 3x-5$	$3x+y-2 = Ce^{x}$
7. $y' = \dfrac{1}{x+2y}$	$x+2y+2 = Ce^{y}$
8. $y' = \sqrt{4x+2y-1}$	$\sqrt{4x+2y-1}-2\ln(\sqrt{4x+2y-1}+2) = x+C$
9. $y' = \dfrac{x+3y}{2x+6y-5}$	$6y-3x-3\ \ln\|x+3y-1\| = C$

2. Consider equation

$$y' = f\left(\frac{y}{x}\right) \tag{1.19}$$

Here, function $f(x,y)$ contains variables x and y in the combination y/x.

Let

$$z = \frac{y}{x} \tag{1.20}$$

be the new variable. Then, $y = zx$. Differentiating this in x, one obtains $y' = z + xz'$, or

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Thus

$$f(z) = z + x\frac{dz}{dx}. \tag{1.21}$$

Separating variables $\frac{dx}{x} = \frac{dz}{f(z)-z}$ and integrating gives

$$\ln|x| + \ln C = \int \frac{dz}{f(z)-z},$$

or

$$x = C\exp\left[\int \frac{dz}{f(z)-z}\right].$$

Finally, replacement of z by y/x in the solution results in the general solution of (1.19).

Example 1.11

$$y' = \frac{x^2 + y^2}{2xy}$$

Division by $2xy$ gives $\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right)$. With the new variable $z = \frac{y}{x}$, we have $y' = z + xz'$. Equating this to $f(z) = \frac{1}{2}\left(\frac{1}{z} + z\right) = \frac{1+z^2}{2z}$ gives $z + xz' = \frac{1+z^2}{2z}$. Next, separate variables, $\frac{dx}{x} = \frac{2zdz}{1-z^2}$, and integrate. This gives $|x| = \frac{C}{|z^2-1|}$, $C > 0$.

Thus, $x = \frac{C}{1-z^2}$, where C is arbitrary constant. Substitution $z = \frac{y}{x}$ gives $x = \frac{Cx^2}{x^2-y^2}$. Finally, $y^2 = x^2 - Cx$ is the general solution.

Example 1.12

$$y' = y/x + \tan(y/x)$$

First, introduce new variable z: $y = xz,\ y' = z + xz'$. Then the equation becomes $x\frac{dz}{dx} + z = z + \tan z$. Next, separate variables: $\frac{dx}{x} = \frac{\cos z}{\sin z}dz$, and integrate. This gives $\ln|\sin z| = \ln|x| + \ln C$, $C>0$. Thus, $\sin z = Cx$, or $\sin(y/x) = Cx$, where C is arbitrary constant.

Problems	Answers
1. $(x + 2y)dx - xdy = 0$	$y = Cx^2 - x$
2. $xy' = y + \sqrt{x^2 + y^2}$	$y - Cx^2 + \sqrt{x^2 + y^2} = 0$
3. $xy' = y - xe^{y/x}$	$y = -x \ln \ln \|Cx\|$
4. $xy' - y = (x + y)\ln\dfrac{x+y}{x}$	$\ln\left(1 + \dfrac{y}{x}\right) = Cx$
5. $xdy = (\sqrt{xy} + y)dx$	$\sqrt{y} = \dfrac{1}{2}\sqrt{x}\ln Cx$
6. $ydx + (2\sqrt{xy} - x)dy = 0$	$\sqrt{x} + \sqrt{y}\ln Cy = 0$
7. $2x^3dy - y(2x^2 - y^2)dx = 0$	$(x/y)^2 = \ln Cx, \quad y = 0$
8. $(y^2 - 2xy)dx + x^2dy = 0$	$y(x-y) = Cx^3, \quad y = 0$
9. $y^2 + x^2y' = xyy'$	$y = Ce^{y/x}$

1.5 LINEAR EQUATIONS AND EQUATIONS REDUCIBLE TO LINEAR FORM

An equation that is linear in y and y',

$$y' + p(x)y = f(x), \tag{1.22}$$

is called *linear first-order differential equation*. We assume that $p(x)$ and $f(x)$ are continuous in a domain where the solution is sought. The coefficient of y' in (1.22) is taken equal to one simply for convenience—if it is not equal to one, then the equation can be divided by this coefficient prior to the solution, which casts the equation in the form (1.22). Linearity of the expression $y' + p(x)y$ in y и y' means that when $y(x)$ is replaced by $C_1y_1(x) + C_2y_2(x)$, the left-hand side of equation (1.22) transforms into $C_1(y_1' + py_1) + C_2(y_2' + py_2)$. For instance, equations $y'y + 2y = x$ and $y' - \sin y = 1$ are nonlinear: the first equation due to the term $y'y$, the second one – due to the term $\sin y$. An example of a linear equation may be $x^2y' + e^xy = 5\sin x$.

If a function $f(x)$ is not identically zero, the first-order equation (1.22) is called *linear inhomogeneous.*

In the opposite case when $f(x) \equiv 0$, equation

$$y' + p(x)y = 0 \tag{1.23}$$

is called *linear homogeneous*. In such equation variables can be separated:

$$\frac{dy}{y} = -p(x)dx.$$

Integration gives

$$\ln|y| = -\int p(x)dx + \ln C,$$

where we chose to denote an arbitrary constant as $\ln C$ (with $C > 0$). Removing the absolute value sign and taking into account that $y(x) = 0$ is also the solution of (5.2), we obtain the *general solution of the linear homogeneous equation*:

$$y(x) = Ce^{-\int p(x)dx}, \tag{1.24}$$

where C is arbitrary constant.

Recall that our goal is to determine the *general* solution of the *inhomogeneous* equation (1.22). Now that the homogeneous equation has been solved, this can be accomplished by using the *method of variation of the parameter.* The idea of the method is that the solution of (1.22) is sought in the form (1.24), *where constant C is replaced by a function $C(x)$*:

$$y(x) = C(x)e^{-\int p(x)dx}. \tag{1.25}$$

To find $C(x)$, we substitute (1.25) in (1.22). First, find the derivative y':

$$y' = C'(x)e^{-\circ p(x)dx} - C(x)p(x)e^{-\circ p(x)dx},$$

and substitute y' and y (see (1.25)) in (1.22):

$$y' + p(x)y = C'(x)e^{-\circ p(x)dx} - C(x)p(x)e^{-\circ p(x)dx} + C(x)p(x)e^{-\circ p(x)dx} = f(x).$$

Terms with $C(x)$ cancel, and we obtain

$$C'(x) = f(x)e^{\circ p(x)dx}.$$

Integrating, we find

$$C(x) = \int f(x)e^{\int p(x)dx}\,dx + C_1.$$

Substituting this $C(x)$ in (1.25), we finally obtain:

$$y(x) = C(x)e^{-\int p(x)dx} = C_1e^{-\int p(x)dx} + e^{-\int p(x)dx}\int f(x)e^{\int p(x)dx}\,dx. \tag{1.26}$$

Expression (1.26) *is the general solution of* (1.22). Notice that the first term at the right-hand side is the general solution (1.24) of the homogeneous equation (of course, notations C_1 and C stand for the same arbitrary constant). The second term in (1.26) is the *particular solution* of the inhomogeneous equation (1.22) (this can be checked by plugging the second term in (1.26) for y in (1.22), and its derivative for y'; also note that particular solution does not involve the arbitrary constant).

In summary, the general solution of the inhomogeneous equation is the sum of the general solution of the homogeneous equation and the particular solution of the inhomogeneous equation. In this book we denote these solutions by $Y(x)$ and $\bar{y}(x)$, respectively. Thus, the general solution of (1.22) is written as

$$y(x) = Y(x) + \bar{y}(x). \tag{1.27}$$

When solving problems, the complicated formula (1.26) usually is not used. Instead, all steps of the solution scheme are applied as described above.

Example 1.13 Solve $y' + 5y = e^x$.

First we solve the homogeneous equation

$$y' + 5y = 0.$$

Separating variables, $dy/y = -5dx$, and integrating gives: $|y| = e^{-5x+C} = e^Ce^{-5x}$, or $y = Ce^{-5x}$ with arbitrary C. This is the general solution of the homogeneous equation.

Next we proceed to find the particular solution of the inhomogeneous equation through the variation of the parameter:

$$y = C(x)e^{-5x}.$$

The derivative is $y' = C'(x)e^{-5x} - 5C(x)e^{-5x}$. Substitution of y and y' in $y' + 5y = e^x$ gives

$$C'(x)e^{-5x} + 5C(x)e^{-5x} - 5C(x)e^{-5x} = e^x,$$

or $C'(x) = e^{6x}$. Integration results in $C(x) = e^{6x}/6$. (We do not add arbitrary constant because we search for a particular solution.) The particular solution of the inhomogeneous equation is $y = C(x)\,e^{-5x} = e^x/6$. Easy to check that $e^x/6$ satisfies the given inhomogeneous equation. The general solution of the inhomogeneous equation is $y(x) = Y(x) + \bar{y}(x) = Ce^{-5x} + e^x/6$.

Example 1.14 Solve $y' - \frac{y}{x} = x$.

First solve the homogeneous equation

$$y' - y/x = 0.$$

Separating variables, we obtain $\frac{dy}{y} = \frac{dx}{x}$. Integration gives $\ln|y| = \ln|x| + \ln C$, $C > 0$. Then $y = Cx$, where C is arbitrary. This is the general solution of the homogeneous equation.

A particular solution of the original inhomogeneous equation is sought in the form:

$$y = C(x)x.$$

Substitution of y and its derivative $y' = C'(x)x + C(x)$ into $y' - y/x = x$ gives $C'(x)x = x$, and then $C(x) = x$, thus $\bar{y}(x) = x^2$. Finally, the general solution of the problem is $y = Cx + x^2$.

Example 1.15 Solve $(2e^y - x)y° = 1$.

This equation, which is not linear for function $y(x)$, becomes linear if we consider it as the equation for function $x(y)$. To do this, let us write it in the form $2e^y - x = \frac{dx}{dy}$, which is a linear inhomogeneous equation for $x(y)$.

The solution of homogeneous equation $dx/dy + x = 0$ is $x = Ce^{-y}$. Then seek a particular solution of inhomogeneous equation in the form $x = C(y)e^{-y}$. Substitution in $2e^y - x = x°$ gives $C'(y) = 2e^{2y}$, and integration gives $C(y) = e^{2y}$. Thus the particular solution of inhomogeneous equation is $x = C(y)e^{-y} = e^y$. Finally, the general solution of the problem is $x = Ce^{-y} + e^y$.

Example 1.16 Solve $(\sin^2 y + x\cot y)y' = 1$.

This equation becomes linear if we consider it as the equation for function $x(y)$:

$$\frac{dx}{dy} - x\cot y = \sin^2 y.$$

This is a linear inhomogeneous equation for $x(y)$. Integration of homogeneous equation $dx/dy - \cot y dy = 0$ gives $\ln|x| = \ln|\sin y| + \ln C$ from which $x = C\sin y$. Then seek a particular solution of inhomogeneous equation in the form $x = C(y)\sin y$. Substitution in inhomogeneous equation gives $C'(y) = \sin y$, thus $C(y) = -\cos y$. Thus the particular solution of inhomogeneous equation is $x = C(y)\sin y = -\sin y\cos y$. The general solution of the problem is the sum of solutions of homogeneous and inhomogeneous equations, thus $x = (C - \cos y)\sin y$.

Another widely used method for the solution of (1.22) is based on using an *integrating factor***.** The integrating factor for a linear differential equation is a function

$$\mu(x) = e^{\int p(x)dx}. \tag{1.28}$$

When (1.21) is multiplied by $\mu(x)$, the equation becomes

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x). \tag{1.29}$$

(You can *check* that the left-hand side in (1.29) equals $\mu y' + \mu p y$.)

Multiplication of (1.29) by dx, integration and division by $\mu(x)$ gives the solution:

$$y(x) = \frac{1}{\mu(x)}\left[\int° \mu(x)f(x)dx + C\right]. \tag{1.30}$$

Substitution of $\mu(x)$ from (1.28) shows that (1.30) coincides with (1.26), which was obtained by the method of variation of the parameter.

Example 1.17 Solve equation $y' + 5y = e^x$ from Example 1.13 using the integrating factor.

Here $\mu(x) = e^{\int 5dx} = e^{5x}$, and solution (1.30) is

$$y(x) = \frac{1}{e^{5x}}\left[\int e^{5x}e^x\,dx + C\right] = \frac{1}{e^{5x}}\left[\frac{1}{6}e^{6x} + C\right] = Ce^{-5x} + \frac{e^x}{6}.$$

Next, consider the *Bernoulli equation*

$$y' + p(x)y = f(x)y^n, \tag{1.31}$$

where n is arbitrary real number (for $n = 0,1$ this equation is already linear). Let show how the Bernoulli equation can be transformed into a linear equation.

First, we divide (1.31) by y^n:

$$y'y^{-n} + p(x)y^{1-n} = f(x). \tag{1.32}$$

Let

$$z(x) = y^{1-n}, \tag{1.33}$$

and find the derivative $z'(x)$ by the chain rule:

$$z'(x) = (1-n)y^{-n}y'.$$

Substitution of $z(x)$ and $z'(x)$ in (5.11) gives the linear inhomogeneous equation for $z(x)$:

$$\frac{1}{1-n}\frac{dz}{dx} + p(x)z = f(x).$$

The general solution of the latter equation can be found by the methods just described above, and then $y(x)$ is determined from (1.33).

Example 1.18 Solve $y' - y = \frac{3}{y}$.

This is Bernoulli equation with $n = -1$. Division by y^{-1} (i.e., multiplication by y) gives $yy' - y^2 = 3$. Let $z = y^2$, then $z' = 2yy'$ and the equation becomes $\frac{1}{2}z' - z = 3$. First we solve the homogeneous equation $\frac{1}{2}z' - z = 0$. Its general solution is $z = Ce^{2x}$. Using variation of the parameter, we seek the particular solution of the inhomogeneous equation in the form $z = C(x)e^{2x}$, which gives $C'(x) = 6e^{-2x}$. Integration gives $C(x) = -3e^{-2x}$, thus the particular solution is $z = -3$ and the general solution of the inhomogeneous equation is $z = Ce^{2x} - 3$. Returning to the variable $y(x)$, the solution of the original equation is obtained: $y^2 = Ce^{2x} - 3$.

Finally, consider the *Riccati equation*

$$y' + p(x)y + q(x)y^2 = f(x). \tag{1.34}$$

This equation cannot be solved analytically in general case, but it can be transformed into the Bernoulli equation if one particular solution, $y_1(x)$, of equation (1.34) is known. In this case let $y = y_1 + z$ and substitute it in equation (1.34):

$$y_1' + z' + p(x)(y_1 + z) + q(x)(y_1 + z)^2 = f(x).$$

Then, because $y_1' + p(x)y_1 + q(x)y_1^2 = f(x)$, for function $z(x)$ we obtain the Bernoulli equation

$$z' + [p(x) + 2q(x)y_1]z + q(x)z^2 = 0.$$

Its general solution plus function $y_1(x)$ gives a general solution of equation (1.34).

Example 1.19 Solve $\frac{dy}{dx} = y^2 - \frac{2}{x^2}$.

Solution. A particular solution of this equation is easy to guess: $y_1 = \frac{1}{x}$. Then $y = z + \frac{1}{x}$ and $y' = z' - \frac{1}{x^2}$. Substituting these y and y' into equation, we obtain $z' - \frac{1}{x^2} = (z + \frac{1}{x})^2 - \frac{2}{x^2}$, or $z' = z^2 + 2\frac{z}{x}$. To solve this Bernoulli equation make the substitution $u = z^{-1}$, which brings to linear equation $\frac{du}{dx} = -\frac{2u}{x} - 1$. A general solution of the corresponding homogeneous equation is

$$\ln|u| = -2\ln|x| + \ln C, \quad \text{thus} \quad u = \frac{C}{x^2}.$$

Then seek a particular solution of inhomogeneous equation in the form $u = c(x)/x$. It leads to $C'(x)/x^2 = -1$, then $C(x) = -x^3/3$ and the particular solution is $u = -x/3$. A general solution is $u = \frac{C}{x^2} - \frac{x}{3}$. It gives $\frac{1}{z} = \frac{C}{x^2} - \frac{x}{3}$, then $\frac{1}{y - \frac{1}{x}} = \frac{C}{x^2} - \frac{x}{3}$ and a general solution of the given equation is

$$y = \frac{1}{x} + \frac{3x^2}{C - x^3}.$$

Problems	Answers
1. $xy' + x^2 + xy - y = 0$	$y = x(Ce^{-x} - 1)$
2. $(2x+1)y' = 4x + 2y$	$y = (2x+1)(\ln\|2x+1\| + C) + 1$
3. $x^2y' + xy + 1 = 0$	$xy = C - \ln\|x\|$
4. $y'\cos x + y\sin x - 1 = 0$	$y = C\cos x + \sin x$
5. $y' = \dfrac{2y + \ln x}{x\ln x}$	$y = C\ln^2 x - \ln x$

6. $(x+y^2)dy = ydx$	$x = y^2 + Cy;\ y = 0$
7. $y'(y^3+2x) = y$	$x = Cy^2 + y^3$
8. $y' + 2y = y^2e^x$	$y = \dfrac{1}{e^x + Ce^{2x}},\quad y = 0$
9. $xy' + 2y + x^5y^3e^x = 0$	$y^{-2} = x^4(2e^x + C),\quad y = 0$
10. $xy^2y' = x^2 + y^3$	$y^3 = Cx^3 - 3x^2$
11. $xy' + 2x^2\sqrt{y} = 4y$	$y = x^4(C - \ln x)^2,\quad y = 0$
12. $xydy = (y^2 + x)dx$	$y^2 = Cx^2 - 2x$
13. $y' = \dfrac{y}{-2x + y - 4\ln y}$	$x = y/3 - 2\ln y + C/y^2 + 1$
14. $(3x - y^2)dy = ydx$	$x = Cy^3 + y^2;\ y = 0$
15. $(1-2xy)y^\circ = y(y-1)$	$(y-1)^2x = y - \ln Cy;\ y = 0;\ y = 1$
16. $2y' - \dfrac{x}{y} = \dfrac{xy}{x^2-1}$	$y^2 = x^2 - 1 + C\sqrt{\lvert x^2 - 1\rvert}$
17. $y'(2x^2y\ln y - x) = y$	$x = \dfrac{1}{y(C - \ln^2 y)}$
18. $y' = \dfrac{x}{x^2 - 2y + 1}$	$x^2 = Ce^{2y} + 2(y-1)$
19. $x^2y' + xy + x^2y^2 = 4$	$y = \dfrac{2}{x} + \dfrac{4}{Cx^3 - x}$
20. $3y' + y^2 + \dfrac{2}{x^2} = 0$	$y = \dfrac{1}{x} + \dfrac{1}{Cx^{2/3} + x}$
21. $xy' - (2x+1)y + y^2 = -x^2$	$y = x + \dfrac{1}{Cx + 1}$
22. $y' - 2xy + y^2 = 5 - x^2$	$y = x + 2 + \dfrac{4}{4Ce^{4x} - 1}$
23. $y' + 2ye^x - y^2 = e^{2x} + e^x$	$y = \dfrac{1}{C - x} + e^x$
24. $y' = \dfrac{1}{2}y^2 + \dfrac{1}{2x^2}$	$y = -\dfrac{1}{x} + \dfrac{2}{x(C - \ln x)}$

1.6 EXACT EQUATIONS

First-order equation $y' = f(x, y)$ always can be written as

$$P(x, y)dx + Q(x, y)dy = 0, \tag{1.35}$$

where $P(x,y)$ and $Q(x,y)$are some functions. If the left-hand side of (1.35) is a total differential of some function $F(x,y)$,

$$dF(x, y) = P(x,y)dx + Q(x, y)dy, \tag{1.36}$$

then equation (1.35) is written as

$$dF(x, y) = 0. \tag{1.37}$$

Such equations are called *exact*. Solution of (1.37) is

$$F(x,y) = C. \tag{1.38}$$

Recall from calculus that in order for $P(x,y)dx + Q(x,y)dy$ be a total differential, it is necessary and sufficient that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1.39}$$

Now write $dF(x,y)$ as

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy. \tag{1.40}$$

Comparison of (1.40) and (1.36) gives (since dx and dy are arbitrary)

$$\frac{\partial F(x, y)}{\partial x} = P(x, y), \tag{1.41}$$

$$\frac{\partial F(x, y)}{\partial y} = Q(x, y). \tag{1.42}$$

Next, integrate (1.41) in x to obtain

$$F(x, y) = \int P(x, y)dx + C(y), \tag{1.43}$$

where $C(y)$ is an arbitrary function of y. (Note that y is considered constant as far as integration in x is a concern.)

Substitution of (1.43) in (1.42) gives

$$\frac{\circ}{\circ y}\left(\int P(x,y)dx\right)+C'(y)=Q(x,y),$$

and thus $C'(y)=Q(x,y)-\frac{\partial}{\partial y}\int P(x,y)dx$. From the last equation $C(y)$ is determined by integration:

$$C(y)=\int\left[Q(x,y)-\frac{\partial}{\partial x}\left(\int P(x,y)dx\right)\right]dy+C_1.$$

Substitution of this $C(y)$ in (1.43) gives $F(x,y)$. Next, equating this $F(x,y)$ to a constant (see (1.38)) gives the final solution of (1.35). Note that the sum of C and C_1 should be combined in a single constant which can be again denoted as C.

Just as we did before for the method of variation of the parameters, in practice it is better not to use the derived formula for the solution. Instead, the solution scheme as described should be applied for each equation to be solved.

Example 1.20

Equation $(3x^2y^2+7)dx+2x^3ydy=0$ is exact because $\frac{d}{dy}(3x^2y^2+7)=\frac{d}{dx}(2x^3y)=6x^2y$. This equation is equivalent to the following equation: $dF(x,y)=0$, which has the solution $F(x,y) = C$.

From $dF(x,y)=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial x}dy$ it follows that $\frac{\partial F(x,y)}{\partial x}=3x^2y^2+7$, thus $F(x,y)=\int(3x^2y^2+7)dx+C(y)$. Since y is considered constant in the integrand, $F(x,y)=x^3y^2+7x+C(y)$.

Substitution of $F(x,y)$ in $\frac{\partial F(x,y)}{\partial y}=2x^3y$ gives $2x^3y+C'(y)=2x^3y$, from which $C'(y)=0$. Thus $C'(y)=C_1$, and the function $F(x,y)$ is $F(x,y)=x^3y^2+7x+C_1$. Equating $F(x,y)$ to C gives the (implicit) solution of the problem:

$$x^3y^2+7x=C.$$

When the left-hand side of (1.35) is not a total differential (and thus the equation is not exact), in some cases it is possible to find the auxiliary function $\mu(x,y)$, called *the integrating factor*. When (1.35) is multiplied by such function, its left-hand side becomes a total differential. That is,

$$dF(x,y) = \mu(x,y)P(x,y)dx + \mu(x,y)Q(x,y)dy.$$

Unfortunately, there is no universally applicable method for the determination of the integrating factor.

As *Reading Exercise*, show that the linear inhomogeneous equation

$$y'+p(x)y=f(x),$$

or $[p(x)y-f(x)]dx+dy=0$, is transformed into an exact equation when it is multiplied by $\mu=e^{\int p(x)dx}$. It must be taken into consideration that the multiplication by $\mu(x,y)$ can lead to extraneous solutions that must be excluded from the final solution. When such extraneous solutions exist, they are simultaneously the solutions of equation $\mu(x,y) = 0$.

Example 1.21

Equation $[x + x^2(x^2 + y^2)]dx + ydy = 0$ is not exact, but after it is multiplied by $\mu = 1/(x^2 + y^2)$ it takes the form

$$\frac{xdx + ydy}{x^2 + y^2} + x^2dx = 0,$$

where the left-hand side is a total differential:

$$dF(x,y) = d\left[\frac{1}{2}\ln(x^2 + y^2) + \frac{x^3}{3}\right].$$

Thus now we are dealing with the equation $dF(x,y) = 0$, which has the solution $F(x,y) = \ln C$, where for convenience we adopt the integration constant $\ln C$ instead of C. Multiplying $F(x,y) = \ln C$ by 2 and choosing to write the arbitrary constant $2\ln C$ as $\ln C$, we obtain

$$\ln(x^2 + y^2) + \frac{2x^3}{3} = \ln C,$$

and then

$$(x^2 + y^2)e^{2x^3/3} = C \quad (C>0).$$

Note that $\mu = 1/(x^2 + y^2) \neq 0$; thus there are no extraneous solutions.

Remark: Equation (1.42):

$$\frac{\partial F(x,y)}{\partial y} = Q(x,y),$$

where $Q(x,y)$ is given and $F(x,y)$ is the unknown function to be determined, is the example of the first-order *partial differential equation* (PDE). Solution of this equation,

$$F(x,y) = \int P(x,y)dx + C(y),$$

contains $C(y)$—an arbitrary function of y. The general solution of a second-order PDE would contain two arbitrary functions, and so on—i.e., the number of arbitrary functions in the general solution is n, where n is the order of the equation.

Example 1.22

Solve second-order PDE

$$\frac{\partial^2 z(x,y)}{\partial x \partial y} = 0.$$

Integration in x gives $\partial z(x,y)/\partial y = \varphi(y)$, where $\varphi(y)$ is an arbitrary function of y. Next, integrate in y to obtain

$$z(x,y) = \int \circ (y)dy + f_1(x),$$

where $f_1(x)$ is an arbitrary function of x. Let denote $\int \circ (y)dy = f_2(y)$—this function is also an arbitrary function of y (since $\varphi(y)$ is arbitrary). At last,

$$z(x,y) = f_1(x) + f_2(y).$$

Problems	Answers
1. $2xydx + (x^2 - y^2)dy = 0$	$3yx^2 - y^3 = C$
2. $(2 - 9xy^2)xdx + (4y^2 - 6x^3)ydy = 0$	$x^2 - 3x^3y^2 + y^4 = C$
3. $3x^2(1 + \ln y)dx = (2y - x^3/y)dy$	$x^3(1 + \ln y) - y^2 = C$
4. $2x(1 + \sqrt{x^2 - y})dx - \sqrt{x^2 - y}dy = 0$	$x^2 + \frac{2}{3}(x^2 - y)^{3/2} = C$
5. $e^{-y}dx - (2y + xe^{-y})dy = 0$	$xe^{-y} - y^2 = C$
6. $(y\cos x - x^2)dx + (\sin x + y)dy = 0$	$y\sin x - x^3/3 + y^2/2 = C$
7. $(e^y + 2xy)dx + (e^y + x)xdy = 0$	$e^y x + x^2 y = C$
8. $(1 + y^2 \sin 2x)dx - 2y\cos^2 xdy = 0$	$x - y^2\cos^2 x = C$
9. $\frac{y}{x}dx + (y^3 + \ln x)dy = 0$	$4y\ln x + y^4 = C$

1.7 EQUATIONS UNRESOLVED FOR DERIVATIVE (INTRODUCTORY LOOK)

Consider first-order equation in its general form, unresolved for y':

$$F(x, y, y') = 0. \tag{1.44}$$

If this equation can be solved for derivative y', we obtain one or several equations

$$y_i' = f_i(x, y), \quad i = \overline{1,m}. \tag{1.45}$$

Integrating these equations we obtain the solution of equation (1.44). The complication comparing to the equation (1.44) is that equations (1.45) gives several direction fields.

The Cauchy problem for equation (1.44) is formulated in the same way as for equation $y' = f(x,y)$: find a particular solution of equation (1.44) satisfying the initial condition

$$y(x_0) = y_0.$$

If the number of solutions of this problem is the same as the number of functions, $f_i(x,y)$, one say that the Cauchy problem has the unique solution. In other words, through point (x_0, y_0) in the given direction (determining by each of the values $f_i(x_0, y_0)$) passes not more than one integral curve of equation (1.44). Such solutions are called *regular*. If in each point of the solution the uniqueness is not valid, such a solution is called *irregular*.

Consider equations quadratic with respect of ***y*′**:

$$(y')^2 + 2P(x,y)y' + Q(x,y) = 0. \tag{1.46}$$

Solve (1.46) for y':

$$y' = -P(x,y) \pm \sqrt{P^2(x,y) - Q(x,y)}. \tag{1.47}$$

This expression is defined for $P^2 - Q \circ 0$. Integrating (1.47) we find a general integral of equation (1.46).

An irregular solution could be only the curve

$$P^2(x,y) - Q(x,y) = 0,$$

which also should satisfy the system of equations for ***y*′**:

$$\begin{cases} (y')^2 + 2P(x,y)y' + Q(x,y) = 0, \\ y' + P(x,y) = 0. \end{cases} \tag{1.48}$$

Example 1.23

Find a general solution of equation

$$(y')^2 - 2(x+y)y' + 4xy = 0.$$

Solving this quadratic equation for y', we obtain $y' = (x+y) \pm (x-y)$, thus $y' = 2x$ and $y' = 2y$. Integrating each of these equations we find

$$y = x^2 + C, \quad \text{and} \quad y = Ce^{2x}.$$

Each of these families of general solutions (shown in Figure 1.7) satisfy the initial equation. Obviously, these solutions are regular. A irregular solution could arise if $(x - y) = 0$, but $y = x$ does not satisfy the given differential equation.

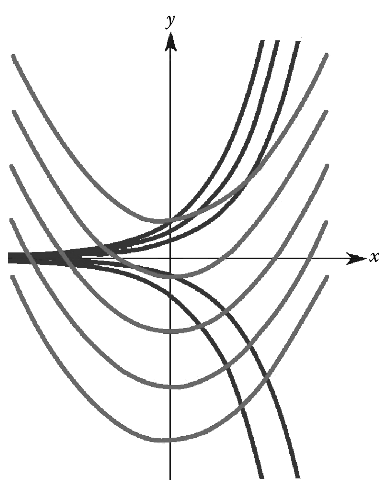


FIGURE 1.7 Two families of solutions for Example 1.23.

Example 1.24

Find a general solution of equation

$$(y')^2 - 4x^2 = 0$$

and find integral curves passing through points *a*) $M(1,1)$, *b*) $O(0,0)$.

From equation one finds $y' = 2x$, $y' = -2x$. Integration gives

$$y = x^2 + C, \ y = -x^2 + C.$$

These general integrals are two families of parabolas and there are no irregular solutions.
Then solve the Cauchy problem:

a) Substituting the initial condition $x_0 = 1$, $y_0 = 1$ in general solution $y = x^2 + C$ we obtain $C = 0$, thus $y = x^2$; substituting the initial condition in $y = -x^2 + C$ we obtain $C = -2$, thus $y = -x^2 + 2$. Therefore, through point $M(1,1)$ pass *two* integral curves: $y = x^2$ and $y = -x^2 + 2$. Because at this point these solutions belong to different direction fields, the uniqueness of Cauchy problem is not violated.
b) Substituting the initial condition $x_0 = 1$, $y_0 = 1$ in general solutions gives $y = x^2$ and $y = -x^2$. Besides that, the solutions are

$$y = \begin{cases} x^2, & x \le 0, \\ -x^2, & x \ge 0 \end{cases} \quad \text{and} \quad y = \begin{cases} -x^2, & x \le 0, \\ x^2, & x \ge 0. \end{cases}$$

The uniqueness of solution in point $O(0,0)$ is violated because the direction field in this through which two integral curves pass, is the same: $y_0' = 0$.

Example 1.25

Find a general solution of equation

$$e^{y'} + y' = x.$$

This is the equation of the type $F(x, y') = 0$, but it is resolved with respect of *x*. In such cases it is useful to introduce a parameter in the following way: $y' = t$. In this example it gives $x = e^t + t$. As the result we have a parametric representation of the given equation in the form

$$x = e^t + t, \ y' = t.$$

This gives

$$dy = y°dx = t(e^t + 1)dt, \quad y = \int t(e^t + 1)dt + C,$$

$$y = e^t(t - 1) + t^2 / 2 + C,$$

from there

$$x = e^t + t, \quad y = e^t(t-1) + t^2/2 + C.$$

Problems	Answers
1. $(y')^2 - 2xy' = 8x^2$	$y = 2x^2 + C;\ y = -x^2 + C$
2. $y'(2y - y') = y^2 \sin^2 x$	$\ln Cy = x \pm \sin x;\ y = 0$
3. $xy'(xy' + y) = 2y^2$	$y = Cx;\ y = \frac{C}{x^2}$
4. $(y')^2 + 2yy' \cot x - y^2 = 0$	$y = C/(1 \pm \cos x)$
5. $(y')^3 + 1 = 0$	$y = -x + C$
6. $(y')^4 - 1 = 0$	$y = x + C, \quad y = -x + C$
7. $x = (y')^3 + 1 = 0$	$x = t^3 + 1, \quad y = \frac{3}{4}t^4 + C$
8. $x(y')^3 = 1 + y'$	$x = \frac{1+t}{t^3}, \quad y = \frac{3}{2t^2} + \frac{2}{t} + C$

1.8 EXAMPLES OF PROBLEMS LEADING TO DIFFERENTIAL EQUATIONS

Example 1.26: Motion of a body under the action of a time-dependent force: slowing down by friction force proportional to velocity.

Newton's law of a linear motion of a body of mass m is the second-order ordinary differential equation for the coordinate $x(t)$:

$$m\frac{dx^2(t)}{dt^2} = F(t, x, x'). \tag{1.49}$$

Consider situation when F is a friction force proportional to the velocity, i.e., $F = -k\text{v}(t)$, where k is a (positive) coefficient. Negative sign in this relation indicates that force and velocity have opposite directions.

Since $x'(t) = \text{v}(t)$, equation (1.49) can be written as the differential equation of first-order for v(t):

$$m\frac{d\text{v}(t)}{dt} = -k\text{v}(t).$$

Velocity decreases; thus the derivative $dv/dt < 0$. In this equation variables are separated:

$$\frac{d\text{v}}{\text{v}} = -\frac{k}{m}dt.$$

Integration gives the general solution:

$$v(t) = Ce^{-kt/m}.$$

Let $t_0 = 0$ be the time instant when the action of the force started, and let v_0 be the velocity at $t = 0$. Then the initial condition is $v(0) = v_0$. Substitution in the general solution gives $C = v_0$, and the particular solution thus is

$$v(t) = v_0 e^{-kt/m}, \tag{1.50}$$

i.e., velocity decreases exponentially with time, and does so faster if k is larger and m is smaller.

The coordinate can be obtained integrating $v(t)$:

$$x(t) = \int_0^t v(t)\,dt = \frac{v_0 m}{k}(1 - e^{-kt/m}). \tag{1.51}$$

From there it is seen that the distance covered before the mass comes to rest is $v_0 m / k$.

Example 1.27: Cooling of an object.

Let the initial object temperature be T_0, and the temperature of the surrounding medium be $T_1 = \text{const}$ (Figure 1.8). The task is to determine the dependence of the object's temperature $T(t)$ on time.

From the experiments it is known that the speed of cooling, dT/dt, is proportional to a difference $T - T_1$ of the object's and the medium's temperatures:

$$\frac{dT}{dt} = -k(T - T_1). \tag{1.52}$$

Here $k > 0$ is a coefficient, and the negative sign in (1.52) is due to the decreasing temperature—the derivative $dT/dt < 0$.

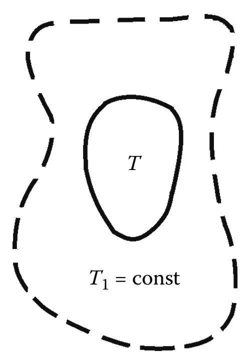


FIGURE 1.8 An object cooling down in the surrounding medium.

In equation (1.52) variables are separated:

$$\frac{dT}{(T - T_1)} = -kdt.$$

Integration gives the general solution:

$$T - T_1 = Ce^{-kt}.$$

Substituting the initial condition $T(0) = T_0$, we find $C = T_0 - T_1$. Substituting this constant in the general solution gives the particular solution:

$$T(t) = T_1 + e^{-kt}(T_0 - T_1). \tag{1.53}$$

Now suppose that the object is heated up. Then $T < T_1$ and the right-hand side of (1.52) has the positive sign. The solution is still given by formula (1.53) where $T_0 - T_1 < 0$.

Equation (1.51) also describes diffusion into a surrounding medium, where concentration of the diffusing substance is constant.

Example 1.28: Radioactive decay.

In radioactive decay, the mass m of a radioactive substance decreases with time, and the speed of decay, $dm(t)/dt$, is proportional to a mass that has not yet disintegrated. Thus, the radioactive decay is described by an equation

$$\frac{dm(t)}{dt} = -km. \tag{1.54}$$

Coefficient $k > 0$ depends on the type of radioactive material, and the negative sign corresponds to the decrease of a mass with time.

Separating variables in (1.54) and integrating, we find the general solution of this equation:

$$m(t) = Ce^{-kt}.$$

Let the initial condition be $m(0) = m_0$, where m_0 is the initial mass of a material. Substitution in the general solution gives $C = m_0$, thus the particular solution is

$$m(t) = m_0 e^{-kt}. \tag{1.55}$$

Half-life is the period of time it takes to decrease the mass by half in radioactive decay. Denoting this time as $T_{1/2}$, we obtain from (1.55)

$$\frac{1}{2} m_0 = m_0 e^{-kT_{1/2}}, \quad \text{and} \quad T_{1/2} = \frac{1}{k} \ln 2.$$

If $T_{1/2}$ is known (or another, shorter time during which a certain quantity of a material decays), one can find the decay constant k.

If equation (1.54) has a positive sign at the right-hand side, then its solution is the expression (1.55) with the positive sign in the exponent. This solution describes, for instance, the exponential growth of the number of neutrons in the nuclear reactions, or the exponential growth of a bacterial colony in the situation when the growth rate of a colony is proportional to its size and also the bacteria do not die.

Problems

1. For a particular radioactive substance 50% of atoms decays in the course of 30 days. When will only 1% of the initial atoms remain?

   Answer: $t \approx 200$ days.

2. According to experiments, during a year from each gram of radium a portion of 0.44 *mg* decays. After what time will one-half of a gram decay?

   $$\textit{Answer}\text{: } T_{1/2} = \frac{\ln 0.5}{\ln(1-0.00044)} \circ 1570 \quad \text{years.}$$

3. A boat begins to slow down due to water resistance, which is proportional to the boat's velocity. Initial velocity of the boat is 1.5 m/s, after 4 s velocity decreased to 1 m/s. After it begins to slow down (a) when will the boat's velocity become 0.5 m/s? (b) What distance will the boat travel during 4 s? (*Hint*: find first the coefficient of friction.)

   Answer: (a) 10.8 s, 4.9 m.

4. The object cooled down from 100°C to 60°C in 10 min. Temperature of the surrounding air is constant 20°C . When will the object cool down to 25°C?

   Answer: t = 40 min.

5. Let m_0 be the initial mass of salt placed in water of mass M. The rate of solution, $\frac{dm(t)}{dt}$, is proportional to a mass of the salt that has not yet dissolved at time t, and the difference between the saturation concentration, $\bar{m}/M$, and the actual concentration, $(m_0-m)/M$. Thus,

   $$\frac{dm}{dt} = -km\left(\circ\,\frac{\bar{m}}{M} - \frac{m_0-m}{M}\right).$$

   Find a solution to this Cauchy problem.

6. In bimolecular chemical reactions substances A and B form molecules of type C. If a and b are the original concentrations of A and B respectively, and x is the concentration of C at time t, then

$$\frac{dx}{dt} = k(a-x)(b-x).$$

Find a solution to this Cauchy problem.

1.9 SOLUTION OF FIRST-ORDER DIFFERENTIAL EQUATIONS WITH ACCOMPANYING SOFTWARE

In this section we describe the solution of Examples 1.13, Chapter 1 and 5.1, Chapter 3 using the program **ODE 1st order** (first-order ODE).

Example 1.29 Solve $y' + 5y = e^x$.

Figure 1.9 shows the first screen of the program **ODE 1st order**.

To do this example start with *"Ordinary Differential Equations of the 1st Order"* and then click on *"Data"* and choose *"New Problem."* Then, as shown in Figure 1.10, you should enter the equation, initial conditions, and interval where you want to obtain the solution of IVP. Also, you can enter the analytical solution of IVP to compare with the numerical.

The program solves equation numerically (the solution can be presented in the form of a graph, Figure 1.11, or table, Figure 1.12) and compares with the reader's analytical solution, $y = (5e^{-5x} + e^x)/6$ (in the interface it is denoted as $y^*(x)$ and should be input as shown in the interface).

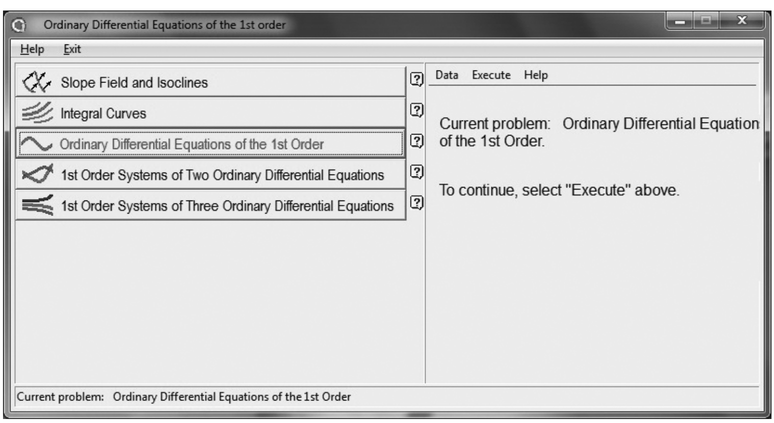


FIGURE 1.9 The first screen of the program ODE 1st order.

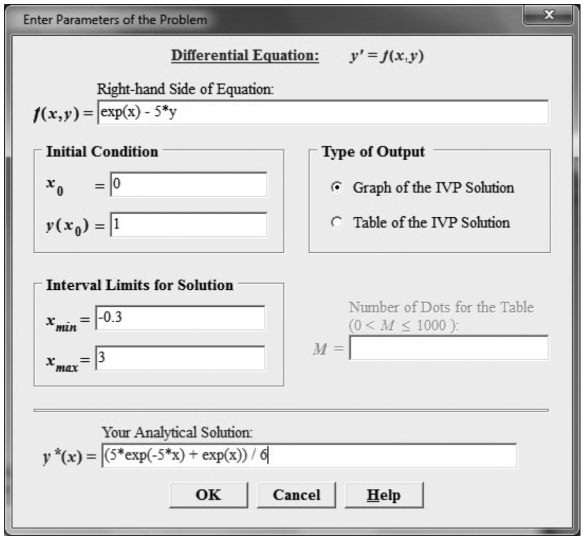


FIGURE 1.10 Equation $y' + 5y = e^x$ with initial condition $y(0) = 1$ in program interface.

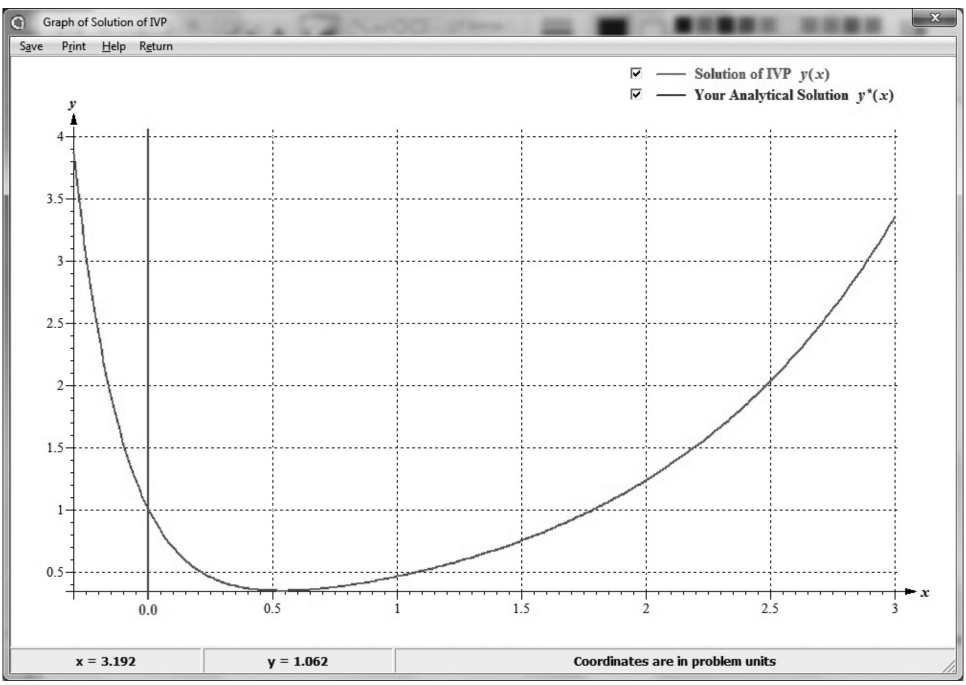


FIGURE 1.11 Solution of equation $y' + 5y = e^x$ with initial condition $y(0) = 1$ presented in graphical form.

k	x_k	$y(x_k)$	$y^*(x_k)$
0	-0.3	3.85821	3.85821
1	-0.2	2.40169	2.40169
2	-0.1	1.52474	1.52474
3	0	1	1
4	0.1	0.689637	0.689637
5	0.2	0.510133	0.510133
6	0.3	0.410918	0.410918
7	0.4	0.361417	0.361417
8	0.5	0.343191	0.343191
9	0.6	0.345176	0.345176
10	0.7	0.36079	0.36079
11	0.8	0.386187	0.386187
12	0.9	0.419191	0.419191
13	1	0.458662	0.458662
14	1.1	0.5041	0.5041
15	1.2	0.555418	0.555418

FIGURE 1.12 Solution of equation $y' + 5y = e^x$ with initial condition $y(0) = 1$ presented in table form.

Example 1.30

Solve system of two linear nonhomogeneous equations

$$\begin{cases} \dfrac{dx}{dt} = -x + 8y + 1, \\ \dfrac{dy}{dt} = x + y + t \end{cases} \quad \text{with initial conditions } x(0) = 1,\ \ y(0) = -1$$

To do this example begin with "*First Order System of Two Ordinary Differential Equations*" in the starting interface of the program **ODE 1**st **order** and then click on "*Data*" and choose "*New Problem*." Then, as shown in Figure 1.13, you should enter the equations, initial conditions, and interval where you want to obtain the solution of IVP. Also, you can enter the analytical solution of IVP to compare with the numerical.

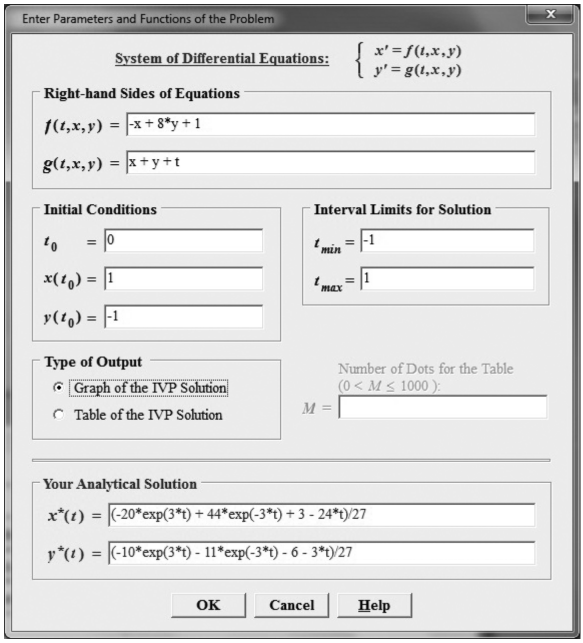


FIGURE 1.13 System of two equations with initial conditions in program interface.

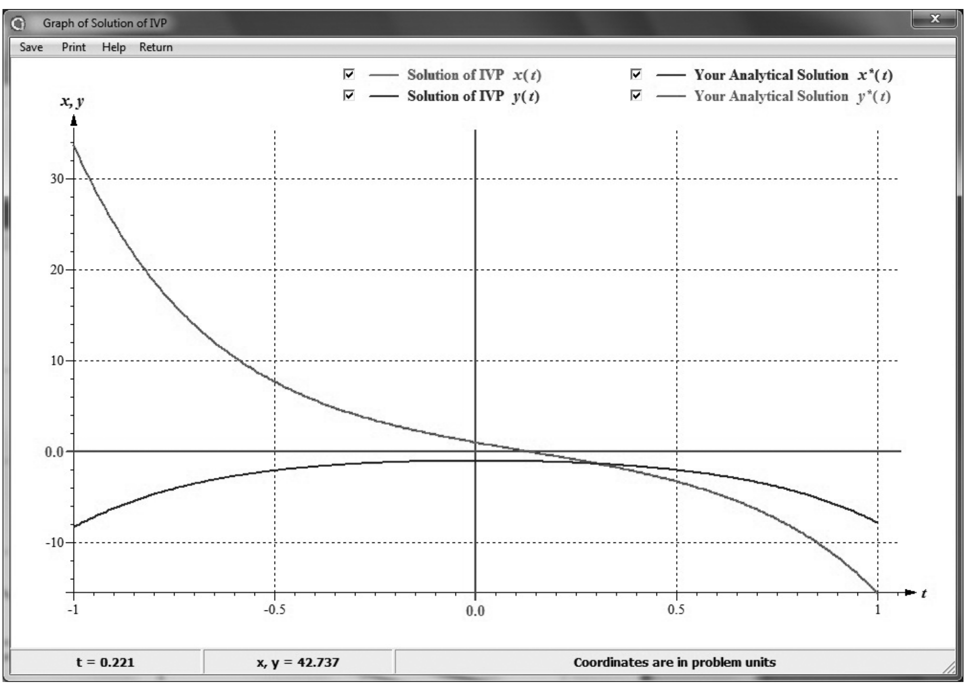
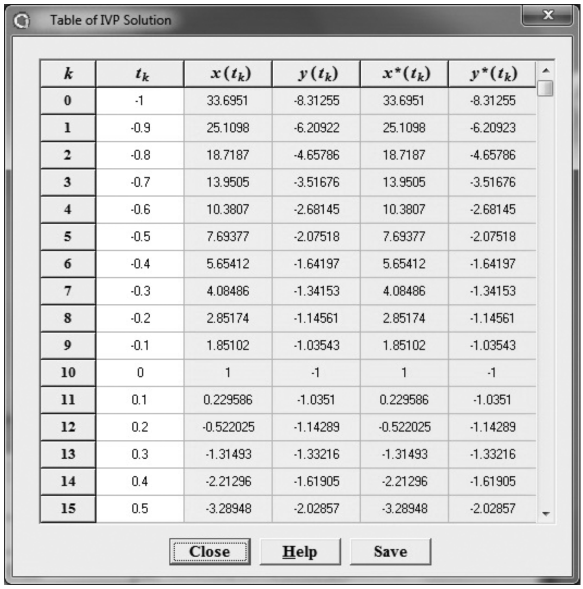


FIGURE 1.14 Solution of IVP problem of Example 5.1 presented in graphical form.



k	t_k	$x(t_k)$	$y(t_k)$	$x^*(t_k)$	$y^*(t_k)$
0	-1	33.6951	-8.31255	33.6951	-8.31255
1	-0.9	25.1098	-6.20922	25.1098	-6.20923
2	-0.8	18.7187	-4.65786	18.7187	-4.65786
3	-0.7	13.9505	-3.51676	13.9505	-3.51676
4	-0.6	10.3807	-2.68145	10.3807	-2.68145
5	-0.5	7.69377	-2.07518	7.69377	-2.07518
6	-0.4	5.65412	-1.64197	5.65412	-1.64197
7	-0.3	4.08486	-1.34153	4.08486	-1.34153
8	-0.2	2.85174	-1.14561	2.85174	-1.14561
9	-0.1	1.85102	-1.03543	1.85102	-1.03543
10	0	1	-1	1	-1
11	0.1	0.229586	-1.0351	0.229586	-1.0351
12	0.2	-0.522025	-1.14289	-0.522025	-1.14289
13	0.3	-1.31493	-1.33216	-1.31493	-1.33216
14	0.4	-2.21296	-1.61905	-2.21296	-1.61905
15	0.5	-3.28948	-2.02857	-3.28948	-2.02857

FIGURE 1.15 Solution of IVP of Example 5.1 presented in the table form.

The program solves the system numerically (the solution can be presented in the form of a graph, Figure 1.14 or table, Figure 1.15) and compares with the reader's analytical solution (in the interface it is denoted as $x^*(t)$ and $y^*(t)$, and should be input as shown in the interface).

CHAPTER 2

Second-Order Differential Equations

2.1 GENERAL CONSIDERATION: *N*TH -ORDER DIFFERENTIAL EQUATIONS

We begin with a general case of the *n*th order differential equation, which is written in the form resolved for the highest derivative of the unknown function $y(x)$:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{2.1}$$

where $f(x, y, y', \ldots, y^{(n-1)})$ is a given function.

Obviously, a general solution (general integral) of equation (2.1) depends on n arbitrary constants. For instance, for the simplest case of equation (2.1):

$$y^{(n)} = f(x),$$

a general solution obtained by consequent integration of the equation n times contains n arbitrary constants as the coefficients in the polynomial of order n–1:

$$y(x) = \int dx \int dx \ldots \int f(x)dx + \sum_{i=0}^{n-1} C_i x^i.$$

When equation (2.1) describes some phenomena, these constants are related to a concrete situation. For instance, consider Newton's second law for a one-dimensional motion of a body of mass m moving under the action of a force F: $m\frac{d^2x}{dt^2} = F$. Integrating, we obtain $x'(t) = \int (F/m)dt + C_1$, then assuming constant values of F and m gives $x(t) = Ft^2/2m + C_1 t + C_2$. This is a *general solution* which gives the answer to the problem: $x(t)$ depends quadratically on time. This general solution can be presented as $x(t) = at^2/2 + \mathrm{v}_0 t + x_0$, where $a = d^2x/dt^2 = F/m$ is the acceleration and two arbitrary constants are denoted as v_0 and x_0.

These constants are the location, $x_0 = x(0)$, and velocity, $v_0 = x'(0)$, at time $t = 0$. For the particular values of $x(0)$ and $x'(0)$ given in the concrete situation, we have the *particular solution* of the problem. Most often, the problems given by differential equations demand a concrete particular solution, but general solutions, like in this example, are often necessary and describe general features of the solution.

The Cauchy problem (or IVP) for the nth order equation (2.1) assumes *n initial conditions* at some point x_0:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad y''(t_0) = y_0'', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)}, \tag{2.2}$$

where y_0, y_0', y_0'', …, $y_0^{(n-1)}$ are n real numbers - the values of the function $y(x)$ and its n–1 derivatives at x_0.

Thus, equation (2.1), along with the initial conditions (2.2), constitute the IVP. The following theorem plays the key role.

Picard's Theorem (Existence and Uniqueness of a Solution to the nth-order differential equation): If function f is a continuous function of all its arguments in the vicinity of the point $(x_0, y_0, y_0', \ldots, y_0^{(n-1)})$, and its partial derivatives with respect to $y, y', \ldots, y^{(n-1)}$ are finite in this vicinity, then the unique solution of the Cauchy problem (2.1), (2.2) exists.

2.2 SECOND-ORDER DIFFERENTIAL EQUATIONS

For the second-order equation

$$y'' = f(x, y, y') \tag{2.3}$$

the Cauchy problem has two initial conditions:

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \tag{2.4}$$

where y_0 and y_0' are the given values of $y(x)$ and $y'(x)$ at x_0. The value $y'(x_0)$ gives the angle of the tangent line (the slope) to the solution curve (integral curve) at the point (x_0, y_0): $\alpha = \arctan y_0^c$ (see Figure 2.1).

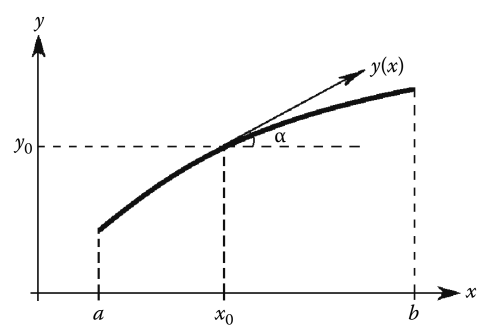


FIGURE 2.1 Solution to the Cauchy problem (2.3), (2.4).

According to the existence and uniqueness theorem, two initial conditions (2.4) are necessary and sufficient to construct a particular solution on some interval $[a,b]$ containing x_0, in which $f(x, y, y')$ is continuous and $\partial f/\partial y$, $\partial f/\partial y°$ are finite. All these restrictions guarantee that the values of y and y' at the point $x_0 + h$ (where h is small) do not differ drastically from y_0 and y_0', respectively.

General solution of (2.3) contains two arbitrary constants, which are usually denoted as C_1 and C_2. The number of initial conditions (2.4) matches the number of arbitrary constants, and plugging the general solution in the initial conditions results in concrete values of C_1 and C_2. Plugging these values in a general solution gives a particular solution of IVP (2.3) and (2.4). The arbitrary constants are often termed parameters, and the general solution is termed *the two-parameter family of solutions*; a particular solution is just one member of this family.

Below we show several simple examples.

Example 2.1

Find the particular solution of IVP $y'' = 1$, $y(1) = 0$, $y'(1) = 1$.

Solution: Integrating the equation twice, we obtain a general solution: $y(x) = x^2/2 + C_1x + C_2$. Substituting this into initial conditions gives $y(1) = 1/2 + C_1 + C_2 = 0$, $y'(1) = 1 + C_1 = 1$, which gives $C_1 = 0, C_2 = -1/2$. Thus, $y(x) = x^2/2 + x$ is the particular solution of IVP for given initial conditions.

Example 2.2

Find the particular solution of IVP $y''' = e^x$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

Solution. Integrating the equation three times, we obtain the general solution: $y(x) = e^x + C_1x^2/2 + C_2x + C_3$. Substituting this into the initial conditions gives $y(0) = 1 + C_3 = 1$, $y'(0) = 1 + C_2 + C_3 = 0$, $y''(0) = 1 + C_1 = 2$. From these three equations we have $C_1 = 1$, $C_2 = -1$, $C_3 = 0$; thus $y(x) = e^x + x^2/2 - x$ is the particular solution of the IVP.

Example 2.3

Find the particular solution of $y'' = 1/(x+2)^2$ for the initial conditions $y(-1) = 1$, $y'(-1) = 2$.

Solution. General solution is $y(x) = -\ln|x+2| + C_1x + C_2$. Notice, that at $x = -2$ it diverges (no solution) since function $f = 1/(x+2)$ is not continuous at $x = -2$, which violates the conditions of the existence and uniqueness theorem. On the other hand, $x = -1$ is a regular point for function f and a particular solution in the vicinity of this point does exist. Substitution of the general solution in the initial conditions gives $y(-1) = -C_1 + C_2 = 1$, $y'(-1) = -1/(-1+2) + C_1 = 2$. Thus, $C_1 = 3$, $C_2 = 4$ and the particular solution is $y(x) = -\ln|x+2| + 3x + 4$ (except the line $x = -2$).

Example 2.4

Find the particular solution of $yy'' - 2y'^2 = 0$ for two sets of initial conditions, (a) $y(0) = 0$, $y'(0) = 2$; (b) $y(0) = 1$, $y'(0) = 0$.

Solution: General solution, $y(x) = 1/(C_1x + C_2)$ will be obtained in Example 2.8—here its correctness can be checked just by substituting this function into equation. The initial condition $y(0) = 0$ obviously contradicts this general solution. This is because when the function $f(x, y, y') = 2y'^2/y$ is written as $y'' = f(x, y, y')$, it has a partial derivative $f_y' = -2y'^2/y^2$, which is infinite at $y(0) = 0$. For the second set of initial conditions f_y' is finite, and for the constants C_1 and C_2 we obtain $C_1 = 0$, $C_2 = 1$; thus $y(x) = 1$.

2.3 REDUCTION OF ORDER

In some cases the order of a differential equation can be reduced—usually this makes the solution easier. In this section we discuss several simple situations that are often encountered in practice.

1. Left side of equation

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{2.5}$$

is a total derivative of some function G, that is, equation (2.5) can be written as

$$\frac{d}{dx} G(x, y, y', \ldots, y^{(n-1)}) = 0.$$

This means that function G is a constant:

$$G(x, y, y', \ldots, y^{(n-1)}) = C_1. \tag{2.6}$$

This is the *first integral* of equation (2.5).

Example 2.5

Solve equation $yy'' + (y')^2 = 0$ using the reduction of order.

Solution. This equation can be written as $\frac{d}{dx}\left(yy'\right) = 0$, which gives $yy' = C_1$, or $ydy = C_1 dx$. Thus, $y^2 = C_1 x + C_2$ is the general solution.

2. Equation does not contain the unknown function y and its first $k - 1$ derivatives:

$$F\left(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}\right) = 0. \tag{2.7}$$

Substitution

$$y^{(k)} = z(x) \tag{2.8}$$

reduces the order of the equation to $(n–k)$.

Example 2.6

Solve $y^{(5)} - y^{(4)}/x = 0$ using the reduction of order.

Solution. Substitution $y^{(4)} = z(x)$ gives $z' - z/x = 0$, thus $z = C_1 x$ and $y^{(4)} = C_1 x$. Integrating four times gives $y = C_1 x^5 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$.

As further illustration, consider the second-order equation

$$F(x, y', y'') = 0. \tag{2.9}$$

Note that y is absent from F. Substitution (2.8) in this case (k = 1) is

$$y' = z(x). \tag{2.10}$$

Then $y'' = z'$ and (2.9) reduces to the first-order equation:

$$F(x, z, z') = 0.$$

Its general solution contains one arbitrary constant, and we can write this solution as $z(x, C_1) = y'$. Integrating this first-order equation, the general solution of (2.9) is $y(x) = \int z(x, C_1)dx + C_2$.

Example 2.7

Solve $y'' - \frac{y'}{x} = x$ using the reduction of order.

Solution. With $y' = z(x)$, we have $y'' = z'(x)$ and the equation becomes $z' - z/x = x$. This is the linear nonhomogeneous first-order equation. Its solution is the sum of the general solution of the homogeneous equation and the particular solution of the non-homogeneous equation. The homogeneous equation is $z' - \frac{z}{x} = 0$, or $\frac{dz}{z} = \frac{dx}{x}$. Its general solution is $z = C_1 x$.

The particular solution of the nonhomogeneous equation can be found using the method of the variation of a parameter: $z = C_1(x)x$. Substituting this into equation $z' - z/x = x$ gives $C_1' x + C_1 - C_1 x/x = x$, $C_1' = 1$, and $C_1(x) = x$. Thus

$$z = C_1\, x + x^2,$$

and finally $y(x) = \displaystyle\int z(x)dx = \int (x^2 + C_1 x)dx = \frac{x^3}{3} + C_1\frac{x^2}{2} + C_2.$

3. Equation does not contain the independent variable (x). For example, the second-order equation is

$$F(y, y', y'') = 0. \tag{2.11}$$

To reduce order, we make the substitution

$$y' = z(y) \tag{2.12}$$

Notice that here z is a function of y, not x. Differentiation using the chain rule gives $y'' = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy}z(y)$. Equation (2.11) becomes $F(y, z'(y), z(y')) = 0$, which has a general solution $z(y, C_1)$. Thus, we arrive to the first-order equation $y'(x) = z(y, C_1)$, where variables can be separated. Then, $\int \frac{dy}{z(y, C_1)} = x + C_2$, which gives the general solution.

Example 2.8

Solve $yy'' - 2y'^2 = 0$ using the reduction of order.

Solution. Differentiation of $z(y) = y'$ gives $y'' = \frac{dz}{dy}\frac{dy}{dx}$, and replacing here y′ by z(y) gives $y'' = z(y)\frac{dz}{dy}$. Equation now reads $yz\frac{dz}{dy} - 2z^2 = 0$. Its solutions are (a) $z(y) = 0$, or $y'=0$, thus y = C; (b) $\frac{dz}{z} = 2\frac{dy}{y}$, then $z = C_1 y^2$, which gives $y' = C_1 y^2$, $dy/y^2 = C_1 dx$, $-1/y = C_1 x + C_2$, $y(x) = -\frac{1}{C_1 x + C_2}$ (sign minus can be omitted).

Example 2.9

Solve IVP, $yy''-2y'^2 = 0$, $y(x_0) = y_0$, $y'(x_0) = y_0'$.

Solution. The general solution (a) y = C of the previous example is a constant continuous function. This solution corresponds to the initial conditions $y(x_0) = C$ (including the value C = 0) and $y'(x_0) = 0$.

General solution (b) $y(x) = -1/\left(C_1 x + C_2\right)$ describes hyperbolas with two asymptotes: vertical, $x = -C_2/C_1$, and horizontal, $y = 0$. If we rewrite the equation in the form $y'' = 2y'^2/y = f$ and find partial derivatives $f_y = -2y'^2/y^2$, $f_{y'} = 4y'/y$, we can notice that f, f_y, $f_{y'}$ are discontinuous at y = 0. At this point the conditions of the existence and uniqueness theorem are not valid.

Treating C_1 and C_2 as parameters, we can think of the solution as a two-parameter set of integral curves, $y(x, C_1, C_2)$. The sets separated by the asymptotes are unrelated, and two initial conditions pick up a particular curve from one of these four sets. If $x_0 < -C_2/C_1$ and $y(x_0) = y_0 > 0$, then we have the solution of IVP in the domain $-\infty < x < -C_2/C_1$, y > 0. There are also three other possibilities.

To illustrate this example, we plot four integral curves, $y(x, C_1, C_2)$, for the case of the vertical asymptote at x = 1.5 (that is, values of C_1 and C_2 in $y(x) = -1/(C_1 x + C_2)$ are chosen such that $-C_2/C_1 = 1.5$). In Figure 2.2, which is obtained using the program ODE 2nd order (second-order ordinary differential equations), the four particular solutions are shown.

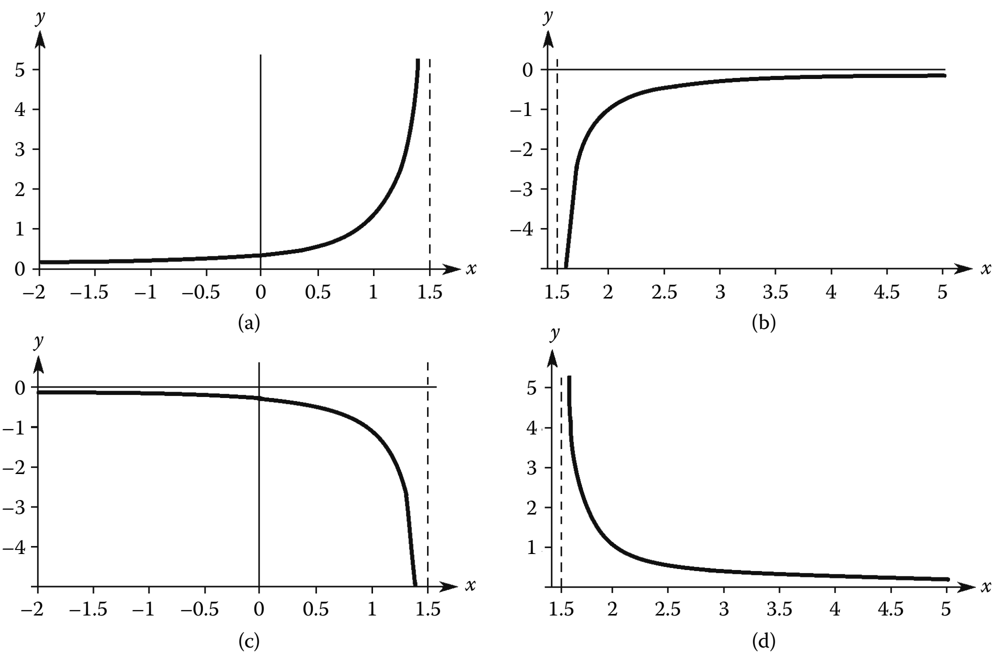


FIGURE 2.2 Four particular solutions for different initial conditions. Dashed line is the vertical asymptote $x = -C_2/C_1 = 1.5$.

1. Using the initial conditions $x_0 = 1$, $y_0 = 1$, $y_0' = 2$ we obtain $C_1 = 2$, $C_2 = -3$, thus the particular solution is $y = -\frac{1}{2x-3}$. This solution exists on the interval $x \subset (-\infty, 1.5)$;
   Similarly:
2. $x_0 = 3$, $y_0 = -1/3$, $y_0^\circ = 2/9$, $C_1 = 2$, $C_2 = -3$, $y' = -\frac{1}{2x-3}$, $x \subset (1.5, \infty)$;
3. $x_0 = 1$, $y_0 = -1$, $y_0^\circ = -2$, $C_1 = -2$, $C_2 = 3$, $y = \frac{1}{2x-3}$, $x \subset (-\infty, 1.5)$;
4. $x_0 = 3$, $y_0 = 1/3$, $y_0' = -2/9$, $C_1 = -2$, $C_2 = 3$, $y = \frac{1}{2x-3}$, $x \subset (1.5, ^\circ)$.

It is obvious from this example how important is to know the analytical solution: without it a numerical procedure may not "detect" the vertical asymptote at $x = -C_2/C_1$, the numerical solution may "slip" to an adjacent integral curve, and thus the obtained solution may be completely unrelated to the initial conditions.

Problems	Answers
1. $(x-3)y^{\infty} + y^\circ = 0$	$y = C_1 \ln\lvert x-3\rvert + C_2$
2. $y^3 y'' = 1$	$C_1 y^2 - 1 = (C_1 x + C_2)^2$
3. $y'^2 + 2yy'' = 0$	$y^3 = C_1 (x + C_2)^2$
4. $y''(e^x + 1) + y' = 0$	$y = C_1(x - e^{-x}) + C_2$
5. $yy'' = y'^2 - y'^3$	$y + C_1 \ln\lvert y\rvert = x + C_2$
6. $x(y'' + 1) + y' = 0$	$y = C_1 \ln\lvert x\rvert - \frac{x^2}{4} + C_2$
7. $y'' = \sqrt{1 - (y')^2}$	$y = C_2 - \cos(x + C_1)$
8. $y''' + 2xy'' = 0$	$y = C_1\left(^\circ \frac{1}{2} e^{-x^2} + x \int e^{-x^2}\, dx\right) + C_2 x + C_3$
9. $y'' + 2y' = e^x y'^2$	$y = -e^{-x} - C_1 x + C_1 \ln\lvert 1 + C_1 e^x\rvert + C_2,\ y = C$
10. $yy'' = (y')^3$	$y \ln\lvert y\rvert + x + C_1 y + C_2 = 0,\ y = C$
11. $(1 - x^2)y^{\infty} + (y^\circ)^2 + 1 = 0$	$y = x + C_1 (x + 2\ln\lvert x-1\rvert) + C_2$
12. $xy''' - y'' = 0$	$y = C_1 x^3 + C_2 x + C_3$

2.4 LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

Linear differential equations arise frequently in applications. We discuss first second-order linear differential equations:

$$y'' + p(x)y' + q(x)y = f(x), \tag{2.13}$$

where $p(x)$, $q(x)$ and $f(x)$are given functions. Linearity of (2.13) with respect to $y'(x)$ and $y(x)$ stems from the observation that this equation does not contain any transcendental