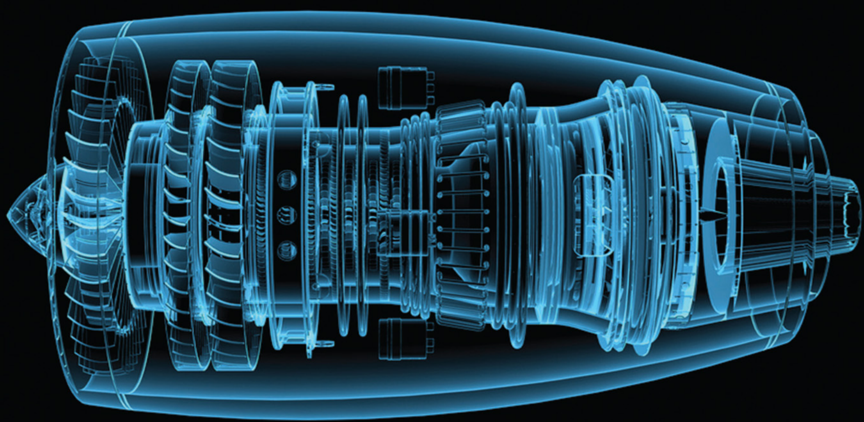
A COURSE IN ORDINARY DIFFERENTIAL EQUATIONS

SECOND EDITION



STEPHEN A. WIRKUS
RANDALL J. SWIFT

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A CHAPMAN & HALL BOOK

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SECOND EDITION

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To our families

Erika Tatiana, Alan, Abdi, and Avani

and

Kelly, Kaelin, Robyn, Erin, and Ryley

for bringing us more joy than math and showing us the true concept

and meaning of

∞ infinity ∞

with their tireless

patience, love, and understanding.

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About the Authors

Stephen A. Wirkus completed his Ph.D. at Cornell University under the direction of Richard Rand. He began guiding undergraduate research projects while in graduate school and came to Cal Poly Pomona in 2000 after being a Visiting Professor at Cornell for a year. He co-founded the Applied Mathematical Sciences Summer Institute (AMSSI), an undergraduate research program jointly hosted by Loyola Marymount University, that ran from 2005 through 2007. He came to Arizona State University in 2007 as a tenured Associate Professor and won the 2013 Professor of the Year Award at ASU as well as the 2011 NSF AGEP Mentor of the Year award. He was a Visiting MLK Professor at the Massachusetts Institute of Technology in 2013-2014. He has guided over 80 undergraduate students in research and has served as Chair for 4 M.S. students, and 2 Ph.D. students. He has over 30 publications and technical reports with over 40 students and has received grants from the NSF and NSA for guiding undergraduate research.

Randall J. Swift completed his Ph.D. at the University of California, Riverside under the direction of M. M. Rao. He began his career at Western Kentucky University and taught there for nearly a decade before moving to Cal Poly Pomona in 2001 as a tenured Associate Professor. He is active in research and teaching, having authored more than 80 journal articles, three research monographs and three textbooks in addition to serving as Chair for 25 M.S. students. Now a Professor, he received the 2011-12 Ralph W. Ames Distinguished Research Award from the College of Science at Cal Poly Pomona. The award honors Swift for his outstanding research in both pure and applied mathematics, and his contributions to the mathematics field as a speaker, journal editor, and principal investigator on numerous grants. He was also a Visiting Professor in 2007-2008 at The Australian National University in Canberra Australia as well as having taught at the Claremont Colleges.

Preface

This book is based on lectures given by the first author at Cal Poly Pomona, Arizona State University (ASU), and the Massachusetts Institute of Technology (MIT), and by the second author at Western Kentucky University (WKU) and California State Polytechnic University–Pomona (Cal Poly Pomona). The text can be used for a traditional one-semester sophomore-level course in ordinary differential equations (such as WKU's MATH 331). However, there is ample material for a two-quarter sequence (such as Cal Poly Pomona's MAT 216-431), as well as sufficient linear algebra in the text so that it can be used for a one-quarter course that combines differential equations and linear algebra (such as Cal Poly Pomona Math 224), or a one-semester course in differential equations that brings in linear algebra in a significant way (such as ASU's MAT 275 or MIT's 18.03 without the PDEs). Most significantly, computer labs are given in MATLAB®,[1] Maple™, and Mathematica at the end of each chapter so the book may be used for a course to introduce and equip the student with a knowledge of the given software (such as ASU's MAT 275). Near the end of this Preface, we give some sample course outlines that will help show the independence of various sections and chapters. The focus of the text is on applications and methods of solution, both analytical and numerical, with emphasis on methods used in the typical engineering, physics, or mathematics student's field of study. We have tried to provide sufficient problems of a mathematical nature at the end of each section so that even the pure math major will be sufficiently challenged.

Key Features

This second edition of the book keeps many of the key features from the first edition:

- MATLAB, Maple, and Mathematica are incorporated at the end of each chapter, helping students with pages of tedious algebra and many of the differential equations topics; the goal of the software is still to show

[1]MATLAB is a registered trademark of The MathWorks, Inc. For product information, please contact:
The Mathworks, Inc.
3 Apple Hill Drive
Natick, MA, 01760-2098 USA
Tel: 508-647-7000
Fax: 508-647-7001
E-mail: info@mathworks.com
Web: www.mathworks.com

students how to make informed use of the relevant software in the field; all three software packages have parallel code and exercises;

- There are numerous problems of varying difficulty for both the applied and pure math major, as well as problems for the nonmathematician (engineers, etc.);
- An appendix that gives the reader a “crash course” in the three software packages; no prior knowledge is assumed;
- Answers to most of the odd problems in the back of the book;
- Chapter reviews at the end of each chapter to help the students review;
- Projects at the end of each chapter that go into detail about certain topics and sometimes introduce new topics that the students are now ready to see;
- An appendix on linear algebra to supplement the treatment within the text, should it be appropriate for the reader/course;
- A full solutions manual for the qualified instructor.

It also incorporates new features, many of which have been suggested by professors and students who have taught/learned from the first edition:

- The computer codes are moved to the end of each chapter as **Computer Labs** to facilitate reading of the book by students and professors who either choose not to use the technology or who do not have access to it immediately;
- The latest software versions are used; significant changes have occurred in certain aspects of MATLAB, Maple, and Mathematica since the first edition in 2006, and the relevant changes are incorporated;
- Much of the linear algebra discussion has been moved to Chapter 5 (from Chapter 3), which deals with linear systems;
- Sections have been added on complex variables (Chapter 3), the exponential response formula for solving nonhomogeneous equations (Chapter 4), forced vibrations (Chapter 4) as well as a subsection on nondimensionalization (Chapter 2), and a combining of the sections on Euler and Runge-Kutta methods (Chapter 2);
- Many rewritten sections highlight applications and modeling within many fields;
- Exercises flow from easiest to hardest;
- Color graphs to help the reader better understand crucial concepts in ordinary differential equations;
- Updated and extended projects at the end of each chapter to reflect changes within the chapters.

Approaches to Teaching Ordinary Differential Equations

The second edition of this book has evolved with our understanding of how to teach the material in the best possible way. Some notable examples from the above list:

1. The structure of the course in covering linear systems in their entirety before covering applications to nonlinear systems (phase plane, etc.) was a direct result of numerous conversations with MIT's Professor Haynes Miller (who frequently teaches MIT's 18.03) as was the incorporation of the new sections on essential topics from complex variables, exponential response and complex replacement (developed by Haynes) for solving nonhomogeneous differential equations, and the s-domain and poles as an important use of Laplace transforms by engineers.
2. Combining the computer codes into Computer Labs at the end of each section rather than having snippets of code embedded throughout the text was a direct result of a switch in ASU's method of teaching this course. Setting aside six class periods for such labs is the way differential equations is now taught at ASU.
3. The presentation of essential linear algebra topics to aid in the understanding of differential equations was helped by discussions with MIT's Professor Gil Strang as well as seeing some of his lectures firsthand.

Most differential equations we have encountered in practice have needed analytical approximations or numerical approximations to gain insight into their behavior. We don't feel that students use technology wisely if they simply ask the computer to solve a given problem. We thus focus on what we consider to be the basics necessary for adequately preparing a student for study in her or his respective fields, including mathematics. We present the syntax from MATLAB, Maple, and Mathematica in Computer Labs at the end of each chapter. We feel that this provides the readers a better understanding of the theory and allows them to gain more insight into real-world problems they are likely to encounter. The vast majority of our students also have *no previous experience with* MATLAB, Maple, or Mathematica and we start from the basics and teach informed use of the relevant mathematical software. The student whom we "typically encounter" has had one year of calculus and is usually a major in a field other than pure mathematics.

Our book is traditional in its approach and coverage of basic topics in ordinary differential equations. However, we cover a number of "modern" topics such as direction fields, phase lines, the Runge-Kutta method, and nondimensionalization in Chapter 2 and epidemiological and ecological models in Chapter 6. As mentioned earlier, we also bring elements from linear algebra, such as eigenvectors, bases, and transformations, in order to best equip the reader of the book for a solid understanding of the material. Besides the Computer Labs there are also Projects at the end of each chapter that give useful insight into past and future topics covered in the book. The topics covered in these projects include a mix of traditional, modeling, numerical, and linear algebra aspects of ordinary differential equations. It is the intent that students who study this book and work *most* of the problems contained in these pages will be very prepared to continue their studies in engineering and mathematics.

Some Sample Course Outlines

While we could not begin to prescribe how this book may best be used for each school, we include some *possible* sections covered for various course outlines. We stress that if you intend to incorporate MATLAB, Maple, or Mathematica into your course, it is crucial to assign Exercises 1-4 (plus a few others) from Appendix A and the Chapter 1 Computer Lab early in the course. Appendix A only requires a knowledge of college algebra and some calculus (Taylor series) while Chapter 1 Computer Lab requires a knowledge of calculus as it is applied to differential equations. Thus both can be assigned within the first 2 weeks of the course (and likely together).

Traditional semester ODE course:

Chap. 1	Chap. 2	Chap. 3	Chap. 4	Chap. 5	Chap. 7	Chap. 8
1.1-1.6	2.1-2.2	3.1-3.3 3.5-3.6	4.1, 4.3 4.5-4.6	5.1 5.4-5.8	7.1-7.4	8.1-8.5

Semester ODE course with modeling or application emphasis:

Chap. 1	Chap. 2	Chap. 3	Chap. 4	Chap. 5	Chap. 6	Chap. 7
1.1-1.4	2.1-2.6	3.1-3.2 3.4-3.7	4.1-4.2 4.4-4.7	5.1, 5.4 5.5, 5.7	6.1-6.5	7.1-7.5

Semester ODE course with linear algebra emphasis and no separate computer labs:

Ch. 1	Ch. 2	Ch. 3	Ch. 4	Ch. 5	Ch. 6	Ch. 7	App. B
1.1-1.4	2.1-2.2 2.5	3.1-3.2 3.4-3.7	4.1-4.2 4.4, 4.7	5.1-5.5 5.7-5.8	6.1	7.1-7.7	B.1-B.4

Semester ODE course with linear algebra emphasis and 6 computer labs:

Ch. 1	Ch. 2	Ch. 3	Ch. 4	Ch. 5	Ch. 7	Comp. Labs
1.1-1.4	2.1-2.2 2.5	3.1-3.2 3.4-3.7	4.1-4.2 4.4, 4.7	5.1-5.5	7.1-7.6	A& 1, 2, 3, 4, 5&B, 7

Quarter ODE course with linear algebra emphasis:

Ch. 1	Ch. 2	Ch. 3	Ch. 4	Ch. 5	App. B
1.1-1.4	2.1-2.2 2.5	3.1-3.2 3.4-3.7	4.1-4.2 4.7	5.1-5.5	B.1-B.4

Acknowledgments

Students, with their questions both in-class and during office hours, helped shaped this second edition as did those professors who used the first edition and/or provided constructive feedback to us, including Erika Camacho, Andrew Knyazev, Luis Melara, Jenny Switkes, Steven Weintraub, and many others. Various chapters were read by Alexandra Churikova, Maytee Cruz-Aponte, Clay Goggil, and Christine Sowa, and their feedback has been of great help. Mike Pappas, in particular, was a big help in proofreading near-final drafts of several chapters. Valerie Cheathon provided a valuable check of all the codes as did Joshua Grosso (MATLAB) and Alan Wirkus-Camacho (Maple and Mathematica). Scott Wilde, again, provided invaluable help in revising the solutions manual.

As texts based upon lecture notes seemingly develop, many of the examples, exercises, and projects have been collected over many years for various courses taught by both authors. Some were taken from others' textbooks and papers. We have tried to give proper credit throughout this text; however, it was not always possible to properly acknowledge the original sources. It is our hope that we repay this explicit debt to earlier writers by contributing our (and their) ideas to further student understanding of differential equations.

We particularly wish to thank our production coordinator, Jessica Vakili, as well as Michele Dimont, Amy Blalock, Hayley Ruggieri, and Sherry Thomas. Bob Stern and Bob Ross, our editors at Chapman & Hall/CRC Press, both deserve a big thanks for believing in this project and for helpful guidance, advice and patience. We sincerely thank all these individuals; without their assistance this text would not have succeeded.

URL for typos and errata:

`http://www.public.asu.edu/~swirkus/ACourseInODEs`

Finally, we would appreciate any comments that you might have regarding this book.

Stephen A. Wirkus (e-mail: `swirkus@asu.edu`*)*
Randall J. Swift (e-mail: `rjswift@csupomona.edu`*)*

From the first edition:

We owe a very special thanks to Erika Camacho (Arizona State University) for her help in writing the MATLAB and Maple code for this book and for detailed suggestions on numerous sections. John Fay and Gary Etgen reviewed earlier drafts of this text and provided helpful feedback. Scott Wilde provided valuable assistance in writing and preparing the solutions manual for the book. We owe a big thanks to our former students David Monarres, for help in preparing portions of this book, and Walter Sosa and Moore Chung, for their help in preparing solutions. We would also like to acknowledge our Cal Poly Pomona colleagues Michael Green, Jack Hofer, Tracy McDonald, Jim McKinney, Siew-Ching Pye, Dick Robertson, Paul Salomaa, Jenny Switkes, Karen Vaughn, and Mason Porter (Caltech/Oxford) for their willingness to use draft versions of this text in their courses and their important suggestions, which improved the overall readability of the text. The faculty and students of AMSSI and MTBI also deserve a special thanks for comments on early drafts of the computer code. Mary Jane Hill assisted us with certain aspects of the text and helped in typesetting some of the chapters of the initial drafts of the book; her effort is greatly appreciated. The production and support staff at Chapman & Hall/CRC Press have been very helpful. We particularly wish to thank our project coordinator Theresa Del Forn and project editor Prudence Board. Our editor Bob Stern deserves a special thanks for believing in this project and for his guidance, advice, and patience. We sincerely thank all these individuals; without their assistance this text would not have succeeded.

A few remarks for students and professors:

This book will succeed if any fears and reservations about learning one of the three computer algebra systems used in this book are put aside. Computers are not here to supplant us, but rather they are here to help illustrate and illuminate concepts and insights that we have. Nothing is foolproof and we stress the importance of *informed use of the relevant mathematical software*. Numerical answers, although quite accurate most of the time, should always be examined carefully because computers are as smart as the programmer allows them to be. There should never be a blind trust in an answer.

It is essential that the technology that you choose—MATLAB, Maple, or Mathematica—be introduced early in the class, just as it is introduced early in the book. While certain mathematical software packages may be better suited for studying differential equations, none have the versatility that the above three programs have to give insight into other areas of mathematics. The two keys to learning the programs are (1) learning the syntax and (2) learning to use the help menus to figure out some of the commands. Setting aside one class, for example, to give a brief tutorial on one of these software packages in the computer lab is a very worthwhile investment. It is by no means necessary and the typical student will be able to learn the material on his/her own by carefully following Appendix A. For reinforcement, it is crucial to include at least one or two technology problems with each homework assignment. The conscientious student will be well prepared to use the same software package in any upper division course in *any* branch of the mathematical sciences and its applications.

It is not necessary to bring computer demonstrations into the classroom. Both authors have taught their courses successfully without classroom demonstrations; handouts sometimes are useful, especially from the appendices. The students, for better or worse, are generally far less afraid of technology than one might expect. If students are sent to the computer lab with an assignment to do and aided with Appendix A, the vast majority will come back with satisfactory answers. Yes, you may bang your head against your desk in frustration at times, but just ask the person next to you for help and also seek the help menus and you will be able to learn MATLAB, Maple, and Mathematica quite well.

Chapter 1

Traditional First-Order Differential Equations

A Very Brief History

The study of Differential Equations began very soon after the invention of Differential and Integral Calculus, to which it forms a natural sequel. In 1676 Newton solved a differential equation by the use of an infinite series, only 11 years after his discovery of the *fluxional* form of differential calculus in 1665. These results were not published until 1693, the same year in which a differential equation occurred for the first time in the work of Leibniz (whose account of the differential calculus was published in 1684).

In the next few years progress was rapid. In 1694–1697 John Bernoulli explained the method of "Separating the Variables," and he showed how to reduce a homogeneous differential equation of the first order to one in which the variables were separable. He applied these methods to problems on orthogonal trajectories. He and his brother Jacob (after whom the "Bernoulli Equation" is named; see Section 1.6.1) succeeded in reducing a large number of differential equations to forms they could solve. Integrating Factors were probably discovered by Euler (1734) and (independently of him) by Fontaine and Clairaut, though some attribute them to Leibniz. Singular Solutions, noticed by Leibniz (1694) and Brook Taylor (1715), are generally associated with the name of Clairaut (1734). The geometrical interpretation was given by Lagrange in 1774, but the theory in its present form was not given until much later by Cayley (1872) and M.J.M. Hill (1888).

Today, differential equations are used in many different fields. They can often accurately capture the behavior of continuous models or a large number of discrete objects where the current state of the system determines the future behavior of the system. Such models are called **deterministic** (as opposed to **stochastic** or **random**). The study of **nonlinear** differential equations is still a very active area of research. Although this text will consider some nonlinear differential equations, here the focus will be on the linear case. We will begin with some basic terminology.

1.1 Introduction to First-Order Equations

Order, Linear, Nonlinear

We begin our study of differential equations by explaining what a differential equation is. From our experience in calculus, we are familiar with some differential equations. For example, suppose that the acceleration of a falling object is $a(t) = -32$, measured in ft/sec^2. Using the fact that the derivative of the velocity function $v(t)$ (measured in ft/sec) is the acceleration function $a(t)$, we can solve the equation

$$v'(t) = a(t) \quad \text{or} \quad \frac{dv}{dt} = a(t) = -32.$$

Many different types of differential equations can arise in the study of familiar phenomena in subjects ranging from physics to biology to economics to chemistry. We give examples from various fields throughout the text and engage the reader with many such applications.

It is clearly necessary (and expedient) to study, independently, more restricted classes of these equations. The most obvious classification is based on the nature of the derivative(s) in the equation. A differential equation involving derivatives of a function of one variable (ordinary derivatives) is called an **ordinary differential equation**, whereas one containing partial derivatives of a function of more than one independent variable is called a **partial differential equation**. In this text, we will focus on ordinary differential equations.

The **order** of a differential equation is defined as the order of the highest derivative appearing in the equation.

Example 1 The following are examples of differential equations with indicated orders:

(a) $dy/dx = ay$ (First-Order)
(b) $x''(t) - 3x'(t) + x(t) = \cos t$ (second order)
(c) $(y^{(4)})^{3/5} - 2y'' = \cos x$ (fourth order)

where the superscript $^{(4)}$ in **(c)** represents the fourth derivative.

Our focus will be on **linear** differential equations, which are those equations that have an unknown function, say y, and each of its higher derivatives appearing in linear functions. That is, we do *not* see them as $y^2, yy', \sin y$, or $(y^{(4)})^{3/5}$.[1] More precisely, a linear differential equation is one in which the dependent variable and its derivatives appear in additive combinations of their first powers. Equations where one or more of y and its derivatives appear in nonlinear functions are called **nonlinear** differential equations. In the above example, only **(c)** is a nonlinear differential equation.

Example 2 Classify the equations as linear or nonlinear.

[1] Most of the equations we consider will involve an unknown function y that depends on x. Two other common variables used are (i) the unknown function y that depends on t and (ii) the unknown function x that depends on t, the latter being used in Example 1b.

(a) $y'' + 3y' - x^2 y = \cos x$
(b) $y'' - 3y' + y^2 = 0$
(c) $y^{(3)} + yy' + \sin y = x^2$

Solution

The first of these equations is linear as it consists of an additive combination of y, y', and y'', each of which is raised to the first power. In contrast to this, the second equation is nonlinear because of the y^2 term. The last equation is nonlinear both because of the yy' term and the $\sin y$ term—either of these terms by itself would have made the equation nonlinear. Our study of nonlinear differential equations will focus on techniques for specific equations or on understanding the qualitative behavior of a nonlinear differential equation, since general techniques of solution are rarely applicable.

Much of this book is concerned with the solutions of linear differential equations. Thus we need to explain what we mean by a solution. First we note that any nth-order differential equation can be written in the form

$$F(x, y, y', ..., y^{(n)}) = 0, \tag{1.1}$$

where n is a positive integer. For example, $y' = x^2 + y^2$ can be written as

$$y' - x^2 - y^2 = 0.$$

Here $F(x, y, y') = y' - x^2 - y^2$. The second-order equation $y'' - 3x^2y' + 5y = \sin x$ can be written as

$$y'' - 3x^2y' + 5y - \sin x = 0$$

and we see that $F(x, y, y', y'') = y'' - 3x^2y' + 5y - \sin x$.

Definition 1.1.1

A *solution* to an nth-order differential equation is a function that is n times differentiable and that satisfies the differential equation. Symbolically, this means that a solution of differential equation (1.1) is a function $y(x)$ whose derivatives $y'(x), y''(x), ..., y^{(n)}(x)$ exist and that satisfies the equation

$$F(x, y(x), y'(x), ..., y^{(n)}(x)) = 0$$

for all values of the independent variable x in some interval (a, b) where

$$F(x, y(x), y'(x), ..., y^{(n)}(x))$$

is defined. (Note that the solution to a differential equation does not contain any derivatives, although the derivatives of this solution exist.) The interval (a, b) may be infinite; that is, $a = -\infty$, or $b = \infty$, or both.

Example 3 The function $y(x) = 2e^{3x}$ is a solution of the differential equation

$$\frac{dy}{dx} = 3y,$$

for $x \in (-\infty, \infty)$ because it satisfies the differential equation by giving an identity:

$$\frac{dy}{dx} = 2\frac{de^{3x}}{dx} = 6e^{3x} = 3y.$$

Initial-Value vs. Boundary-Value Problems

We will soon see that solving a general differential equation gives rise to a solution that has constants. These constants can be eliminated by specifying the initial state of the system or conditions that the solution must satisfy on its domain of definition or "boundary." An example of the first situation is specifying the position and velocity of a mass on a spring. An example of the second is a rope hanging from two supports, given the location of these two supports.

Consider a first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

and suppose that the solution $y(x)$ was subject to the condition that $y(x_0) = y_0$. This is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is called an **initial condition** and x_0 is called the **initial point**. More generally, we have the following:

Definition 1.1.2

An *initial-value problem* consists of an nth-order differential equation together with n initial conditions of the form

$$y(x_0) = a_0, \quad y'(x_0) = a_1, ..., \quad y^{(n-1)}(x_0) = a_{n-1}$$

that must be satisfied by the solution of the differential equation and its derivatives at the initial point x_0.

Example 4 The following are examples of initial-value problems:
(a) $dy/dx = 2y - 3x$, $y(0) = 2$ (here $x = 0$ is the initial point)
(b) $x''(t) + 5x'(t) + \sin(tx(t)) = 0$, $x(1) = 0, x'(1) = 7$ (here $t = 1$ is the initial point).
(Note that the differential equation in **(a)** is linear, whereas the equation in **(b)** is nonlinear.) We define a *solution* to an nth-order initial-value problem as a function that is n times differentiable on an interval (a, b); this satisfies the given differential equation on that interval, and satisfies the n, given initial conditions with the requirement that $x_0 \in (a, b)$. As before, the interval (a, b) might be infinite.

In contrast to an initial-value problem, a **boundary-value problem** consists of a differential equation and a set of conditions *at different x-values* that the solution $y(x)$ must satisfy. Although any number of conditions (≥ 2) may be specified, usually only two are given. Rather than specifying the initial state of the system, we can think of a boundary-value problem as specifying the state of the system at two different physical locations, say $x_0 = a, x_1 = b, a \neq b$.

Example 5 The following are examples of boundary-value problems:
(a) $d^2y/dx^2 + 5xy = \cos x, y(0) = 0, y'(\pi) = 2$
(b) $dy/dx + 5xy = 0, y(0) = y(1) = 2$

Although a boundary-value problem may not seem too different from an initial-value problem, methods of solution are quite varied. We will focus on initial-value problems. We ask whether an initial-value problem has a unique solution. Essentially this is two questions:
1. Is there a solution to the problem?
2. If there is a solution, is it the only one?

As we see in the next two examples, the answer may be “no” to each question.

Example 6 *An initial-value problem with no solution.*

The initial-value problem

$$\left(\frac{dy}{dx}\right)^2 + y^2 + 1 = 0$$

with $y(0) = 1$ has no real-valued solutions, since the left-hand side is always positive for real-valued functions.

Example 7 *An initial-value problem with more than one solution.*

The initial-value problem

$$\frac{dy}{dx} = xy^{1/3}$$

with $y(0) = 0$ has at least two solutions in the interval $-\infty < x < \infty$. Note that the functions

$$y = 0 \text{ and } y = \frac{x^3}{3\sqrt{3}}$$

both satisfy the initial condition and the differential equation.

Two Important Models

One of the most fundamental models in biology deals with population growth and one of the most fundamental models in physics deals with a mass on a spring. In the next two examples, we examine how differential equations describe the behavior of these two phenomena.

Example 8 The change in the population of bacteria with sufficient nutrients and space to grow is known to be proportional to its current population. The differential equation can be written as

$$\frac{dP}{dt} = kP \tag{1.2}$$

where $P(t)$ is the current population of bacteria and k is a constant determined by its growth rate. We can verify that

$$P(t) = P(0)e^{kt} \tag{1.3}$$

is a solution to this differential equation. Because of the presence of the constant $P(0)$, we say that Equation (1.3) is a family of solutions parameterized by the constant $P(0)$. To verify this is indeed a solution we take the derivative to get $P(0)ke^{kt}$. Subsituting this into the left side of the differential equation and the supposed solution into the right side:

$$\underbrace{\frac{dP}{dt}}_{P(0)ke^{kt}} = k \cdot \underbrace{P}_{P(0)e^{kt}}$$

we see that with a slight rearrangement of the expressions underneath, we have equality for all t. Thus (1.3) is a solution to differential equation (1.2) for all t and we see the solution describes the *exponential growth* of the population.

Example 9 In a later chapter we will learn that a mass on a spring moving along a slippery[2] surface can be described by the differential equation

$$mx'' + kx = 0$$

where $x(t)$ is the distance the spring has stretched from its resting length, k is the spring constant, and m is the mass, as shown in Figure 1.1. We can verify that $x = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$ is a solution. To do so we take the second derivative to get $x'' = -\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}\, t\right)$ and substitute it into the equation along with the assumed form of x:

$$m \cdot \underbrace{\left[-\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}\, t\right)\right]}_{x''} + \underbrace{k \cdot \cos\left(\sqrt{\frac{k}{m}}\, t\right)}_{x} = 0.$$

Simplification shows that it is indeed a solution and it holds for all t.

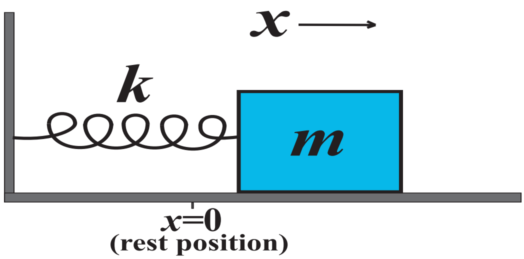

FIGURE 1.1: Model of spring system for Example 9. $x = 0$ marks the position if the spring were at its natural (unstretched) length and we will take x to the right as positive.

In the next several sections we will develop methods for finding solutions to first-order differential equations. We will then discuss existence and uniqueness of solutions in Chapter 2.

Problems

In Problems **1–12**, classify the differential equations by specifying (i) the order, (ii) whether it is linear or nonlinear, and (iii) whether it is an initial-value or boundary-value problem (where appropriate).

[2]Physicists use the word "slippery" to mean "ignore frictional forces."

1. $3y'' + y = \sin x$

2. $y'' = \sin x$

3. $y'' + y' - y = 0$

4. $y'' + 3y' = 0, \quad y(0) = 1, y'(1) = 0$

5. $y^{(3)} + (\sin x)y^{(2)} + y = x, \quad y(0) = 1, y'(0) = 0, y''(0) = 2$

6. $y'' = 0, \quad y(1) = 1, y'(1) = 2$

7. $y' + e^x y = y^4, \quad y(0) = 0$

8. $y'' - 3yy' = x$

9. $y'' + \sin y = 0$

10. $y'' - 4y' + 4y = 0, \quad y(0) = 1, y'(0) = 1$

11. $y'' + e^x y' + y^2 = 0, \quad y(0) = 1, y(\pi) = 0$

12. $x^2 y'' + y' + (\ln x)y = 0$

In Problems **13–24**, verify that the given function is a solution to the differential equation by substituting it into the differential equation and showing that the equation holds true. Assume the interval is $(-\infty, \infty)$ unless otherwise stated. Do NOT attempt to solve the differential equation.

13. $y(x) = 2x^3,\ x\dfrac{dy}{dx} = 3y$

14. $y(x) = x,\ y'' + y = x$

15. $y = 2,\ \dfrac{dy}{dx} = x^3(y-2)^2$

16. $y(x) = x^3,\ \dfrac{dy}{dx} = 3y^{2/3}$

17. $y(x) = e^x - x,\ \dfrac{dy}{dx} + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$

18. $y(x) = \sin x + 2\cos x,\ y'' + y = 0$

19. $y(x) = x^2 - x^{-1},\ x^2 y'' = 2y,\ x \neq 0$

20. $y(x) = x + C\sin x,\ y'' + y = x,\ C =$constant

21. $y(x) = \dfrac{-1}{x-3},\ \dfrac{dy}{dx} = y^2,\ (-\infty, 3)$

22. $y(x) = \dfrac{-1}{5x+4},\ \dfrac{dy}{dx} = 5y^2,\ (-4/5, \infty)$

23. $y_1(x) = e^x$ and $y_2(x) = e^{2x},\ y'' - 3y' + 2y = 0$

24. $y_1(x) = e^x$ and $y_2(x) = xe^x,\ y'' - 2y' + y = 0$

In Problems **25–28**, detetermine which of the functions solve the given differential equation.

25. $y'' + 6y' + 9y = 0$: (a) e^x, (b) e^{-3x}, (c) xe^{-3x}, (d) $4e^{3x}$, (e) $2e^{-3x} + xe^{-3x}$

26. $y'' + 9y = 0$: (a) $\sin 3x$, (b) $\sin x$, (c) $\cos 3x$, (d) e^{3x}, (e) x^3

27. $y'' - 7y' + 12y = 0$: (a) e^{2x}, (b) e^{3x}, (c) e^{4x}, (d) e^{5x}, (e) $e^{3x} + 2e^{4x}$

28. $y'' + 4y' + 5y = 0$: (a) e^{-2x}, (b) $e^{-2x}\sin 2x$, (c) $e^{-2x}\cos 2x$, (d) $\cos 2x$

In Problems **29–32**, find values of r for which the given function solves the differential equation on $(-\infty, \infty)$.

29. $y(x) = e^{rx}, \quad y'' + 3y' + 2y = 0$

30. $y(x) = e^{rx}, \quad y'' + 3y' - 4y = 0$

31. $y(x) = xe^{rx}, \quad y'' + 6y' + 9y = 0$

32. $y(x) = xe^{rx}, \quad y'' + 4y' + 4y = 0$

1.2 Separable Differential Equations

We will now introduce the simplest first-order differential equations. Although these are the simplest class of differential equations we will encounter, they appear in numerous applications and aspects of subsequent theory. We make the following definition:

Definition 1.2.1

A first-order differential equation that can be written in the form

$$g(y)\,y' = f(x) \quad \text{or} \quad g(y)\,dy = f(x)\,dx,$$

where $y = y(x)$, is called a separable differential equation.

Separable differential equations are solved by collecting all the terms involving the dependent variable y on one side of the equation and all the terms involving the independent variable x on the other side. Once this is completed (it may require some algebra), both sides of the resulting equations are integrated. That is, the equation

$$g(y)\,y' = f(x)$$

can be written in "differential form"

$$g(y)\,\frac{dy}{dx} = f(x)$$

so that treating dy/dx as a fraction, we have

$$g(y)\,dy = f(x)\,dx.$$

Here the variables are separated, so that integrating both sides gives

$$\int g(y)\,dy = \int f(x)\,dx. \tag{1.4}$$

The Method of Separation of Variables, which we just applied to (1.4), is the name given to the method we use to solve Separable Equations—it is one of the simplest and most useful methods for solving differential equations. (Incidentally, it is an important technique for solving certain classes of partial differential equations, too.)

Sometimes we will be able to solve (1.4) for y. When we can do so, we will say we can express the **explicit solution** and will write $y = h(x)$. Other times, we will not be able to solve (1.4) or it will not be worth our time and efforts to do so. In these situations, we say that we are giving the **implicit solution** with (1.4). When our solution can be written explicitly, it will be easy to plot solutions in the x-y plane, by hand or with the computer; however, when the solution is implicit, plotting solutions by hand is challenging at best. The various computer programs, discussed in Appendix A and the end of each chapter, will allow us to view plots in the x-y plane without much additional work. We now consider a number of examples.

Example 1 Solve $y' = ky$ where k is a constant.

Solution

Writing y' as $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = ky.$$

Treating $\frac{dy}{dx}$ as a "fraction" and rearranging terms gives

$$\frac{dy}{y} = k\,dx.$$

This step will only be valid if $y \neq 0$. We note that $y = 0$ is also a solution to the original differential equation. Integrating gives

$$\int \frac{dy}{y} = \int k\,dx,$$

which is

$$\ln|y| = kx + C_1, \Longrightarrow |y| = e^{kx+C_1}.$$

This gives

$$y = \pm e^{kx} e^{C_1}.$$

Now e^{C_1} is a positive constant, so that we may let $C = \pm e^{C_1}$. In the above process, we encountered the constant solution $y = 0$, which also gives us the possibility that $C = 0$. Thus, we have

$$y = Ce^{kx} \tag{1.5}$$

as our solution, where $x \in (-\infty, \infty)$ and C is any real constant. We say that (1.5) defines a **one-parameter family of solutions** of $y' = ky$. It is also important to remember the "trick" used above for getting rid of the absolute values—it will come up quite often in practice! We will consider a few more examples with similar standard "tricks."

Example 2 Solve $\dfrac{dx}{dt} = e^{t-x}$, $\quad x(0) = \ln 2$, for $x(t)$.

Solution

Separating the variables gives

$$\frac{dx}{dt} = e^t e^{-x}$$

and thus

$$e^x\,dx = e^t\,dt.$$

Integrating both sides of this equation gives

$$e^x = e^t + C.$$

Solving for x, we have

$$x = \ln|e^t + C|.$$

Applying the initial condition $x(0) = \ln 2$ yields

$$\ln 2 = \ln|1 + C|, \quad \text{so that} \quad C = 1.$$

Thus

$$x = \ln(e^t + 1),$$

which is defined for all t. Note that $e^t + 1$ is always positive so that we can drop the absolute value signs. *We should also note that after integrating, we could have applied the initial condition to determine C and then proceeded to solve for x instead of first solving for x and then applying the initial condition to determine C. Both methods will result in the same final answer.* See Figure 1.2 for a plot of the solution.

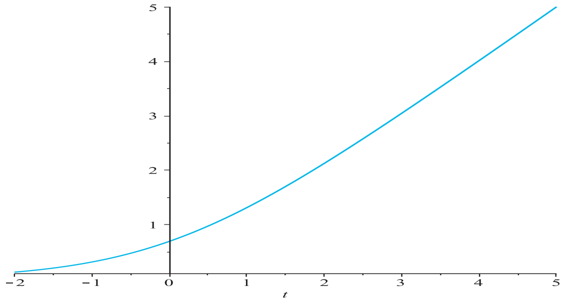

FIGURE 1.2: Plot of solution for Example 2.

Example 3 Solve $(x - 4)\, y^4 - x^3\,(y^2 - 3)\,\dfrac{dy}{dx} = 0.$

Solution

To separate variables, we divide by x^3y^4, which implicitly assumes that $x \neq 0$ and $y \neq 0$. Doing so gives

$$\frac{x-4}{x^3}\,dx = \frac{y^2-3}{y^4}\,dy.$$

This simplifies to $(x^{-2} - 4x^{-3})\,dx = (y^{-2} - 3y^{-4})\,dy$. Integrating gives

$$\frac{-1}{x} + \frac{2}{x^2} = \frac{-1}{y} + \frac{1}{y^3} + C$$

as the general solution, which is valid when $x \neq 0$ and $y \neq 0$. This is definitely a case where giving the solution in an implicit representation is acceptable! See Figure 1.3 for a plot of the implicit solution. We refer the reader to the end of this chapter for the computer code used to plot these types of solutions with one of the software packages. There is, however, a more important idea that is illustrated by this example. If we assume $x \neq 0$ and $y^2 - 3 \neq 0$, we can rewrite the original differential equation as

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)},$$

and then one can clearly see that $y = 0$ is a solution. (That is, when $y = 0$ is substituted into both sides of the equation we get an identity for all x.) This problem shows that the separation process can lose solutions.

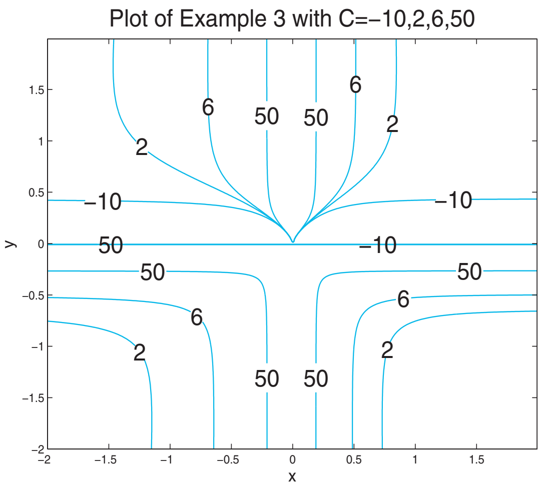

FIGURE 1.3: Implicit plot for Example 3. The curves plotted here satisfy the implicit solution. We note here that the C-values superimposed on the curves were good for this problem, but it often takes ingenuity, experience, trial and error, or some combination of these to get a "nice" picture.

How can we verify that

$$\frac{-1}{x} + \frac{2}{x^2} = \frac{-1}{y} + \frac{1}{y^3} + C$$

is a solution? We need to substitute it into the differential equation as before. This will require us to find y' and we will do so with implicit differentiation. Taking the derivative of both sides of the equation gives

$$\frac{1}{x^2} - \frac{4}{x^3} = \frac{1}{y^2}y' - \frac{-3}{y^4}y'.$$

We solve for y' and then simplify the complex fraction to obtain

$$y' = \frac{y^4(x-4)}{x^3(y^2-3)},$$

which is an equivalent form of our original differential equation.

Although the separation process will work on any differential equation in the form of Definition 1.2.1, evaluating the integrals in (1.4) can sometimes be a daunting, if not impossible, task. As discussed in calculus, certain indefinite integrals such as

$$\int e^{x^2}\,dx$$

cannot be expressed in finite terms using elementary functions. When such an integral is encountered while solving a differential equation, it is often helpful to use definite integration by assuming an initial condition $y(x_0) = y_0$.

Example 4 Solve the initial-value problem

$$\frac{dy}{dx} = e^{x^2}y^2, \quad y(2) = 1$$

and use the solution to give an approximate answer for $y(3)$.

Solution

We would like to divide both sides by y^2 and we note that $y = 0$ is a solution. We set this solution aside and now assume $y \neq 0$, divide by y^2, and integrate from $x = 2$ to $x = x_1$ to obtain

$$\begin{aligned}\int_2^{x_1} [y(x)]^{-2}\,\frac{dy}{dx}\,dx &= -[y(x)]^{-1}\big|_2^{x_1}\\ &= \frac{-1}{y(x_1)} + \frac{1}{y(2)}\\ &= \int_2^{x_1} e^{x^2}\,dx.\end{aligned}$$

If we let t be the variable of integration and replace x_1 by x and $y(2)$ by 1, then we can express the solution to the initial-value problem by

$$y(x) = \frac{1}{1 - \displaystyle\int_2^x e^{t^2}\,dt}.$$

With an explicit solution, we often want to be able to find the corresponding y-value given any x. The right-hand side still cannot be solved exactly but can be approximated if x is given. For example, $y(3) \approx -0.0007007$. We note that we will have a point $x > 2$ that will make the denominator zero (and thus is not in the domain of our solution) and our function will become unbounded.

It is sometimes the case that a substitution or other "trick" will convert the given differential equation into a form that we can solve. A differential equation of the form

$$\frac{dy}{dx} = f(ax + by + k),$$

where a, b, and k are constants, is separable if $b = 0$; however, if $b \neq 0$ the substitution

$$u(x) = ax + by + k$$

makes it a separable equation.

Example 5 Solve

$$\frac{dy}{dx} = (x + y - 4)^2$$

by first making an appropriate substitution.

Solution

We let $u = x + y - 4$ and thus $\frac{dy}{dx} = u^2$. We need to calculate $\frac{du}{dx}$. For this example, taking the derivative with respect to x gives

$$\frac{du}{dx} = 1 + \frac{dy}{dx}.$$

Substitution into the original differential equation gives

$$\frac{du}{dx} - 1 = u^2.$$

This equation is separable. Dividing by $1 + u^2$, we obtain

$$\frac{du}{1 + u^2} = dx$$

and integrating gives

$$\arctan(u) = x + c.$$

Thus $u = \tan(x + c)$. Since $u = x + y - 4$, we then have

$$y = -x + 4 + \tan(x + c),$$

which is defined wherever $\tan(x + c)$ is defined. ■

Problems

In Problems, **1–20**, solve each of the following differential equations. Explicitly solve for $y(x)$ or $x(t)$ when possible.

1. $\frac{dy}{dx} = \cos x$
2. $x\frac{dy}{dx} = (1+y)^2$
3. $x\frac{dx}{dt} + t = 1$
4. $(1+x)\frac{dy}{dx} = 4y$
5. $\tan x \; dy + 2y\, dx = 0$
6. $\frac{dy}{dx} = 2\sqrt{xy}$
7. $4xydx + (x^2+1)dy = 0$
8. $\frac{dy}{dx} = \frac{x^2}{1+y^2}$
9. $y' = 10^{x+y}$
10. $xy' = \sqrt{1-y^2}$
11. $y' = xye^{x^2}, \quad y(0) = 1$. Explain why this differential equation guarantees that its solution is symmetric about $x = 0$.
12. $y' = 2x^2(y^2+1), \quad y(0) = 1$
13. $(e^x+1)\cos y\, dy + e^x(\sin y + 1)\, dx = 0, \quad y(0) = 3$
14. $(\tan x)y' = y, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$
15. $2x(y^2+1)\, dx + (x^4+1)\, dy = 0, \quad y(1) = 1$
16. $(x^2-1)y' + 2xy^2 = 0, \qquad y(\sqrt{2}) = 1$
17. $(y+2)\, dx + y(x+4)\, dy = 0, \quad y(-3) = -1$
18. $8\cos^2 y\, dx + \csc^2 x\, dy = 0, \quad y(\pi/12) = \pi/4$
19. $y' = e^{x^2}, \quad y(0) = 0$
20. $\dfrac{dy}{dx} = \dfrac{y^3+2y}{x^2+3x}, \quad y(1) = 1$

21. Find the solution of the following equation that satisfies the given conditions for $x \to +\infty$: $x^2y' - \cos 2y = 1, \quad y(+\infty) = \frac{9\pi}{4}$.
22. Find the solution of the following equation that satisfies the given conditions for $x \to +\infty$: $3y^2y' + 16x = 2xy^3, \quad y(x)$ is bounded for $x \to +\infty$.

In Problems **23–27** make an appropriate substitution to solve each of the following differential equations. Explicitly solve for $y(x)$ or $x(t)$ when possible.

23. $xydx + (x+1)dy = 0$
24. $y' - y = 2x - 3$
25. $(x+2y)y' = 1 \quad y(0) = -2$
26. $y' = \cos(y-x)$
27. $y' = \sqrt{4x+2y-1}$

28. Suppose that the population $N(t)$ of a given species (bacteria, elves, Toolie birds, college students, etc.) is not always zero and varies at a rate proportional to its current value. That is,

$$\frac{dN}{dt} = rN,$$

where $r \in \mathbb{R}$ is some measured constant proportionality factor. If the initial population is assumed to be $N(0) = N_0 > 0$, solve this exponential differential equation and discuss the behavior of the solution as $t \to \infty$ for different values of r.

29. An equivalent way of thinking of the exponential growth problem 28 is to assume the per capita growth rate, $\frac{1}{N}\frac{dN}{dt}$, is constant. That is, we assume $\frac{1}{N}\frac{dN}{dt} = r$. It is more realistic to assume that the per capita growth rate decreases as the population grows. If we assume this decrease is linear and agrees with the exponential growth model for small populations, we can write the equation

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

where the left-hand side is the per capita growth rate and the right-hand side is a linearly decreasing function in N that has y-intercept r and x-intercept K. Multiplying both sides by N gives

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N,$$

which is the well-known *logistic* differential equation. If the initial population is given as $N(0) = N_0 > 0$, solve this differential equation and discuss the behavior of the solution as $t \to \infty$. From this behavior, why is K called a *carrying capacity*?

1.3 Linear Equations

Linear first-order differential equations are perhaps the most commonly arising class of differential equations in applications. A linear differential equation is defined as follows:

Definition 1.3.1

A first-order ordinary differential equation is linear in the dependent variable y and the independent variable x if it can be written as

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1.6}$$

More generally, we often see equations of the form

$$a_1(x)y' + a_0(x)y = b(x)$$

but, provided $a_1(x) \neq 0$ for all x, we can always divide by $a_1(x)$ and define $P(x) = a_0/a_1$ and $Q(x) = b/a_1$ to obtain an equation of the form of (1.6).

In our work to follow, specifically in Chapters 3 and 4 we will refer to an equation of this form as a "linear nonhomogeneous equation." In the case when $Q(x) = 0$, we refer to the equation as "homogeneous," but we caution the reader to be careful with the word "homogeneous" as it can also have other meanings; see Section 1.6. While it is an unfortunate fact that mathematicians often use the same term for different mathematical notions, our use of it should be clear by context.

In the following pages, we present two techniques for solving linear differential equations. It is likely the case that only one of these methods will be presented in class depending on the emphasis of your course. The first is variation of parameters while the second is the integrating factor technique.

Variation of Parameters

The first method of solving linear equations that we consider has a nice generalization for higher order equations. If we consider (1.6) with $Q(x) = 0$:

$$\frac{dy}{dx} + P(x)\,y = 0$$

we can solve this linear homogeneous equation by using separation of variables. We obtain y_c, the **complementary solution**.[3] We know that we can multiply y_c by any constant and it will still be a solution; however, we instead consider uy_c where u is a function of x and try to find a function u that will make this work. In order for $u(x)y_c$ to be a solution, it needs to satisfy the differential equation. Substituting the assumed solution into (1.6) we obtain

$$(u'(x)y_c + u(x){y_c}') + P(x)u(x)y_c = Q(x), \tag{1.7}$$

which we can regroup and then simplify:

$$\begin{aligned} u'(x)y_c + u(x)\underbrace{[{y_c}' + P(x)y_c]}_{=0} &= Q(x) \\ \implies \quad u'(x)y_c &= Q(x) \end{aligned}$$

since y_c is a solution to the homogeneous equation. We then solve for $u'(x)$ and integrate to obtain:

$$u(x) = \int \frac{Q(x)}{y_c}dx. \tag{1.8}$$

As we only care about finding one function $u(x)$ that will work, we don't introduce the typical $+C$ upon integration. Thus we have found a function

[3]This solution is sometimes called the **homogeneous solution** and is denoted y_h. The terms are used interchangeably.

$u(x)$ that makes uy_c a solution—we call this a particular solution and denote it y_p. Our general solution to (1.6), with $y_p = u(x)y_c$, is then

$$y = Cy_c + y_p, \tag{1.9}$$

where C is a constant that is determined by the initial condition.

Example 1 Solve $\dfrac{dy}{dx} + 2xy = 3x$ using variation of parameters.

Solution

This equation is linear with $P(x) = 2x$ and $Q(x) = 3x$. We solve the homogeneous equation first:

$$\begin{aligned}\frac{dy}{dx} = -2xy &\Rightarrow \int \frac{dy}{y} = \int -2x dx \\ &\Rightarrow \ln|y| = -x^2 + C_1 \\ &\Rightarrow y = Ce^{-x^2}. \end{aligned} \tag{1.10}$$

We now assume that a particular solution can be written as

$$y_p = u(x)y_c = u(x)e^{-x^2}.$$

The function $u(x)$ that will allow this to be a solution of the original linear equation is

$$\begin{aligned} u(x) &= \int \frac{Q(x)}{y_c} dx \\ &= \int \frac{3x}{e^{-x^2}} dx \\ &= 3\int xe^{x^2} dx \\ &= \frac{3}{2}e^{x^2}. \end{aligned} \tag{1.11}$$

Recalling that $y_p = u(x)y_c$, our solution is then given by

$$\begin{aligned} y &= Cy_c + y_p \\ &= \underbrace{Ce^{-x^2}}_{y_c} + \underbrace{\frac{3}{2}e^{x^2}}_{u(x)}\underbrace{e^{-x^2}}_{y_c} \end{aligned} \tag{1.12}$$

which simplifies to

$$y = \frac{3}{2} + Ce^{-x^2}.$$

We can easily check that this is a solution of the original differential equation.

Example 2 Solve

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x},$$

using variation of parameters.

Solution

This is clearly linear and we first solve the homogeneous equation

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = 0.$$

Separation of variables gives us

$$y_c = C\frac{e^{-2x}}{x}.$$

We now assume a particular solution of the form $y_p = u(x)\frac{e^{-2x}}{x}$. From the derivation, we know that things will cancel out so that we need to solve for u in (1.8):

$$u(x) = \int \frac{e^{-2x}}{e^{-2x}/x} = \int x dx$$

so that

$$u(x) = \frac{x^2}{2} \Longrightarrow y_p = u(x)y_c = \frac{x^2}{2}\frac{e^{-2x}}{x} = \frac{1}{2}xe^{-2x}.$$

Our general solution is $y_c + y_p$:

$$y = \frac{C}{x}e^{-2x} + \frac{1}{2}xe^{-2x}.$$

Superposition

A key idea in the study of linear differential equations is that of **superposition**. We have been studying the basic linear equation (1.6)

$$\frac{dy}{dx} + P(x)y = Q(x).$$

We can state a very useful theorem that will serve as an important tool in our further study.

THEOREM 1.3.1 Superposition

Suppose that y_1 is a solution to $y' + P(x)y = Q_1(x)$ and y_2 is a solution to $y' + P(x)y = Q_2(x)$. Then

$$c_1 y_1 \qquad \text{is a solution to} \qquad y' + P(x)y = c_1 Q_1(x)$$

for any constant c_1. For any constants c_1, c_2, we also have that

$$c_1 y_1 + c_2 y_2 \qquad \text{is a solution to} \qquad y' + P(x)y = c_1 Q_1(x) + c_2 Q_2(x).$$

Example 3 Verify that e^{2x} is a solution to $\dfrac{dy}{dx} + y = 3e^{2x}$ and $5x - 5$ is a solution to $\dfrac{dy}{dx} + y = 5x$. Then find a solution to

$$\frac{dy}{dx} + y = e^{2x} + 4x.$$

Solution

We can easily verify that $y_1 = e^{2x}$ and $y_2 = 5x - 5$ are the solutions of the respective differential equations. Let $Q_1(x) = 3e^{2x}$, $Q_2(x) = 5x$, and $Q(x) = e^{2x} + 4x$ denote the right-hand sides of the three differential equations. We observe that

$$Q(x) = \frac{1}{3}Q_1(x) + \frac{4}{5}Q_2(x).$$

By superposition, it follows that

$$y = \frac{1}{3}y_1 + \frac{4}{5}y_2 = \frac{1}{3}(e^{2x}) + \frac{4}{5}(5x - 5)$$

is a solution of $y' + y = Q(x)$.

Integrating Factor Technique

In studying separable equations, we put all the terms of one variable on the left side of the equation and the terms of the other variable on the right side of the equation. This allowed us to integrate functions of just one variable. Another trick that we will use is to rewrite the left side so that it looks like the result of the product rule (from Calculus). To remind ourselves, for y, μ that are both functions of the same variable, the product rule states that

$$(y\mu)' = y'\mu + \mu' y.$$

We know how to integrate the left hand side so the goal is to somehow rewrite part of our equation so that it looks like the right-hand side. Looking at $y'\mu + \mu' y$ and recalling our basic linear equation $y' + Py = Q$, we want to multiply the left-hand side by a function μ that satisfies

$$\mu' = \mu P. \tag{1.13}$$

In this equation, $P = P(x)$ is known, whereas $\mu = \mu(x)$ (called the **integrating factor**) is unknown. We can find $\mu(x)$ because Equation (1.13) is separable. Thus

$$\frac{d\mu}{\mu} = P(x)\,dx.$$

Integrating gives

$$\mu(x) = e^{\int P(x)dx}. \tag{1.14}$$

Since (1.14) is an integrating factor, we have

$$\underbrace{e^{\int P(x)dx}}_{\mu}\underbrace{\frac{dy}{dx}}_{y'} + \underbrace{e^{\int P(x)dx}P(x)}_{\mu'}y = Q(x)e^{\int P(x)dx},$$

which is the same as

$$\frac{d}{dx}\left(\underbrace{e^{\int P(x)dx}}_{\mu}\,y\right) = Q(x)e^{\int P(x)dx}.$$

So

$$e^{\int P(x)dx}y = \int Q(x)e^{\int P(x)dx}dx + C,$$

which gives

$$y = e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)dx}\,dx + C\right) \tag{1.15}$$

as the solution of the differential equation (1.6). Note that we have explicitly written the constant of integration even though the integral has not yet been evaluated. Depending upon your situation, one can memorize the formula (1.15) for the solution of a first-order linear equation; however, it is just as easy (if not out right preferable) to simply apply the method of solution each time.

Summary: Solving linear equations via an integrating factor

1. Write the linear equation in the form of Equation (1.6).

2. Calculate the integrating factor $e^{\int P(x)\,dx}$.

3. Evaluate the integral $\int Q(x)e^{\int P(x)dx}dx$ and then multiply this result by $e^{-\int P(x)\,dx}$.

4. The general solution to (1.6) is

$$y = Ce^{-\int P(x)\,dx} + e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)dx}\,dx.$$

In the event that we are given an initial condition $y(x_0) = y_0$, we would apply it at the time of integration, going from x_0 to a final (general) value x. If we let $\bar{p}(x) = \int P(x)dx$, then the general formula becomes

$$y = Ce^{-\bar{p}(x)} + e^{-\bar{p}(x)} \int Q(x)e^{\bar{p}(x)}\, dx,$$

and applying the initial condition gives us the solution

$$y = y_0 e^{\bar{p}(x_0)-\bar{p}(x)} + e^{-\bar{p}(x)} \int_{x_0}^{x} Q(t)e^{\bar{p}(t)}\, dt, \tag{1.16}$$

where the variable of integration has changed to a dummy variable t.

Example 4 Solve

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}.$$

This is linear with

$$P(x) = \frac{2x+1}{x} \quad \text{and} \quad Q(x) = e^{-2x}$$

so that an integrating factor is

$$\begin{aligned} e^{\int P(x)dx} &= e^{\int \frac{2x+1}{x}\, dx} \\ &= e^{(2x+\ln|x|)} \\ &= |x|e^{2x}. \end{aligned}$$

We note that integrating factors are not unique. For instance, dropping the absolute value to obtain xe^{2x} gives another integrating factor of the differential equation. Thus, multiplying the original equation by this expression gives

$$xe^{2x}\frac{dy}{dx} + e^{2x}(2x+1)y = x.$$

If we had multiplied by $-xe^{2x}$, we would have obtained the *same* equation. This equation can be simplified to give

$$\frac{d}{dx}(xe^{2x}y) = x.$$

Integrating this equation gives

$$xe^{2x}y = \frac{1}{2}x^2 + C,$$

which becomes

$$y = \frac{1}{2}xe^{-2x} + \frac{C}{x}e^{-2x}.$$

These last few steps could have been avoided by using (1.15).

Example 5 Solve $(x^2+1)\dfrac{dy}{dx} + 4xy = x$ with the initial condition $y(0) = 10$.

Solution

Rewriting this equation gives

$$\frac{dy}{dx} + \left(\frac{4x}{x^2+1}\right)y = \frac{x}{x^2+1},$$

hence

$$P(x) = \frac{4x}{x^2+1} \quad \text{and} \quad Q(x) = \frac{x}{x^2+1}$$

so that an integrating factor is

$$\begin{aligned} e^{\int P(x)dx} &= e^{\int \frac{4x}{x^2+1}\,dx} \\ &= e^{\ln(x^2+1)^2} \\ &= (x^2+1)^2. \end{aligned}$$

Once we have our integrating factor, we can use the solution as given in (1.15), first noting that

$$\begin{aligned} e^{-\int P(x)dx} &= e^{-\ln(x^2+1)^2} \\ &= (x^2+1)^{-2}. \end{aligned} \tag{1.17}$$

Then

$$\begin{aligned} y &= \frac{1}{(x^2+1)^2}\left(\int x(x^2+1)\,dx\right) \\ &= \frac{1}{(x^2+1)^2}\left(\frac{1}{4}x^4 + \frac{1}{2}x^2 + C\right). \end{aligned}$$

Now the initial condition, $y(0) = 10$, gives $C = 10$ and thus

$$y = \frac{\frac{1}{4}x^4 + \frac{1}{2}x^2 + 10}{(x^2+1)^2}$$

is the solution we seek.

Now that we know the techniques of solving linear equations, we consider some applications. In Section 1.4, we will consider **Newton's law of cooling** that describes how the temperature of an object changes due to the constant temperature of the medium surrounding it. This is not always realistic, as in some settings the temperature of the surroundings varies. For example, determining the temperature inside a building over a span of a 24-hour day is complicated because the outside temperature varies. If we assume that the building has no heating or air conditioning, the differential equation that needs to be solved to find the temperature $u(t)$ at time t inside the building is

$$\frac{du}{dt} = k(C(t) - u(t)), \tag{1.18}$$

where $C(t)$ is a function that describes the outside temperature and $k > 0$ is a constant that depends on the insulation of the building. Note that (1.18) is a linear equation. According to this equation, if $C(t) > u(t)$, then

$$\frac{du}{dt} > 0,$$

which implies that $u(t)$ increases, and if $C(t) < u(t)$, then

$$\frac{du}{dt} < 0,$$

so that $u(t)$ decreases.

Example 6 Suppose that on a given day during the month of April in Pomona, California, the outside temperature in degrees Fahrenheit is given by

$$C(t) = 70 - 10\cos\left(\frac{\pi t}{12}\right)$$

for $0 \le t \le 24$. Determine the temperature in a building that has an initial temperature of 60°F if $k = 1/4$. See Figure 1.4.

Solution

We see that the average temperature (i.e., the average of $C(t)$) is 70°F because

$$\int_0^{24} \cos\left(\frac{\pi t}{12}\right) dt = 0.$$

The initial-value problem that we must solve is

$$\frac{du}{dt} = k\left(70 - 10\cos\left(\frac{\pi t}{12}\right) - u\right)$$

with initial condition $u(0) = 60$. The differential equation can be rewritten as

$$\frac{du}{dt} + ku = k\left(70 - 10\cos\left(\frac{\pi t}{12}\right)\right),$$

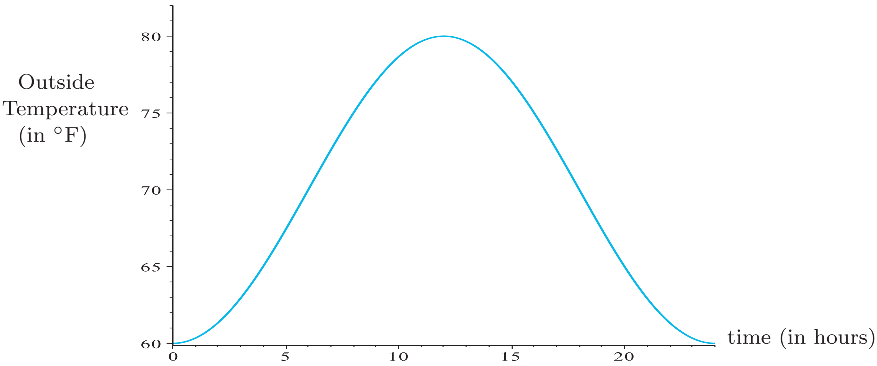

FIGURE 1.4: Outside temperature over 24 hours for Example 6.

which is a linear equation and is thus solvable. This gives (check it!)

$$u(t) = \frac{10}{9+\pi^2}\left(63 + 7\pi^2 - 9\cos\left(\frac{\pi t}{12}\right) - 3\pi\sin\left(\frac{\pi t}{12}\right)\right) + C_1 e^{-t/4}.$$

We then apply the initial condition $u(0) = 60$ to determine the arbitrary constant C_1 and obtain the solution

$$u(t) = \frac{10}{9+\pi^2}\left(63 + 7\pi^2 - 9\cos\left(\frac{\pi t}{12}\right) - 3\pi\sin\left(\frac{\pi t}{12}\right)\right) - \frac{10\pi^2}{9+\pi^2}e^{-t/4}.$$

A graph of this solution is shown in Figure 1.5. The graph shows that the temperature reaches its maximum of about 77°F near $t = 15.5$, which is about 3:30 p.m.

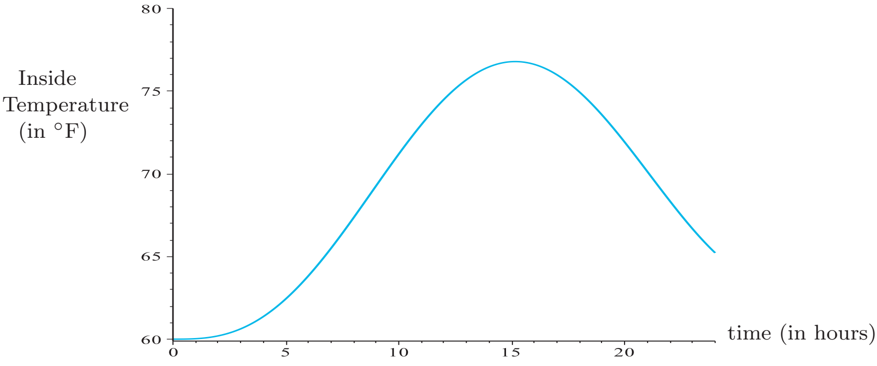

FIGURE 1.5: Inside temperature over 24 hours for Example 6.

Sometimes an equation may not immediately appear to be linear.

Example 7 Consider the differential equation

$$y^2\,dx + (3xy - 1)\,dy = 0.$$

This equation is <u>not</u> linear in y. What do we do? Look harder. If we consider y as the independent variable and x as the dependent variable, we can write

$$\frac{dx}{dy} = \frac{1 - 3xy}{y^2},$$

which is

$$\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2},$$

and we see that it is in the form

$$\frac{dx}{dy} + P(y)\,x = Q(y),$$

which <u>is</u> linear in x, so that this equation can be solved using the theory we have just developed.

Hence, an integrating factor is

$$e^{\int P(y)dy} = e^{\int \frac{3}{y}\,dy} = e^{\ln|y|^3} = y^3.$$

We also have $\exp\left(-\int P(y)dy\right) = 1/y^3$. Then our solution is

$$\begin{aligned} x &= \frac{1}{y^3}\left(\int \frac{1}{y^2}(y^3)dy\right) + \frac{C}{y^3} \\ &= \frac{1}{y^3}\left(\frac{y^2}{2}\right) + \frac{C}{y^3}. \end{aligned}$$

This becomes

$$x = \frac{1}{2y} + \frac{C}{y^3}$$

which is defined for all $y \neq 0$.

Problems

Solve the linear equations in Problems **1–18** by considering y as a function of x, that is, $y = y(x)$.

1. $y' + y = e^x$

2. $y' + 2y = 4$

3. $y' + 2y = -3x$

4. $y' - 2xy = e^{x^2}$

5. $y' - 3x^2 y = x^2$

6. $3xy' + y = 12x$

7. $\dfrac{dy}{dx} + \dfrac{1}{x} y = x$

8. $y' + \dfrac{1}{x} y = e^x$

9. $\dfrac{dy}{dx} - \dfrac{2x}{1+x^2} y = x^2$

10. $xy' + (1+x)y = e^{-x} \sin 2x$

11. $\dfrac{dy}{dx} + y = \cos x$

12. $(2x+1)y' = 4x + 2y$

13. $\dfrac{dy}{dx} - y = 4e^x, \quad y(0) = 4$

14. $y' + 2y = xe^{-2x}, \quad y(1) = 0$

15. $y' + y \tan x = \sec x, \quad y(\pi) = 1$

16. $y' = (1-y)\cos x, \quad y(\pi) = 2$

17. $\frac{dy}{dx} + \frac{y}{x} = \frac{\cos x}{x}$, $y(\frac{\pi}{2}) = \frac{4}{\pi}$, $x > 0$

18. $xy' + 2y = \sin x$, $y\left(\frac{\pi}{2}\right) = 1$, $x > 0$

Solve the linear equations in Problems **19–21** by considering x as a function of y, that is, $x = x(y)$.

19. $(x + y^2)dy = ydx$

20. $(2e^y - x)y' = 1$

21. $(\sin 2y + x \cot y)y' = 1$

Problems **22–23** address aspects of superposition.

22. Recall that a linear equation is called **homogeneous** if $Q(x) = 0$, i.e., if it can be written as

$$\frac{dy}{dx} + P(x)\,y = 0.$$

(a) Show that $y = 0$ is a solution (called the *trivial* solution).
(b) Show that if $y = y_1(x)$ is a solution and k is a constant, then $y = ky_1(x)$ is also a solution.
(c) Show that if $y = y_1(x)$ and $y = y_2(x)$ are solutions, then $y = y_1(x) + y_2(x)$ is a solution.

23. (a) If $y = y_1(x)$ satisfies the homogeneous linear equation $\dfrac{dy}{dx} + P(x)\,y = 0$ and $y = y_2(x)$ satisfies the nonhomogeneous linear equation $\dfrac{dy}{dx} + P(x)\,y = r(x)$, show that $y = y_1(x) + y_2(x)$ is a solution to the nonhomogeneous linear equation

$$\frac{dy}{dx} + P(x)\,y = r(x).$$

(b) Show that if $y = y_1(x)$ is a solution of $\dfrac{dy}{dx} + P(x)\,y = r(x)$, and $y = y_2(x)$ is a solution of $\dfrac{dy}{dx} + P(x)\,y = q(x)$, then $y = y_1(x) + y_2(x)$ is a solution of

$$\frac{dy}{dx} + P(x)\,y = q(x) + r(x).$$

(c) Use the results obtained in parts (a) and (b) to solve

$$\frac{dy}{dx} + 2y = e^{-x} + \cos x.$$

24. A pond that initially contains 500,000 gal of unpolluted water has an outlet that releases 10,000 gal of water per day. A stream flows into the pond at 12,000 gal/day containing water with a concentration of 2 g/gal of a pollutant. Find a differential equation that models this process and determine what the concentration of pollutant will be after 10 days.

25. When wading in a river or stream, you may notice that microorganisms like algae are frequently found on rocks. Similarly, if you have a swimming pool, you may notice that in the absence of maintaining appropriate levels of chlorine and algaecides, small patches of algae take over the pool surface, sometimes overnight. Underwater surfaces are attractive environments for microorganisms because water removes waste and provides a continuous supply of nutrients. On the other hand, the organisms must spread over the surface without being washed away. If conditions become unfavorable, they must be able to free themselves from the surface and recolonize on a new surface.

The rate at which cells accumulate on a surface is proportional to the rate of growth of the cells and the rate at which the cells attach to the surface. An equation describing this situation is given by

$$\frac{dN(t)}{dt} = r(N(t) + A),$$

where $N(t)$ represents the cell density, r the growth rate, A the attachment rate, and t time.
(a) If the attachment rate, A, is constant, solve

$$\frac{dN(t)}{dt} = r(N(t) + A)$$

with the initial condition $N(0) = 0$.
(b) If $A = 3$ in a particular colony of cells, use the following table to find the growth rate at the end of each hour:

t	1	2	3	4
$N(t)$	3	9	21	45

.
Using this growth rate, estimate the algae population size at the end of 24 hours and 36 hours.

26. In Section 3.7, you will learn about electric circuits as an application of a second order differential equation. However, consider the circuit with an inductor and resistor only, whose differential equation is first-order and linear and is given by

$$LI' + RI = V,$$

where I is the to-be-determined current in the circuit, L measures the inductance, R measures the resistance, and V is the constant applied voltage. Find an equation describing the current in the circuit.

27. Suppose $a(t) > 0$, and $f(t) \to 0$ for $t \to \infty$. Show that every solution of the equation
$$\frac{dx}{dt} + a(t)x = f(t)$$
approaches 0 for $t \to \infty$.

28. In the same equation suppose that $a(t) > 0$, and let $x_0(t)$ be the solution for which the initial condition $x(0) = b$ is satisfied. Show that for every positive $\varepsilon > 0$ there is a $\delta > 0$, such that if we perturb the function $f(t)$ and the number b by a quantity less than δ, then the solution $x(t), t > 0$, is perturbed by less than ε. The word **perturbed** is understood in the following sense: $f(t)$ is replaced by $f_1(t)$ and b is replaced by b_1 where
$$|f_1(t) - f(t)| < \varepsilon, \quad |b_1 - b| < \delta.$$
This property of the solution $x(t)$ is called **stability for persistent disturbances**.

1.4 Some Physical Models Arising as Separable Equations

Now that we have studied separable equations in detail, we consider some applications. The wide variety of application problems that we will consider all lead to equations in which variables can be separated.

Free Fall, Neglecting Air Resistance

We will begin this application section with an easy problem from elementary physics. This application should be very familiar.

If $x(t)$ represents the position of a particle at time t, then the velocity of the particle is given by
$$v(t) = \frac{dx}{dt}.$$
Similarly, the acceleration of the particle is
$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Thus, if we consider a particle that is in free fall, where the acceleration of the particle is due to gravity alone, we have
$$a(t) = -g.$$

Here g is assumed to be a constant and we use $-g$ as gravity acts downward. For the moment, we ignore the effects of air resistance. Thus,

$$\frac{dv}{dt} = -g,$$

which is a simple separable equation, so that

$$v(t) = -gt + c.$$

If we assume that the particle has an initial velocity v_0, so that $v(0) = v_0$, then $v(t) = -gt + v_0$. Now this gives the separable equation

$$\frac{dx}{dt} = -gt + v_0$$

which has solution

$$x(t) = \frac{-g}{2}t^2 + v_0 t + C_1.$$

If the particle has initial position x_0, then

$$x(0) = x_0$$

which gives

$$x(t) = \frac{-g}{2}t^2 + v_0 t + x_0 \tag{1.19}$$

as the position $x(t)$ of the particle in free fall, at time t.

Example 1 A man standing on a cliff 60 m high hurls a stone upward at a rate of 20 m/sec. How long does the stone remain in the air and with what speed does it hit the ground below the cliff?

Solution

Here $x_0 = 60$ and $v_0 = 20$. We take $g = 9.8$ m/sec^2. Thus,

$$x(t) = -\frac{9.8}{2}t^2 + 20t + 60$$

and

$$v(t) = -9.8t + 20.$$

The stone is in the air while $x(t) > 0$, so to find the time t that the stone is in the air, we set $x(t) = 0$ and solve for t. Using the quadratic equation,

$$t = \frac{-20 \pm \sqrt{(20)^2 - 4(-4.9)(60)}}{2(-4.9)} = -2.01,\ 6.09.$$

The stone is thus in the air for about 6.1 sec. We use this time to find the velocity upon impact:

$$v(6.1) = -9.8(6.1) + 20 = -39.78 \quad \text{m/sec}.$$

Air Resistance

We will now consider the effects of **air resistance**. The amount of air resistance (sometimes called the **drag force**) depends upon the size and velocity of the object, but there is no general law expressing this dependence. Experimental evidence shows that at very low velocities for small objects it is best to approximate the resistance R as proportional to the velocity, while for larger objects and higher velocities it is better to consider it as proportional to the square of the velocity [38].

By Newton's second law $F = ma$, so that if $v(t)$ is the velocity of the object, we have

$$m\frac{dv}{dt} = F_1 + F_2$$

where F_1 is the weight of the object,

$$F_1 = mg,$$

and F_2 is the force of the air resistance on the object as it falls, so

$$F_2 = k_1 v \quad \text{or} \quad F_2 = k_2 v^2$$

where k_1, k_2 are proportionality constants. Note that $k_i < 0$ because air resistance is always opposite the velocity; see examples 2 and 3 below. We also point out that the units of k_1 and k_2 are different. In SI units, force has units of Newtons = N = kg$\cdot$ m/sec^2. Thus k_1 must have the units of kg/sec. On the other hand, k_2 can be written as

$$k_2 = -\frac{1}{2}C\rho A$$

where ρ is the air density (SI units of kg/m^3), A is the cross-sectional area of the object (SI units of m^2), and C is the drag coefficient (unitless) [38].

Example 2 An object weighing 8 pounds falls from rest toward earth from a great height. Assume that air resistance acts on it with a force equal to $2v$. Calculate the velocity $v(t)$ and position $x(t)$ at any time. Find and interpret $\lim_{t\to\infty} v(t)$.

Solution

Remembering that pounds is a **force** (not a mass), we see that we need to calculate the mass of the object in order to apply Newton's second law. Using $g = 32$ ft/sec^2 gives $m = w/g = 8/32 = 1/4$. Thus by Newton's second law

$$m\frac{dv}{dt} = F_1 + F_2,$$

that is

$$\frac{1}{4}\frac{dv}{dt} = 8 - 2v.$$

This is a separable equation and can be written as

$$\frac{dv}{8-2v} = 4dt$$

so that upon integrating both sides we have

$$-\frac{1}{2}\ln|8-2v| = 4t + c.$$

Using the condition that the object fell from rest, so that $v(0) = 0$, we can determine the constant c and solve for $v(t)$. We have

$$v(t) = 4 - 4e^{-8t}$$

as the velocity of the object at any time. A graph of this velocity is shown in Figure 1.6. Analytically, we see that $v(t)$ approaches 4 as $t \to \infty$. This value is known as the **limiting** or **terminal velocity** of the object.

Now since $\frac{dx}{dt} = v(t)$, we have

$$\frac{dx}{dt} = 4 - 4e^{-8t}.$$

This is easily integrated to obtain $x(t) = 4t + \frac{1}{2}e^{-8t} + c$. If we take the initial position of the object as zero, so that $x(0) = 0$, then

$$x(t) = 4t + \frac{1}{2}e^{-8t} - \frac{1}{2}.$$

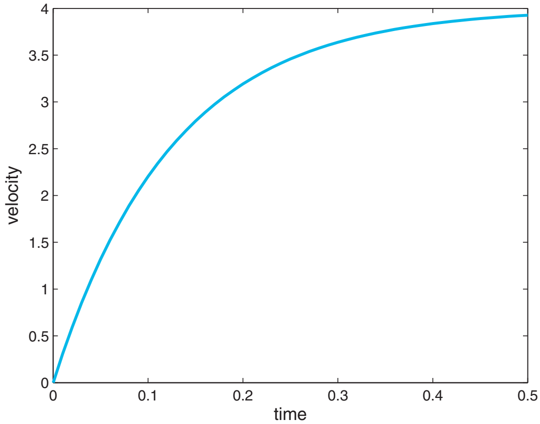

FIGURE 1.6: Approach to terminal velocity of free-falling object of Example 2.

A Cool Problem

In addition to free-fall problems, separable equations arise in some simple thermodynamics applications. One such application is the following example.

Suppose that a pie is removed from a hot oven and placed in a cool room. After a given period of time the pie has a temperature of $150°F$*. We want to determine the time required to cool the pie to a temperature of* $80°F$*, when we can finally enjoy eating it.*

This example is an application of **Newton's law of cooling**, which states

> *the rate at which the temperature* $T(t)$ *changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature* T_s *of the surrounding medium.*

Symbolically we know the rate of change is the derivative and the statement is expressed as

$$\frac{dT}{dt} = k(T - T_s), \tag{1.20}$$

with the initial temperature of the body $T(0) = T_0$ and k a constant of proportionality. We observe that if the initial temperature T_0 is larger than the temperature of the surrounding T_s, then $T(t)$ will be a decreasing function of t (as the body is cooling), so $dT/dt < 0$, but $T_0 - T_s > 0$ so that the proportionality constant k must be negative. A similar analysis with $T_0 < T_s$ also gives $k < 0$. This condition on k also follows by noting that the temperature of the body will approach that of the surrounding medium as time gets large.

To solve (1.20), we seek a function $T(t)$ that describes the temperature at time t. For this equation, separating the variables we have

$$\frac{dT}{T - T_s} = k\,dt.$$

Integrating both sides of this equation gives

$$\int \frac{dT}{T - T_s} = \int k dt.$$

Evaluating both integrals, we obtain

$$\ln |T - T_s| = kt + C,$$

where C is the constant of integration. Exponentiating both sides and simplifying gives

$$|T - T_s| = e^{kt} e^{C} \Longrightarrow T - T_s = \pm e^{C} e^{kt}.$$

Solving for the temperature, we see that

$$T(t) = C_1 e^{kt} + T_s$$

where $C_1 = \pm e^C$. We can then apply the initial condition $T(0) = T_0$, which implies $T_0 = C_1 + T_s$, so that $C_1 = T_0 - T_s$ and the solution is then

$$T(t) = (T_0 - T_s)e^{kt} + T_s. \tag{1.21}$$

We know that the temperature of the body approaches that of its surroundings and this can be seen mathematically as

$$\lim_{t\to\infty} T(t) = T_s,$$

which is true because $k < 0$.

Let's now consider a specific pie-cooling example.

Example 3 Suppose that a pie is removed from a 350°F oven and placed in a room with a temperature of 75°F. In 15 min the pie has a temperature of 150°F. We want to determine the time required to cool the pie to a temperature of 80°F, when we can finally enjoy eating it.

Solution

Comparing with the above derivation, we see that $T_0 = 350$ and $T_s = 75$. Substituting these values in (1.21) gives

$$T(t) = 275e^{kt} + 75.$$

We still need to find k or equivalently e^k, which quantifies how fast the cooling of the pie occurs. We were given the temperature after 15 min, i.e., $T(15) = 150$. Thus

$$275e^{15k} + 75 = 150,$$

and solving for e^k gives

$$e^k = \left(\frac{3}{11}\right)^{1/15},$$

or $k = -0.08662$. Thus

$$T(t) = 275\left(\frac{3}{11}\right)^{t/15} + 75,$$

and this can be used to find the temperature of the pie at any given time. We can also calculate the time it takes to cool to any given temperature. We want to know when $T(t) = 80$°F. Thus we solve

$$275\left(\frac{3}{11}\right)^{t/15} + 75 = 80$$

for t to obtain

$$t = \frac{-15 \ln 55}{\ln 3 - \ln 11} \approx 46.264.$$

Thus, the pie will reach a temperature of 80°F after approximately 46 min.

It is interesting to note that the first term in our equation for the pie temperature satisfies

$$275 \left(\frac{3}{11}\right)^{t/15} > 0$$

for all $t > 0$. Thus

$$T(t) = 275 \left(\frac{3}{11}\right)^{t/15} + 75 > 75.$$

The pie never actually reaches room temperature! This is an artifact of our model; we do note, however, that

$$\lim_{t \to \infty} 275 \left(\frac{3}{11}\right)^{t/15} + 75 = 75,$$

which can also be seen in Figure 1.7.

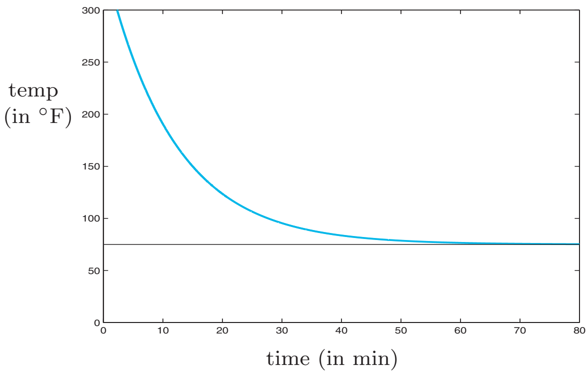

FIGURE 1.7: Graph of pie temperature vs. time of Example 3.

We present another example of Newton's law of cooling from forensic science.

Example 4 In the investigation of a homicide, the time of death is important. The normal body temperature of most healthy people is 98.6°F. Suppose that when a body is discovered at noon, its temperature is 82°F. Two hours later it is 72°F. If the temperature of the surroundings is 65°F, what was the approximate time of death?

Solution

This problem is solved as the last example. Here $T(0)$ represents the temperature when the body was discovered and $T(2)$ is the temperature of the body 2 hours later.

Thus, $T_0 = 82$ and $T_s = 65$ so that (1.21) becomes

$$T(t) = 17e^{kt} + 65.$$

Using $T(2) = 72$, we solve $17e^{2k} + 65 = 72$ for e^k to find

$$e^k = \left(\frac{7}{17}\right)^{1/2}$$

so that

$$T(t) = 17\left(\frac{7}{17}\right)^{t/2} + 65.$$

This equation gives us the temperature of the body at any given time. To find the time of death, we use the fact that the body temperature was at 98.6°F at this time. Thus we solve

$$17\left(\frac{7}{17}\right)^{t/2} + 65 = 98.6$$

for t and find that

$$t = \frac{2\ln(1.97647)}{\ln 7 - \ln 17} \approx -1.53569.$$

This means that the time of death occurred approximately 1.53 hours before being discovered. Therefore, the time of death was approximately 10:30 a.m. because the body was found at noon.

Mixture Problems

Problems involving mixing typically give rise to separable differential equations. A typical mixture problem is given in the following example.

Example 5 A bucket contains 10 L of water and to it is being added a salt solution that contains 0.3 kg of salt per liter. This salt solution is being poured in at the rate of 2 L/min. The solution is being thoroughly mixed and drained off. The mixture is drained off at the same rate so that the bucket contains 10 L at all times. How much salt is in the bucket after 5 min?

Solution

Let $y(t)$ be the number of kilograms of salt in the bucket at the end of t minutes. We need to derive a differential equation for this problem and we do so by considering change in this system over a small time interval. We first find the amount of salt added to the bucket between time t and time $t + \Delta t$. Each minute, 2 L of solution is added so that in Δt minutes, $2\Delta t$ liters is added.

In these $2\Delta t$ liters the amount of salt is

$$0.3 \text{ kg/L} \times (2\Delta t) \text{ L} = (0.6\Delta t) \text{ kg}.$$

On the other hand, $2\Delta t$ liters of solution is withdrawn from the bucket in an interval Δt. Now at time t the 10 L in the flask contains $y(t)$ kilograms of salt. Then $2\Delta t$ of these liters contains approximately $(0.2\Delta t)(y(t))$ kilograms of salt if we suppose that the change in the amount of salt $y(t)$ is small in the short period of time Δt.

We have computed the amount of salt added in the interval $(t, t + \Delta t)$, as well as the amount subtracted in the same interval. But the difference between the amounts of salt present at times $t + \Delta t$ and t is $y(t + \Delta t) - y(t)$, so that we have obtained the equation

$$y(t + \Delta t) - y(t) = 0.6\Delta t - (0.2\Delta t)(y(t)).$$

We now divide by Δt and let $\Delta t \to 0$. The left side approaches the derivative $y'(t)$, and the right side is $0.6 - 0.2y(t)$. The differential equation is thus

$$y'(t) = 0.6 - 0.2y(t), \tag{1.22}$$

which can be thought of as *the rate of change in the number of kilograms of salt in the bucket* $y'(t)$ *being equal to the rate of salt (in kg) flowing into the bucket* 0.6 $(= 0.3$ *kg/L* $\times$ 2 *L) minus the rate of salt flowing out of the bucket* $0.2y(t)$.

Equation (1.22) is a separable equation and can be written as

$$\frac{dy}{0.6 - 0.2y} = dt.$$

Integrating both sides gives

$$\ln|0.6 - 0.2y| = -0.2t + c$$

so that solving for $y(t)$ we obtain

$$y(t) = 3 - Ce^{-0.2t}. \tag{1.23}$$

When t is zero, the amount of salt in the bucket is zero, that is, $y(0) = 0$. Equation (1.23) shows that when $t = 0$, we have

$$y(0) = 3 - C;$$

or $C = 3$. The value of C is now known, so that Equation (1.23) becomes

$$y(t) = 3 - 3e^{-0.2t}.$$

To find y at the end of 5 min, we simply substitute $t = 5$ so that the amount of salt in the bucket is $y(5) \approx 1.9$ kg.

Problems

In Problems **1–7** it will be convenient to take the velocity to be the unknown function.

1. A ball dropped from a building falls for 4.00 sec before it hits the ground. If air resistance is neglected, answer the following questions:
(a) What was its final velocity just as it hit the ground?
(b) What was the average velocity during the fall?
(c) How high was the building?

2. You drop a rock from a cliff, and 5.00 sec later you see it hit the ground. Neglecting air resistance, how high is the cliff?

3. A ball thrown straight up climbs for 3.0 sec before falling. Neglecting air resistance, with what velocity was the ball thrown?

4. Iron Man is flying at treetop level near Paris when he sees the Eiffel Tower elevator start to fall (the cable snapped). He knows Pepper Potts is inside. If Iron Man is 2 km away from the tower, and the elevator falls from a height of 350 m, how long does he have to save Pepper, and what must be his average velocity? Solve this problem assuming no air resistance. (Of course, Tony Stark instantly does the calculations required, as he is an expert in differential equations!)

5. The mass of a football is 0.4 kg. Air resists passage of the ball, the resistive force being proportional to the square of the velocity, and being equal to 0.004 N when the velocity is 1 m/sec. Find the height to which the ball will rise, and the time to reach that height if it is thrown upward with a velocity of 20 m/sec. How is the answer altered if air resistance is neglected?

6. The football of the preceding exercise is released (from rest) at an altitude of 17.1 m. Find its final velocity and time of fall.

7. Assume that air resistance is proportional to the square of velocity. The terminal velocity of a 75-kg human in air of standard density is 60 m/sec [38]. Neglecting the variation of air density with altitude and assuming that the 75-kg parachutist falls from an altitude of 1.8 km, find the velocity. Hint: use the terminal velocity to find the coefficient of v^2.

Problems **8–12** concern Newton's law of cooling.

8. At the request of their children, Randy and Stephen make homemade popsicles. At 2:00 p.m., Kaelin asks if the popsicles are frozen (0°C), at which time they test the temperature of a popsicle and find it to be 5°C. If they put the popsicles with a temperature of 15°C in the freezer at 12:00 noon and the temperature of the freezer is -2°C, when will Erin, Kaelin, Robyn, Ryley, Alan, Abdi, and Avani be able to enjoy the popsicles?

9. An object cools in 10 min from 100°C to 60°C. The surroundings are at a temperature of 20°C. When will the object cool to 25°C?

10. Determine the time of death if a corpse is 79°F when discovered at 3:00 p.m. and 68°F 3 hours later. Assume that the temperature of the surroundings is 60°F and that normal body temperature is 98.6°F.

11. A thermometer is taken from an inside room to the outside, where the air temperature is 5°F. After 1 minute the thermometer reads 55°F, and after 5 minutes it reads 30°F. Determine the initial temperature of the inside room.

12. A slug of metal at a temperature of 800°F is put in an oven, the temperature of which is gradually increased during an hour from a° to b°. Find the temperature of the metal at the end of an hour, assuming that the metal warms kT degrees per minute when it finds itself in an oven that is T degrees warmer.

In Problems **13–17** it is supposed that the amount of gas (or liquid) contained in any fixed volume is constant. Also, thorough mixing is assumed.

13. A 20-L vessel contains air (assumed to be 80% nitrogen and 20% oxygen). Suppose 0.1 L of nitrogen is added to the container per second. If continual mixing takes place and material is withdrawn at the rate at which it is added, how long will it be before the container holds 99% nitrogen?

14. A 100-L beaker contains 10 kg of salt. Water is added at the constant rate of 5 L/min with complete mixing, and drawn off at the same rate. How much salt is in the beaker after 1 hour?

15. A tank contains 25 lb of salt dissolved in 50 gal of water. Brine containing 4 lb/gal is allowed to enter at a rate of 2 gal/min. If the solution is drained at the same rate find the amount of salt as a function $S(t)$ of time t. Find the concentration of salt at time. Suppose the rate of draining is modified to be 3 gal/min. Find the amount of salt and the concentration at time t.

16. Consider a pond that has an initial volume of $10,000$ m^3. Suppose that at time $t = 0$, the water in the pond is clean and that the pond has two streams flowing into it, stream A and stream B, and one stream flowing out, stream C. Suppose 500 m^3/day of water flows into the pond from stream A, 750 m^3/day flows into the pond from stream B, and 1250 m^3 flows out of the pond via stream C. At $t = 0$, the water flowing into the pond from stream A becomes contaminated with road salt at a concentration of 5 kg/1000 m^3. Suppose the water in the pond is well mixed so the concentration of salt at any given time is constant. To make matters worse, suppose also that at time $t = 0$ someone begins dumping trash into the pond at a rate of 50 m^3/day. The trash settles to the bottom of the pond, reducing the volume by 50 m^3/day. To adjust for the incoming trash, the rate that water flows out via stream C increases to 1300 m^3/day and the banks of the pond do not overflow. Determine how the amount of salt in the pond changes over time. Does the amount of salt in the pond reach 0 after some time has passed?

17. A large chamber contains 200 m^3 of gas, 0.15% of which is carbon dioxide (CO_2). A ventilator exchanges 20 m^3/min of this gas with new gas containing only 0.04% CO_2. How long will it be before the concentration of CO_2 is reduced to half its original value?

Problems **18–20** concern radioactive decay. The decay law states that the amount of radioactive substance that decays is proportional at each instant to the amount of substance present.

18. The strength of a radioactive substance decreases 50% in a 30-day period. How long will it take for the radioactivity to decrease to 1% of its initial value?

19. It is experimentally determined that every gram of radium loses 0.44 mg in 1 year. What length of time elapses before the radioactivity decreases to half its original value?

20. A tin organ pipe decays with age as a result of a chemical reaction that is catalyzed by the decayed tin. As a result, the rate at which the tin decays is proportional to the product of the amount of tin left and the amount that has already decayed. Let M be the total amount of tin before any has decayed. Find the amount of decayed tin $p(t)$.

Problems **21–22** deal with geometric situations where the derivative arises and yields a separable equation.

21. Find a curve for which the area of the triangle determined by the tangent, the ordinate to the point of tangency, and the x-axis has a constant value equal to a^2.

22. Find a curve for which the sum of the sides of a triangle constructed as in the previous problem has a constant value equal to b.

23. On an early Monday morning in February in rural Kentucky (not far from Western Kentucky University) it started to snow. There had been no snow on the ground before. It was snowing at a steady, constant rate so that the thickness of the snow on the ground was increasing at a constant rate. A snowplow began clearing the snow from the streets at noon. The speed of the snowplow in clearing the snow is inversely proportional to the thickness of the snow. The snowplow traveled two miles during the first hour after noon and traveled one mile during the second hour after noon. At what time did it begin snowing?

1.5 Exact Equations

We will now introduce another type of differential equation. Exact equations are not separable equations nor are they necessarily linear. They come up in higher level math in fields such as potential theory and harmonic analysis.

Consider the first-order differential equation $\frac{dy}{dx} = f(x, y)$. We observe that it can always be expressed in the differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

or equivalently as

$$M(x, y)\ + N(x, y)\,\frac{dy}{dx} = 0$$

and vice versa. We will now consider a type of differential equation that is not separable, but, nevertheless, has a solution. We need a definition from multivariable calculus to proceed:

Definition 1.5.1

Let $F(x, y)$ be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of F is defined by

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\,dx + \frac{\partial F(x, y)}{\partial y}\,dy$$

for all $(x, y) \in D$.

Example 1 Suppose $F(x, y) = xy^2 + 2x^3y$; then

$$\frac{\partial F}{\partial x} = y^2 + 6x^2y \quad \text{and} \quad \frac{\partial F}{\partial y} = 2xy + 2x^3$$

so that the total differential dF is given by

$$\begin{aligned} dF(x, y) &= \frac{\partial F(x, y)}{\partial x}\,dx + \frac{\partial F(x, y)}{\partial y}\,dy \\ &= (y^2 + 6x^2y)\,dx + (2xy + 2x^3)\,dy. \end{aligned}$$

Definition 1.5.2
The expression

$$M(x, y)\, dx + N(x, y)\, dy \tag{1.24}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$. That is, (1.24) is an exact differential in D if there exists a function F such that

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y)$$

for all $(x, y) \in D$.

If $M(x, y)\, dx + N(x, y)\, dy$ is an exact differential, then the differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1.25}$$

is called an *exact differential equation*. As long as $x = C$ (a constant) is not a solution, we consider the equivalent form

$$M(x, y) + N(x, y)\, \frac{dy}{dx} = 0 \tag{1.26}$$

as the standard form for an exact equation.

Example 2 The differential equation

$$y^2 + 2xy\, \frac{dy}{dx} = 0$$

is exact, since if $F(x, y) = xy^2$ then

$$\frac{\partial F}{\partial x} = y^2 \text{ and } \frac{\partial F}{\partial y} = 2xy.$$

Not all differential equations, however, are exact. Consider

$$y + 2x\, \frac{dy}{dx} = 0.$$

We cannot find an $F(x, y)$ so that

$$\frac{\partial F}{\partial x} = y \text{ and } \frac{\partial F}{\partial y} = 2x.$$

Numerous trials and errors may be enough to convince us that this is the case. What we really need is a method for testing a differential equation for exactness and for constructing the corresponding function $F(x, y)$. Both are contained in the following theorem and its proof.

THEOREM 1.5.1
Consider the differential equation

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{1.27}$$

where M and N have continuous first partial derivatives at all points (x, y) in a rectangular domain D. Then the differential equation (1.27) is exact in D, **if and only if**

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{1.28}$$

for all (x, y) in D.

Remark: The proof of this theorem is rather important, as it not only provides a test for exactness, but also a method of solution for exact differential equations.
Proof: To prove one direction of the theorem, we first suppose the differential equation (1.27) is exact in D and show that (1.28) must hold as a result. If (1.27) is exact, then there is a function F such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y).$$

So

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

by differentiation. Now we have assumed the continuity of the first partials of M and N in D, so that

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

This means that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

which is the same as (1.28).

To prove the other direction, we assume (1.28) and show that (1.27) must be exact. (Proving this direction will also show us how to construct the solution for a given exact equation.) Thus, we assume

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and find an F so that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y). \tag{1.29}$$

It is clear that we can find an F that satisfies either of these equations, but can we find an F that satisfies both? Let's proceed and see what happens. Suppose that F satisfies

$$\frac{\partial F}{\partial x} = M(x, y).$$

We can integrate both sides of this equation to get

$$F(x, y) = \int M(x, y)\, dx + \phi(y) \tag{1.30}$$

where $\int M(x, y)\, dx$ is the partial integration with respect to x holding y constant. Note that our "constant" of integration, $\phi(y)$, is a function but is a function of y only (it might also include an additive constant, but definitely no x). This is because the expression $\partial F/\partial x$ would result in the loss of any "only y functions." Now we need to find an $F(x, y)$ that satisfies both equations in (1.29). We thus need to make sure the $F(x, y)$ in (1.30) also satisfies $\frac{\partial F}{\partial y} = N(x, y)$. We calculate $\partial F/\partial y$ by differentiating (1.30) with respect to y:

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \int M(x, y)\, dx + \frac{d\phi(y)}{dy}.$$

Equating with $N(x, y)$ gives

$$N(x, y) = \left(\frac{\partial}{\partial y} \int M(x, y)\, dx \right) + \phi'(y),$$

where $\phi'(y) = d\phi(y)/dy$. Solving for $\phi'(y)$ gives

$$\phi'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx.$$

Since $\phi(y)$ is a function of <u>only</u> y, it must also be the case that $\phi'(y)$ is a function of only y. We can see this by showing

$$\frac{\partial}{\partial x} \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx \right) = 0.$$

Evaluating the left-hand side and simplifying give

$$\begin{aligned}
\frac{\partial}{\partial x}\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx\right) &= \frac{\partial N}{\partial x}-\frac{\partial^2}{\partial x\partial y}\int M(x,y)\,dx \\
&= \frac{\partial N}{\partial x}-\frac{\partial^2 F}{\partial x\partial y} \text{ (by noting what } F \text{ is)} \\
&= \frac{\partial N}{\partial x}-\frac{\partial^2 F}{\partial y\partial x} \text{ (by continuity)} \\
&= \frac{\partial N}{\partial x}-\frac{\partial^2}{\partial y\partial x}\int M(x,y)\,dx \\
&= \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} \\
&= 0,
\end{aligned}$$

where the last equality holds since we have assumed that

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

What this means is that

$$N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx$$

cannot depend on x since its derivative with respect to x is zero. Hence,

$$\phi(y)=\int\left(N(x,y)-\frac{\partial}{\partial y}\int M(x,y)\,dx\right)\,dy$$

and thus

$$F(x,y)=\int M(x,y)\,dx+\phi(y)$$

is a function that satisfies both

$$\frac{\partial F}{\partial x}=M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y}=N(x,y).$$

Thus,

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0$$

is exact in D. ∎

In short, the criterion for exactness is (1.28):

$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}.$$

If this equation holds, then the differential equation is exact. If this is not true, the differential equation is not exact.

Example 3 We considered the differential equation

$$y^2 + 2xy\,\frac{dy}{dx} = 0 \tag{1.31}$$

earlier. We see that

$$M(x,y) = y^2 \quad \text{and} \quad N(x,y) = 2xy.$$

Thus,

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x},$$

so that the differential equation is exact. On the other hand,

$$y + 2x\,\frac{dy}{dx} = 0 \tag{1.32}$$

gives $M(x,y) = y$ and $N(x,y) = 2x$ so that

$$\frac{\partial M}{\partial y} = 1 \neq 2 = \frac{\partial N}{\partial x}.$$

Hence $y + 2x\,\frac{dy}{dx} = 0$ is not exact.

Example 4 Consider the differential equation

$$(2x\sin y + y^3e^x) + (x^2\cos y + 3y^2e^x)\,\frac{dy}{dx} = 0.$$

Here

$$M(x,y) = 2x\sin y + y^3e^x \quad \text{and} \quad N(x,y) = x^2\cos y + 3y^2e^x;$$

hence

$$\frac{\partial M}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial N}{\partial x}.$$

Thus the differential equation is exact.

Remark: The test for exactness applies to equations in the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0. \tag{1.33}$$

If the left-hand side is an exact differential, then we can solve the exact differential equation (1.33) by finding a function $F(x,y)$ so that

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0.$$

More simply, using the total differential, we obtain $dF(x, y) = 0$. Thus,

$$F(x, y) = C$$

is a solution to (1.33).

THEOREM 1.5.2

Suppose the differential equation

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0$$

is exact. Then the general solution of this differential equation is given implicitly by

$$F(x, y) = C,$$

where $F(x, y)$ is a function such that

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y).$$

Remark 1: As with separable and homogeneous equations, the constant in Theorem 1.5.2 is determined by an initial condition.

Remark 2: We have an explicit form for $F(x, y)$, namely,

$$F(x, y) = \int M(x, y)\,dx + \phi(y),$$

where $\phi(y) = \displaystyle\int \left(N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\,dx \right) dy$. This form, however, is not always useful. We will see by example how to solve exact differential equations.

Remark 3: We integrated $\partial F/\partial x = M$ and substituted this into $\partial F/\partial y = N$. We instead could have solved $\partial F/\partial y = N$ first (by integrating with respect to y and obtaining a "constant" $\psi(x)$) and then substituted into $\partial F/\partial x = M$. The resulting F is the same but would be written

$$F(x, y) = \int N(x, y)\,dy + \psi(x), \tag{1.34}$$

where $\psi(x) = \displaystyle\int \left(M(x, y) - \frac{\partial}{\partial x} \int N(x, y)\,dy \right) dx$. See Problem 19 at the end of this section.

Example 5 Show that

$$(3x^2 + 4xy) + (2x^2 + 2y)\,\frac{dy}{dx} = 0$$

is exact and then solve it by the methods discussed in this section.

Solution

We have

$$M(x,y) = 3x^2 + 4xy \quad \text{and} \quad N(x,y) = 2x^2 + 2y$$

so that the equation is exact, since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}.$$

Our goal is to find an $F(x,y)$ that simultaneously satisfies the equations

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y).$$

That is, F must satisfy

$$\frac{\partial F}{\partial x} = 3x^2 + 4xy \quad \text{and} \quad \frac{\partial F}{\partial y} = 2x^2 + 2y.$$

Integrating $\partial F/\partial x$ with respect to x gives

$$\begin{aligned} F(x,y) &= \int (3x^2 + 4xy)\,dx \\ &= x^3 + 2x^2y + \phi(y). \end{aligned}$$

This same F must also satisfy $\partial F/\partial y = N$ and we then have

$$2x^2 + \phi'(y) = \frac{\partial F}{\partial y} = 2x^2 + 2y.$$

Thus, $\phi'(y) = 2y$. Integrating with respect to y gives

$$\phi(y) = y^2 + C_0$$

so that

$$F(x,y) = x^3 + 2x^2y + y^2 + C_0.$$

Thus, a one-parameter family of solutions is given by

$$x^3 + 2x^2y + y^2 = C.$$

We now solve an exact equation by first integrating with respect to y; see Remark 3 above.

Example 6 Show that

$$(2x\cos y + 3x^2 y) + (x^3 - x^2 \sin y - y)\,\frac{dy}{dx} = 0$$

is exact and solve it subject to the initial condition $y(0) = 2$. Plot the solution.

Solution

We have $M(x, y) = 2x\cos y + 3x^2 y$ and $N(x, y) = x^3 - x^2 \sin y - y$. The equation is exact because

$$\frac{\partial M}{\partial y} = 3x^2 - 2x\sin y = \frac{\partial N}{\partial x}.$$

Now we find an $F(x, y)$ so that

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y).$$

This time we will integrate $\partial F/\partial y = N$ with respect to y. Thus

$$\begin{aligned} F(x, y) &= \int N(x, y)\,dy \\ &= \int (x^3 - x^2 \sin y - y)\,dy \\ &= x^3 y + x^2 \cos y - \frac{y^2}{2} + \psi(x). \end{aligned}$$

This must also satisfy $\partial F/\partial x = M$. Calculating $\partial F/\partial x$ gives

$$\frac{\partial F}{\partial x} = 3x^2 y + 2x\cos y + \psi'(x).$$

Substituting into

$$\frac{\partial F}{\partial x} = M(x, y)$$

gives $\psi'(x) = 0$, which is easily integrated to obtain $\psi(x) = C_1$. Thus,

$$F(x, y) = x^3 y + x^2 \cos y - \frac{1}{2}y^2 + C_1,$$

and a one-parameter family of solutions is

$$x^3 y + x^2 \cos y - \frac{1}{2}y^2 = C.$$

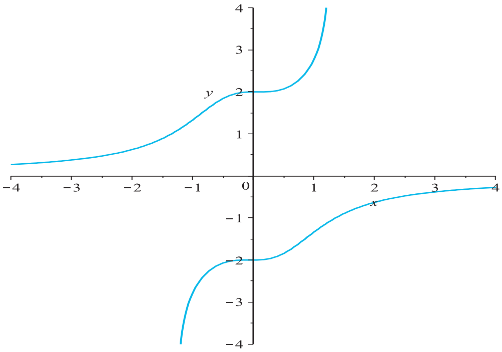

FIGURE 1.8: Implicit plot for Example 6. The upper curve is the solution curve because it passes through the initial condition.

The initial condition $y(0) = 2$ gives $C = -2$. Hence

$$x^2 \cos y + x^3 y - \frac{1}{2} y^2 = -2$$

is the implicit solution that satisfies the given initial condition. The solution curves can be plotted as shown in Figure 1.8.

Note that although both curves in Figure 1.8 satisfy the implicit equation, only one of these curves passes through the given initial condition and thus is the correct solution.

Solution by Grouping

There is a much slicker method for solving exact differential equations and it is known as the **method of grouping.** For better or worse, it requires a "working knowledge" of differentials and a certain amount of ingenuity. We again consider Example 5, this time in its differential form:

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0.$$

We rewrite it in the form

$$3x^2\,dx + (4xy\,dx + 2x^2\,dy) + 2y\,dy = 0$$

which is

$$d(x^3) + d(2x^2 y) + d(y^2) = d(C).$$

That is,

$$d(x^3 + 2x^2 y + y^2) = d(C)$$

so that

$$x^3 + 2x^2 y + y^2 = C.$$

Clearly, this procedure is much quicker if we can find the appropriate grouping. Let's try this method one more time by again considering Example 6. We group the terms as

$$(2x\cos y\,dx - x^2\sin y\,dy) + (3x^2y\,dx + x^3\,dy) - y\,dy = 0.$$

Thus, we have

$$d(x^2\cos y) + d(x^3y) - d\left(\frac{y^2}{2}\right) = d(C)$$

and so

$$x^2\cos y + x^3y - \frac{1}{2}y^2 = C$$

is a one-parameter family of solutions.

Important Note: If we use the method of grouping, we still need to check that the equation is exact for our first step.

Problems

In Problems **1–13**, check to see if the equation is exact. If it is, solve it by the methods of this section. If an initial condition is given, graph the solution.

1. $(1+xy^2)+(1+x^3y)\frac{dy}{dx}=0$

2. $(1+y^2\sin 2x)-2y\cos 2x\frac{dy}{dx}=0$

3. $2xy+(x^2-y^2)\frac{dy}{dx}=0$

4. $(1+y^2\sin 2x)-y\cos 2x\frac{dy}{dx}=0$

5. $2xy^3+(1+3x^2y^2)\frac{dy}{dx}=0$

6. $(2+\frac{y}{x^2})\,dx+(y-\frac{1}{x})\,dy=0$

7. $3x^2(1+\ln y)=(2y-\frac{x^3}{y})\frac{dy}{dx}$

8. $(\frac{x}{\sin y}+2)+\frac{(x^2+1)\cos y}{\cos 2y-1}\frac{dy}{dx}=0$

9. $(2xy+1)+(x^2+4y)\frac{dy}{dx}=0,\ y(0)=1$

10. $(2y\sin x\cos x+y^2\sin x)+(\sin^2 x-2y\cos x)\frac{dy}{dx}=0,\ y(0)=3$

11. $(2-9xy^2)x+(4y^2-6x^3)y\frac{dy}{dx}=0,\ y(1)=1$

12. $(y\sec^2 x+\sec x\tan x)+(\tan x+2y)\frac{dy}{dx}=0,\ y(0)=1$

13. $e^{-y}-(2y+xe^{-y})\frac{dy}{dx}=0,\ y(1)=3$

In Problems **14–15**, determine the constant A such that the equation is exact. Then solve the resulting exact equation.

14. $(x^2+3xy)+(Ax^2+4y)\frac{dy}{dx}=0$

15. $\left(\frac{Ay}{x^3}+\frac{y}{x^2}\right)+\left(\frac{1}{x^2}-\frac{1}{x}\right)\frac{dy}{dx}=0$

In Problems **16–17**, determine the most general function ($N(x,y)$ or $M(x,y)$) that makes the equation exact.

16. $M(x,y)+(2ye^x+y^2e^{3x})\frac{dy}{dx}=0$

17. $(x^3+xy^2)+N(x,y)\frac{dy}{dx}=0$

18. Let x represent the units of labor and y represent the units of capital. If $f(x,y)$ measures the number of units produced, a differential equation satisfied by a level curve of it is

$$ax^{a-1}y^{1-a} + (1-a)x^ay^{-a}\frac{dy}{dx}=0.$$

Solve this equation as (i) a separable equation and (ii) an exact equation. In doing (ii), we obtain the well-known **Cobb-Douglas production function** $f(x,y)=Cx^ay^{1-a}$.

19. By following the proof of Theorem 1.5.1, show that an equivalent formulation of $F(x, y)$ is given by

$$F(x,y) = \int N(x,y)\,\frac{dy}{dx} + \int \left(M(x,y) - \frac{\partial}{\partial x} \int N(x,y)\,\frac{dy}{dx} \right).$$

Although this could easily be obtained by rearranging the previously obtained expression for F (1.30), do *not* simply rearrange terms.

20. By using the substitution $y = vx$, show that the homogeneous equation

$$(Ax + By) + (Cx + Ey)\,\frac{dy}{dx} = 0,$$

where A, B, C, and E are constants, is exact if and only if $B = C$.

21. By using the substitution $y = vx$, show that the homogeneous equation

$$(Ax^2 + Bxy + Cy^2) + (Ex^2 + Fxy + Gy^2)\,\frac{dy}{dx} = 0,$$

where A, B, C, E, F, and G are constants, is exact if and only if $B = 2E$ and $F = 2C$.

1.6 Special Integrating Factors and Substitution Methods

Special Integrating Factors

In solving linear equations, we learned that we could multiply by an appropriate **integrating factor**, thus transforming the equation into a form we can solve. Besides the one we learned, there are other integrating factors that we will now consider.

Definition 1.6.1

If the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1.35}$$

is not exact in a domain D but the differential equation

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{1.36}$$

is exact in D, then $\mu(x, y)$ is called an integrating factor of the differential equation (1.35).

Example 1 The differential equation

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0 \tag{1.37}$$

is not exact since

$$\frac{\partial M}{\partial y} = 3 + 8xy \neq 2 + 6xy = \frac{\partial N}{\partial x}.$$

If we let $\mu(x, y) = x^2y$, we can use (1.36) to rewrite (1.37) as

$$(x^2y)(3y + 4xy^2)\,dx + (x^2y)(2x + 3x^2y)\,dy = 0.$$

Expanding gives

$$M = 3x^2y^2 + 4x^3y^3 \quad \text{and} \quad N = 2x^3y + 3x^4y^2.$$

Then

$$\frac{\partial}{\partial y}(3x^2y^2 + 4x^3y^3) = 6x^2y + 12x^3y^2 = \frac{\partial}{\partial x}(2x^3y + 3x^4y^2).$$

Thus the new equation is exact and hence $\mu(x, y) = x^2y$ is an integrating factor.

We saw above how multiplying by an appropriate integrating factor converted a linear equation into an exact equation, which we could then solve. Multiplying by an appropriate integrating factor is a technique that will work in other situations as well.

We have seen that if the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is not exact and if $\mu(x, y)$ is an integrating factor, then the differential equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$$

is exact. Using the criterion for exactness, we must have

$$\frac{\partial}{\partial y}(\mu(x, y)M(x, y)) = \frac{\partial}{\partial x}(\mu(x, y)N(x, y)).$$

To simplify notation, we will write M, N instead of $M(x, y), N(x, y)$ when taking the partial derivatives, even though both M and N are functions of x and y. The criterion for exactness can then be written

$$\frac{\partial \mu}{\partial y}\,M(x, y) + \mu(x, y)\,\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}\,N(x, y) + \mu(x, y)\,\frac{\partial N}{\partial x}.$$

Rearranging gives

$$\frac{\partial \mu}{\partial y} M(x,y) - \frac{\partial \mu}{\partial x} N(x,y) = \mu(x,y)\frac{\partial N}{\partial x} - \mu(x,y)\frac{\partial M}{\partial y}. \tag{1.38}$$

Thus $\mu(x,y)$ is an integrating factor if and only if it is a solution of the partial differential equation (1.38). We will not consider the solution of this partial differential equation. We will instead consider (1.38) in the case where μ only depends on x, i.e., $\mu(x,y) = \mu(x)$. (We can also consider the case when $\mu(x,y) = \mu(y)$ and the analogous formulation is left as one of the exercises.) In this situation, (1.38) reduces to

$$-\mu'(x)\, N(x,y) = \mu(x)\frac{\partial N}{\partial x} - \mu(x)\frac{\partial M}{\partial y}.$$

That is,

$$\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right). \tag{1.39}$$

If the right-hand side of (1.39) involves two dependent variables, we run into trouble. If, however, it depends only upon x, then Equation (1.39) is separable, in which case we obtain

$$\mu(x) = \exp\left[\int \frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\right]$$

as an integrating factor.

Example 2 Solve the differential equation

$$(2x^2 + y)\, dx + (x^2 y - x)\, dy = 0.$$

Solution

In this equation,

$$M(x,y) = 2x^2 + y \quad \text{and} \quad N(x,y) = x^2 y - x$$

so that

$$\frac{\partial M}{\partial y} = 1 \neq 2xy - 1 = \frac{\partial N}{\partial x}$$

and the equation is not exact. It can also be shown (try it!) that the differential equation is not separable, homogeneous, or linear. Now

$$\begin{aligned}\frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) &= \frac{1}{x^2y - x}(1 - (2xy - 1)) \\ &= \frac{-2}{x}\end{aligned}$$

depends only upon x. Thus,

$$\mu(x) = \exp\left(-\int \frac{2}{x}\,dx\right) = e^{-2\ln|x|} = \frac{1}{x^2}$$

is an integrating factor. If we multiply the equation through by this factor we have

$$\left(2 + \frac{y}{x^2}\right)\,dx + \left(y - \frac{1}{x}\right)\,dy = 0.$$

Now this equation is exact since

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} = \frac{\partial N}{\partial x}.$$

We can thus solve this differential equation using the exact method to obtain

$$2x + \frac{y^2}{2} - \frac{y}{x} = C.$$

1.6.1 Bernoulli Equation

We will now consider a class of differential equations that can be reduced to linear equations by an appropriate transformation. These equations are called *Bernoulli* equations and often arise in applications.

Definition 1.6.2
A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \qquad n \in \mathbb{R} \tag{1.40}$$

is called a Bernoulli differential equation.

Note that when $n = 0$ or $n = 1$, the Bernoulli equation is actually a linear equation and can be solved as such. When $n \neq 0$ or 1, then we must consider an additional method.

THEOREM 1.6.1
Suppose $n \neq 0$ or 1, then the transformation

$$v = y^{1-n}$$

reduces the Bernoulli equation (1.40) to

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x), \tag{1.41}$$

which is a linear equation in v.

Proof: Multiply the Bernoulli equation by y^{-n} and thus obtain

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)\,. \tag{1.42}$$

Now let $v = y^{1-n}$ so that

$$\frac{dv}{dx} = (1-n)y^{-n}\,\frac{dy}{dx}.$$

Hence, Equation (1.42) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)\,v = Q(x),$$

that is,

$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x).$$

Letting

$$P_1(x) = (1-n)P(x) \quad \text{and} \quad Q_1(x) = (1-n)Q(x)$$

gives

$$\frac{dv}{dx} + P_1(x)\,v = Q_1(x),$$

a linear differential equation in v. ∎

Example 3 Solve the differential equation

$$\frac{dy}{dx} + y = xy^3.$$

Solution

This is a Bernoulli equation with $n = 3$. We thus let $v = y^{1-3} = y^{-2}$, so that

$$\frac{dv}{dx} = -2y^{-3}\,\frac{dy}{dx}.$$

Using (1.41) we obtain

$$\frac{dv}{dx} - 2v = -2x. \tag{1.43}$$

This is a linear differential equation with integrating factor

$$\exp\left(\int P(x)\,dx\right) = \exp\left(\int -2\,dx\right) = e^{-2x}.$$

We also calculate $\exp\left(-\int P(x)\,dx\right) = e^{2x}$. Thus the solution of (1.43) can be written

$$v = e^{2x}\left(\int -2xe^{-2x}\;dx\right).$$

Integrating by parts gives

$$v = e^{2x}\left(xe^{-2x} + \frac{1}{2}e^{-2x}\right) + Ce^{2x}.$$

Simplifying gives

$$v = x + \frac{1}{2} + Ce^{2x}.$$

But our original problem was in the variable y. We know $v = y^{-2}$ and thus the solution is

$$\frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$$

which can be written as

$$y = \pm\left(\frac{1}{x + \frac{1}{2} + Ce^{2x}}\right)^{1/2}.$$

This solution is defined as long as the denominator is not equal to zero.

1.6.2 Homogeneous Equations of the Form $g(y/x)$

We have now been introduced to separable differential equations and their relative ease of solution. We will now consider a class of differential equations that can be reduced to separable equations by a change of variables.

Remark: Before proceeding, we alert the reader that the use of the word *homogeneous* in this section must not be confused with its use as the type of linear ordinary differential equation whose right-hand side is zero (as in Chapters 3 and 4). Its use in the latter chapters is more common but both have their place.

Example 4 Consider the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

Solution

After a minute or so of reflection, we see that this is not a separable equation. We can, however, rewrite the equation as

$$\frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \tag{1.44}$$

so that we can isolate the fraction y/x. This suggests we consider the change of variable

$$v = \frac{y}{x}$$

or equivalently

$$y = vx.$$

Our original problem has dy/dx and thus we take the derivative of both sides of the above equation with respect to x to get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution of this and $y = vx$ into (1.44) gives

$$v + x\frac{dv}{dx} = \frac{1 - v}{1 + v}.$$

Simplifying results in the separable equation

$$x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1 + v},$$

and we separate its variables as

$$\frac{1 + v}{1 - 2v - v^2}dv = \frac{dx}{x},$$

and integrate to give

$$\ln|1 - 2v - v^2| = -2\ln x + C_1.$$

Exponentiation of both sides yields

$$|1 - 2v - v^2| = e^{C_1}x^{-2} = C_2x^{-2}.$$

But, $v = y/x$ so that substitution gives

$$1 - \frac{2y}{x} - \left(\frac{y}{x}\right)^2 = \pm C_2x^{-2} = Cx^{-2}.$$

Multiplying by x^2 to clear the fraction gives

$$x^2 - 2xy - y^2 = C$$

as the implicit solution to the differential equation.

This is an example of a general method of reducing a class of differential equations to that of a separable equation. We need some terminology.

Definition 1.6.3
The first-order differential equation

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0$$

is said to be of homogeneous type (or homogeneous) if, when written in the derivative form

$$\frac{dy}{dx} = f(x, y),$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$.

By classifying the equation as homogeneous, we will be able to apply the above technique in order to reduce the differential equation to one that is separable. It is sometimes not obvious that a given equation can be rewritten as a homogeneous equation. We present two examples now to help clarify this concept.

Example 5 The differential equation

$$(x^2 - 3y^2) + 2xy\,\frac{dy}{dx} = 0$$

is homogeneous, since the equation can be written in derivative form as

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy},$$

and we can rearrange this as

$$\frac{3y^2 - x^2}{2xy} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

so that

$$\frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right).$$

The right-hand side is of the form $g(y/x)$ for the function

$$g(z) = \frac{3z}{2} - \frac{1}{2z},$$

and so the differential equation is homogeneous.

Example 6 The differential equation

$$\left(y + \sqrt{x^2 + y^2}\right)\,dx - x\,dy = 0$$

can be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

For $x > 0$, we have

$$\frac{y + \sqrt{x^2 + y^2}}{x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2},$$

so

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

which is of the form $g(y/x)$ for a function of the form

$$g(z) = z + \sqrt{1 + z^2}.$$

If we had considered $x < 0$, we would have obtained

$$g(z) = z - \sqrt{1 + z^2}.$$

In either case, we see that the differential equation is homogeneous.

As we mentioned, we have introduced homogeneous differential equations because they are related to separable equations; in fact, we have the following theorem which formalizes the method used in Example 4.

THEOREM 1.6.2

If

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{1.45}$$

is a homogeneous equation, then the change of variables

$$y = vx$$

transforms (1.45) into a separable equation in the variables v and x.

Note that this change of variables implies that

$$y' = v + xv'$$

by the product rule.

Example 7 Solve

$$y + (x - 2y)\,\frac{dy}{dx} = 0.$$

Solution

We first observe that this can be rewritten as

$$\frac{dy}{dx} = \frac{y}{2y - x}.$$

Dividing numerator and denominator by x gives

$$\frac{dy}{dx} = \frac{y/x}{2y/x - 1}.$$

The right-hand side is then of the form $g(y/x)$ and making the change of variables $y = vx$ gives

$$v + x\frac{dv}{dx} = \frac{v}{2v - 1},$$

which becomes

$$x\frac{dv}{dx} = \frac{2(v - v^2)}{2v - 1}.$$

This equation is separable! Rearranging gives

$$\frac{2v - 1}{2(v - v^2)}\, dv = \frac{1}{x}\, dx,$$

and integrating both sides yields

$$-\frac{1}{2}\ln|v - v^2| = \ln|x| + C_1.$$

We then use $v = y/x$ to reintroduce the y-variable. Thus

$$-\frac{1}{2}\ln\left|\frac{y}{x} - \left(\frac{y}{x}\right)^2\right| = \ln|x| + C_1,$$

but we can let $C_1 = \ln C$ for an arbitrary constant C, so that

$$-\frac{1}{2}\ln\left|\frac{y}{x} - \left(\frac{y}{x}\right)^2\right| = \ln|Cx|$$

is the implicit solution. We could obtain an explicit solution with a bit more work but choose not to. Again, we could plot these solutions for various C-values with our favorite software package.

Problems

Solve Problems **1–8** by first finding an integrating factor of suitable form.

1. $ydx + (e^x - 1)dy = 0$

2. $(x^2 + y^2 + x)dx + ydy = 0$

3. $y(x + y)dx + (xy + 1)dy = 0$

4. $(x^2 - y^2 + y)dx + x(2y - 1)dy = 0$

5. $ydx - xdy = 2x^3 \sin x\ dx$

6. $(3x^2 + y)\, dx + (x^2 y - x)\, dy = 0$

7. $(3x^2 y - x^2)dx + dy = 0$

8. $(x^2 + 2x + y)dx = (x - 3x^2 y)dy$

9. Show that if $(\partial N/\partial x - \partial M/\partial y)/(xM - yN)$ depends only on the product xy, that is,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} = H(xy),$$

then the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

has an integrating factor of the form $\mu(xy)$. Find the general formula for $\mu(xy)$.

10. We derived a formula for an integrating factor if $\mu(x, y) = \mu(x)$. If $\mu(x, y) = \mu(y)$, derive the integrating factor formula

$$\mu(y) = \exp\left[\int \frac{1}{M(x,y)}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\right]. \tag{1.46}$$

Solve the Bernoulli equations given in Problems **11–21**.

11. $y' + y = xy^2$
12. $y' + 3y = y^4$
13. $y' + 2xy = 4y$
14. $y' - xy = xy^3$
15. $xydy = (y^2 + x)dx$
16. $xy' + 2y + x^5y^3e^x = 0$
17. $xy' - 2x^2\sqrt{y} = 4y$
18. $y' = y^4 \cos x + y \tan x$
19. $xy^2y' = x^2 + y^3$
20. $(x + 1)(y' + y^2) = -y$
21. Solve the logistic equation $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$.

For Problems **22–34**, solve the homogeneous differential equation analytically.

22. $(x + y)\, dx - x\, dy = 0$
23. $(x + 2y)dx - xdy = 0$
24. $(y^2 - 2xy)dx + x^2dy = 0$
25. $2x^3y' = y(2x^2 - y^2)$
26. $2x^2 \frac{dy}{dx} = x^2 + y^2$
27. $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$
28. $xy' - y = x\tan(\frac{y}{x})$
29. $(x^2 + y^2)y' = 2xy$
30. $ydx = (2x + y)dy$
31. $(x - y)dx + (x + y)dy = 0$
32. $y' = 2(\frac{y}{x+y})^2$
33. $y^2 + x^2y' = xyy'$
34. $(x + 4y)y' = 2x + 3y$

35. A function F is called **homogeneous of degree** n if

$$F(tx, ty) = t^n F(x, y) \text{ for all } x \text{ and } y.$$

That is, if tx and ty are substituted for x and y in $F(x, y)$ and if t^n is then factored out, we are left with $F(x, y)$. For instance, if $F(x, y) = x^2 + y^2$, we note that

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2 F(x, y)$$

so that F is homogeneous of degree 2. Homogeneous differential equations and functions that are homogeneous of degree n are related in the following manner. Suppose the functions M and N in the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

are both homogeneous of the same degree n.
(a) Show, using the change of variables $t = 1/x$, that

$$M\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^n M(x,y),$$

which implies that

$$M(x,y) = \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right).$$

(b) Show, using a similar calculation, that

$$N(x,y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right),$$

so that the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

becomes

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = -\frac{\left(\frac{1}{x}\right)^{-n} M(1, \frac{y}{x})}{\left(\frac{1}{x}\right)^{-n} N(1, \frac{y}{x})}.$$

Simplifying gives

$$\frac{dy}{dx} = -\frac{M(1, \frac{y}{x})}{N(1, \frac{y}{x})}.$$

(c) Show that both numerator and denominator of the right-hand side of

$$\frac{dy}{dx} = -\frac{M(1, \frac{y}{x})}{N(1, \frac{y}{x})}$$

are in the form $g(y/x)$ and conclude that if M and N are both homogeneous functions of the same degree n, then the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is a homogeneous differential equation.

36. Using the idea presented in Problem **35**, show that each of the equations in Problems **22–34** are homogeneous.

37. Suppose that the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous. Show that the transformation $x = r\cos t,\ y = r\sin t$ reduces this equation to a separable equation in the variables r and t.

Use the method of Problem **37** to solve Problems **38**–**39**.
38. $(x-y)dx + (x+y)dy = 0$ **39**. $(x+y)\,dx - x\,dy = 0$

40. (a) Solve

$$\frac{dy}{dx} = \frac{y-x}{y+x}.$$

(b) Now consider

$$\frac{dy}{dx} = \frac{y-x+1}{y+x+5}. \tag{1.47}$$

(i) Show that this equation is NOT homogeneous.
How can we solve this? Consider the equations $y-x=0$ and $y+x=0$. They represent two straight lines through the origin. The intersection of $y-x+1=0$ and $y+x+5=0$ is $(-2,-3)$. Check it! Let $x = X-2$ and $y = Y-3$. This amounts to taking new axes parallel to the old with an origin at $(-2,-3)$.
(ii) Use this transformation to obtain the differential equation

$$\frac{dY}{dX} = \frac{Y-X}{Y+X}.$$

(iii) Using the solution from part (a), obtain the solution to (1.47).

Use the technique of Problem **40** to solve Problems **41**–**45**.
41. $(2x+y+1)dx - (4x+2y-3)dy = 0$
42. $x-y-1+(y-x+2)y' = 0$
43. $(x+4y)y' = 2x+3y-5$
44. $(y+2)dx = (2x+y-4)dy$
45. $y' = 2(\frac{y+2}{x+y-1})^2$

Chapter 1 Review

In Problems **1**–**7**, determine whether the statement is true or false. If it is true, give reasons for your answer. If it is false, give a counterexample or other explanation of why it is false.

1. The equation $y'' + xy' - y = x^2$ is a linear ordinary differential equation that is considered an initial-value problem.