

A M E R I C A N U N I V E R S I T Y S T U D I E S

The Concept of Logical Consequence

An Introduction to Philosophical Logic

MATTHEW W. McKEON

The Concept of Logical Consequence is a critical evaluation of the model-theoretic and proof-theoretic characterizations of logical consequence that proceeds from Alfred Tarski's characterization of the informal concept of logical consequence. This study evaluates and expands upon ideas set forth in Tarski's 1936 article on logical consequence, and appeals to his 1935 article on truth. Classical logic, as well as extensions and deviations are considered. Issues in the philosophy of logic such as the nature of logical constants, the philosophical significance of completeness, and the metaphysical and epistemological implications of logic are discussed in the context of the examination of the concept of logical consequence.

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For Beth

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Chapter 1

Introduction

This book is an inquiry into the concept of logical consequence, arguably the central concept of logic. We take logical consequence to be a relation between a given set of sentences and the sentences that logically follow. One sentence is said to be a logical consequence of a set of sentences, if and only if, in virtue of logic alone, it is impossible for the sentences in the set to be all true without the other sentence being true as well. The central question to be investigated here is: what conditions must be met in order for a sentence to be a logical consequence of others?

One historically significant answer derives from the work of Alfred Tarski, one of the greatest logicians of the twentieth century. In Chapter 2, we distinguish features of the ordinary, informal concept of logical consequence using some of Tarski's work, particularly his seminal (1936) paper on logical consequence. Here Tarski uses his observations of the salient features of what he calls the common concept of logical consequence to guide his theoretical development of it. We shall develop Tarski's observations of the criteria by which we intuitively judge what follows from what, and which Tarski thinks must be reflected in any theory of logical consequence.

After presenting his theoretical definition of logical consequence, which is the forerunner of the modern, model-theoretic definition, Tarski asserts in his (1936) paper that it reflects the salient features of the common concept of logical consequence. This assertion is not obvious, and Tarski defends it nowhere in his published writings. This raises the particular issues of whether Tarski's informal characterization of the common concept of logical consequence is correct, and whether it is reflected in his theoretical definition. The more general issues raised are: how do we justify a theoretical definition of logical consequence? What role should the informal concept play?

We shall answer these questions with respect to the model-theoretic and the deductive-theoretic characterizations of logical consequence for first-order languages. They represent two major theoretical approaches to making the common concept of logical consequence more precise. Chapter 2 shall motivate both approaches by considering them as natural developments of the ordinary, informal characterization. This shall set the context for our critical evaluation of these two approaches to characterizing logical conse-

quence. After introducing some set-theoretic concepts used in the book and a simple first-order (extensional) language M in Chapter 3, (classical) logical consequence shall be defined for M model-theoretically in Chapter 4, and deductive-theoretically (a natural deduction system N is given) in Chapter 5. I account for their status as definitions, and sketch how they work in determining what follows from what. Also, there are accounts of what models and deductive apparatuses are, and, what, exactly, they represent when used to fix the logical consequence relation.

Both Chapters 4 and 5 consider methodological criticism of the model-theoretic and deductive-theoretic approaches. In particular, we consider the adequacy of models and deductive apparatuses as tools for defining logical consequence, and these considerations are used to answer the two questions posed above: how do we justify a theoretical definition of logical consequence? What role should the informal concept play? Also, in Chapters 4 and 5, there is some criticism of classical logic. Both types of criticism (methodological and logical) not only motivate consideration of alternative logics, but also suggest revisions to the Tarskian understanding of the informal concept of logical consequence introduced in Chapter 2.

While most logicians accept the model-theoretic and deductive-theoretic characterizations of logical consequence for extensional languages, there is less agreement on the pre-theoretic notion these technical definitions are supposed to represent, and little discussion about whether they actually do represent it adequately. Almost all of the formal logic textbooks written for the book's intended audience give an ordinary, informal characterization of logical consequence either in the introduction or at the beginning of the first chapter. Unfortunately, the informal characterization of logical consequence typically amounts to a mere sketch which is either insufficient for clarifying the status of the technical characterizations that follow or conflicts with them. The book's focus on the concept of logical consequence, its introductory manner of presentation, and its monograph-length, make it ideal for the intended audience as a means for clarifying the status and aims of the technical characterizations of logical consequence, and for highlighting their relationship to the informal concept of logical consequence which motivates them. This enhances understanding of not only the status of the model-theoretic and deductive-theoretic characterizations of logical consequence, but also deepens our understanding of criteria for evaluating them.

The book's intended audience matches the audiences of other introductions to philosophical logic such as Haack (1978), Sainsbury (1991), and

Read (1995). Like these classics, this book is written at a level that makes it beneficial to advanced undergraduates with exposure to introductory formal logic, graduate students, and professional philosophers planning to self-educate themselves about the philosophy of logical consequence and for whom this book is only a first step. What distinguishes this book is its approach to thinking about logical consequence. It is tightly organized around the informal concept of logical consequence, and its relationship to the more technical model-theoretic and deductive-theoretic characterizations. I am unaware of any introduction to philosophical logic devoted to motivating the technical characterizations of logical consequence by appealing to the informal concept of logical consequence, and evaluating the former in terms of how successfully they capture the central features of the latter. As with the above three books when first published, the freshness of this book's approach to studying logical consequence and its engagement with themes in the recent literature should make it of interest to specialists working in the philosophy of logic.

The goal of realizing the envisioned length of the book has, of course, had an expository impact. In order to forgo lengthy exegetical analysis, ideas and arguments from the literature are typically presented in a distilled form. Also, since references and discussion are confined to the main text, there are no footnotes. More importantly, I have been careful in choosing where to be argumentative (as in my defense of the Tarskian model-theoretic characterization of logical consequence against criticism) and where to remain agnostic (as with respect to the issue of the nature of a logical constant and whether the meanings of logical constants should be identified with their truth-conditional properties or their inferential properties). I've chosen to be argumentative in those places where I believe that I have the space to be persuasive. In many places where the discussion is more expository and less argumentative, I have developed topics to the point that satisfies the stated goals of the relevant section. I certainly realize that on pretty much every topic covered in the book, much more can be usefully said. I have at points in the text provided the reader with references that extend the book's discussion in various ways.

There are references to Tarski's work throughout the book's discussion of logical consequence. While I do not believe that my reading of Tarski is controversial, the desire not to lengthen the book prohibits defense of my interpretation of Tarski. To be clear, this is not a book about Tarski. Rather, some of Tarski's writings are used as a platform for the book's discussion of

logical consequence. The thoughts of other logicians such as Dummett, Gentzen, and Frege, are also used towards this end. The trajectory of the discussion is squarely aimed at the model-theoretic and deductive-theoretic approaches to logical consequence, and their relationship to the informal concept of logical consequence. Even though the focus of the book is on logical consequence, it is studied in a way that allows it to serve as an introduction to philosophical logic. Its emphasis on the informal concept of logical consequence and its relationship to the more technical model-theoretic and deductive-theoretic approaches highlights and sharpens in a unique way other issues central in the philosophy of logic such as the nature of logic, logical constants, and logical necessity. Introducing issues in the philosophy of logic from the perspective of a study of logical consequence will illustrate how these issues are related, and why they are significant for understanding logical consequence.

Chapter 2

The Concept of Logical Consequence

Tarski's Characterization of the Common Concept of Logical Consequence

Tarski begins his article, "On the Concept of Logical Consequence," by noting a challenge confronting the project of making precise the common concept of logical consequence.

The concept of *logical consequence* is one of those whose introduction into a field of strict formal investigation was not a matter of arbitrary decision on the part of this or that investigator; in defining this concept efforts were made to adhere to the common usage of the language of everyday life. But these efforts have been confronted with the difficulties which usually present themselves in such cases. With respect to the clarity of its content the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree. ((1936), p. 409)

Not every feature of a precise definition of logical consequence will be reflected in the common concept of logical consequence, and we should not expect any precise definition to reflect all of its features. Nevertheless, despite its vagueness, Tarski believes that there are identifiable, essential features of the common concept of logical consequence.

...consider any class K of sentences and a sentence X which follows from this class. From an intuitive standpoint, it can never happen that both the class K consists of only true sentences and the sentence X is false. Moreover, since we are concerned here with the concept of logical, i.e., formal consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of class K refer. The consequence relation cannot be affected by replacing designations of the objects referred to in these sentences by the designations of any other objects. (1936, pp.414-415)

According to Tarski, the logical consequence relation is (1) necessary, (2) formal, and (3) not influenced by empirical knowledge. We now elaborate on (1)-(3).

The logical consequence relation has a modal element

Tarski countenances an implicit modal notion in the common concept of logical consequence. If X is a logical consequence of K , then not only is it the case that not all of the sentences of K are true and X is false, but it can *never* happen that both the class K consists of only true sentences and the sentence X is false. That is, X logically follows from K only if it is necessarily true that if all the sentences in K are true, then X is true, i.e., it is not possible for all the K -sentences to be true with X false. For example, the supposition that *All West High School students are football fans* and that *Kelly is not a West High School student* does not rule out the possibility that Kelly is a football fan. Hence, the sentences *All West High School students are football fans* and *Kelly is not a West High School student* do not entail *Kelly is not a football fan*, even if she, in fact, isn't a football fan. Also, *Most of Kelly's male classmates are football fans* does not entail *Most of Kelly's classmates are football fans*. What if the majority of Kelly's class is composed of females who are not fond of football?

The sentences *Kelly is not both at home and at work* and *Kelly is at home* jointly imply that *Kelly is not at work*. Note that it doesn't seem possible for the first two sentences to be true and *Kelly is not at work* false. But it is hard to see what this comes to without further clarification of the relevant notion of possibility. For example, consider the following pairs of sentences.

Kelly is a female.

Kelly is not the US President.

Kelly kissed her sister at 2:00pm.

2:00pm is not a time during which Kelly and her sister were ten miles apart.

There is a chimp in Paige's house.

There is a primate in Paige's house.

Ten is greater than nine.

Ten is a prime number.

For each pair, there is a sense in which it is not possible for the first to be true and the second false. At the very least, an account of logical consequence must distinguish logical possibility from other types of possibility. Should truths about physical laws, US political history, zoology, and mathematics constrain what we take to be possible in determining whether or not the first sentence of each pair could logically be true with the second sentence false? If not, then this seems to mystify logical possibility (e.g., how could ten be a prime number?). Given that I know that Barack Obama is US President and that he is not a female named *Kelly*, isn't it inconsistent for me to grant the logical possibility of the truth of *Kelly is a female* and the falsity of *Kelly is not the US President*? Or should I ignore my present state of knowledge in considering what is logically possible? Tarski does not derive clear notions of the logical modalities (i.e., logical necessity and possibility) from the common concept of logical consequence. Perhaps there is none to be had, and we should seek the help of a proper theoretical development in clarifying these modal notions. With this end in mind, let's turn to the other features of logical consequence highlighted by Tarski, starting with the formality criterion of logical consequence.

The logical consequence relation is formal

Tarski observes that logical consequence is a formal consequence relation. And he tells us that a formal consequence relation is a consequence relation that is uniquely determined by the sentential forms of the sentences between which it holds. Consider the following pair of sentences.

1. Some children are both lawyers and peacemakers
2. Some children are peacemakers

Intuitively, (2) is a logical consequence of (1). It appears that this fact does not turn on the subject matter of the sentences. Replace 'children', 'lawyers', and 'peacemakers' in (1) and (2) with the variables *S*, *M*, and *P* to get the following.

- 1'. Some *S* are both *M* and *P*
- 2'. Some *S* are *P*

(1') and (2') are sentential forms (*sentential functions*, in Tarski's terminology) of (1) and (2), respectively. Note that there is no interpretation of *S*, *M*, and *P* according to which the sentence that results from (1') is true and the

resulting instance of (2') is false. Hence, (2) is a formal consequence of (1), and on each meaningful interpretation of *S*, *M*, and *P* the resulting (2') is a formal consequence of the sentence that results from (1') (e.g., *some clowns are sad* is a formal consequence of *some clowns are lonely and sad*). Tarski's observation is that, relative to a language *L*, for any sentence *X* and class *K* of sentences, *X* is a logical consequence of *K* only if *X* is a formal consequence of *K*. The formality criterion of logical consequence can work in explaining why one sentence doesn't entail another in cases where it seems impossible for the first to be true and the second false.

For example, to think that (3) *Ten is a prime number* does not entail (4) *Ten is greater than nine* does not require one to think that ten could be a prime number and less than or equal to nine, which is a good thing since it is hard to see how this is possible. Rather, we take (3') *a* is a *P* and (4') *a* is *R b* to be the forms of (3) and (4) and note that there are interpretations of 'a', 'b', 'P', and 'R' according to which the first is true and the second false (e.g., let 'a' and 'b' name the numbers two and ten, respectively, and let 'P' mean *prime number*, and 'R' *greater than*). Note that the claim here is not that formality is sufficient for a consequence relation to qualify as logical, but only that it is a necessary condition. I now elaborate on this last point by saying a little more about sentential forms and formal consequence.

Distinguishing between a term of a sentence replaced with a variable and one held constant determines a form of the sentence. In *Some children are both lawyers and peacemakers* we may replace 'Some' with a variable and treat all the other terms as constant. Then

1". *D* children are both lawyers and peacemakers

is a form of (1), and each sentence generated by assigning a meaning to *D* shares this form with (1). For example, the following three sentences are instances of (1"), produced by interpreting *D* as 'No', 'Many', and 'Few'.

No children are both lawyers and peacemakers

Many children are both lawyers and peacemakers

Few children are both lawyers and peacemakers

Whether *X* is a formal consequence of *K* then turns on a prior selection of terms as constant and others uniformly replaced with variables. Relative to such a determination, *X* is a formal consequence of *K* if and only if (hereafter we abbreviate 'if and only if' to 'iff') there is no *interpretation* of the variables according to which each of the *K*-sentences are true and *X* is false.