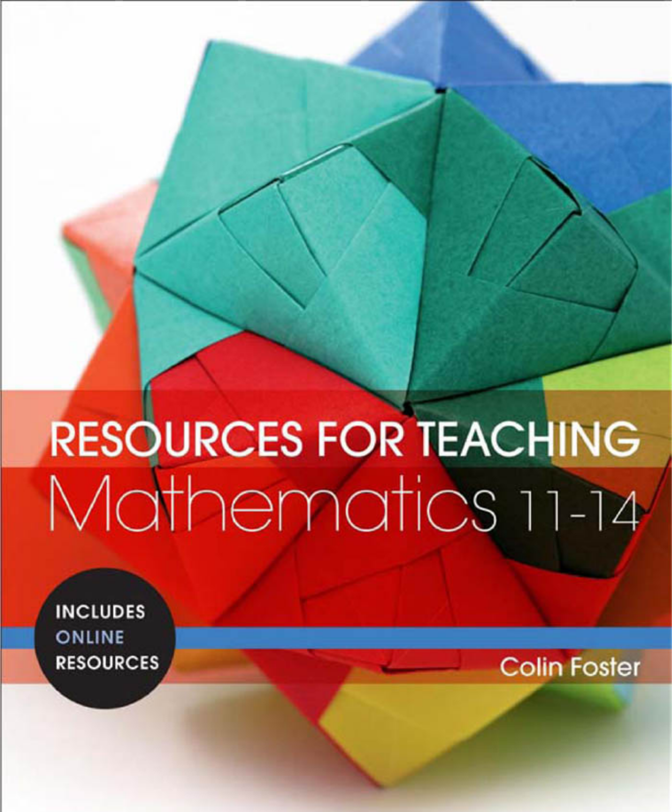

RESOURCES FOR TEACHING Mathematics 11-14

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Colin Foster

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Introduction

Resources for Teaching Mathematics 11–14 comprises 50 lesson plans covering many of the topics commonly encountered at this age. The lessons seek to draw learners into thinking about the various ideas by involving them in tasks that are not straightforward or routine. Learners of any level of prior attainment can gain satisfaction from thinking carefully about worthwhile mathematics and, in the hands of a skilful teacher, these tasks can work well for all learners, however they may be grouped or setted. Much of the material is designed to be divergent, with learners self-differentiating according to their current skills, interest, speed or mood. It is to be expected that different learners, beginning with the same starting point, will finish in quite different places. 'Answers' are given for the teacher's convenience rather than to suggest that they should be the optimum endpoint for everyone. It helps if the teacher is not too rigid in his/her mind about what the 'proper' route should be or how far learners must get during a particular lesson. The notion of 'finishing off' is problematic for many mathematics teachers when working with rich tasks, since when a mathematician answers one question they are likely to find themselves asking others.

The lessons are described under headings that indicate a possible beginning of the lesson ('starter'), a middle (some kind of 'main lesson') and a suggested whole-class discussion to end ('plenary'), but clearly a successful mathematics lesson does not have to follow that pattern slavishly. Beginning with a plenary or having one or more mini-plenaries in the middle, or omitting a starter, etc., are all viable options, so there is no need to be too regimented about such matters. The teacher should not feel dictated to by resources such as this book and must be free to adapt material, responding to learners' progress, comments, questions and interests, and using their skills to lead the lesson in the most beneficial way possible. A photocopiable Task Sheet is provided for each lesson to help learners engage with the tasks at their own pace. Likely outcomes and objectives are indicated to help the teacher to think in advance about what may arise and what opportunities are available. Possible homework tasks are suggested, as are ways of extending the tasks for early-finishers or those very confident with the ideas. There are also suggestions of ways in which a learner who finds the work particularly difficult might begin.

On many occasions in these lessons, learners are asked to generate their own mathematical examples. This can be beneficial by encouraging them to see what possibilities are available within the particular mathematical structure. Additionally, they also give the teacher valuable information, enabling him/her to assess the depth and breadth of a learner's understanding at that moment. What learners choose to create tells you about what possibilities they are aware of and able to access. Techniques such as specializing and generalizing in a problem-solving context are central to working on mathematics and are encouraged on many occasions.

Human beings possess the capacity to solve demanding problems and work with complex situations. The mathematical tasks presented in this book seek to draw on learners' innate abilities to make sense out of situations – to bring order out of chaos. Some of the material will be highly challenging, but if learners can be encouraged to struggle without giving up, and are given the time and freedom necessary to dig into the ideas, they can get a lot out of working with rich problems. Sometimes teachers feel under pressure to over-simplify and to try to turn every area of mathematics into a stepwise list of rules, so that 'methods' come from the teacher and the learners' responsibility is merely to memorize and reproduce them on demand. If this happens, lessons are likely to degenerate into 'demonstration by the teacher' followed by 'practice' by the learners. Such a structure does not harness the natural mathematical talents that young people possess and tends to be unrewarding for everyone involved.

Some of the questions and prompts on the resource sheets are linguistically demanding and learners who find this a barrier are likely to need in-class support in order to access the materials fully. At times the wording may appear vague, but this is in an attempt to move away from rigid instructions and provide space for learners to interpret things in their own way and structure their own work. These lessons will succeed if learners and teachers are not trying to finish them as quickly as possible but instead are attempting to look around for interesting avenues and generate questions and ideas of their own as they go.

The questions and prompts intended directly for learners are displayed in *italics* throughout the teacher notes. All of the Task Sheets are available online and other online resources are indicated by the mouse symbol.

Colin Foster
March 2011

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1 Amazing Ways

Introduction

'Maze' word puzzles, similar to the ones used in this lesson, appear frequently in puzzle books and newspapers. One way to tackle them is simply to try to count all the possibilities. However, being more systematic can save a lot of time, improve accuracy and enable an appreciation of the structure. The one presented on the Task Sheet is intended to be large enough to discourage simple counting and provoke a more thoughtful approach.

Aims and outcomes

- Count systematically and reason logically.
- Investigate a structure related to Pascal's triangle.

Lesson starter (15 minutes)

What do you see when you look at this?

S

S H S

S H T H S

S H T A T H S

S H T A M A T H S

It may take some time before learners see more than one occurrence of the word 'MATHS'. Learners could come to the board to show a spelling of 'MATHS' with their finger.

How many ways are there of 'doing MATHS' here?

Different learners could show different routes. There are 31 ways, but this answer is likely to come later.

Can you be sure that you have found them all?

One way would be to simplify the problem – it would be easier in countries which call the subject 'MATH', so we could start by losing the 'S'. Learners could begin by finding ways of doing 'MA', then 'MAT', then 'MATH'.

Main lesson (25 minutes)

Give out the Task Sheets. 'MATHEMATICS' is much too long a word to do by counting. You could focus learners' attention on the number of ways of arriving at any particular *letter* on the grid (shown with numbers over the page), or leave learners to find this for themselves.

$$
\begin{array}{ccccccccc}
 & & & & S & & & & \\
 & & & S^4 & H^1 & S^4 & & & \\
 & & S^6 & H^3 & T^1 & H^3 & S^6 & & \\
 & S^4 & H^3 & T^2 & A^1 & T^2 & H^3 & S^4 & \\
S^1 & H^1 & T^1 & A^1 & M^1 & A^1 & T^1 & H^1 & S^1
\end{array}
$$

$1 + 4 + 6 + 4 = 2^4 - 1$ $\qquad$ $1 + 4 + 6 + 4 = 2^4 - 1$

To arrive at any of the letters in the right-hand side of the grid, all movements must be either upwards (U) or right (R), so instructions such as UUR would represent a $\binom{2}{1}$ shift from the starting M to an H. Since there are ${}^3C_1$ ways of permuting two Us and an R, there are this many (3) ways of arriving at this H. So the positions in the grid are the binomial coefficients ${}^nC_r$ from Pascal's triangle. (The left-hand side works similarly using upwards and *left*.) Learners may see that to arrive at a particular letter necessitates having come from one position to the side or one position down, and therefore the number of ways of getting there must be the sum of the number of ways of getting to these two positions. Alternatively, looking at the right-hand side of the diagram, there are two ways of getting to either of the As, and then, whichever A you are on, there are two ways of getting to a T, and so on, giving 2^4 ways of getting to any of the six Ss on the right. Since there will also be 2^4 ways of getting to the six Ss on the left, we just need to subtract 1 for the top S (which we have counted twice) to get $2(2^4) - 1 = 2^5 - 1 = 31$ ways.

Plenary (15 minutes)

Which words did you try? Why? How many ways did you find? What did you notice?

Learners may observe symmetries that make it easier to calculate the total number of ways – such as dividing the diagram into halves or quarters. Learners may comment on the fact that the letters M, A, T and H all have a vertical line of symmetry, and so a diagram based on MA, MAT or MATH will have a vertical line of symmetry through the M. Learners may notice that the total number of ways must be an odd number, since all routes have a partner found by reflecting in the vertical line through the M; however, there is also the straight-up path from M, which adds an additional 1, making the total number of ways odd.

For MATHS, taking this approach gives that the total number of ways $= 2(2^4 - 1) + 1 = 2^5 - 1 = 31$, as before. In general, for an n-letter word, number of ways $= 2^n - 1$.

n	**Number of ways**
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1023
11	2047

So, the answer for MATHEMATICS (with 11 letters) is 2047. Learners may notice that each answer is the previous answer multiplied by 2 and add 1.

If we allow the letters to continue *downwards* as well, we obtain:

S
S H S
S H T H S
S H T A T H S
S H T A M A T H S
S H T A T H S
S H T H S
S H S
S

Here, number of ways $= 4(2^4 - 1) = 2^6 - 4 = 60$. In general, for an n-letter word, the number of ways $= 4(2^{n-1} - 1)$.

n	**Number of ways**
1	1
2	4
3	12
4	28
5	60
6	124
7	252
8	508
9	1020
10	2044
11	4092

So the answer for MATHEMATICS this time is 4092. In this situation, from the second value onwards, each term is the previous one multiplied by 2 and add 4.

Do all five-letter words, say, have the same number of ways?

Double letters may lead to some discussion, since *diagonal* moves now (perhaps) become possible, if this is allowed, leading to many more routes (two shown below, for the word MEET):

T
T E T
T E E E T
T E E M E E T
T E E E T
T E T
T

Including diagonal moves, there are now $8 \times 3.5 \times 2.5 = 70$ ways.

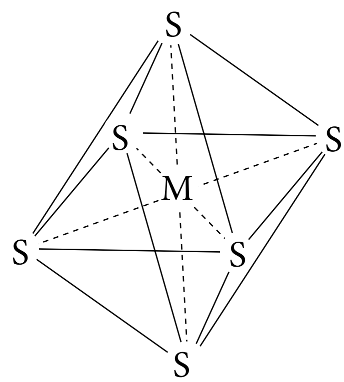
Palindromic words (words which read the same forwards and backwards) are also interesting. Learners may think that they will simply have twice as many ways, since each route can be traced in either direction, but there are more than this, since the word now does not have to begin or end at the centre of the diagram. A three-letter palindrome, such as TOT, has 48 ways, but a four-letter palindrome such as TOOT (which necessarily has a double letter in the middle) has, ignoring diagonal moves, exactly twice as many ways as MATH (i.e. 56 ways). A five-letter palindrome, such as LEVEL, however, has 228 ways, if you count cyclic routes in which the first and final L are the same position.

Homework (5 minutes)

Learners could attempt to make up and solve similar problems on *isometric* grids.

To make it harder

Confident learners could extend the idea into three dimensions, creating an octahedron of letters.



The contribution of the 'equatorial' Ss will be $2^6 - 4$, as for the 2D case. There are eight more edges, each contributing $2^4 - 2$ ('subtract 2' since we have already counted the 'equatorial' S of each one). Then there are two further Ss at the apexes. So the total number of ways of reading MATHS will be $2^6 - 4 + 8(2^4 - 2) + 2 = 174$. The same approach with an n-letter word, shows that, in general, the total number of ways $= 6(2^n - 3)$.

n	**Number of ways**
1	1
2	6
3	30
4	78
5	174
6	366
7	750
8	1518
9	3054
10	6126
11	12 270

To make it easier

Starting with a short word, such as MAT, should be accessible to all learners.

Amazing Ways

```
                    S
                  S C S
                S C I C S
              S C I T I C S
            S C I T A T I C S
          S C I T A M A T I C S
        S C I T A M E M A T I C S
      S C I T A M E H E M A T I C S
    S C I T A M E H T H E M A T I C S
  S C I T A M E H T A T H E M A T I C S
S C I T A M E H T A M A T H E M A T I C S
  S C I T A M E H T A T H E M A T I C S
    S C I T A M E H T H E M A T I C S
      S C I T A M E H E M A T I C S
        S C I T A M E M A T I C S
          S C I T A M A T I C S
            S C I T A T I C S
              S C I T I C S
                S C I C S
                  S C S
                    S
```

What do you see when you look at this diagram?
How many ways do you think there are of finding MATHEMATICS in this diagram? Why?
How can you be sure that you have found them all?

What do you think would happen if you used a shorter or longer word? Why?
What happens with *double* letters?
What if you used a *palindrome* (a word that reads the same forwards and backwards), like ROTAVATOR?

2 Dividing Cakes

Introduction

Does $\frac{1}{2} + \frac{1}{2} = 1$? Mathematically, of course, yes, but in real life two broken half computers do not make one whole working machine; two half-accurate statements do not add up to one accurate statement. Similarly, with many classic 'real-life mathematics' sharing problems, although the fractional pieces may, in theory, add up to the same amount, they are not necessarily equally acceptable to those involved. You expect to pay less for a bag of broken biscuits than you do for the same mass of undamaged product! This lesson uses this idea as an opportunity for finding different decompositions of a certain fractional quantity.

Aims and outcomes

- Add fractions to obtain desired fractional amounts.
- Understand how to partition fractions into different amounts.

Lesson starter (10 minutes)

You have 10 cakes to share equally among 9 people. How would you do it? How many cuts would you have to make? Why?

Learners are likely to suggest something like this: First give everyone one whole cake. Then cut the leftover cake into 9 equal pieces (this is not too hard; you can do thirds and then thirds of thirds). So give each person $\frac{1}{9}$, then everyone has a whole cake and $\frac{1}{9}$ of a cake (i.e. two pieces).

Let's do it the other way round. This time you have 10 people and 9 cakes. What would you do this time?

Learners could think about this in groups. You could cut $\frac{1}{10}$ out of each of the nine cakes. Then 9 of the people each get the remaining $\frac{9}{10}$ of these nine cakes. The ninth person gets all 9 of the separate $\frac{1}{10}$ pieces.

Is this fair?

'Fairness' means different things to different people. Maybe some people are bigger, older or more important (or all three!) or are allergic to or don't like cake, etc. So it would be good to ask learners what they think counts as fair. Here everyone gets the same amount of cake, but nine of the people get almost a complete cake, and one person just gets a heap of crumbs! Learners might suggest cutting each cake into eighths, say, so that 9 people get $\frac{7}{8}$ each and the person with the bits gets $\frac{9}{8} = 1\frac{1}{8}$, which is a little more, so as to compensate for the 'crumbiness' of their portion.

Main lesson (30 minutes)

Come up with a better solution, so that everyone gets exactly *the same – not just the same amount of cake but the same number and size of pieces: indistinguishable portions. Try to make as little mess as possible (as few cuts as you can).*

Give out the Task Sheets and encourage learners to think hard about this problem. Learners may suggest cutting *all* nine cakes into tenths and giving each person nine separate tenths. This satisfies the fairness aspect but is obviously messy, wasteful and time-consuming. So you might want to encourage learners also to minimise the number of cuts.

Plenary (15 minutes)

What did you work out? What were your ideas?

Learners may suggest things like baking another cake or drawing lots to decide as fairly as possible who should get the inferior portion. But we are aiming to make nine equal-in-every-respect portions.

One solution is to give each person $\frac{1}{2} + \frac{2}{5}$. Other possible solutions, for 10 cakes but different numbers of people, are shown in the table below

$\frac{\text{Number of cakes}}{\text{number of people}}$	Possible fractions given to each person
$\frac{9}{10}$	$\frac{1}{2} + \frac{1}{5}$ $\frac{1}{2} + \frac{1}{3} + \frac{1}{15}$
$\frac{8}{10}\left(= \frac{4}{5}\right)$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{20}$ $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$
$\frac{7}{10}$	$\frac{1}{2} + \frac{1}{5}$
$\frac{6}{10}\left(= \frac{3}{5}\right)$	$\frac{1}{2} + \frac{1}{10}$
$\frac{5}{10}\left(= \frac{1}{2}\right)$	$\frac{1}{2}$

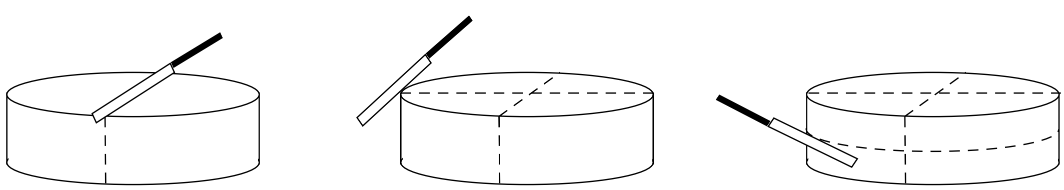
Learners may comment on the 'sacrifice' that some people are making, in terms of the quality of their cake portions, for the sake of equality for everyone, and this could relate to learners' views on society and egalitarianism.

Homework (5 minutes)

Find out about *Egyptian fractions* and what they have to do with this problem. Learners could be asked to make a poster about it. A good website is: www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/egyptian.html

To make it harder

There should be plenty of depth in this problem for all learners. A related problem to consider is to work out the maximum number of pieces you can cut a circular cake into with n straight-line cuts. The answer is the so-called *lazy caterer's sequence*: 2, 4, 7, 11, 16, . . . which fit the formula ${}^nC_2 + {}^nC_1 + {}^nC_0 = \frac{1}{2}(n^2 + n + 2)$. In *three* dimensions, you have a *cylindrical* cake divided into n plane slices, and this gives ${}^nC_3 + {}^nC_2 + {}^nC_1 + {}^nC_0 = \frac{1}{6}(n^3 + 5n + 6)$ pieces as the maximum number you can make; the so-called *cake numbers*.



2 4 8

To make it easier

Learners who find this hard could start with dividing 3 cakes among 4 people, in which case each person can have one half and one quarter. Then move to 3 cakes among 5 people ($\frac{1}{2} + \frac{1}{10}$ each).

Dividing Cakes

Share 9 cakes among 10 people so that everybody gets *exactly the same portion*, not just the same amount of cake.

Try to use as few cuts as possible.

What happens with a different number of cakes?

What happens with a different number of people?

Draw a Picture

Introduction

This lesson relies on each learner (or pair of learners) having access to a computer running graph-drawing software, such as *Omnigraph* or *Autograph*, or a graphical calculator. Learners will work on making pictures using mathematical equations, particularly circle equations. Often learners will get close to what they want and then have to tweak it – moving a curve up a little or to the left or enlarging it. On other occasions, the curve will not appear at all as the learner wished and they will have to work out what has happened. This is where their sense of how the equations relate to the drawings will develop.

Aims and outcomes

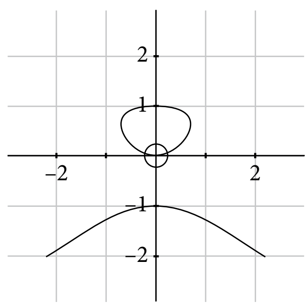
- Know how the equation of a curve relates to its appearance.
- Know how varying a, b and r in the equation $(x - a)^2 + (y - b)^2 = r^2$ of a circle affects the circle.

Lesson starter (10 minutes)

If you have access to graph-drawing software and a data projector, enter $y = x^2 + y^3$ and ask learners to describe what they see.

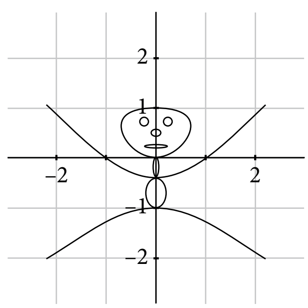
It is likely that someone will suggest that it looks like a person; if not, you could do so.

What do we need to add?



Take learners' suggestions and, by adding suitable curves (e.g. circles, ellipses or whatever learners are familiar with), you can end up with something like this:

(For more information, see Foster, C. (2007) 'Introducing . . . Maths-man!', *Mathematics in School*, 36, (1), 15.)



Main lesson (35 minutes)

Give out the Task Sheets. Learners can use any equations that they know about, or experiment to see what is possible. However, circles are likely to be very useful, particularly if learners realize (or are told) that adding constants p and q into the standard equation $(x - a)^2 + (y - b)^2 = r^2$, creating $p(x - a)^2 + q(y - b)^2 = r^2$, generates *ellipses*. (Strictly speaking, only one of p and q is needed, since the other is equivalent to varying r; however, it may be more convenient to have both.)
You may need to assist learners in finding their curves if they are off the screen, using the zoom in/out features.

Plenary (10 minutes)

Rather than having a whole-class discussion, it might be better to invite learners to print out or save their completed drawings and use these for display purposes. If it is possible to print them out during the lesson, they could be placed around the room and learners could have a 'walking plenary', looking at and commenting on each other's work.

How did you do that? How did you change the equation to make that happen?

Homework (5 minutes)

Write about the types of equations you used and how you varied them. Summarize what happens when you change the constants in the equation of a circle.

Learners could also find out more about ellipses and their equations.

To make it harder

There are no limits to the opportunities here for more complicated equations and curves.

To make it easier

Learners who find this task difficult can be encouraged to experiment with the computer, entering different equations and seeing what happens. Reassure them that they can always 'undo' anything that they don't like or that doesn't work. Drawing different-sized circles in different positions on a blank set of axes could be a useful way to begin.