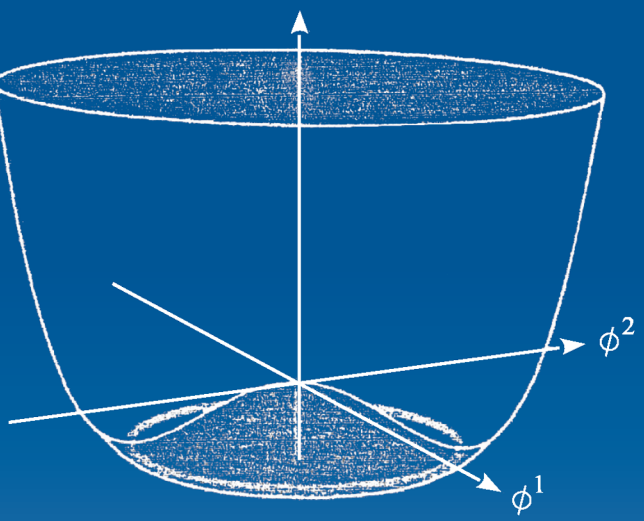
Series in High Energy Physics, Cosmology, and Gravitation

Group Theory for the Standard Model of Particle Physics and Beyond



Ken J. Barnes

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Group Theory for the Standard Model of Particle Physics and Beyond

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Group Theory for the Standard Model of Particle Physics and Beyond

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Preface

This book emerged out of lectures to first year postgraduate students at the then Department of Physics and Astronomy, University of Southampton, before I retired. It is hoped that this book will be appropriate for similar groups of readers in many other institutions across the world. Experimenters in this subject would probably gain much from reading this book, although some may find it difficult.

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This book could never have been written without the consistently excellent help of Mrs. Hannah Williams, who handled LaTeX, figures, and packaging apparently with ease. My son, Dr. Geoffrey W. Morton, is also thanked for some of the figures and general advice. Dr. Jason Hamilton-Charlton is thanked for his generosity in providing both LaTeX and English electronic copies of my supersymmetry notes. Finally, I thank my wife, Jacky, for her continual support and help when writing anything seemed quite impossible.

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Introduction

This book is definitely not a book on mathematics. It is a book on the use of symmetries, mainly described by the techniques of Lie groups and Lie algebras. Although no proofs of theorems and the like are given, except in special cases, the ideas are very firmly based on a lifetime of lecturing experience.

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1

Symmetries and Conservation Laws

You may already be familiar with the ideas of conserved quantities, such as charge in electromagnetism, but it will not hurt to go through this once more, and there may be students for whom it is quite new. Since we are dealing with elementary particles, we may as well think of conserved numbers carried on particles, and indeed we will start with the charge e on the proton. If we consider the charge of the electron $(-e)$, which carries electric currents, what do we mean by "it is conserved" and what consequences might this have? We might as well, for simplicity, start with the problem in classical physics and turn to quantum mechanics later. Well, the first thing is that it cannot simply vanish or appear. Of course it can vanish by having equal but opposite charges annihilate it (producing, for example, the photons of light), or it can appear in the reverse of this. All other conserved quantities such as energy, and linear and angular momentum must be conserved—in our picture carried on the photons. Already we see that this must happen at the same time and at the same spatial point, but this is natural when the charges are carried on the particles.

You may well be familiar with the idea of conservation of charge being associated with the four divergence of the current carrying that charge. Calling j^μ the current carried by an electron (of charge $(-)e$) we can write

$$\partial_\mu j^\mu = 0 \,. \tag{1.1}$$

Then we have

$$\partial\rho\partial t + \underline{\nabla}.\underline{j} = 0 \tag{1.2}$$

where ρ is the time component of j^μ and $\underline{j}$ is the spatial part of this current. If we integrate over a fixed volume we find

$$\frac{\partial\rho}{\partial t} + \text{ flow of current normally into the volume}$$
$$- \text{ flow of current normally out of the volume} = 0. \tag{1.3}$$

This means that the rate of increase of charge in the volume is equal to the rate of flow of charge into the volume minus the rate of flow out of the volume. A very natural feature of the model we use is where the charges are carried on the

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particles. Of course, this concept needs slight changing in the world of special relativity where there is apparent contraction of lengths and dilation of times in different reference frames. Similarly in quantum mechanics further modifications are needed, which are yet further changed in quantum field theory. But we are getting too far ahead of ourselves. Let us ask what symmetries have to do with these conservation laws as our title of this chapter suggests. There is a theorem by E. Noether [1] to the effect that this is precisely what happens. It is not appropriate to prove this theorem at this stage, but it is very powerful and extends to all types of description of the physics discussed earlier. (Students note that Noether was a woman doing important work of this type at a time when there were nowhere near as many women working in science.)

The point that is necessary to understand at this stage is that all conserved quantities in physics are linked to symmetries in this way. We shall meet examples of this later. The mathematics underlying this structure is that of group theory, both discrete groups and continuous groups as described by Lie. But for the moment we move on to simple examples in the next two chapters.

Lagrangian and Hamiltonian Mechanics

Although it has been made clear that the reader is expected to be competent in quantum field theory, an exception is made at this point to be sure that the readers really can cope.

It is one of those curious quirks of history that long before quantum theory was developed this version of classical mechanics established a framework that was capable of treating both fields and particles in both classical and quantum aspects. You are strongly urged to read Chapter 19 of Volume II of *The Feynman Lectures on Physics* [2] as an introduction to the deep and fascinating approach to physics in terms of the "principle of least action," if you have not met it previously. We shall approach the topic in a more pedestrian manner than Feynman, partly because I am not so brave a teacher and partly because I want to get you calculating for yourself as soon as possible. It is my firm belief that the best way to get on top of a subject like this is to lose your fear of it by getting your hands dirty and actually doing the real calculations in detail yourself.

Suppose we have a one-dimensional system—yes, it is going to be the harmonic oscillator. We shall call the displacement from equilibrium $q(t)$ rather than $x(t)$ because later on we shall want "displacements of the fields" at various points x and we do not wish to confuse the "displacements" with the spatial positions. Then Newton's second law is replaced by the Euler–Lagrange [3] equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \tag{1.4}$$

where $\dot{q}$ is the time derivative of q. The Lagrangian, L, is the difference between the kinetic energy (T) and the potential energy (V), that is,

$$L(q, \dot{q}) = T - V \tag{1.5}$$

and is to be regarded as a function of the independent variables q and $\dot{q}$ for the purposes of partial differentiation. For the harmonic oscillator with mass m and spring constant k we have

$$V = \frac{k}{2}q^2 = \frac{m\omega^2}{2}q^2 \tag{1.6}$$

where $\omega^2 = \frac{k}{m}$. So that

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \tag{1.7}$$

and the Euler–Lagrange equation yields

$$\frac{d}{dt}(m\dot{q}) = -m\omega^2 q \tag{1.8}$$

and we retrieve

$$\ddot{q} = -\omega^2 q \tag{1.9}$$

as expected.

Now that we have a little experience with this formalism, we can take a look at the principle of least action. You will have noticed perhaps that the concept of force (which was primary in Newton's approach) has become secondary to the idea of potential. The least action principle makes the equation of motion itself something that is derived from the minimization of the action

$$S = \int_{t_i}^{t_f} L(q, \dot{q})dt \tag{1.10}$$

where t_i and t_f are initial and final times. The principle postulates that the actual path (often alternatively called trajectory) followed by the particle is that which minimizes S. Imagine that, given L as an explicit function of q and $\dot{q}$, you evaluate S for a few paths. These are just fictitious paths and none of them is likely to be the Newtonian one. I have drawn the three from the problem on the q–t diagram in Figure 1.1.

These must start and finish at the same places and times. According to the principle, only if one of these coincides with the Newtonian path will the value of S be the minimum possible. You need a calculus approach to get a general answer. Notice, however, S is a function of the function $q(t)$. We say it is a "functional" of $q(t)$. We need to find the particular function, $q_0(t)$, that minimizes S.

Suppose there is a small variation $\delta q(t)$ in a path $q(t)$ from $q(t_i)$ to $q(t_f)$. When $q(t) = q_0(t)$, the variation δS caused by this change δq must vanish.

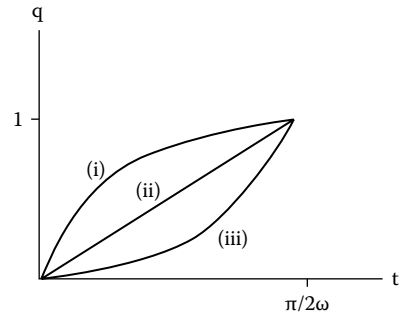

FIGURE 1.1
q–t diagram.

Now we can work out the change of action for any path as

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\right) dt \\ &= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\delta q\right) dt \\ &= \int_{t_i}^{t_f} \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]\right) dt + \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_i}^{t_f}\end{aligned}$$

where we used $\delta\dot{q} = \frac{d}{dt}\delta q$ in the second step. But we are considering paths with fixed end points, so that $\delta q(t_i) = 0 = \delta q(t_f)$ for any variation, and the final term vanishes. Hence, since δS must vanish for arbitrary δq, we need

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q},$$

which retrieves the Euler–Lagrange equation of motion. The solution of this is the $q_0(t)$, which gives the path actually followed by the particle.

As we shall see later, this formalism is well suited to treat systems of the many (indeed infinitely many) linked dynamical variables found in field theories. But the transition from classical to quantum mechanics is made more transparent by considering the Hamiltonian formulation. The idea, in the first place, is to find a change in variables (from q and $\dot{q}$) which will replace the second order Euler–Lagrange equation by two linked first order equations. This piece of magic is performed by introducing

$$p = \frac{\partial L}{\partial \dot{q}} \tag{1.11}$$

as a "generalized momentum conjugate to the generalized coordinate q." (When q is a Cartesian coordinate, p will frequently be the usual linear momentum, as we shall see.) Then the Hamiltonian is introduced by the

Legendre transformation

$$H(q, p) = p\dot{q} - L(q, \dot{q}) \tag{1.12}$$

and the Euler–Lagrange equation is replaced by the pair of equations

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1.13}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{1.14}$$

which are known as Hamilton's canonical equations. To get a feel for this formulation we return to our old friend the harmonic oscillator. From Equation (1.7) we see that

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}, \tag{1.15}$$

which is reassuring, and we can then see that from Equation (1.12)

$$\begin{aligned} H &= \frac{p^2}{m} - \left\{ \frac{p^2}{2m} - \frac{m\omega^2}{2} q^2 \right\} \\ &= \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \end{aligned}$$

is the form of the Hamiltonian in the new variables. Notice that the Hamiltonian is the total energy, $T + V$. This is a very general feature, and provided that time does not appear explicitly then

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\left[-\frac{\partial H}{\partial q} \right] = 0, \tag{1.16}$$

which reflects energy conservation. In the present case the equations of motion, Equations (1.13) and (1.14), yield

$$\dot{q} = \frac{p}{m} \tag{1.17}$$

$$\dot{p} = -m\omega^2 q \tag{1.18}$$

when Equation (1.16) is used directly. The first of these reconfirms the definition of the momentum, and on substitution into the second retrieves Equation (1.9) as the second order equation of motion. It turns out, however, to be instructive to solve the first order Equations (1.17) and (1.18) directly. Consider the linear combination

$$A = \frac{1}{\sqrt{2}} \left(x\sqrt{m\omega} + ip \frac{1}{\sqrt{m\omega}} \right), \tag{1.19}$$

which is so designed that

$$\dot{A} = -i\omega A \tag{1.20}$$

$$A = ae^{-i\omega t} \tag{1.21}$$

as the obvious solution, where a is constant. Taking the complex conjugate of Equation (1.19), we immediately find

$$x = \frac{1}{\sqrt{2m\omega}}(A + A^*) = \frac{1}{\sqrt{2m\omega}}(ae^{-i\omega t} + a^*e^{i\omega t}), \tag{1.22}$$

which is equivalent to the previous solution.

Quantum Mechanics

The passage to quantum mechanics in this formalism is facilitated by introducing the Poisson bracket notation. The Poisson bracket of any two functions f and g, of q and p, is simply

$$\{f, g\} \equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{1.23}$$

and we see that

$$\{q, H\} = \dot{q} \tag{1.24}$$

$$\{p, H\} = \dot{p} \tag{1.25}$$

are alternative ways of writing Equations (1.13) and (1.14), the equations of motion. Moreover, if F is any function of q and p, then

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} \\ &= \frac{\partial F}{\partial q}\{q, H\} + \frac{\partial F}{\partial p}\{p, H\} = \{F, H\}\end{aligned} \tag{1.26}$$

while

$$\begin{aligned}\{q, q\} &= 0 \\ \{p, p\} &= 0 \\ \{q, p\} &= 1\end{aligned} \tag{1.27}$$

follow directly from the definition (Equation (1.24)) of the Poisson bracket. The transition to quantum mechanics is now effected by the correspondence $\{\alpha, \beta\} \to -i[\hat{\alpha}, \hat{\beta}] = -i(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha})$ between the classical dynamical variables

and their hatted quantum mechanical operator correspondences. (We use "natural" units with $\hbar = 1$.) In particular, Equation (1.27) yields

$$[\hat{q}(t),\, \hat{p}(t)] = i \tag{1.28}$$

expressing the Heisenberg uncertainty principle [5], and Equation (1.26) gives

$$\frac{d\hat{F}(t)}{dt} = -i[\hat{F}(t),\, H] \tag{1.29}$$

as the Heisenberg equation of motion. The time dependence has been exhibited to draw the reader's attention to the fact that this is quantum mechanics expressed in the Heisenberg picture [6], where states are time independent but the dynamical variables contain the time dependence.

The alternative Schrödinger picture, in which the variables are time independent, has the time dependence of state vectors given by the Schrödinger equation

$$\hat{H}|\psi(t) > = i\frac{\partial}{\partial t}|\psi(t) > \tag{1.30}$$

with the formal solution

$$|\psi(t) > = e^{-i\hat{H}}|\psi > \tag{1.31}$$

where we have identified the Schrödinger state at time zero with $|\psi(0) >$, with $|\psi >$ the time independent Heisenberg state. Of course, Equation (1.31) is just a unitary transformation between the two pictures, with

$$\hat{F}(t) + e^{i\hat{H}}(t) = e^{i\hat{H}}(t)\hat{F}\,e^{-i\hat{H}}(t) \tag{1.32}$$

as the corresponding transformation between operators. The important feature of this is that

$$[\hat{q},\, \hat{p}] = i \tag{1.33}$$

follows immediately from Equation (1.28) as an expression of the uncertainty principle in the Schrödinger picture. In quantum field theory we shall find the Heisenberg picture very convenient.

In the quantum case we have the operator version of Equation (1.15)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2 \tag{1.34}$$

$$= \frac{\hat{p}^2(t)}{2m} + \frac{m\omega^2}{2}\hat{q}^2(t) \tag{1.35}$$

with Equation (1.32) giving trivially the equality of these alternate forms. From the Heisenberg equation of motion (Equation (1.29)) we can easily see that

$$\dot{\hat{q}}(t) = \frac{\hat{p}(t)}{m} \tag{1.36}$$

$$\dot{\hat{p}}(t) = -m\omega^2 \dot{\hat{q}}(t) \tag{1.37}$$

so that we get

$$\ddot{\hat{q}}(t) = -m\omega^2 \hat{q}(t) \tag{1.38}$$

by combining these. Now, please notice that this is not just the classical equation of motion (Equation (1.9)) again. What Equation (1.38) tells us is the behavior of the operator with time, not where the particle can be found. If we take the expectation value of Equation (1.38) between (time independent) Heisenberg states, then we learn that the mean position of the particle does follow the classical path. This is very reassuring, but there will be quantum fluctuations about the classical path, of course.

The Oscillator Spectrum: Creation and Annihilation Operators

This subtopic is of such central importance later that it deserves a section all to itself. You have no doubt all been exposed to this material before, but I want to stress the operator treatment that we shall see again in our field theory. (If you already know this method, it will at least serve as a review and to establish notation.)

We seek a set of states $|E_n>$, $n = 0, 1, \ldots$, to serve as a complete basis in which to expand any general state, and thus must solve the time independent Schrödinger equation

$$\hat{H}|E_n> E_n|E_n> \tag{1.39}$$

for the eigenvalues and eigenvectors. The Hamiltonian is given in Equation (1.34) as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2$$

but our classical treatment suggests Equation (1.19)

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{q}\sqrt{m\omega} + i\,\hat{p}\frac{1}{\sqrt{m\omega}}\right) \tag{1.40}$$

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(\hat{q}\sqrt{m\omega} - i\,\hat{p}\frac{1}{\sqrt{m\omega}}\right) \tag{1.41}$$

as preferable dynamical variables. It is straightforward to see that

$$\begin{aligned}\hat{a}^{\uparrow}\hat{a} &= \frac{m\omega}{2}\hat{q}^2 + \frac{1}{2m\omega}\hat{p}^2 + \frac{i}{2}[\hat{q},\,\hat{p}] \\ &= \frac{1}{\omega}\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2\right) - \frac{1}{2}\end{aligned}$$

where Equation (1.33) is used in the last term. Hence we have

$$\hat{H} = \omega\hat{a}^{\uparrow}\hat{a} + \frac{\omega}{2} \tag{1.42}$$

$$\hat{H} = \omega\hat{a}\hat{a}^{\uparrow} - \frac{\omega}{2} \tag{1.43}$$

so that

$$\hat{H} = \frac{\omega}{2}(\hat{a}^{\uparrow}\hat{a} + \hat{a}\hat{a}^{\uparrow}) \tag{1.44}$$

$$[\hat{a},\,\hat{a}^{\uparrow}] = 1 \tag{1.45}$$

follow by adding and subtracting. Notice that (from Equation (1.42)),

$$[\hat{H},\,\hat{a}^{\uparrow}] = \omega\hat{a}^{\uparrow}[\hat{a},\,\hat{a}^{\uparrow}]$$

$$[\hat{H},\,\hat{a}^{\uparrow}] = \omega\hat{a}^{\uparrow} \tag{1.46}$$

$$[\hat{H},\,\hat{a}] = -\omega\hat{a}\,. \tag{1.47}$$

(In Equations (1.45)–(1.47) we now have the algebraic information in a suitable form to find the spectrum. I urge you to do Problem 1.14 before continuing.) We are now in a position to see exactly why $\hat{a}$ and $\hat{a}^{\uparrow}$ are so important. They have the magical property in that they take you from one energy eigenstate into another, rather than into some arbitrary linear combination of states. To see this, consider the effect of the Hamiltonian on an eigenstate that has been changed by the action of $\hat{a}^{\uparrow}$

$$\begin{aligned}\hat{H}\hat{a}^{\uparrow}|E_n> &= ([\hat{H},\,\hat{a}^{\uparrow}] + \hat{a}^{\uparrow}\hat{H})|E_n> \\ &= (\omega\hat{a}^{\uparrow} + \hat{a}^{\uparrow}E_n)|E_n> \\ &= (E_n + \omega)\hat{a}^{\uparrow}|E_n>\end{aligned} \tag{1.48}$$

so we see that $\hat{a}^{\uparrow}|E_n>$ is indeed an eigenstate of $\hat{H}$ and $(E_n + \omega)$ is the eigenvalue. In a similar way we can establish that $\hat{a}|E_n>$ is an eigenstate with $(E_n - \omega)$ as the eigenvalue this time. Of course, you cannot lower the energy until it becomes negative, so there must be a ground state of lowest energy E_0 with

$$\hat{a}|E_0> = 0 \tag{1.49}$$

as its definition to maintain consistency. (Beware! In relativistic physics such reasoning will not be true.) But here you can prove it. From Equation (1.42) we see that

$$\hat{H}|E_0> = 0 + \frac{1}{2}\omega|E_0>$$

establishing $E_0 = \frac{1}{2}\omega$ as the ground state (or zero point) energy. Then, by raising, we see that the energy spectrum is

$$E_n = \left(n + \frac{1}{2}\right)\omega \qquad n = 0, 1, \ldots \tag{1.50}$$

and the corresponding eigenstates are given by

$$|E_n> = \frac{(\hat{a}^\uparrow)^n}{\sqrt{n!}}|E_0> \tag{1.51}$$

where the exact factor follows from the requirement

$$< E_n|E_n> = 1 \tag{1.52}$$

of normalization. It is now natural to speak of a vacuum rather than a ground state, and then to envisage the "creation of particles" (or "excitation of quanta") into that vacuum. Indeed if we define a number operator

$$\hat{N} = \hat{a}^\uparrow\hat{a} \tag{1.53}$$

to conform to our notation in Equations (1.42) and (1.50), then the change of notation to

$$\hat{N}|n> = n|n> \tag{1.54}$$

$$\hat{H}|n> = E_n|n> = (n + 1/2)\omega|n> \tag{1.55}$$

becomes irresistible.

Coupled Oscillators: Normal Modes

Before we launch into an attack on the quantum field theory of infinitely many degrees of freedom, it is probably sensible to try a finite number of variables. Let's start with the classical theory of two equal masses in a one-dimensional space (e.g., in a straight slot on a horizontal table) tied together by a spring of spring constant g, and tied to fixed points by springs of spring constant k.

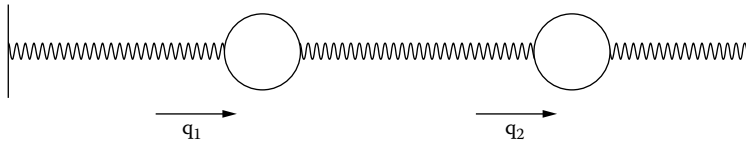

FIGURE 1.2
Three spring forces.

I have in mind the picture in Figure 1.2, where q_1 and q_2 are displacements from equilibrium, and the Lagrangian takes the form

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}g(q_2 - q_1)^2 \tag{1.56}$$

if none of the springs are stretched or compressed in the equilibrium position. You can think of this as a model of a (very) small solid. One advantage of the Lagrangian approach is that we never have to introduce the forces in the springs and then eliminate them again; constraints are handled very neatly in this formalism. The Euler–Lagrange equations yield

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{g}{m}(q_2 - q_1) \tag{1.57}$$

$$\ddot{q}_2 = -\frac{k}{m}q_2 - \frac{g}{m}(q_2 - q_1), \tag{1.58}$$

which are sufficiently simple that we do not need formal methods to solve them. We spot the relevant combinations of variables by adding and subtracting to obtain

$$(\ddot{q}_1 + \ddot{q}_2) = -\frac{k}{m}(q_1 + q_2) \tag{1.59}$$

$$(\ddot{q}_2 - \ddot{q}_1) = -\left(\frac{k}{m} + \frac{2g}{m}\right)(q_2 - q_1), \tag{1.60}$$

which we recognize as uncoupled simple harmonic oscillators. The solutions are then obvious. We have one normal mode of oscillation with frequency

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{1.61}$$

and Equation (1.60) is satisfied trivially by having the two displacements equal. The second normal mode has frequency

$$\omega_2 = \sqrt{\frac{k + 2g}{m}} \tag{1.62}$$

and Equation (1.59) is satisfied trivially by the two displacements being equal but opposite in sense. The general solution is then obtained by superposition