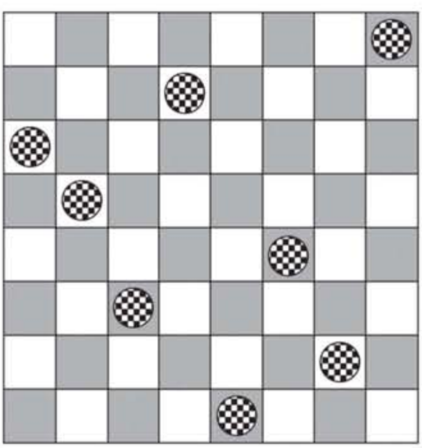
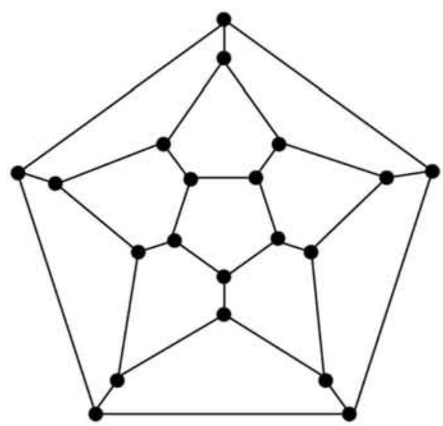
DISCRETE MATHEMATICS AND ITS APPLICATIONS

Series Editor KENNETH H. ROSEN

HOW TO COUNT
AN INTRODUCTION TO COMBINATORICS

Second Edition





R.B.J.T. Allenby

Alan Slomson

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A CHAPMAN & HALL BOOK

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Kenneth H. Rosen, Ph.D.

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Preface to the Second Edition

We explain the aims and range of this book in Chapter 1, which we strongly urge you to read before any of the subsequent chapters.

This second edition is a considerably expanded version of the first edition, originally published in 1991. It has 100% more authors, and correspondingly more material. Nonetheless, it does not cover the whole of the subject, and the coverage of the topics that are included is not comprehensive. Our choice of topics has mainly been determined by personal taste, but the emphasis is on *counting problems*, that is, questions about how many different arrangements there are of a particular type. We will not have fully succeeded in our aim of providing an attractive introduction to these topics if the book does not leave you wanting more. To this end we have added a list of suggestions for further reading.

The first edition was based on a course of 22 lectures given to students at the University of Leeds. We think the material included in this expanded edition could be covered in around 40 lectures. We emphasize, however, that it is our hope that the book can be read independently by anyone wishing for an accessible introduction to the topics that it covers. To this end, and unlike most other texts, we have, as far as possible, set exercises *in pairs* and provided an extensive answer section in which a *complete solution* is given to one exercise in each pair. Of course, diligent readers will tackle the exercises before looking to the answer section (this being an essential part of the learning process), but, if they are *really* stuck, the full solution should prove to be of value.

R.B.J.T Allenby
Alan Slomson

Acknowledgments

We have not originated any of the mathematics in this book. We have tried to give references to the papers where the main theorems were first proved, but a lot of the book covers standard material that has been known for some time, and where locating original sources is not easy. In writing biographical notes about many of the mathematicians whose work we discuss, we have made much use of the excellent Web site: *The MacTutor History of Mathematics Archive*, http://www-history.mcs.st-andrews.ac.uk, based at the University of St Andrews.

We have also benefited from the expository texts listed in our suggestions for further reading. Each of us has taught much of the material to students from the University of Leeds. We are grateful to our colleagues in Leeds who first allowed us to teach in an area of mathematics away from our research interests and to the students whose comments helped to shape our teaching.

Modern technology has made the writing of this book much easier than it was for the first edition twenty years ago. However, technology does have some disadvantages. It turns out that the template used to typeset the book has a mind of its own. This has meant that in some places the layout of the pages, and, in particular, the position of the diagrams and tables is not as we would have wished. We apologize for this, but it has been outside our control.

Previous experience leads us to believe that not everything in the book is correct. We hope that the errors are mostly minor misprints that have escaped our attention and that there are no major blunders. We encourage readers who detect errors or have questions about anything in this book to contact us.

Authors

R.B.J.T. Allenby received his PhD from the University of Wales and MSc Tech from the University of Manchester. He taught mathematics at the University of Leeds from 1965 to 2007. He is the author of *Rings, Fields and Groups: An Introduction to Abstract Algebra* (Edward Arnold, 1983); *Linear Algebra* (Edward Arnold, 1995), and *Number and Proofs* (Edward Arnold, 1997) and, with E. J. Redfern, of *Introduction to Number Theory with Computing* (Edward Arnold, 1989).

Alan Slomson received his MA and DPhil from the University of Oxford. He taught mathematics at the University of Leeds from 1967 to 2008. He is the author of the first edition of this book (Chapman and Hall, 1991) and, with John Bell, of *Models and Ultraproducts* (North Holland Publishing Company, 1969). He is currently the secretary of the United Kingdom Mathematics Trust.

CHAPTER 1

What's It All About?

1.1 WHAT IS COMBINATORICS?

Mathematics is a problem-solving activity, and the ultimate source of most mathematics is the external, nonmathematical world. Mathematical concepts are developed to help us tackle problems arising in this way. The abstract mathematical ideas that we use soon assume a life of their own and generate further problems, but these are more technical problems whose connection with the external world is more remote.

Since the time of Isaac Newton and until quite recently, almost the entire emphasis of applied mathematics has been on continuously varying processes, modeled by the mathematical continuum and using methods derived from the differential and integral calculus. In contrast, combinatorics concerns itself mainly with *finite* collections of *discrete* objects. With the growth of digital devices, especially computers, discrete mathematics has become more and more important.

The way mathematics has developed creates a difficulty when it comes to teaching and learning the subject. It is generally thought best to begin with the most basic ideas and then gradually work your way up to more and more complicated mathematics. This seems more sensible than being thrown in at the deep end and hoping you will learn to swim before you drown. However, this logical approach often obscures the historical reasons why a particular mathematical idea was developed. This means that it can be difficult for the student to understand the real point of the subject.

Fortunately, combinatorics is different. The starting point usually consists of problems that are easy to understand even if finding their solutions is not straightforward. They tend to be concrete problems that can be understood by those who do not know any technical mathematics. In this chapter we list a number of these problems that gave rise to much of the mathematics explained in the remainder of this book.

So what sorts of problems does combinatorics address? As combinatorics finds its origins in statistical, gambling, and recreational problems, a rough answer is: anywhere where knowing "How many?" (the answer to which may be "zero") is of interest. So, for example, the techniques we describe (or generalized versions of them) have been used in design of experiments, for example, the testing of crops (statistics), design of traffic routes (graph theory), construction of codes, arrangements of meetings (permutations and combinations), placing of people in jobs (rook polynomials), determination of certain chemical compounds (graph theory), production and school teaching schedules, as well as increasingly in mathematical biology in relation to the DNA code. There are also applications within mathematics, especially in the theory of numbers, and within computing, where questions of speed and complexity of working are important. Among recreational puzzles with a combinatorial flavor are the well-known Rubik's Cube, "magic" squares, Lucas's Tower of Hanoi, and the famous problem that was the origin of graph theory, that of crossing the bridges of Königsberg.

In many problems, not only a solution but an "optimal" solution is sought. For example, it would seem desirable to seek to maximize factory output or traffic flow or to minimize factory or computing costs or travel cost (by choosing the shortest route when visiting a succession of towns; this is the well-known traveling salesman problem). These aspects indicate the three basic problems of combinatorics: counting the number of solutions, checking existence (is there even *one* solution?), and searching for an optimal solution.

In this general introductory book we are not able to go into any of these topics in any great depth. Our aim is to give the reader the flavor of a broad range of interesting combinatorial ideas. At the end of the book we make suggestions for further reading where many of these ideas can be followed up.

1.2 CLASSIC PROBLEMS

In this section we list a number of classic and other interesting combinatorial problems that, later in the book, we show you how to solve. Many of these problems have been around for a long time. Accordingly, in most cases we have not tried to attribute the problems to their authors. Nevertheless, the present authors would be grateful for any enlightenment readers wish to provide in this regard.

We have selected at least one problem for all but one of the subsequent chapters, and our numbering of them matches the chapter numbers. So our first problem is called "Problem 2A" to indicate that it relates to Chapter 2 and is the first problem listed here from that chapter.

Many combinatorial problems arise from questions about probabilities. For, if a certain event can produce a finite set of equally likely different outcomes, of which some are deemed *favorable*, then we say that the probability of a favorable outcome is the fraction

$$\frac{\text{the number of favorable outcomes}}{\text{the total number of all outcomes}}.$$

So we can work out probabilities by counting the number of outcomes in the two sets occurring in this fraction. This sort of counting is mostly what this book is about. For example, consider the probability problem: "When throwing two standard dice, what is the probability that the total shown on the dice is 6?" The total number of outcomes when two dice are thrown is 36 as each face 1,2,3,4,5,6 on one die* can appear partnered by each one of the faces

* Being mathematicians, naturally we are pedantic. Although *dice* is often used as the singular term, strictly speaking, it is *one die* and *two or more dice*. Remember Julius Caesar's remark as he crossed the Rubicon, "The die is cast" ("Iacta alea est").

1,2,3,4,5,6 on the other die. The outcome 6 can be achieved in five different ways, namely, $1 + 5$, $2 + 4$, $3 + 3$, $4 + 2$, and $5 + 1$. So the answer to the probability problem is 5/36.

The first problem we list is rather more complicated but one of direct interest to anyone who "invests" their money in a lottery.

PROBLEM 2A
What Is the Chance of a Jackpot?
The operators of "Lotto," the British National Lottery, advertize that each ticket has a 1 in 13,983,816 chance of winning a share of the jackpot. How is this probability calculated?

Next we have an old classic, whose answer usually comes as a surprise.

PROBLEM 2B
The Birthdays Problem
How many people do you need to have in a room before there is a better than 50% chance that at least two people share a birthday?

While solving Problems 2A and 2B is fairly straightforward, some counting problems are a bit more perplexing. Let us have a look at two involving food and money, the second being of rather more general interest than the first!

PROBLEM 3A
Hot Chocolates
A manufacturer of high-quality (and therefore high-priced) chocolates makes just six different flavors of chocolate and sells them in boxes of 10. He claims he can offer over 3000 different "selection boxes." If he is wrong, he will fall foul of the advertizing laws. Should he fear prosecution?

PROBLEM 3B
A Common Opinion
There is a widely held view that, in a truly random selection of six distinct numbers from among the numbers 1 to 49 (as in the British lottery), the chance that two consecutive numbers will be chosen is extremely small. Has this opinion any credibility?

The solution to both of these problems relies on the same general principles used to solve the following purely arithmetic problem. Notice how much less "cluttered" and more transparent the arithmetic problem is without the "real-life" trimmings!

PROBLEM 3C
Counting Solutions
How many solutions does the equation $x + y + z + t = 60$ have where x, y, z, and t are positive integers?

If, in this problem, there were only, say, three unknowns x, y, and z and the 60 were replaced by a 6, there would be little difficulty, as we could easily list all the solutions systematically, that is, (1,1,4), (1,2,3), (1,3,2), (1,4,1), (2,1,3), (2,2,2), (2,3,1), (3,1,2), (3,2,1), and (4,1,1). So, there are exactly 10 of them.

That was easy. But what if we return to four unknowns and reintroduce the 60 or change it to 600? How many solutions then? It is clear that the same technique of listing all possible answers could be tried; the difficulty would seem to be in making sure you count all the solutions once and once only. To be sure, in the case where there are only three unknowns, it is not too difficult, even by the above method, to determine the *number* of solutions of each equation $x + y + z + = n$ *without actually listing any of them*. But with more variables we surely need some new ideas. And after these new ideas are introduced in Chapter 3 you will be able to write down the answer to *every* problem of this sort (that is, with any number of variables and any n on the right-hand side) *immediately*.

In Chapter 3 we shall also see how Problem 3A can be reinterpreted as a problem of placing identical marbles in distinguishable boxes. This new point of view then generates a host of fascinating problems known as *occupancy problems*, one slight variant of which is the intriguing but very difficult problem involving the *partitions of an integer* (see Problem 6 and Chapters 6 and 8).

Here is a probability problem whose answer would confound most people's intuition.

PROBLEM 4
Snap!
Two fully shuffled standard packs of 52 cards are placed face down and side by side. One after another, pairs of cards, one from each pack, are turned over. What is the probability that, as all 52 pairs are turned over, at least one pair of cards will be identical?

If you have not seen this problem before, you will surely be intrigued by the answer. And for those who are happy to engage in a little gambling (neither of the authors does) there is a chance for readers with no conscience to use the counterintuitive result to make a little money on the side from their more susceptible friends! In Chapter 4 we introduce the *inclusion-exclusion principle* and show how it may be used to solve this problem.

Certain numbers arise in combinatorics so frequently that they often bear the names of their originators. The numbers that arise on putting distinct balls into identical cells, with no cell left empty, are named after James Stirling, who defined the *Stirling numbers* in a completely different context. Here is a problem from calculus (don't worry if you haven't yet met the ideas involved) to which the Stirling numbers provide an answer. We let θ be the operator $x(d/dx)$. Thus $\theta y = x(dy/dx)$, and

$$\theta^2 y = x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx},$$

and you can check that, similarly,

$$\theta^3 y = x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}.$$

PROBLEM 5A
Differential Operators
Find a formula for the coefficient of the term $x^k(d^ky/dx^k)$ in $\theta^n y$, for $1 \le k \le n$.

In Chapter 5 we also introduce the *Catalan numbers*, which arise in very many seemingly unrelated problems.

PROBLEM 5B
Walking East and North!
Suppose we have an $n \times n$ grid. How many paths are there, following edges of the grid, from the bottom left corner to the top right corner that may touch, but not go above, the diagonal shown in Figure 1.1?

Here is a problem that engaged the Swiss mathematician Leonhard Euler.* By a *triangulation* of a polygon, we mean a way of dividing the polygon in triangles by non-intersecting diagonals, that is, lines joining two vertices. For example, Figure 1.2 is an example of a triangulation of a regular hexagon.

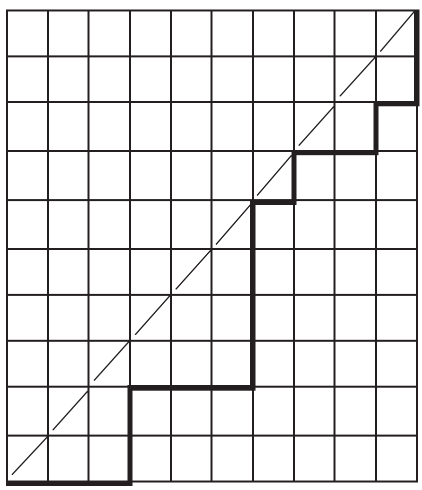

FIGURE 1.1

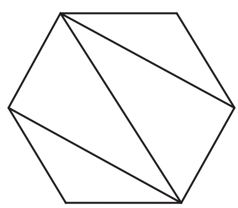

FIGURE 1.2

* See Chapter 8, Section 8.2, for a brief biography of Euler.

PROBLEM 5C
Chopping Up a Hexagon
How many different triangulations are there of a regular hexagon?

A classic combinatorial problem is that of counting *partitions*. Here is a problem of this type.

PROBLEM 6
Partitions
In how many different ways can the number 100 be expressed as the sum of positive integers?

To make this question precise we explain that we are here interested only in which numbers make up the sum and not the order in which they are written. So, for example, we regard the sum 50 + 24 + 13 + 13 as representing the same way of expressing 100 as a sum of positive integers as does 13 + 24 + 50 + 13. We call 50 + 24 + 13 + 13 a *partition* of 100, and we let $p(n)$ be the number of different partitions of n. When n is small, the value of $p(n)$ can be calculated by listing all the possibilities. For example, we can see that $p(6) = 11$, from the list

$$6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1,$$
$$2+1+1+1+1, 1+1+1+1+1+1$$

of all the different ways of writing 6 as the sum of positive integers. Of course, to evaluate $p(6)$ in this way, we need to be satisfied that we have included all the possibilities. And while it is feasible to evaluate $p(6)$ in this way, it is hardly practicable for $p(100)$.

All this raises the question of finding a formula for $p(n)$. This problem was solved by two giants of mathematics, the Indian Srinivasa Ramanujan and the Englishman G. H. Hardy. Their proof is far too involved to reproduce in this book, but, in Chapters 6 and 8, we can experience a more modest sense of achievement by using some elementary calculus, believe it or not, to obtain fairly reasonable upper and lower bounds for $p(n)$ and related functions. We shall also prove a lovely theorem that will enable you to calculate quite a large number of the smaller $p(n)$ fairly quickly *by hand* and avoiding brute force!

Almost every mathematically inclined student will have heard of Fibonacci and his rabbits. At the heart of the story is the *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, 21, 34, … in which each integer after the first two is the sum of the previous two. If we let f_n be the nth term of this sequence, we can describe the sequence by saying that

$$f_1 = f_2 = 1 \text{ and, generally, for } n \geq 3, f_n = f_{n-2} + f_{n-1}.$$

The Fibonacci sequence has so many wonderful properties that a quarterly journal is produced to publicize them. One famous application is the test devised by Lucas* to

* Francois-Edouard-Anatole Lucas was born in Amiens on April 4, 1842, and died in Paris on October 3, 1891. He introduced the recreational game The Tower of Hanoi, which we discuss in Chapter 7. He died of an infection after being hit by a flying shard of a broken dinner plate.

check whether or not a number of the form $2^n - 1$ is a prime. In 1876 he proved, with only pencil and paper but lots of patience, that the 39-digit number

$$2^{127}-1 = 170{,}141{,}183{,}460{,}469{,}231{,}731{,}687{,}303{,}715{,}884{,}105{,}727$$

is prime. This number stood as the largest prime known to anyone until the 1950s when electronic calculators were applied to the task!

PROBLEM 7
The Fibonacci Numbers
Is there a formula explicitly giving the values of the Fibonacci numbers?

The Fibonacci numbers are defined by the *recurrence relation* $f_n = f_{n-2} + f_{n-1}$, which relates later numbers in the sequence to earlier numbers in the sequence. We study recurrence relations systematically in Chapter 7. One technique that is useful here is that of *generating functions*, which we describe in Chapter 8. This technique is particularly useful with some intriguing problems that arise if we place restrictions on the numbers that are allowed in a partition. Here is one suggested by experiments with small numbers.

PROBLEM 8
Special Partitions
Is it true that for each integer n the number of ways of writing n as the sum of odd positive integers is the same as the number of ways of writing n as the sum of positive integers that are all different?

In the following table we have listed the partitions of 12 into odd numbers and into different numbers.

11+1	12
9+3	11+1
9+1+1+1	10+2
7+5	9+3
7+3+1+1	9+2+1
7+1+1+1+1+1	8+4
5+5+1+1	8+3+1
5+3+3+1	7+5
5+3+1+1+1+1	7+4+1
5+1+1+1+1+1+1+1	7+3+2
3+3+3+3	6+5+1
3+3+3+1+1+1	6+4+2
3+3+1+1+1+1+1+1	6+3+2+1
3+1+1+1+1+1+1+1+1+1	5+4+3
1+1+1+1+1+1+1+1+1+1+1+1	5+4+2+1

Not many partitions occur in both lists, but both lists contain the same number of partitions. Is this a coincidence? It comes as a surprise that the answer is “no” and that the proof uses only rather simple manipulations of power series.

We now come to one of the best known of all combinatorial problems. The river Pregel flows through Königsberg* and divides the town, flowing around an island called the Kneiphof (“Beer Garden”). The city was connected by seven bridges as shown in the sketch map in Figure 1.3. It is said that the residents considered the following problem during their Sunday stroll.

PROBLEM 9A
The Bridges of Königsberg
Is there a route that would have taken the burghers of Königsberg over each of the bridges exactly once?

This problem was solved by Leonhard Euler in 1735. Euler’s solution of this problem is regarded as the origin of the branch of mathematics known as *graph theory*. The “graphs” of graph theory are rather different from the “graphs” of functions you will be familiar with, but in both cases, *graph* abbreviates *graphical representation*. In the case of the Königsberg bridges problem, all that really matters is how the different parts of the city are connected by the bridges. So we can replace Figure 1.3 with the representation of the same situation shown in Figure 1.4, where the dots represent the land areas and the lines represent the bridges.

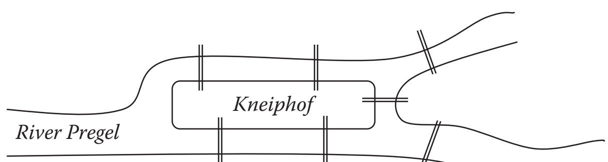

FIGURE 1.3

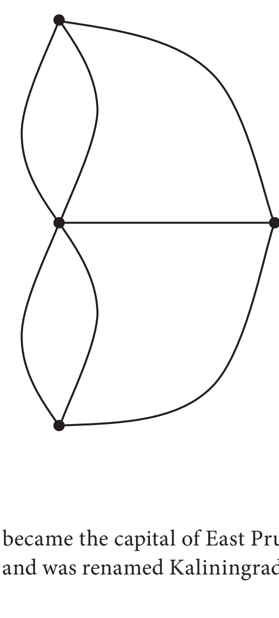

FIGURE 1.4

* Originally a medieval city, Königsberg became the capital of East Prussia in the fifteenth century. In 1945 it was ceded by Germany to the Soviet Union and was renamed Kaliningrad. It remains part of modern-day Russia.

Another classic problem of graph theory is the *utilities problem*. We state it in a slightly modernized form.

PROBLEM 9B
The Utilities Problem
Is it possible to connect up a house, a cottage, and a bungalow to supplies of electricity, gas, and cable television so that the pipes, wires, and cables do not cross each other?

In Figure 1.5 we show a failed attempt to solve this problem. But one failure does not prove that the problem does not have a solution. The issue is whether the graph consisting of two sets of three dots, where each dot in the first set is joined to each dot in the second graph, can be drawn in the plane so that the lines meet only at dots.

An even more famous problem in this area is whether when drawing maps we always need only four colors to ensure that countries with a common boundary can be colored differently. The question was first asked in 1852, but it took 124 years for a solution to emerge—although a "proof" proposed in 1879 stood for 11 years before a flaw in it was found.

PROBLEM 9C
The Four-Color Problem
Is it possible to color every map drawn in the plane with at most four colors so that adjacent countries are colored differently?

We show in Chapter 9 how this problem may be reworded as a problem about coloring the vertices of a graph. The *four-color theorem* says that the answer to this question is "yes." It was proved to most people's satisfaction by Kenneth Appel and Wolfgang Haken in 1977. Their proof involved a considerable use of computer time to check a large number of cases and so has not been accepted as a "mathematical proof" by everyone. We are not able to go into the technicalities in this book, but we are able to give a proof of the less ambitious claim that every map may be colored using at most five colors.

Graphs can also be used to represent chemical molecules, with the vertices representing atoms and the edges representing valency bonds. Two examples are shown in Figure 1.6.

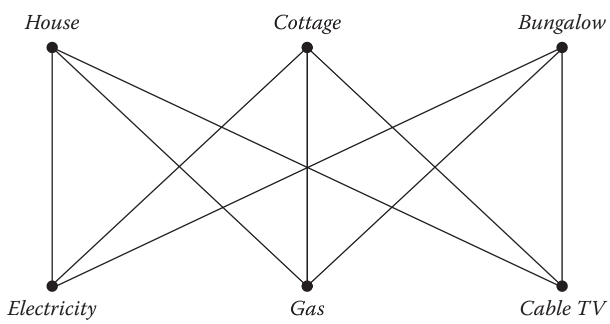

FIGURE 1.5

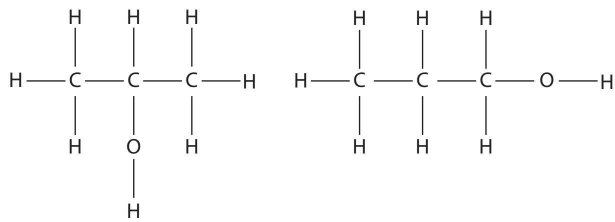

FIGURE 1.6

The graphs used to represent these molecules are of a special kind called *trees*. These graphs have lots of applications, and we study these in Chapter 10. The classic problem in this area is the following.

PROBLEM 10
Labeled Trees
How many different trees are there with n vertices labeled with the numbers $1,2,\ldots,n$?

In Chapter 10 we give the answer originally found by the English mathematician Arthur Cayley.

Much of mathematics is about finding patterns. In order to prepare for our study of patterns, we look at symmetries of geometric figures. Symmetries give rise to mathematical structures called *groups*. You may already have met this concept, but in case not, we introduce groups from scratch in Chapter 11. Ideas from group theory can be used to solve a large range of problems. Here is one example.

PROBLEM 11
Shuffling Cards
What is the most effective way to shuffle a pack of cards?

We next look at the relationship between the coloring of figures and the symmetries of these figures. We prove an important theorem, the *orbit-stabilizer theorem*, in Chapter 12, but we find we have to develop the theory further in Chapter 13 in order to answer questions such as the next three.

PROBLEM 13A
Coloring a Chessboard
How many different ways are there to color the squares of a chessboard using two colors?

On a standard 8×8 chessboard, the squares are colored alternately black and white as shown in Figure 1.7i, but clearly they could be colored in many other ways. Alternative colorings are shown in Figure 1.7ii and iii. The problem is to decide exactly how many different colorings are possible.

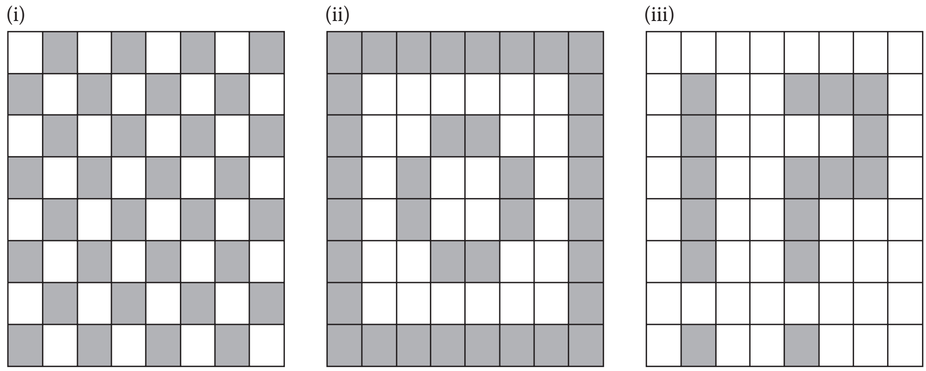

FIGURE 1.7

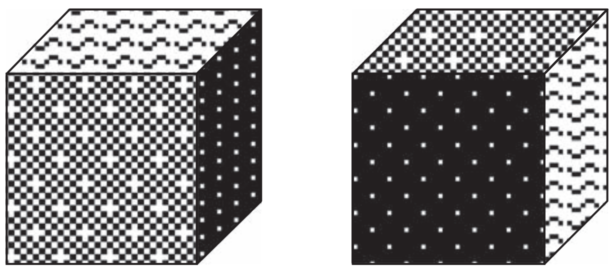

FIGURE 1.8

PROBLEM 13B
Varieties of Colored Cubes
In how many different ways can you color a cube using three colors?

Before we can answer this question we need to make clear what "different" means. The cube on the right in Figure 1.8 looks different from the one on the left. But perhaps they could be made to look the same by rotating the cube on the right about an appropriate axis. This is where the *symmetries* of a geometric figure come in.

This eventually leads us to a very powerful technique due to George Pólya for answering more complicated questions, such as the following.

PROBLEM 14
Counting Patterns Again
In how many different ways can you color a cube using one red, two white, and three blue faces?

We also show in Chapter 14 that Pólya's Counting Theorem enables us to count fairly readily the number of different simple graphs with a given number of vertices. Our next combinatorial principle arises in the context of number theory where it was used by Dirichlet to solve the following problem.

PROBLEM 15A
Rational Approximations to Irrational Numbers
Show that for each irrational number a, there exists a rational number p/q such that

$$\left|a-\frac{p}{q}\right|<\frac{1}{q^2}.$$

The principle used to solve this problem is called *Dirichlet's pigeonhole principle*, which is of deceptive simplicity but has surprising consequences. Here is one of a more recreational kind.

It is obvious that, using 18 dominoes, each of size 1 by 2, we can "cover" a 6 by 6 (square) board completely and without any two dominoes overlapping. An example is given in Figure 1.9.

Note that, in the figure, the two leftmost *columns* of dominoes are separated from the other four columns by a "fault line." That is, a knife could be dragged down the board cutting it in two *without* having to cut through any domino. At the same time, the board cannot be similarly split along any other row or column. So, the problem is:

PROBLEM 15B
Placing Dominoes "Faultlessly"
Can 18 dominoes be placed on a 6 by 6 board "faultlessly"?

We next come to another existence problem that is the starting point for a large area of study called *Ramsey theory* and that can also be reinterpreted in terms of graphs:

PROBLEM 16A
Friends at a Party
There are six people at a party. Each pair are either friends or strangers. We claim that, among the six, there are (at least) three people who are either all friends or all strangers. Are we correct?

A nice way to picture this problem is by using graphs, this time with *colored* edges. If we ask each pair of friends to hold opposite ends of a red string and each pair of

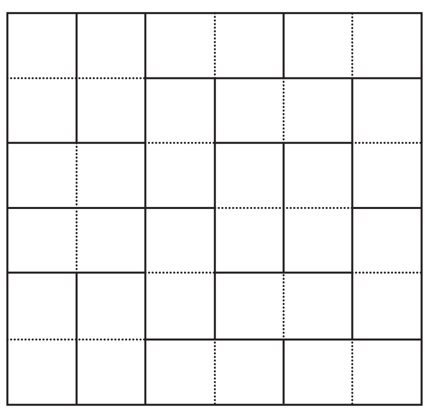

FIGURE 1.9

strangers (if they would be so kind) to hold opposite ends of a blue string, the problem becomes: Is it true that if the 15 edges of a graph with six vertices are colored either red or blue, then there is always either a red "triangle," that is, three edges forming a triangle that are all red, or a blue "triangle"? In brief, must there always exist a *monochromatic* triangle?

There are many other fascinating problems that are concerned with coloring (even infinitely many) *points*. For example:

PROBLEM 16B
Plane Colors
Let us be given a red/blue coloring of the points of the plane. That is, imagine that with each point of the plane there is associated one of those two colors. Suppose that someone now draws a particular triangle T in the plane. Is it always possible to move T to a position in the plane so that its vertices lie over three points *all of the same color*? In other words, can we find, in the plane, a *monochromatic* triangle congruent to T? If not, can we find one *similar* to T? We attempt no picture here, for obvious reasons!

In the final chapter of this book we deal with questions of the following kind.

PROBLEM 17A
Nonattacking rooks
Given the 5×5 board in Figure 1.10, in how many ways can 0, 1, 2, 3, 4, 5, or more nonattacking rooks (that is, no two rooks in the same row or column) be placed on the board so that *none of them lies on a black square?*

At first sight this seems rather a frivolous problem, but actually it is one with many practical applications. As you can see from Figure 1.11, it is equivalent to asking in how

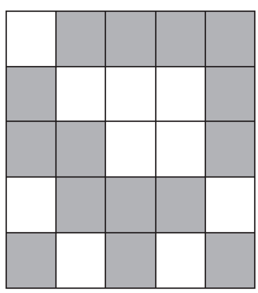

FIGURE 1.10

	Job 1	Job 2	Job 3	Job 4	Job 5
Employee A					
Employee B					
Employee C					
Employee D					
Employee E					

FIGURE 1.11

many ways the five people A, B, C, D, and E can each be allocated one of the jobs 1 to 5, where a black square indicates that the person named in its row is unable to do the job named in its column.

Finding the answer leads to the introduction of *rook polynomials*. In most cases the key question is whether there is any solution at all rather than the number of solutions. One particularly intriguing version of this problem is the following, whose answer is certainly not obvious one way or the other.

PROBLEM 17B
Selecting Cards
Let 40 cards—10 red, 10 blue, 10 green, and 10 yellow, each set being numbered 1 to 10—be well shuffled and then dealt out into 10 groups of four. Is it always possible to pick one card from each group so that the 10 cards chosen will include exactly one of each number 1,2,…,10? (Obviously we do not insist that all the chosen cards are also of the same color.)

This problem is solved using Hall's marriage theorem with which we end Chapter 17.

1.3 WHAT YOU NEED TO KNOW

One of the advantages of following a course in combinatorics is that only a modest mathematical background is necessary in order to get started. This modest amount includes some basic set theory, which we remind you of below, a bit of elementary algebra, and, perhaps surprisingly (to help with two or three estimation problems), some calculus. In Chapter 11 we use *groups*, but we assume no previous knowledge of group theory, which we introduce as we need it.

Suppose A and B are sets. The *union* of A and B, written as $A \cup B$, is the set of elements that are in A or B or both. The *intersection* of A and B, written as $A \cap B$, is the set of elements that are both in A and in B. The *difference*, written $A \setminus B$, is the set of elements in A but not in B. That is, $A \setminus B = \{x{:}x \in A \text{ and } x \notin B\}$. This may be represented pictorially as in Figure 1.12, where the shading represents the set $A \setminus B$.

There is, unfortunately, no standard notation for the number of elements in a set, X. The notations $\overline{\overline{X}}$, $|X|$, and *card*(X) are all used. However, our preferred notation is #(X).

We use the symbols N, N^+, Z, Q, R, and C for the following sets of numbers:

N is the set of *natural numbers*, that is, N = {0,1,2,...}. N^+ is the set of *positive integers*, that is, N = {1,2,3,...}. Z is the set of *integers*, that is, Z = {...–2,–1,0,1,2,...}. Q is the set of *rational numbers*, R is the set of *real numbers*, and C is the set of *complex numbers*. We use R^2 for the points of the plane.

We use the "arrow" notation for functions. For example, the function that squares each number will be written as $x \mapsto x^2$. If the function f maps elements of the set D to the set C, we write $f: D \to C$ and call the set D the *domain* of f, and the set C the *codomain* of f.

We say that a function $f: D \to C$ is *injective* if f does not repeat values, that is, if for all x, $y \in D$, $x \neq y$ implies that $f(x) \neq f(y)$. We say that f is *surjective* if each element of C is a value of f, that is, if for each $y \in C$, there is some $x \in D$ such that $f(x) = y$. We say that f is *bijective*

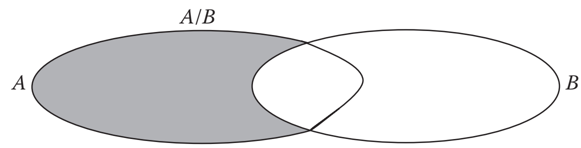

FIGURE 1.12

if it is both injective and surjective. (The alternative nomenclature that you may have met is *one-one* for injective, *onto* for surjective, and *one-one correspondence* for bijective.)

We assume that you are already familiar with the "sigma" notation for sums. For example, $\sum_{i=1}^{n} a_i$ represents the sum $a_1 + a_2 + \ldots + a_n$. On occasion we also, in a similar way, use the "pi" notation for products. So $\prod_{i=1}^{n} a_i$ represents the product $a_1 \times a_2 \times \ldots \times a_n$, or, omitting the multiplication signs, $a_1a_2\ldots a_n$.

Although we approach most of the topics discussed in this book through concrete problems, a strong aim of this book is to emphasize the importance of proof in mathematics. While this is a book that focuses on combinatorial methods, these are worthless if we are unable to prove that employing them will give the correct answer. Mathematics is in the very privileged position of being the only area of human knowledge where assertions made have the chance of being verified by unassailable proof – or shot down by counterexample! A course in combinatorics provides an ideal opportunity for paying special attention to methods of proof since, often, the reader will not have to make a huge mental effort to understand the meaning of the statements themselves. Accordingly, we offer no apology for paying careful attention to the majority of proofs themselves, the odd exceptions being proofs that are beyond the scope of this book or where even the most fastidious mathematician might say, "Clearly, this very same proof goes over to the general case."

We therefore largely assume that the reader is already familiar with the standard methods of proof. In particular, as we are frequently concerned with results that hold for all natural numbers, or for all positive integers, *proof by mathematical induction* is used a good deal. If you need an introduction to this topic, or a reminder about it, please consult one of the many books that cover this topic.*

1.4 ARE YOU SITTING COMFORTABLY?

Once upon a time there was a program on the radio called *Listen with Mother*. (In those days it was assumed that it would be the mother who would be at home with young children.) In the first program in 1950 the storyteller, Julia Lang, introduced the story she was about to tell by saying, "Are you sitting comfortably? Then we'll begin." Apparently this introduction was not planned, but it caught on and was used regularly until the program came to an end in 1982.†

* We mention just two, R. B. J. T. Allenby, *Numbers and Proofs*, Arnold, London, 1997, and Kevin Houston, *How to Think like a Mathematician*, Cambridge University Press, Cambridge, 2009.

† See Nigel Rees, *Sayings of the Century*, Allen & Unwin, London, 1984.

When it comes to reading mathematics, however, this is not an appropriate beginning. A mathematics book cannot be read like a novel, sitting in a comfortable chair, with a glass at your side. Reading mathematics requires you to be active. You need to be sitting at a table or a desk, with pencil and paper, both to work through the theory and to tackle the problems. A good guide is the amount of time it takes you to read the book. A novel can be read at a rate of about 60 pages an hour, whereas with most mathematics books you are doing well if you can read 5 pages an hour. (It follows that, even at 12 times the price, a mathematics book is good value for the money!)

Since the approach the book takes is to begin with problems and usually to use them to lead into the theory, we have posed a good number of *problems* in the text. The *b*th problem in Chapter *a* is labeled **Problem a.b.** These problems are immediately followed by their solutions, but you are strongly encouraged to try the problems for yourself before reading our solution.

At the end of most of the chapter sections, there are *exercises*. Some of these are routine problems to help consolidate your understanding, and some take the theory a bit further or are designed to challenge you. In most cases these problems occur in pairs, labeled A and B. Usually the B question is quite similar to the A question. The difference is that we have included solutions for the A questions at the back of the book but not for the B questions. The solutions to the A questions are usually written out in detail. This is intended to be helpful, but it will not achieve their purpose of helping you to learn the subject, if you give in to the temptation to read the solutions before making your own attempt at the exercises. The B questions, with no solutions, are there for those who cannot resist this temptation!

CHAPTER 2

Permutations and Combinations

2.1 THE COMBINATORIAL APPROACH

In Chapter 1 we gave examples of counting problems that we hope convinced you of their interest and importance. In this chapter we introduce two of the most basic ideas, counting *permutations* and counting *combinations*. These occur over and over again throughout this book. You may have already met these ideas in algebra in connection with the binomial theorem, but the combinatorial approach may be new to you. It can be hard to relearn a topic you are already familiar with but using a different approach. However, we encourage you to adopt the combinatorial approach, which gives more importance to counting methods than to algebraic manipulation, as this is the key to much of the rest of this book.

2.2 PERMUTATIONS

We begin with some problems that are very simple, but the ideas behind their solutions are of fundamental importance in many counting problems.

PROBLEM 2.1

Cayley's Café has the following menu:

Cayley's Café
Starters
Tomato Soup
Fruit Juice
Mains
Lamb Chops
Battered Cod
Nut Bake
Desserts
Apple Pie
Strawberry Ice

How many different three-course meals could you have?

Solution

You have two choices for your starter, and, whichever choice you make, you have three choices for your main course. This makes $2 \times 3 = 6$ choices for the first two courses.

Soup	Soup	Soup	Juice	Juice	Juice
|	|	|	|	|	|
Chops	Cod	Bake	Chops	Cod	Bake

In each of these six cases you have two choices for your dessert, making $6 \times 2 = 12$ possibilities altogether. We can set them out in Figure 2.1, which makes it clear why the number of cases multiplies at each stage and why the final answer is the product of the number of choices at each stage.

So we obtain $2 \times 3 \times 2 = 12$ as the total number of possible meals.

PROBLEM 2.2

In a race with 20 horses, in how many ways can the first three places be filled? (For simplicity, assume that there cannot be a dead heat.)

Solution

There are 20 horses, each of which could come first. Whichever horse comes first, there are 19 other horses that can come second. So there are $20 \times 19 = 380$ ways in which the first two places can be filled. In each of these 380 cases, there are 18 remaining horses that can come third. So there are $380 \times 18 = 20 \times 19 \times 18 = 6840$ ways in which the first three places can be filled.

We now consider the way in which these two problems are different and the way in which they are similar. In Problem 2.1 your choice of a starter did not affect the choice of the main course. Whether you chose the tomato soup or the fruit juice, you still have the choice of lamb chops, battered cod, or nut bake for your main course. And whatever your choices of starter and main course, you still have the same choices, apple pie or strawberry ice, for your dessert.

In Problem 2.2, the horse that wins the race cannot also come in second. So the possibilities for which horse comes in second vary according to which horse wins the race. However, the *number* of possibilities remains the same. Whichever horse wins the race, there are 19 horses that can come second, though which 19 horses these are varies according to which the winner is. Likewise, the possibilities for the third horse vary according to which two horses come in first and second, but, whichever these horses are, there always remain 18 horses each of which can come in third. It is

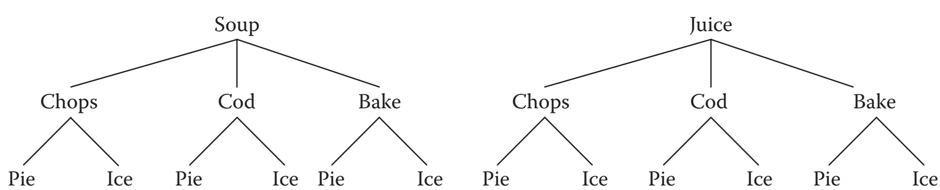

FIGURE 2.1

because the *number of choices* at each stage does not depend on the particular choices made earlier that we could use the same multiplication method to solve Problem 2.2 as we used to solve Problem 2.1, and thus $20 \times 19 \times 18$ does indeed give the number of ways in which the first three positions in the race can be filled.

The multiplication principle we have used in these two problems is sufficiently important to be worth stating explicitly.

> **THE PRINCIPLE OF MULTIPLICATION OF CHOICES**
>
> If there are r successive choices to be made, and for $1 \le i \le r$, the ith choice can be made in n_i ways, then the total number of ways of making these choices is $n_1 \times n_2 \times \ldots \times n_r$.

Note that we can use the "pi" notation to write the product in the box as $\prod_{i=1}^{r} n_i$.

Although the principle of multiplication of choices applies equally to Problems 2.1 and 2.2, Problem 2.2 has an additional feature that frequently occurs in problems of this type. The successive choices were all being made from the set of 20 horses taking part in the race. So the number of horses left to choose from goes down by one at each successive stage. That is, in the notation we are using,

$$n_{i+1} = n_i - 1, \quad \text{for } 1 \le i < r.$$

In such a case, if $n_1 = n$, then for $2 \le i \le r$, $n_i = n - i + 1$, so that the product $\prod_{i=1}^{r} n_i$ is $n(n-1)(n-2)\ldots(n-r+1)$. We can express this product more succinctly by making use of factorial notation. We have that

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots\ldots\times 2\times 1}{(n-r)(n-r-1)\ldots\times 2\times 1} = \frac{n!}{(n-r)!}.$$

Since this situation occurs very frequently, we introduce some special terminology and notation to describe it. We call a choice of r objects from a set of n objects in which the *order* of choice is to be taken into account, a *permutation* of r objects from n. We let $P(n,r)$ be the number of different permutations of r objects from n. Of course, this makes sense only in the case where r and n are nonnegative integers with $r \le n$. The above remarks yield the general formula for $P(n,r)$.

THEOREM 2.1

For all nonnegative integers r, n with $r \le n$, $P(n,r) = n!/(n-r)!$

It is important to remember that $P(n,r)$ counts the number of ways of choosing r objects *in order* from a set of n objects. If the order does not matter, the number of choices is smaller, as we shall see in the next section.

In Problem 2.2 we considered only the number of different ways in which the first three positions could be filled. Suppose now we are interested in the number of different ways all 20 horses can finish in order (again, assuming no dead heats). We can see that this number is $20 \times 19 \times 18 \times \ldots \times 2 \times 1$, that is, 20! Note that this is the same as the number of ways of choosing 20 horses in order from a set of 20 horses, and so we could have obtained this by using Theorem 2.1, which tells us that this number is $P(20,20) = 20!/0! = 20!$ (Recall the standard convention that $0! = 1$.) Thus we have:

THEOREM 2.2

The number of different permutations of n objects is $n!$

The values of $n!$ grow very rapidly. Even for quite small values of n, the factorial $n!$ is very large. For example, $10! = 3,628,800$ and $100!$ is larger than 10^{157}.

PROBLEM 2.3

In how many ways can eight counters be placed on a square 8×8 chessboard in such a way that no two counters lie either in the same row or in the same column? Note that we can reword this problem as: In how many ways can eight rooks be placed on a chessboard so that no two rooks are "attacking" each other? This latter problem is generalized in Chapter 17.

Solution

Let us place one counter in each row in turn. For the first row there are eight columns in which the counter may be put. Having placed this counter, when it comes to the second row, there are just seven columns where we may place a counter, as it must not be in the same column as the counter in the first row. As we place counters in successive rows, the number of possible columns where the next counter may be placed goes down by one at each stage. So the total number of permissible arrangements of the counters is $8 \times 7 \times \ldots \times 2 \times 1$, that is, $8!(= 40,320)$. One of these arrangements is shown in Figure 2.2.

Clearly there are, more generally, $n!$ different ways to place n counters on the squares of an $n \times n$ chessboard so that there are neither two counters in the same row nor in the same column. In Exercise 2.2.4A you are asked to generalize this to the case of placing any number of counters on rectangular boards of any size.

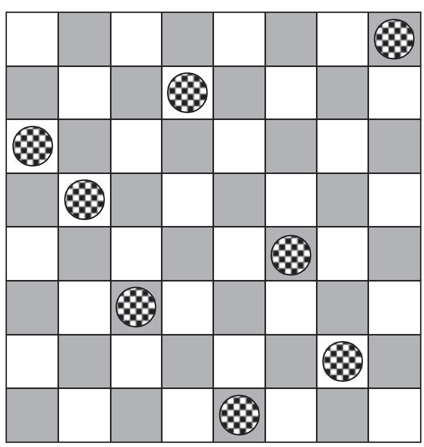

FIGURE 2.2

Exercises

2.2.1A Currently a €10 note has a "serial number" of the form X19298164502, that is, a letter followed by 11 digits. How many different serial numbers of this form are there? A Bank of England £10 note has a serial number of the form CD49000372, that is, two letters followed by eight digits. Are there more of these serial numbers than there are for a €10 note?

2.2.1B A personal identification number (PIN) consists of a sequence of four digits, each drawn from the set {0,1,2,3,4,5,6,7,8,9}, except that the first digit of a PIN cannot be 0. How many different PINs are there? How many different PINs are there in which no digit is repeated?

2.2.2A How many different sequences of length 10 are there in which each of the digits 0,1,2,3,4,5,6,7,8,9 is used once? How long would it take you to list them all if each sequence took one second to write down?

2.2.2B
i. How many sequences are there of n digits in which all the digits are different?
ii. How many sequences are there of n digits in which no two consecutive digits are the same?

2.2.3A In three races there are 10, 8, and 6 horses running, respectively. You win a jackpot prize if you correctly predict the first 3 horses, in the right order (assuming no dead heats), in each race. How many different predictions can be made?

2.2.3B A password is a sequence of six characters, the first three being either an upper or a lowercase letter, the next being a digit, and the final two coming from the set {!,£,$,%,^,&,*,(,),_, + , = ,{,},[,], @,#,?} of 19 other symbols occurring on a standard keyboard. How many different passwords are there? How many are there if consecutive characters must be different? How many are there if all the characters must be different?

2.2.4A Let k, m, and n be positive integers with $k \leq m$, $k \leq n$, and $m \leq n$. In how many different ways may k counters be placed on the squares of an $m \times n$ grid so that no two counters are in the same row or in the same column?

2.2.4B In how many different ways may eight red and eight green counters be placed on the squares of an 8×8 chessboard so that there are not two counters on any one square and there is one red counter and one green counter in each row and column?

2.3 COMBINATIONS

Let us now count the number of ways of choosing a specified number of objects from a set when the order of selection does not matter. We tackle this problem by relating it to the problem of counting permutations, which we have already solved. (Reducing a new problem to a case that has already been solved is a common mathematical technique. It is said that many a mathematician who has learned how to make a cup of tea starting with an empty kettle will, when given a full kettle and asked to make tea, first empty the kettle to reduce the problem to one that they already know how to solve.) A couple of examples will make the line of approach clear.

PROBLEM 2.4

A team of three bowls players is to be selected from a squad of six players. How many different teams can be selected?

Solution

We have seen that we can choose three players, in order, from a squad of six players in $P(6,3) = 6 \times 5 \times 4 = 120$ ways. But, *and this is the key point*, there are not 120 different teams of three players. This is because the order in which we pick the members of the team does not matter. For example, choosing first Pat, then Chris, and then Sam leads to the same team as first choosing Chris, then Sam, and then Pat. Thus, each team can be chosen in more than one way. The number of ways in which three given players can be chosen in order is 3!, that is, 6. Since we get 120 when we count each team six times, the number of different teams is $120/6 = 20$. Put another way, the number of different ways to pick three bowls players from six is $P(6,3)/3!$

The technique that we have used in this problem is used again, not only in the next problem, but in many other counting problems. We count the number of arrangements of a particular kind by counting them in such a way that each arrangement is counted more than once. We then adjust our answer to allow for the duplicate counting.

PROBLEM 2.5

How many different hands of 5 cards can be chosen from a pack of 52 cards?

Solution

We can choose 5 cards, in order, from a pack of 52 cards, in $P(52,5)$ different ways. But the order in which the cards are chosen does not affect the hand we end up with. The same hand of 5 cards can be arranged in order in 5! ways and so can be chosen, in order, in 5! ways. Thus $P(52,5)$ gives the number of 5-card hands when each hand is counted 5! times. Hence the number of different 5-card hands is

$$\frac{P(52,5)}{5!} = \frac{(52!/47!)}{5!} = \frac{52!}{5!47!} = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1} = 2{,}598{,}960.$$

We can now generalize the method used in these last two problems. We call a selection of r objects chosen from n objects, *when the order in which they are chosen does not matter*, a *combination* of r objects from n. We use the notation $C(n,r)$ for the number of different combinations of r objects from n. (Notice that the mathematical usage of *permutation*, where the order matters, and *combination*, where it does not, does not correspond to all the uses of these words in everyday life. In football pools permutations or “perms” are selections of football (otherwise known as "soccer") matches where the order does not matter. In a combination lock, the order of the numbers is important.)

The method that we used to solve Problems 2.4 and 2.5 leads us to the general formula for $C(n,r)$.

THEOREM 2.3

For all nonnegative integers r, n with $r \le n$,

$$C(n,r) = \frac{n!}{r!(n-r)!}.$$

Proof

By Theorem 2.2, we know that a set of r objects can be ordered in $r!$ ways. Thus $P(n,r)$, the number of ways in which r objects can be chosen in order from a set of n objects, counts each set of r objects chosen from the given set of n objects $r!$ times. Hence, $C(n,r) = P(n,r)/r!$, and therefore, by Theorem 2.1, $C(n,r) = n!/[r!(n-r)!]$.

The numbers $C(n,r)$ are the well-known *binomial coefficients* that occur in the binomial theorem that we give as Theorem 2.6. There are several common alternative notations for these binomial coefficients. The number for which we have used the notation $C(n,r)$ is often written as $\binom{n}{r}$, or ${}_nC_r$ or C_r^n. We have chosen $C(n,r)$ because it is less cumbersome to print than $\binom{n}{r}$ and, unlike ${}_nC_r$ and C_r^n, it makes the numbers n and r easier to read. It also fits in with the standard mathematical notation, $f(x,y)$, for a function of two variables and also with the notation for two-dimensional arrays in many programming languages. Its disadvantage is that it ties the letter C to a particular meaning. To avoid this, the alternative notation $(n!r)$ was once suggested.*

It is worth noting that since $C(n,r)$ is the number of ways of choosing r objects from n, we must have $0 \le r \le n$. We allow the case $r = n$. In this case the formula gives $C(n,n) = n!/(0!n!) = 1$. This corresponds to the fact that an n-element set A has just one n-element subset, namely, the set A itself. We also have $C(n,0) = n!/(0!n!) = 1$, corresponding to the fact that there is just one subset of A that has zero elements, namely, the empty set $\varnothing$.

The formula for $C(n,r)$ given by Theorem 2.3 can be used to give algebraic proofs of many properties of the binomial coefficients. We prefer, however, to emphasize the combinatorial meaning of these numbers and to give combinatorial proofs whenever this is convenient. In line with this approach, we have given a combinatorial definition of the number $C(n,r)$. The alternative would have been to define $C(n,r)$ by the formula of Theorem 2.3. It would then have been necessary to prove that the number of r-element subsets of an n-element set is indeed $C(n,r)$. Our combinatorial approach is illustrated by our proofs of the next four theorems.

THEOREM 2.4

For all positive integers r, n with $r \le n$, $rC(n,r) = nC(n-1,r-1)$.

Proof

Let X be an n-element set. We evaluate the sum of the numbers of elements in all the r-element subsets of X in two different ways.

* In *The Printing of Mathematics*, by T. W. Chaundy, P. R. Barrett, and Charles Batey, Oxford University Press, London, 1954. This book is out of date technologically as it was written in the days of hot-metal typesetting, but its advice to mathematical authors is still valuable.

First, as there are $C(n,r)$ subsets of X, each containing r elements, this sum is $C(n,r) \times r$, that is, $rC(n,r)$. Second, consider one particular object, say a, from the n-element set X. To obtain an r-element subset of X containing a we need to choose a further $r-1$ elements from the remaining $n-1$ elements of X. This can be done in $C(n-1,r-1)$ ways. Therefore, each of the n elements of X occurs in $C(n-1,r-1)$ different r-element subsets of X. Consequently, the sum of the numbers of elements in these sets is $n \times C(n-1,r-1)$. As these two different ways of obtaining this sum must lead to the same answer, it follows that $rC(n,r) = nC(n-1,r-1)$.

Note that we can deduce from Theorem 2.4 that $C(n,r) = (n/r)[C(n-1,r-1)]$. This enables us to give a direct, combinatorial proof that $C(n, r) = n!/[r!(n-r)!]$ without the need to consider permutations.

THEOREM 2.5

For all nonnegative integers r, n with $r \le n$, $C(n,r) = C(n,n-r)$.

Proof

Deciding which r objects to select from a set of n objects amounts to exactly the same thing as deciding which $n-r$ objects *not* to select. Hence the number of ways of choosing r objects from n is the same as the number of ways of choosing $n-r$ objects from n.

The next theorem explains how the binomial coefficients get their name.

THEOREM 2.6
The Binomial Theorem

For all variables a, b, and each positive integer n,

$$(a + b)^n = a^n + C(n,1)a^{n-1}b + C(n,2)a^{n-2}b^2 + \dots + b^n,$$

that is,

$$(a+b)^n = \sum_{r=0}^{n} C(n,r)a^{n-r}b^r, \text{ as } C(n,0) = C(n,n) = 1.$$

Proof

Consider the product

$$(a + b)(a + b) \dots (a + b)$$

with n pairs of brackets. When we multiply out this product, each separate term that arises comes from choosing either a or b from each pair of brackets and then multiplying these a's and b's together. We obtain the term $a^{n-r}b^r$ each time we choose b from r of these pairs of brackets and a from the remaining $n-r$ pairs. Thus the number of terms of the form $a^{n-r}b^r$ that we obtain equals the number of ways of choosing r pairs of brackets from which to pick b, and this number is $C(n,r)$. Hence when we gather similar terms together, the coefficient of $a^{n-r}b^r$ is $C(n,r)$.

Of course, selecting r b's forces us to select $n-r$ a's. Repeating the argument with a and b interchanged shows that the coefficient of $a^{n-r}b^r$ is also $C(n,n-r)$, as Theorem 2.5 tells us it should be.

The idea that we have used in this combinatorial proof of the binomial theorem will play an important role later in this book (in Chapter 7). The algebraic expression $(a + b)^n$ is called a *binomial* (from the Latin *binomius* meaning "having two names"), and this is why the binomial coefficients were given their name. The binomial theorem is sometimes attributed to Isaac Newton, though the binomial coefficients were known and tabulated long before Newton's time. His main contribution in this area was to prove the form of this theorem that applies when the exponent n is not a positive integer.

Our next theorem about binomial coefficients leads to a very well-known method for calculating their values. We again emphasize that we give a combinatorial proof of this theorem. An algebraic proof, using the formula for $C(n,r)$, is very straightforward but hides the combinatorial meaning of the result.

THEOREM 2.7

For all positive integers r, n with $r \leq n$,

$$C(n+1,r) = C(n,r-1) + C(n,r).$$

Proof

Let X be a set containing $n + 1$ objects, and let a be one of the objects in the set X. We count the number of subsets of X containing r elements by separating them into the set, say $\boldsymbol{Y}$, of those r-element subsets that include a and the set, say $\boldsymbol{Z}$, of those r-element subsets that do *not* include a.

A subset of X in $\boldsymbol{Y}$ contains r elements one of which is a and a further $r-1$ elements chosen from the n-element set $X \setminus \{a\}$. Thus, there are $C(n,r-1)$ subsets in $\boldsymbol{Y}$.

A subset of X in $\boldsymbol{Z}$ contains r elements none of which is a, and hence consists of r elements chosen from $X \setminus \{a\}$ and hence there are $C(n,r)$ of these.

Each r-element subset of X is either in $\boldsymbol{Y}$ or in $\boldsymbol{Z}$, and none of them is in both. Hence the number of r-element subsets of X is the sum of the number of subsets in $\boldsymbol{Y}$ and the number in $\boldsymbol{Z}$, that is, $C(n+1,r) = C(n,r-1) + C(n,r)$.

The numbers $C(n,r)$, for $0 \leq r \leq n$, are often displayed in a triangle formation, as in Figure 2.3. It then follows from Theorem 2.7 that each number in the $(n + 1)$th row (apart from those at the ends) is the sum of the two adjacent numbers in the row above. For example, the number 21 in the eighth row is the sum of 6 and 15 from the row above. This triangle is usually called *Pascal's triangle*, after the seventeenth-century French mathematician Blaise Pascal, although it was not originated by him.* The first 11 rows of Pascal's triangle are shown in Figure 2.3.

* We quote the following account of the matter from *The Backbone of Pascal's Triangle* by Martin Griffiths, United Kingdom Mathematics Trust (UKMT), Leeds, 2008 p. 10: "Pascal himself called it 'the arithmetical triangle', but after the mathematicians Pierre Rémond de Montmort and Abraham de Moivre referred to it in writing as 'the combinatorial triangle of Mr. Pascal' (in 1708) and 'Pascal's arithmetical triangle' (in 1730) respectively, the name stuck. However the Italian mathematician Nicolo Tartaglia actually published these numbers in 1556, and there is evidence that the Chinese mathematician Yang Hui was working with these numbers in the thirteenth-century (the Chinese do indeed use the term 'Yang Hui's triangle')."

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

FIGURE 2.3

There are innumerable relationships between the binomial coefficients that correspond to patterns that can be found within Pascal's triangle. For example, it follows from Theorem 2.5 that Pascal's triangle is symmetrical about its central vertical axis. Some other relationships are given in the next theorem and in the exercises at the end of this section.

THEOREM 2.8

For all positive integers k, n with $k \leq n$,

$$C(n+1,k+1) = C(n,k) + C(n-1,k) + \ldots + C(k,k).$$

Proof

Let $X = \{x_1,x_2,\ldots,x_n, x_{n+1}\}$. $C(n+1,k+1)$ is the number of subsets of X that contain $k+1$ of the elements of X. We can also count the number of these subsets in the following way. For $k+1 \leq r \leq n+1$ we let X_r be the set of all those $(k+1)$-element subsets, Y, of X such that r is the largest integer for which $x_r \in Y$. Thus, if $Y \in X_r$, then $x_r \in Y$ but $x_{r+1},\ldots, x_{n+1} \notin Y$.

Clearly, the sets $X_{k+1},\ldots, X_n, X_{n+1}$ are pairwise disjoint. Also, between them they include all the $(k+1)$-element subsets of X, as each subset of X containing $k+1$ elements must include at least one of the elements $\{x_{k+1},\ldots, x_{n+1}\}$. Hence

$$\#(X) = \sum_{r=k+1}^{n+1} \#(X_r). \tag{2.1}$$

The sets in X_r contain x_r and k elements chosen from the set $\{x_1,\ldots,x_{r-1}\}$. Therefore, $\#(X_r) = C(r-1,k)$. We can therefore deduce from Equation 2.1 that $C(n+1,k+1) = \sum_{r=k+1}^{n+1} C(r-1,k)$, which, when we rewrite the terms on the right-hand side in reverse order, gives $C(n+1,k+1) = C(n,k) + C(n-1,k) + \ldots + C(k,k)$.

We conclude this section with a simple but intriguing application of Theorem 2.3 to number theory.

THEOREM 2.9

For each positive integer r, the product of any r consecutive positive integers is divisible by $r!$

Proof

We need to prove that for all positive integers k, r the product of the r consecutive integers $k, k+1, k+2, \ldots, k+r-1$ is divisible by r. Now,

$$\frac{k(k+1)(k+2)\ldots(k+r-1)}{r!} = C(k+r-1, r),$$

by Theorem 2.3. Since this binomial coefficient gives the number of r-element subsets of a set of $k+r-1$ elements, it must be an integer. So $k(k+1)(k+2)\ldots(k+r-1)$ is divisible by $r!$

Exercises

2.3.1A A mathematics course offers students the choice of three options from 12 courses in pure mathematics, two options from 10 courses in applied mathematics, two options from 6 courses in statistics, and one option from 4 courses in computing. In how many different ways can the students choose their eight options?

2.3.1B A cricket squad consists of six batsmen, eight bowlers, three wicketkeepers, and four all-rounders. The selectors wish to pick a team made up of four batsmen, four bowlers, one wicketkeeper, and two all-rounders. How many different teams can they pick?

The next three pairs of questions can all be answered by using the binomial theorem. However, you are encouraged to give combinatorial proofs in the style of those we have given for the theorems in this section.

2.3.2A Prove that a set of n elements has 2^n different subsets, and deduce that for each positive integer n, $\sum_{r=0}^{n} C(n,r) = 2^n$.

2.3.2B Prove that, for each positive integer n, $\sum_{r=0}^{n} C(n,r)^2 = C(2n,n)$.

2.3.3A Prove that, for all positive integers n, k, s with $s \le k \le n$, $C(n,k)\, C(k,s) = C(n,s)$ $C(n-s,k-s)$.

2.3.3B Let X be a finite set. Prove that the number of subsets of X that contain an even number of elements is equal to the number of subsets of X that contain an odd number of elements. Deduce that for each positive integer n, $\sum_{r=0}^{n} (-1)^r C(n,r) = 0$.

2.3.4A Prove that, for each positive integer n, $\sum_{r=0}^{n} rC(n,r) = n2^{n-1}$.
[*Hint:* Let X be an n-element set. Note that as X has $C(n, r)$ subsets containing r elements, $\sum_{r=0}^{n} rC(n,r) = \sum_{A \subseteq X} \#(A)$. Also, we can calculate $\sum_{A \subseteq X} \#(A)$ by pairing off each subset A of X with its complement $X \backslash A$. How many pairs are there, and what is $\#(A) + \#(X \backslash A)$?]

2.3.4B Prove that for each positive integer n, $\sum_{r=0}^{n} r^2 C(n,r) = n(n+1)2^{n-2}$.
(*Hint:* First find $\sum_{r=0}^{n} r(r-1)C(n,r)$ by counting the ordered pairs (a, b), where a and b are chosen from an n-element set, in two ways. Then use the result of Exercise 2.3.4A.)