



FLUID DYNAMICS

VIA EXAMPLES AND SOLUTIONS

SERGEY NAZARENKO



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University of Warwick, UK



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Contents

Preface	xiii
Acknowledgements	xv
List of Figures	xvii
Author Biography	xxi
1 Fluid equations and different regimes of fluid flows	1
1.1 Background theory	1
1.1.1 Incompressible flows	1
1.1.2 Inviscid flows	2
1.1.3 Rotating flows	3
1.2 Further reading	3
1.3 Problems	3
1.3.1 Reynolds number	4
1.3.2 Mach number	4
1.3.3 Rossby number	4
1.3.4 Richardson number	5
1.3.5 Prandtl number	5
1.3.6 Stokes number	6
1.4 Solutions	7
1.4.1 Model solution to question 1.3.1	7
1.4.2 Model solution to question 1.3.2	7
1.4.3 Model solution to question 1.3.3	8
1.4.4 Model solution to question 1.3.4	8
1.4.5 Model solution to question 1.3.5	9
1.4.6 Model solution to question 1.3.6	10
2 Conservation laws in incompressible fluid flows	11
2.1 Background theory	11
2.1.1 Velocity-vorticity form of the Navier-Stokes equation	11
2.1.2 Bernoulli theorems	12
2.1.3 The vorticity form of the flow equation	12

2.1.4	Energy balance and energy conservation	13
2.1.5	Momentum balance and momentum conservation	14
2.1.6	Circulation: Kelvin's theorem	15
2.1.7	Vorticity invariants in 2D flows	15
2.2	Further reading	16
2.3	Problems	17
2.3.1	Conservation of potential vorticity	17
2.3.2	Tap water	17
2.3.3	Discharge into a drainage pipe	18
2.3.4	Oscillations in a U-tube	19
2.3.5	Force on a bent garden hose	20
2.3.6	Firehose flow	20
2.3.7	Shear flow in a strain field	21
2.3.8	Rankine vortex in a strain field	22
2.3.9	Forces produced by a vortex dipole	23
2.3.10	Torque produced by a vortex	24
2.3.11	Jammed garden hose	24
2.3.12	Flow through a Borda mouthpiece	26
2.3.13	Water barrel on wheels	27
2.3.14	Vortex lift	28
2.3.15	Water clock	29
2.3.16	Reservoir with regulated water level	29
2.3.17	Energy of ideal irrotational flows	30
2.4	Solutions	31
2.4.1	Model solution to question 2.3.1	31
2.4.2	Model solution to question 2.3.2	32
2.4.3	Model solution to question 2.3.3	32
2.4.4	Model solution to question 2.3.4	33
2.4.5	Model solution to question 2.3.5	34
2.4.6	Model solution to question 2.3.6	35
2.4.7	Model solution to question 2.3.7	35
2.4.8	Model solution to question 2.3.8	37
2.4.9	Model solution to question 2.3.9	38
2.4.10	Model solution to question 2.3.10	39
2.4.11	Model solution to question 2.3.11	41
2.4.12	Model solution to question 2.3.12	42
2.4.13	Model solution to question 2.3.13	42
2.4.14	Model solution to question 2.3.14	43
2.4.15	Model solution to question 2.3.15	44
2.4.16	Model solution to question 2.3.16	44
2.4.17	Model solution to question 2.3.17	45

3	Fluid with free surface	47
3.1	Background theory	47
3.1.1	Pressure boundary condition	47
3.1.2	Kinematic boundary condition	48
3.1.3	Axially and spherically symmetric flows	48
3.2	Further reading	49
3.3	Problems	49
3.3.1	Water surface distortion due to vortex	49
3.3.2	Free surface shape of water in an annular flow	50
3.3.3	Water-filled turntable	51
3.3.4	Cavity implosion	52
3.3.5	Flow in an expanding air bubble	53
3.3.6	Flow in wire coating die	54
3.4	Solutions	54
3.4.1	Model solution to question 3.3.1	54
3.4.2	Model solution to question 3.3.2	56
3.4.3	Model solution to question 3.3.3	57
3.4.4	Model solution to question 3.3.4	58
3.4.5	Model solution to question 3.3.5	60
3.4.6	Model solution to question 3.3.6	60
4	Waves and instabilities	63
4.1	Background theory	63
4.1.1	Waves	63
4.1.2	Instabilities	65
4.2	Further reading	68
4.3	Problems	68
4.3.1	Motion of a wave packet	68
4.3.2	Gravity waves on the water surface	69
4.3.3	Gravity and capillary waves: dimensional analysis	70
4.3.4	Inertial waves in rotating fluids	71
4.3.5	Internal waves in stratified fluids	72
4.3.6	Sound waves in compressible fluids	73
4.3.7	Sound rays in shear flows: acoustic mirage	74
4.3.8	Sound rays in stratified flows: wave guides, Snell's law	75
4.3.9	Sound rays in a vortex flow: a black hole effect	76
4.3.10	Kelvin-Helmholtz instability	76
4.3.11	Rayleigh-Taylor instability	77
4.3.12	Rapid distortion theory	78
4.4	Solutions	80
4.4.1	Model solution to question 4.3.1	80
4.4.2	Model solution to question 4.3.2	81
4.4.3	Model solution to question 4.3.3	83

4.4.4	Model solution to question 4.3.4	84
4.4.5	Model solution to question 4.3.5	85
4.4.6	Model solution to question 4.3.6	86
4.4.7	Model solution to question 4.3.7	87
4.4.8	Model solution to question 4.3.8	88
4.4.9	Model solution to question 4.3.9	90
4.4.10	Model solution to question 4.3.10	91
4.4.11	Model solution to question 4.3.11	93
4.4.12	Model solution to question 4.3.12	95
5	Boundary layers	99
5.1	Background theory	99
5.2	Problems	100
5.2.1	The boundary layer equations	100
5.2.2	A boundary layer over a semi-infinite plate	101
5.2.3	Boundary layer produced by a pure strain flow	102
5.2.4	A flow near an oscillating wall	104
5.2.5	A boundary layer in a rotating fluid	104
5.3	Solutions	106
5.3.1	Model solution to question 5.2.1	106
5.3.2	Model solution to question 5.2.2	107
5.3.3	Model solution to question 5.2.3	109
5.3.4	Model solution to question 5.2.4	110
5.3.5	Model solution to question 5.2.5	111
6	Two-dimensional flows	115
6.1	Background theory	115
6.2	Problems	118
6.2.1	Pure strain flow	118
6.2.2	Couette flow	118
6.2.3	Poiseuille flow	119
6.2.4	“Turbulent” shear flows	119
6.2.5	Jet flow	121
6.2.6	Mixing layer	121
6.2.7	Stream function for a 2D flow	122
6.2.8	Round vortices: Rankine vortex and a point vortex	123
6.2.9	Flow bounded by two intersecting planes	123
6.2.10	“Binary star system”	124
6.2.11	Complex potential for the gravity water waves	125
6.2.12	Aeroplane lift and trailing vortices	125
6.2.13	Finding drag and lift using dimensional analysis	125
6.2.14	A laminar jet flow	126
6.2.15	Flow in a cylinder with an elliptical cross-section	127

6.2.16	Rain flow over an inclined roof	127
6.2.17	Flow over an inclined plane	128
6.3	Solutions	129
6.3.1	Model solution to question 6.2.1	129
6.3.2	Model solution to question 6.2.2	129
6.3.3	Model solution to question 6.2.3	130
6.3.4	Model solution to question 6.2.4	131
6.3.5	Model solution to question 6.2.5	133
6.3.6	Model solution to question 6.2.6	134
6.3.7	Model solution to question 6.2.7	135
6.3.8	Model solution to question 6.2.8	136
6.3.9	Model solution to question 6.2.9	137
6.3.10	Model solution to question 6.2.10	139
6.3.11	Model solution to question 6.2.11	141
6.3.12	Model solution to question 6.2.12	142
6.3.13	Model solution to question 6.2.13	142
6.3.14	Model solution to question 6.2.14	143
6.3.15	Model solution to question 6.2.15	144
6.3.16	Model solution to question 6.2.16	145
6.3.17	Model solution to question 6.2.17	146
7	Point vortices and point sources	149
7.1	Background theory	149
7.2	Further reading	152
7.3	Problems	152
7.3.1	Energy, momentum, and angular momentum of a point vortex set	152
7.3.2	Motion of two point vortices	153
7.3.3	Vortex “molecules”	153
7.3.4	Motion of three point vortices	155
7.3.5	Point vortex in a channel	155
7.3.6	Point vortices and their images	157
7.3.7	Clustering in the gas of point vortices	157
7.3.8	Discharge through a hole	159
7.3.9	Submerged pump near a wall	159
7.3.10	Flows past a zeppelin and a balloon	159
7.4	Solutions	161
7.4.1	Model solution to question 7.3.1	161
7.4.2	Model solution to question 7.3.2	162
7.4.3	Model solution to question 7.3.3	163
7.4.4	Model solution to question 7.3.4	165
7.4.5	Model solution to question 7.3.5	167
7.4.6	Model solution to question 7.3.6	167
7.4.7	Model solution to question 7.3.7	168

7.4.8	Model solution to question 7.3.8	169
7.4.9	Model solution to question 7.3.9	170
7.4.10	Model solution to question 7.3.10	171
8	Turbulence	173
8.1	Background theory	173
8.2	Further reading	176
8.3	Problems	176
8.3.1	Kolmogorov spectrum of turbulence	176
8.3.2	Dual cascade in steady two-dimensional turbulence	177
8.3.3	Dual cascade in evolving two-dimensional turbulence	178
8.3.4	Spectra of two-dimensional turbulence	179
8.3.5	Dispersion of particles in turbulence	179
8.3.6	Near-wall turbulence	180
8.3.7	Dissipative anomaly in turbulence	181
8.4	Solutions	182
8.4.1	Model solution to question 8.3.1	182
8.4.2	Model solution to question 8.3.2	182
8.4.3	Model solution to question 8.3.3	183
8.4.4	Model solution to question 8.3.4	183
8.4.5	Model solution to question 8.3.5	184
8.4.6	Model solution to question 8.3.6	185
8.4.7	Model solution to question 8.3.7	186
9	Compressible flow	187
9.1	Background theory	187
9.1.1	One-dimensional gas dynamics	187
9.1.2	Two-dimensional gas dynamics	190
9.1.2.1	Irrotational isentropic flows	190
9.1.2.2	Steady hypersonic flow past a slender body	191
9.2	Further reading	193
9.3	Problems	193
9.3.1	Characteristic equations	193
9.3.2	Flow due to a piston withdrawal	194
9.3.3	Gas expansion into vacuum	195
9.3.4	Dam break flow	195
9.3.5	Momentum conservation in a viscous compressible flow	195
9.3.6	Energy conservation in compressible flow	195
9.3.7	Rankine-Hugoniot conditions for jumps across shocks	196
9.3.8	Hypersonic collision of two gas masses	196
9.3.9	Zhukovskiy's theorem for the subsonic flow	198
9.3.10	Flow around a cone-nosed rocket	199
9.3.11	Flow around a wedge	200

9.3.12	Lift force on a hypersonic wing	202
9.3.13	Formation of a blast wave by a very intense explosion	203
9.3.14	Balloon in polytropic atmosphere	203
9.4	Solutions	204
9.4.1	Model solution to question 9.3.1	204
9.4.2	Model solution to question 9.3.2	204
9.4.3	Model solution to question 9.3.3	207
9.4.4	Model solution to question 9.3.4	207
9.4.5	Model solution to question 9.3.5	207
9.4.6	Model solution to question 9.3.6	208
9.4.7	Model solution to question 9.3.7	209
9.4.8	Model solution to question 9.3.8	210
9.4.9	Model solution to question 9.3.9	211
9.4.10	Model solution to question 9.3.10	212
9.4.11	Model solution to question 9.3.11	214
9.4.12	Model solution to question 9.3.12	216
9.4.13	Model solution to question 9.3.13	217
9.4.14	Model solution to question 9.3.14	218
Bibliography		219
Index		223

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Preface

As an applied subject, fluid dynamics is best studied via considering specific examples and solving problems dealing with various phenomena and effects in fluids. This is well recognised in most Fluid Dynamics courses, which often have support classes devoted to considering physically motivated exercises. Also, such Fluid Dynamics courses are typically assessed via solution of specific problems more often than via reproducing mathematical proofs or general abstract constructions. However, original Fluid Dynamics problems are rather hard to invent, and the Fluid Dynamics lecturers are often “on their own” having to reinvent successful ideas, tricks and representative examples. The present book addresses these issues by systematically providing such ideas and model examples.

A distinct feature of the present book is that it is problem oriented. Of course, there are many wonderful fluid dynamics textbooks, classical and more recent, which contain exercises and examples, to name just a few: *Hydrodynamics* by H. Lamb [13], *Essentials of Fluid Dynamics* L. Prandtl [21], *An Introduction to Fluid Dynamics* by G.K. Batchelor [4], *Fluid Dynamics* by L.D. Landau and E.M. Lifshitz [14], *Prandtl-Essentials of Fluid Mechanics* by Oertel et al. [17], *Elementary Fluid Dynamics* by D.J. Acheson [1], *Fluid Mechanics* by P.K. Kundu and I.M. Cohen [12], *Physical Fluid Dynamics* by D.J. Tritton [28], *A First Course in Fluid Dynamics* by A.R. Paterson [19], *Elementary Fluid Mechanics* by T. Kambe [9], *Fluid Mechanics: A Short Course for Physicists* by G. Falkovich [6], *Fundamentals of Geophysical Fluid Dynamics* by J.C. McWilliams [16] and *Waves in Fluids* by J. Lighthill [15]. However, most of the existing books make an accent on the theory or expositions. There has been a clear lack of a text which would contain a sizeable set of example problems and detailed model solutions. The present book is intended to fill this gap by presenting a number of fluid dynamics problems organised in chapters dealing with several sub-areas, types of flows and applications. The problems form a “skeleton” of the book structure. Throughout this book, we include supplementary theoretical material when necessary, with an extended list of references for suggested further reading material at the end of each chapter. We also provide a complete set of model solutions.

The book is designed to be used in problem solving support classes and for exam revision in undergraduate and graduate fluid dynamics courses. Also, the book will aid lecturers by offering a pool of possible exam questions for such fluid dynamics courses. It is my hope that the book could be useful also

to students and lecturers in related subjects, such as continuum mechanics, turbulence, ocean and atmospheric sciences, etc. More broadly, the provided set of example problems should help an effective hands-on study of fluid dynamics, within or outside of a university course, including an independent study by specialists in other scientific areas who would like to learn basics of fluid dynamics.

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List of Figures

2.1	Tap water.	18
2.2	Drain pipe.	19
2.3	U-tube.	20
2.4	Bent garden hose.	21
2.5	Water flow in firehose.	22
2.6	Vortex pair in a cylindrical container.	24
2.7	Vortex in a cylindrical container.	25
2.8	Sudden change of a pipe cross-section.	25
2.9	Flow through a Borda mouthpiece.	27
2.10	Flow from a moving barrel.	28
2.11	Vortices over sharply swept plane wings.	28
2.12	Vortex over an infinite flat plate located at $y = 0$. The image vortex is below the plate (dashed line).	29
2.13	Water clock.	30
2.14	Reservoir with a sluice gate.	30
3.1	Vortex in a layer of fluid with free surface.	50
3.2	Water in an annular flow.	51
3.3	Rotating tank of water.	52
3.4	Cavity implosion.	53
3.5	Expanding air bubble.	54
3.6	Flow in a wire coating die.	55
4.1	Wave packet.	64
4.2	Stable (left) and unstable (right) fixed points.	66
4.3	Plane-parallel shear flow.	67
4.4	Fourier distribution in a wave packet.	69
4.5	Sound propagation in a shear flow.	74
4.6	Kelvin-Helmholtz instability setup: piecewise constant shear flow with a tangential discontinuity.	77
4.7	Constant-shear flow.	79
4.8	Sound rays in a shear flow.	88
4.9	Sound rays in a stratified flow.	89
4.10	Sound rays in a stratified flow.	90
4.11	Sound rays in a point vortex flow.	92

5.1	Boundary layer profile.	100
5.2	Boundary layer over a semi-infinite plate.	102
5.3	Boundary layer near a stagnation point.	103
5.4	Boundary layer near an oscillating wall.	104
5.5	Boundary layer in a rotating fluid.	105
6.1	Couette flow.	118
6.2	Poiseuille flow.	119
6.3	Jet flow.	121
6.4	Mixing layer.	122
6.5	Flow down a roof.	127
6.6	Flow down a slope.	128
6.7	“Turbulent” Couette flow.	132
6.8	“Turbulent” Poiseuille flow.	133
6.9	Calculation of the mass flux.	136
6.10	Point vortex.	136
6.11	Case $n = 4$	138
6.12	Case $n = 4/3$	138
6.13	Case $n = 2/3$	139
6.14	Case $n = 1/2$	139
6.15	“Binary star system”.	140
7.1	“Gas” of point vortices. Counter-clockwise and clockwise pointing arrows mark vortices with positive and negative circulations respectively.	150
7.2	A point source flow.	151
7.3	Three-vortex of molecule.	153
7.4	Four-vortex of molecule.	154
7.5	N -vortex of molecule for $N = 6$	154
7.6	Scattering of a point vortex dipole at an isolated vortex.	156
7.7	Vortex in channel.	157
7.8	Infinite chain of image vortices.	158
7.9	Water discharging through a hole at the bottom.	159
7.10	Submerged pump near a wall.	160
7.11	Flow around a zeppelin.	160
7.12	Motion of two like-signed point vortices. Both vortices are counter-clockwise in this figure, i.e. they have positive circulations $\Gamma_1 > 0$ and $\Gamma_2 > 0$	163
7.13	Motion of two opposite-signed point vortices. Counter-clockwise and clockwise pointing arrows mark vortices with positive and negative circulations respectively. In this example $\Gamma_1 > 0$ and $\Gamma_2 < 0$	164
7.14	Dipole of point vortices. Counter-clockwise and clockwise pointing arrows mark vortices with positive and negative circulations respectively.	165

7.15	Three-vortex of molecule.	166
8.1	Richardson cascade in the physical space.	174
8.2	Richardson's cascade in the k -space.	175
8.3	2D turbulence: dual cascade in the k -space space.	175
9.1	Hypersonic flow past a slender body ("hypersonic plane"). The bold solid line is a shock, dashed lines indicate an expansion fan.	192
9.2	Gas motion after a collision of two clouds. The dashed lines mark positions of the three jumps: two shocks (on the right and on the left) and a contact discontinuity (in the middle). The solid line is the density profile. The pressure profile is similar, except that there is no pressure jump across the contact discontinuity.	197
9.3	Flow past a thin rocket.	199
9.4	Flow past a wedge.	201
9.5	Hypersonic flow past a flat wing.	202
9.6	Characteristics in a piston-generated flow. C^0 – solid lines, C^- – dashed lines, C^+ – dash-dotted lines. The bold straight line passing through $(0, 0)$ is separating the C^+ -characteristics originating on the x -axis and the ones originating at the piston's trajectory $x = X(t)$	205
9.7	C^+ -characteristics in a piston-generated flow in the case when the piston's speed is constant and negative, $\dot{X}(t) = -V < 0$	207
9.8	Flow past a thin rocket.	212
9.9	Flow past a wedge.	214
9.10	Hypersonic flow past a flat wing. The wing is shown by the very bold line. The bold line marks the shocks, and the dashed lines mark the expansion fans. The streamlines are marked by the thin solid lines.	216

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Author Biography

Sergey Nazarenko's research is in the areas of fluid dynamics, turbulence and waves arising in different applications. This includes wave turbulence, magneto-hydrodynamic turbulence, superfluid turbulence, water waves, Rossby waves, vortices and jets in geophysical fluids, drift waves and zonal jets in plasmas, optical vortices and turbulence, turbulence in Bose-Einstein condensates.

Sergey Nazarenko has been working at the University of Warwick since 1996, where presently he holds a position of full professor. Prior to Warwick, Sergey Nazarenko worked as visiting assistant professor at the Department of Mathematics, University of Arizona in 1993–1996, postdoc at the Department of Mechanical and Aerospace Engineering, Rutgers University, New Jersey in 1992–1993, and researcher at the Landau Institute for Theoretical Physics, Moscow in 1991–1992.

Sergey Nazarenko wrote *Wave Turbulence* published by Springer in 2011. He was a co-editor of the books *Non-equilibrium Statistical Mechanics and Turbulence*, CUP 2008, and *Advances in Wave Turbulence*, World Scientific, 2013. Sergey Nazarenko has organised twelve international scientific conferences and workshops in fluid dynamics and turbulence.

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Chapter 1

Fluid equations and different regimes of fluid flows

1.1 Background theory

A fluid is a continuous medium whose state is characterised by its velocity field, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, pressure and density fields, $p = p(\mathbf{x}, t)$ and $\rho = \rho(\mathbf{x}, t)$ respectively, and possibly other relevant fields (e.g. temperature). Here, t is time and $\mathbf{x} \in R^d$ is the physical coordinate in the d -dimensional space, $d = 1, 2$ or 3 . Respectively, $\mathbf{u} \in R^d$, although in some special flows the dimensions of \mathbf{x} and \mathbf{u} may be different from each other.

Most of the fluid dynamics results have been obtained starting from the *Navier-Stokes equations*. These equations have many variations depending on the forces acting on the fluid, as well as the properties of the fluid itself—compressibility, thermoconductivity, viscosity, density homogeneity/inhomogeneity, chemical composition, etc. We will consider a relatively small but important subset of idealised cases.

1.1.1 Incompressible flows

The *Navier-Stokes equation* for incompressible flow is given by

$$D_t \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (1.1)$$

which is a momentum balance equation for fluid particles (a continuous medium version of Newton's second law). We introduced notation for the fluid particle acceleration,

$$D_t \mathbf{u} \equiv \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}. \quad (1.2)$$

Operator $D_t \equiv \partial_t + (\mathbf{u} \cdot \nabla)$ is a time derivative along the fluid particle trajectory—a Lagrangian time derivative. Possible external forces acting on the fluid (per unit mass) are denoted by term \mathbf{f} in the right-hand side of equation (1.1); these could be gravity, electrostatic force, etc.

The momentum balance equation (1.1) has to be complemented by a mass

balance equation, which for an incompressible fluid is

$$\nabla \cdot \mathbf{u} = 0, \quad (1.3)$$

and is usually called the *incompressibility condition*.

Note that fluids can be incompressible and yet the density of the fluid particles may vary in physical space. In this case we need an extra equation describing the conservation of ρ along the fluid particle trajectories,

$$D_t \rho = 0. \quad (1.4)$$

1.1.2 Inviscid flows

When the Reynolds number is large one can ignore viscosity (see problem 1.3.1), and the momentum balance equation (1.1) reduces to

$$D_t \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{f}, \quad (1.5)$$

which is known as the *Euler equation*. It is valid for both compressible and incompressible fluids. However, for compressible fluids, the mass balance equation is now different:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.6)$$

(This equation remains the same in presence of viscosity). Also, since there is an extra unknown field, $\rho(\mathbf{x}, t)$, we need an extra evolution equation for the model to be complete. Generally, such an equation is provided by the energy balance relation. In particular, assuming that different fluid particles are thermally insulated from each other, one can write the additional equation in the form of conservation of *entropy* $S \equiv S(\mathbf{x}, t)$ along the fluid paths,

$$D_t S = 0. \quad (1.7)$$

For the polytropic gas model

$$S = C_v \ln \frac{p}{\rho^\gamma}, \quad (1.8)$$

where constants C_v and γ are called the specific heat constant and the adiabatic index respectively. Obviously, C_v drops out of the equation (1.8) and therefore it is irrelevant in this case. For monatomic ideal gas (e.g. helium, neon, argon) $\gamma = 5/3$; for diatomic gas $\gamma = 7/5$ (e.g. oxygen, nitrogen).

In the simplest case of *isentropic gas*, $S = \text{const}$, i.e.

$$p \propto \rho^\gamma. \quad (1.9)$$

In incompressible fluids the equation (1.7) implies conservation of temperature T along the fluid paths,

$$D_t T = 0. \quad (1.10)$$