# FUNDAMENTALS OF FRACTURE MECHANICS 

## TRIBIKRAM KUNDU



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## Dedication

To my wife, Nupur, our daughters, Auni and Ina, and our parents, Makhan Lal Kundu, Sandhya Rani Kundu,

Jyotirmoy Naha, and Rubi Naha

## Contents

1 Fundamentals of the Theory of Elasticity ..... 1
1.1 Introduction ..... 1
1.2 Fundamentals of Continuum Mechanics and the Theory of Elasticity ..... 1
1.2.1 Deformation and Strain Tensor ..... 1
1.2.1.1 Interpretation of $\varepsilon_{i j}$ and $\omega_{i j}$ for Small Displacement Gradient .....  3
1.2.2 Traction and Stress Tensor ..... 6
1.2.3 Traction-Stress Relation ..... 8
1.2.4 Equilibrium Equations ..... 9
1.2.4.1 Force Equilibrium ..... 9
1.2.4.2 Moment Equilibrium ..... 11
1.2.5 Stress Transformation. ..... 12
1.2.5.1 Kronecker Delta Symbol ( $\delta_{i j}$ ) and Permutation Symbol ( $\varepsilon_{i j k}$ ) ..... 14
1.2.5.2 Examples of the Application of $\delta_{i j}$ and $\varepsilon_{i j k}$ ..... 14
1.2.6 Definition of Tensor ..... 15
1.2.7 Principal Stresses and Principal Planes ..... 15
1.2.8 Transformation of Displacement and Other Vectors ..... 19
1.2.9 Strain Transformation ..... 20
1.2.10 Definition of Elastic Material and Stress-Strain Relation. ..... 20
1.2.11 Number of Independent Material Constants ..... 24
1.2.12 Material Planes of Symmetry ..... 25
1.2.12.1 One Plane of Symmetry ..... 25
1.2.12.2 Two and Three Planes of Symmetry ..... 26
1.2.12.3 Three Planes of Symmetry and One Axis of Symmetry ..... 27
1.2.12.4 Three Planes of Symmetry and Two or Three Axes of Symmetry ..... 28
1.2.13 Stress-Strain Relation for Isotropic Materials- Green's Approach ..... 30
1.2.13.1 Hooke's Law in Terms of Young's Modulus and Poisson's Ratio ..... 32
1.2.14 Navier's Equation of Equilibrium ..... 33
1.2.15 Fundamental Equations of Elasticity in Other Coordinate Systems ..... 36
1.2.16 Time-Dependent Problems or Dynamic Problems ..... 36
1.3 Some Classical Problems in Elasticity ..... 36
1.3.1 In-Plane and Out-of-Plane Problems ..... 38
1.3.2 Plane Stress and Plane Strain Problems ..... 39
1.3.2.1 Compatibility Equations for Plane Stress Problems ..... 41
1.3.2.2 Compatibility Equations for Plane Strain Problems ..... 42
1.3.3 Airy Stress Function ..... 42
1.3.4 Some Classical Elasticity Problems in Two Dimensions ..... 45
1.3.4.1 Plate and Beam Problems ..... 45
1.3.4.2 Half-Plane Problems ..... 51
1.3.4.3 Circular Hole, Disk, and Cylindrical Pressure Vessel Problems ..... 59
1.3.5 Thick Wall Spherical Pressure Vessel ..... 72
1.4 Concluding Remarks ..... 75
References ..... 75
Exercise Problems ..... 75
2 Elastic Crack Model ..... 85
2.1 Introduction ..... 85
2.2 Williams' Method to Compute the Stress Field near a Crack Tip ..... 85
2.2.1 Satisfaction of Boundary Conditions ..... 88
2.2.2 Acceptable Values of $n$ and $\lambda$ ..... 90
2.2.3 Dominant Term ..... 92
2.2.4 Strain and Displacement Fields ..... 96
2.2.4.1 Plane Stress Problems ..... 96
2.2.4.2 Plane Strain Problems ..... 98
2.3 Stress Intensity Factor and Fracture Toughness ..... 100
2.4 Stress and Displacement Fields for Antiplane Problems ..... 101
2.5 Different Modes of Fracture ..... 102
2.6 Direction of Crack Propagation ..... 102
2.7 Mixed Mode Failure Curve for In-Plane Loading ..... 105
2.8 Stress Singularities for Other Wedge Problems ..... 107
2.9 Concluding Remarks ..... 107
References ..... 108
Exercise Problems ..... 108
3 Energy Balance ..... 113
3.1 Introduction ..... 113
3.2 Griffith's Energy Balance ..... 113
3.3 Energy Criterion of Crack Propagation for Fixed Force and Fixed Grip Conditions ..... 115
3.3.1 Soft Spring Case ..... 118
3.3.2 Hard Spring Case ..... 119
3.3.3 General Case ..... 120
3.4 Experimental Determination of $G_{c}$ ..... 120
3.4.1 Fixed Force Experiment ..... 122
3.4.2 Fixed Grip Experiment ..... 122
3.4.3 Determination of $G_{c}$ from One Specimen ..... 123
3.5 Relation between Strain Energy Release Rate $(G)$ and Stress Intensity Factor (K) ..... 123
3.6 Determination of Stress Intensity Factor ( $K$ ) for Different Problem Geometries ..... 126
3.6.1 Griffith Crack ..... 126
3.6.2 Circular or Penny-Shaped Crack ..... 129
3.6.3 Semi-infinite Crack in a Strip ..... 130
3.6.4 Stack of Parallel Cracks in an Infinite Plate ..... 131
3.6.5 Star-Shaped Cracks ..... 133
3.6.6 Pressurized Star Cracks ..... 135
3.6.7 Longitudinal Cracks in Cylindrical Rods ..... 138
3.7 Concluding Remarks ..... 141
References ..... 142
Exercise Problems ..... 143
4 Effect of Plasticity ..... 147
4.1 Introduction ..... 147
4.2 First Approximation on the Plastic Zone Size Estimation ..... 147
4.2.1 Evaluation of $r_{p}$ ..... 148
4.2.2 Evaluation of $\alpha r_{p}$ ..... 149
4.3 Determination of the Plastic Zone Shape in Front of the Crack Tip ..... 150
4.4 Plasticity Correction Factor ..... 155
4.5 Failure Modes under Plane Stress and Plane Strain Conditions ..... 157
4.5.1 Plane Stress Case ..... 157
4.5.2 Plane Strain Case ..... 158
4.6 Dugdale Model ..... 159
4.7 Crack Tip Opening Displacement ..... 161
4.8 Experimental Determination of $K_{c}$ ..... 164
4.8.1 Compact Tension Specimen ..... 164
4.8.1.1 Step 1: Crack Formation ..... 165
4.8.1.2 Step 2: Loading the Specimen. ..... 166
4.8.1.3 Step 3: Checking Crack Geometry in the Failed Specimen ..... 166
4.8.1.4 Step 4: Computation of Stress Intensity Factor at Failure ..... 167
4.8.1.5 Step 5: Final Check ..... 168
4.8.2 Three-Point Bend Specimen ..... 168
4.8.3 Practical Examples ..... 170
4.8.3.1 7075 Aluminum ..... 170
4.8.3.2 A533B Reactor Steel ..... 170
4.9 Concluding Remarks ..... 171
References ..... 172
Exercise Problems ..... 172
5 J-Integral ..... 175
5.1 Introduction ..... 175
5.2 Derivation of J-Integral ..... 175
5.3 J-Integral over a Closed Loop ..... 178
5.4 Path Independence of J-Integral ..... 180
5.5 J-Integral for Dugdale Model ..... 182
5.6 Experimental Evaluation of Critical J-Integral Value, $J_{c}$ ..... 183
5.7 Concluding Remarks ..... 187
References ..... 188
Exercise Problems ..... 188
6 Fatigue Crack Growth ..... 189
6.1 Introduction ..... 189
6.2 Fatigue Analysis-Mechanics of Materials Approach ..... 189
6.3 Fatigue Analysis-Fracture Mechanics Approach ..... 189
6.3.1 Numerical Example ..... 193
6.4 Fatigue Analysis for Materials Containing Microcracks ..... 193
6.5 Concluding Remarks ..... 195
References ..... 195
Exercise Problems ..... 195
7 Stress Intensity Factors for Some Practical Crack Geometries ..... 197
7.1 Introduction ..... 197
7.2 Slit Crack in a Strip ..... 197
7.3 Crack Intersecting a Free Surface ..... 199
7.4 Strip with a Crack on Its One Boundary ..... 200
7.5 Strip with Two Collinear Identical Cracks on Its Two Boundaries ..... 201
7.6 Two Half Planes Connected over a Finite Region Forming Two Semi-infinite Cracks in a Full Space ..... 202
7.7 Two Cracks Radiating Out from a Circular Hole ..... 203
7.8 Two Collinear Finite Cracks in an Infinite Plate ..... 204
7.9 Cracks with Two Opposing Concentrated Forces on the Surface ..... 206
7.10 Pressurized Crack ..... 206
7.11 Crack in a Wide Strip with a Concentrated Force at Its Midpoint and a Far Field Stress Balancing the Concentrated Force ..... 207
7.12 Circular or Penny-Shaped Crack in a Full Space ..... 209
7.13 Elliptical Crack in a Full Space ..... 212
7.13.1 Special Case 1—Circular Crack ..... 213
7.13.2 Special Case 2—Elliptical Crack with Very Large Major Axis ..... 214
7.13.3 SIF at the End of Major and Minor Axes of Elliptical Cracks ..... 214
7.14 Part-through Surface Crack ..... 214
7.14.1 First Approximation ..... 215
7.14.2 Front Face Correction Factor ..... 215
7.14.3 Plasticity Correction ..... 215
7.14.4 Back Face Correction Factor ..... 216
7.15 Corner Cracks ..... 216
7.15.1 Corner Cracks with Almost Equal Dimensions ..... 217
7.15.2 Corner Cracks at Two Edges of a Circular Hole ..... 218
7.15.3 Corner Crack at One Edge of a Circular Hole. ..... 218
7.16 Concluding Remarks ..... 219
References ..... 219
Exercise Problems ..... 220
8 Numerical Analysis ..... 221
8.1 Introduction ..... 221
8.2 Boundary Collocation Technique ..... 221
8.2.1 Circular Plate with a Radial Crack ..... 223
8.2.2 Rectangular Cracked Plate ..... 223
8.3 Conventional Finite Element Methods ..... 224
8.3.1 Stress and Displacement Matching ..... 224
8.3.2 Local Strain Energy Matching ..... 228
8.3.3 Strain Energy Release Rate ..... 229
8.3.4 J-Integral Method ..... 232
8.4 Special Crack Tip Finite Elements ..... 233
8.5 Quarter Point Quadrilateral Finite Element ..... 236
8.6 Concluding Remarks ..... 239
References ..... 239
9 Westergaard Stress Function ..... 241
9.1 Introduction ..... 241
9.2 Background Knowledge ..... 241
9.3 Griffith Crack in Biaxial State of Stress ..... 242
9.3.1 Stress and Displacement Fields in Terms of Westergaard Stress Function ..... 243
9.3.2 Westergaard Stress Function for the Griffith Crack under Biaxial Stress Field ..... 244
9.3.3 Stress Field Close to a Crack Tip ..... 250
9.4 Concentrated Load on a Half Space ..... 252
9.5 Griffith Crack Subjected to Concentrated Crack Opening Loads $P$ ..... 255
9.5.1 Stress Intensity Factor ..... 256
9.6 Griffith Crack Subjected to Nonuniform Internal Pressure ..... 257
9.7 Infinite Number of Equal Length, Equally Spaced Coplanar Cracks ..... 258
9.8 Concluding Remarks ..... 259
References ..... 259
Exercise Problems ..... 260
10 Advanced Topics ..... 261
10.1 Introduction ..... 261
10.2 Stress Singularities at Crack Corners. ..... 261
10.3 Fracture Toughness and Strength of Brittle Matrix Composites ..... 263
10.3.1 Experimental Observation of Strength Variations of FRBMCs with Various Fiber Parameters. ..... 265
10.3.2 Analysis for Predicting Strength Variations of FRBMCs with Various Fiber Parameters. ..... 267
10.3.2.1 Effect of Fiber Volume Fraction ..... 268
10.3.2.2 Effect of Fiber Length ..... 271
10.3.2.3 Effect of Fiber Diameter ..... 274
10.3.3 Effect on Stiffness ..... 276
10.3.4 Experimental Observation of Fracture Toughness Increase in FRBMCs with Fiber Addition ..... 276
10.4 Dynamic Effect. ..... 277
10.5 Concluding Remarks ..... 278
References ..... 278
Exercise Problems ..... 280
Index ..... 283

## Preface

My students motivated me to write this book. Every time I teach the course on fracture mechanics my students love it and ask me to write a book on this subject, stating that my class notes are much more organized and easy to understand than the available textbooks. They say I should simply put together my class notes in the same order I teach so that any entry level graduate student or senior undergraduate student can learn fracture mechanics through self-study. Because of their encouragement and enthusiasm, I have undertaken this project.

When I teach this course I start my lectures reviewing the fundamentals of continuum mechanics and the theory of elasticity relevant to fracture mechanics. Chapter 1 of the book does this. Students lacking a continuum mechanics background should first go through this chapter, solve the exercise problems, and then start reading the other chapters. The materials in this book have been carefully selected and only the topics important enough to be covered in the first course on fracture mechanics have been included. Except for the last chapter, no advanced topics have been covered in this book. Therefore, instructors of elementary fracture mechanics courses should have a much easier time covering the entire book in a three-unit graduate level course; they will not have to spend too much time picking and choosing appropriate topics for the course from the vast knowledge presented in most fracture mechanics books available today.

A professor who has never taught fracture mechanics can easily adopt this book as the official textbook for his or her course and simply follow the book chapters and sections in the same order in which they are presented. A number of exercise problems that can be assigned as homework problems or test problems are also provided. At the end of the semester, if time permits, the instructor can cover some advanced topics presented in the last chapter or topics of his or her interest related to fracture mechanics.

From over 20 years of my teaching experience I can state with confidence that if the course is taught in this manner, the students will love it. My teaching evaluation score in fracture mechanics has always been very high and often it was perfect when I taught the course in this manner. Since many students of different backgrounds over the last two decades have loved the organization of the fracture mechanics course presented in this book, I am confident that any professor who follows this book closely will be liked by his or her students.

The book is titled Fundamentals of Fracture Mechanics because only the essential topics of fracture mechanics are covered here. Because I was motivated by my students, my main objective in writing this book has been to
keep the materials and explanations very clear and simple for the benefit of students and first-time instructors. Almost all books on fracture mechanics available in the market today cover the majority of the topics presented in this book and often much more. These books are great as reference books but not necessarily as textbooks because the materials covered are not necessarily presented in the same order as most instructors present them in their lectures. Over half of the materials presented in any currently available fracture mechanics book is not covered in an introductory fracture mechanics course. For this reason, the course instructors always need to go through several fracture mechanics books' contents carefully and select appropriate topics to cover in their classes. It makes these books expensive and difficult for self-study. Often, instructors find that some important topics may be missing or explained in a complex manner in the fracture mechanics books currently available. For this reason, they are forced to follow several books in their course or provide supplementary class notes for clearer explanations of difficult topics. Fundamentals of Fracture Mechanics overcomes this shortcoming. Since it only covers the essential topics for an introductory fracture mechanics course, it is the right book for first-time learners, students, and instructors.

## The Author

Tribikram Kundu, Ph.D., is a professor in the Department of Civil Engineering and Engineering Mechanics and the Aerospace and Mechanical Engineering Department at the University of Arizona, Tucson. He is the winner of the Humboldt Research Prize (senior scientist award) and Humboldt Fellowship from Germany. He has been an invited professor in France, Sweden, Denmark, Russia, and Switzerland. Dr. Kundu is the editor of 14 books and 3 research monographs, as well as author or coauthor of 2 textbooks and over 200 scientific papers, 100 of which have been published in refereed scientific journals; 3 have received "best paper" awards. Among his noteworthy recognitions are receipt of the President of India gold medal for ranking first in his graduating class in IIT Kharagpur and the regents' fellowship and outstanding MS graduate award from UCLA. He is a fellow of ASME, ASCE, and SPIE.

## 1

## Fundamentals of the Theory of Elasticity

### 1.1 Introduction

It is necessary to have a good knowledge of the fundamentals of continuum mechanics and the theory of elasticity to understand fracture mechanics. This chapter is written with this in mind. The first part of the chapter (section 1.2) is devoted to the derivation of the basic equations of elasticity; in the second part (section 1.3), these basic equations are used to solve some classical boundary value problems of the theory of elasticity. It is very important to comprehend the first chapter fully before trying to understand the rest of the book.

### 1.2 Fundamentals of Continuum Mechanics and the Theory of Elasticity

Relations among the displacement, strain, and stress in an elastic body are derived in this section.

### 1.2.1 Deformation and Strain Tensor

Figure 1.1 shows the reference state R and the current deformed state D of a body in the Cartesian $x_{1} x_{2} x_{3}$ coordinate system. Deformation of the body and displacement of individual particles in the body are defined with respect to this reference state. As different points of the body move, due to applied force or change in temperature, the configuration of the body changes from the reference state to the current deformed state. After reaching equilibrium in one deformed state, if the applied force or temperature changes again, the deformed state also changes. The current deformed state of the body is the equilibrium position under current state of loads. Typically, the stressfree configuration of the body is considered as the reference state, but it is not necessary for the reference state to always be stress free. Any possible configuration of the body can be considered as the reference state. For simplicity, if it is not stated otherwise, the initial stress-free configuration of the body, before applying any external disturbance (force, temperature, etc.), will be considered as its reference state.


FIGURE 1.1
Deformation of a body: R is the reference state and D is the deformed state.

Consider two points $P$ and Q in the reference state of the body. They move to $\mathrm{P}^{*}$ and $\mathrm{Q}^{*}$ positions after deformation. Displacement of points P and Q is denoted by vectors $\mathbf{u}$ and $\mathbf{u}+\mathbf{d u}$, respectively. (Note: Here and in subsequent derivations, vector quantities will be denoted by boldface letters.) Position vectors of $P, Q, P^{*}$, and $Q^{*}$ are $\mathbf{r}, \mathbf{r}+\mathbf{d r}, \mathbf{r}^{*}$, and $\mathbf{r}^{*}+\mathbf{d r} \mathbf{r}^{*}$, respectively. Clearly, displacement and position vectors are related in the following manner:

$$
\begin{align*}
\mathbf{r}^{*} & =\mathbf{r}+\mathbf{u} \\
\mathbf{r}^{*}+\mathrm{dr}^{*} & =\mathbf{r}+\mathbf{d r}+\mathbf{u}+\mathbf{d u}  \tag{1.1}\\
\therefore \mathbf{d r}^{*} & =\mathrm{dr}+\mathbf{d u}
\end{align*}
$$

In terms of the three Cartesian components, the preceding equation can be written as:

$$
\begin{equation*}
\left(d x_{1}^{*} \mathbf{e}_{\mathbf{1}}+d x_{2}^{*} \mathbf{e}_{2}+d x_{3}^{*} \mathbf{e}_{3}\right)=\left(d x_{1} \mathbf{e}_{1}+d x_{2} \mathbf{e}_{2}+d x_{3} \mathbf{e}_{3}\right)+\left(d u_{1} \mathbf{e}_{1}+d u_{2} \mathbf{e}_{2}+d u_{3} \mathbf{e}_{3}\right) \tag{1.2}
\end{equation*}
$$

where $\mathbf{e}_{1}, \mathbf{e}_{2}$, and $\mathbf{e}_{3}$ are unit vectors in $x_{1}, x_{2}$, and $x_{3}$ directions, respectively.
In index or tensorial notation, equation (1.2) can be written as

$$
\begin{equation*}
d x_{i}^{*}=d x_{i}+d u_{i} \tag{1.3}
\end{equation*}
$$

where the free index $i$ can take values 1,2 , or 3 .
Applying the chain rule, equation (1.3) can be written as

$$
\begin{align*}
\quad d x_{i}^{*}=d x_{i}+\frac{\partial u_{i}}{\partial x_{1}} d x_{1}+\frac{\partial u_{i}}{\partial x_{2}} d x_{2}+\frac{\partial u_{i}}{\partial x_{3}} d x \\
\therefore d x_{i}^{*}=d x_{i}+\sum_{j=1}^{3} \frac{\partial u_{i}}{\partial x_{j}} d x_{j}=d x_{i}+u_{i, j} d x_{j} \tag{1.4}
\end{align*}
$$

In the preceding equation, the comma (,) means "derivative" and the summation convention (repeated dummy index means summation over 1,2, and 3) has been adopted.

Equation (1.4) can also be written in matrix notation in the following form:

$$
\left\{\begin{array}{l}
d x_{1}^{*}  \tag{1.5}\\
d x_{2}^{*} \\
d x_{3}^{*}
\end{array}\right\}=\left\{\begin{array}{l}
d x_{1} \\
d x_{2} \\
d x_{3}
\end{array}\right\}+\left[\begin{array}{lll}
\frac{\partial u_{1}}{\partial x_{1}} & \frac{\partial u_{1}}{\partial x_{2}} & \frac{\partial u_{1}}{\partial x_{3}} \\
\frac{\partial u_{2}}{\partial x_{1}} & \frac{\partial u_{2}}{\partial x_{2}} & \frac{\partial u_{2}}{\partial x_{3}} \\
\frac{\partial u_{3}}{\partial x_{1}} & \frac{\partial u_{3}}{\partial x_{2}} & \frac{\partial u_{3}}{\partial x_{3}}
\end{array}\right]\left\{\begin{array}{l}
d x_{1} \\
d x_{2} \\
d x_{3}
\end{array}\right\}
$$

In short form, equation (1.5) can be written as

$$
\begin{equation*}
\left\{\mathbf{d r}^{*}\right\}=\{\mathbf{d r}\}+[\nabla \mathbf{u}]^{\mathrm{T}}\{\mathbf{d r}\} \tag{1.6}
\end{equation*}
$$

If one defines

$$
\begin{equation*}
\varepsilon_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right) \tag{1.7a}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega_{i j}=\frac{1}{2}\left(u_{i, j}-u_{j, i}\right) \tag{1.7b}
\end{equation*}
$$

then equation (1.6) takes the following form:

$$
\begin{equation*}
\left\{\mathbf{d r}^{*}\right\}=\{\mathbf{d r}\}+[\varepsilon]\{\mathbf{d r}\}+[\omega]\{\mathbf{d} \mathbf{r}\} \tag{1.7c}
\end{equation*}
$$

### 1.2.1.1 Interpretation of $\varepsilon_{i j}$ and $\omega_{i j}$ for Small Displacement Gradient

Consider the special case when $\mathbf{d r}=d x_{1} \mathbf{e}_{1}$. Then, after deformation, three components of $\mathbf{d} \mathbf{r}^{*}$ can be computed from equation (1.5):

$$
\begin{align*}
& d x_{1}^{*}=d x_{1}+\frac{\partial u_{1}}{\partial x_{1}} d x_{1}=\left(1+\varepsilon_{11}\right) d x_{1} \\
& d x_{2}^{*}=\frac{\partial u_{2}}{\partial x_{1}} d x_{1}=\left(\varepsilon_{21}+\omega_{21}\right) d x_{1}  \tag{1.8}\\
& d x_{3}^{*}=\frac{\partial u_{3}}{\partial x_{1}} d x_{1}=\left(\varepsilon_{31}+\omega_{31}\right) d x_{1}
\end{align*}
$$

In this case, the initial length of the element PQ is $d S=d x_{1}$, and the final length of the element $\mathrm{P}^{*} \mathrm{Q}^{*}$ after deformation is

$$
\begin{align*}
d S^{*} & =\left[\left(d x_{1}^{*}\right)^{2}+\left(d x_{2}^{*}\right)^{2}+\left(d x_{3}^{*}\right)^{2}\right]^{\frac{1}{2}}=d x_{1}\left[\left(1+\varepsilon_{11}\right)^{2}+\left(\varepsilon_{21}+\omega_{21}\right)^{2}+\left(\varepsilon_{31}+\omega_{31}\right)^{2}\right]^{\frac{1}{2}}  \tag{1.9}\\
& \approx d x_{1}\left[1+2 \varepsilon_{11}\right]^{\frac{1}{2}}=d x_{1}\left(1+\varepsilon_{11}\right)
\end{align*}
$$

In equation (1.9) we have assumed that the displacement gradients $u_{i, j}$ are small. Hence, $\varepsilon_{i j}$ and $\omega_{i j}$ are small. Therefore, the second-order terms involving $\varepsilon_{i j}$ and $\omega_{i j}$ can be ignored.
From its definition, engineering normal strain $\left(\mathrm{E}_{11}\right)$ in $x_{1}$ direction can be written as

$$
\begin{equation*}
\mathrm{E}_{11}=\frac{d S^{*}-d S}{d S}=\frac{d x_{1}\left(1+\varepsilon_{11}\right)-d x_{1}}{d x_{1}}=\varepsilon_{11} \tag{1.10}
\end{equation*}
$$

Similarly one can show that $\varepsilon_{22}$ and $\varepsilon_{33}$ are engineering normal strains in $x_{2}$ and $x_{3}$ directions, respectively.
To interpret $\varepsilon_{12}$ and $\omega_{12}$, consider two mutually perpendicular elements PQ and PR in the reference state. In the deformed state these elements are moved to $P^{*} Q^{*}$ and $P^{*} R^{*}$ positions, respectively, as shown in Figure 1.2.
Let the vectors PQ and PR be $(\mathbf{d r})_{\mathrm{PQ}}=d x_{1} \mathbf{e}_{1}$ and $(\mathbf{d r})_{\mathrm{PR}}=d x_{2} \mathbf{e}_{2}$, respectively. Then, after deformation, three components of $\left(\mathbf{d r}^{*}\right)_{\mathrm{PQ}}$ and $\left(\mathbf{d r}^{*}\right)_{\mathrm{PR}}$ can be written in the forms of equations (1.11) and (1.12), respectively:

$$
\begin{align*}
& \left(d x_{1}^{*}\right)_{P Q}=d x_{1}+\frac{\partial u_{1}}{\partial x_{1}} d x_{1}=\left(1+\varepsilon_{11}\right) d x_{1} \\
& \left(d x_{2}^{*}\right)_{P Q}=\frac{\partial u_{2}}{\partial x_{1}} d x_{1}=\left(\varepsilon_{21}+\omega_{21}\right) d x_{1}  \tag{1.11}\\
& \left(d x_{3}^{*}\right)_{P Q}=\frac{\partial u_{3}}{\partial x_{1}} d x_{1}=\left(\varepsilon_{31}+\omega_{31}\right) d x_{1}
\end{align*}
$$



## FIGURE 1.2

Two elements, PQ and PR, that are mutually perpendicular before deformation are no longer perpendicular after deformation.

$$
\begin{align*}
& \left(d x_{1}^{*}\right)_{P R}=\frac{\partial u_{1}}{\partial x_{2}} d x_{2}=\left(\varepsilon_{12}+\omega_{12}\right) d x_{1} \\
& \left(d x_{2}^{*}\right)_{P R}=\frac{\partial u_{2}}{\partial x_{2}} d x_{2}=\left(1+\varepsilon_{22}\right) d x_{2}  \tag{1.12}\\
& \left(d x_{3}^{*}\right)_{P R}=\frac{\partial u_{3}}{\partial x_{2}} d x_{2}=\left(\varepsilon_{32}+\omega_{32}\right) d x_{1}
\end{align*}
$$

Let $\alpha_{1}$ be the angle between $\mathrm{P}^{*} \mathrm{Q}^{*}$ and the horizontal axis, and $\alpha_{2}$ the angle between $\mathrm{P}^{*} \mathrm{R}^{*}$ and the vertical axis as shown in Figure 1.2. Note that $\alpha+\alpha_{1}+$ $\alpha_{2}=90^{\circ}$. From equations (1.11) and (1.12), one can show that

$$
\begin{align*}
& \tan \alpha_{1}=\frac{\left(\varepsilon_{21}+\omega_{21}\right) d x_{1}}{\left(1+\varepsilon_{11}\right) d x_{1}} \approx \varepsilon_{21}+\omega_{21}=\varepsilon_{12}+\omega_{21} \\
& \tan \alpha_{2}=\frac{\left(\varepsilon_{12}+\omega_{12}\right) d x_{2}}{\left(1+\varepsilon_{22}\right) d x_{2}} \approx \varepsilon_{12}-\omega_{21} \tag{1.13}
\end{align*}
$$

In the preceding equation, we have assumed a small displacement gradient and therefore $1+\varepsilon_{i j} \approx 1$. For a small displacement gradient, $\tan \alpha_{i} \approx \alpha_{i}$ and one can write:

$$
\begin{align*}
\alpha_{1} & =\varepsilon_{12}+\omega_{21} \\
\alpha_{2} & =\varepsilon_{12}-\omega_{21}  \tag{1.14}\\
\therefore \varepsilon_{12} & =\frac{1}{2}\left(\alpha_{1}+\alpha_{2}\right) \quad \& \quad \omega_{21}=\frac{1}{2}\left(\alpha_{1}-\alpha_{2}\right)
\end{align*}
$$

From equation (1.14) it is concluded that $2 \varepsilon_{12}$ is the change in the angle between the elements PQ and PR after deformation. In other words, it is the engineering shear strain and $\omega_{21}$ is the rotation of the diagonal PS (see Figure 1.2) or the average rotation of the rectangular element PQSR about the $x_{3}$ axis after deformation.

In summary, $\varepsilon_{i j}$ and $\omega_{i j}$ are strain tensor and rotation tensor, respectively, for small displacement gradients.

## Example 1.1

Prove that the strain tensor satisfies the relation $\varepsilon_{i j, k \ell}+\varepsilon_{k \ell, i j}=\varepsilon_{i k, j \ell}+\varepsilon_{j \ell, i k}$. This relation is known as the compatibility condition.

## Solution

$$
\begin{aligned}
& \text { Left-hand side }=\varepsilon_{i j, k \ell}+\varepsilon_{k \ell, i j}=\frac{1}{2}\left(u_{i, j k \ell}+u_{j, i k \ell}+u_{k, \ell i j}+u_{\ell, k i j}\right) \\
& \text { Right-hand side }=\varepsilon_{i k, j \ell}+\varepsilon_{j \ell, i k}=\frac{1}{2}\left(u_{i, k j \ell}+u_{k, i j \ell}+u_{j, \ell i k}+u_{\ell, j k}\right)
\end{aligned}
$$

Since the sequence of derivative should not make any difference, $\mathrm{u}_{i, j k \ell}=$ $u_{i, k j j}$ similarly, the other three terms in the two expressions can be shown as equal. Thus, the two sides of the equation are proved to be identical.

## Example 1.2

Check if the following strain state is possible for an elasticity problem:

$$
\varepsilon_{11}=k\left(x_{1}^{2}+x_{2}^{2}\right), \quad \varepsilon_{22}=k\left(x_{2}^{2}+x_{3}^{2}\right), \quad \varepsilon_{12}=k x_{1} x_{2} x_{3}, \quad \varepsilon_{13}=\varepsilon_{23}=\varepsilon_{33}=0
$$

## Solution

From the compatibility condition, $\varepsilon_{i j, k \ell}+\varepsilon_{k \ell, i j}=\varepsilon_{i k, j \ell}+\varepsilon_{j e, i k}$, given in example
1.1, one can write

$$
\begin{gathered}
\varepsilon_{11,22}+\varepsilon_{22,11}=2 \varepsilon_{12,12} \text { by substituting } i=1, j=1, k=2, \ell=2 . \\
\qquad \varepsilon_{11,22}+\varepsilon_{22,11}=2 k+0=2 k \\
2 \varepsilon_{12,12}=2 k x_{3}
\end{gathered}
$$

Since the two sides of the compatibility equation are not equal, the given strain state is not a possible strain state.

### 1.2.2 Traction and Stress Tensor

Force per unit area on a surface is called traction. To define traction at a point $P$ (see Figure 1.3), one needs to state on which surface, going through that point, the traction is defined. The traction value at point $P$ changes if the orientation of the surface on which the traction is defined is changed.

Figure 1.3 shows a body in equilibrium under the action of some external forces. If it is cut into two halves by a plane going through point $P$, in general, to keep each half of the body in equilibrium, some force will exist at the cut plane. Force per unit area in the neighborhood of point $P$ is defined as the traction at point $P$. If the cut plane is changed, then the traction at the same point will change. Therefore, to define traction at a point, its three components must be given and the plane on which it is defined must be identified. Thus, the traction can be denoted as $\mathbf{T}^{(\mathbf{n})}$, where the superscript $\mathbf{n}$ denotes the unit


FIGURE 1.3
A body in equilibrium can be cut into two halves by an infinite number of planes going through a specific point $P$. Two such planes are shown in the figure.


FIGURE 1.4
Traction $\mathbf{T}^{(n)}$ on an inclined plane can be decomposed into its three components, $T_{n i}$, or into two components: normal and shear stress components ( $\sigma_{n n}$ and $\sigma_{n s}$ ).
vector normal to the plane on which the traction is defined and where $\mathrm{T}^{(\mathrm{n})}$ has three components that correspond to the force per unit area in $x_{1}, x_{2}$, and $x_{3}$ directions, respectively.

Stress is similar to traction; both are defined as force per unit area. The only difference is that the stress components are always defined normal or parallel to a surface, while traction components are not necessarily normal or parallel to the surface. A traction $\mathbf{T}^{(n)}$ on an inclined plane is shown in Figure 1.4. Note that neither $\mathbf{T}^{(\mathbf{n})}$ nor its three components $T_{n i}$ are necessarily normal or parallel to the inclined surface. However, its two components $\sigma_{n n}$ and $\sigma_{n s}$ are perpendicular and parallel to the inclined surface and are called normal and shear stress components, respectively.

Stress components are described by two subscripts. The first subscript indicates the plane (or normal to the plane) on which the stress component is defined and the second subscript indicates the direction of the force per unit area or stress value. Following this convention, different stress components in the $x_{1} x_{2} x_{3}$ coordinate system are defined in Figure 1.5.


FIGURE 1.5
Different stress components in the $x_{1} x_{2} x_{3}$ coordinate system.

Note that on each of the six planes (i.e., the positive and negative $x_{1}, x_{2}$, and $x_{3}$ planes), three stress components (one normal and two shear stress components) are defined. If the outward normal to the plane is in the positive direction, then we call the plane a positive plane; otherwise, it is a negative plane. If the force direction is positive on a positive plane or negative on a negative plane, then the stress is positive. All stress components shown on positive $x_{1}, x_{2}$, and $x_{3}$ planes and negative $x_{1}$ plane in Figure 1.5 are positive stress components. Stress components on the other two negative planes are not shown to keep the Figure simple. Dashed arrows show three of the stress components on the negative $x_{1}$ plane while solid arrows show the stress components on positive planes. If the force direction and the plane direction have different signs, one positive and one negative, then the corresponding stress component is negative. Therefore, in Figure 1.5, if we change the direction of the arrow of any stress component, then that stress component becomes negative.

### 1.2.3 Traction-Stress Relation

Let us take a tetrahedron OABC from a continuum body in equilibrium (see Figure 1.6). Forces (per unit area) acting in the $x_{1}$ direction on the four surfaces of OABC are shown in Figure 1.6. From its equilibrium in the $x_{1}$ direction one can write

$$
\begin{equation*}
\sum F_{1}=T_{n 1} A-\sigma_{11} A_{1}-\sigma_{21} A_{2}-\sigma_{31} A_{3}+f_{1} V=0 \tag{1.15}
\end{equation*}
$$

where $A$ is the area of the surface $\mathrm{ABC} ; A_{1}, A_{2}$, and $A_{3}$ are the areas of the other three surfaces $\mathrm{OBC}, \mathrm{OAC}$, and OAB , respectively; and $f_{1}$ is the body force per unit volume in the $x_{1}$ direction.


FIGURE 1.6
A tetrahedron showing traction components on plane ABC and $x_{1}$ direction stress components on planes $\mathrm{AOC}, \mathrm{BOC}$, and AOB .

If $n_{j}$ is the $j$ th component of the unit vector $\mathbf{n}$ that is normal to the plane ABC , then one can write $A_{j}=n_{j} A$ and $V=(A h) / 3$, where $h$ is the height of the tetrahedron measured from the apex $O$. Thus, equation (1.15) is simplified to

$$
\begin{equation*}
T_{n 1}-\sigma_{11} n_{1}-\sigma_{21} n_{2}-\sigma_{31} n_{3}+f_{1} \frac{h}{3}=0 \tag{1.16}
\end{equation*}
$$

In the limiting case when the plane $A B C$ passes through point $O$, the tetrahedron height $h$ vanishes and equation (1.16) is simplified to

$$
\begin{equation*}
T_{n 1}=\sigma_{11} n_{1}+\sigma_{21} n_{2}+\sigma_{31} n_{3}=\sigma_{j 1} n_{j} \tag{1.17}
\end{equation*}
$$

In this equation the summation convention (repeated index means summation) has been used.
Similarly, from the force equilibrium in $x_{2}$ and $x_{3}$ directions, one can write

$$
\begin{align*}
& T_{n 2}=\sigma_{j 2} n_{j}  \tag{1.18}\\
& T_{n 3}=\sigma_{j 3} n_{j}
\end{align*}
$$

Combining equations (1.17) and (1.18), the traction-stress relation is obtained in index notation:

$$
\begin{equation*}
T_{n i}=\sigma_{j i} n_{j} \tag{1.19}
\end{equation*}
$$

where the free index $i$ takes values 1,2, and 3 to generate three equations and the dummy index $j$ takes values 1,2 , and 3 and is added in each equation.

For simplicity, the subscript $n$ of $T_{n i}$ is omitted and $T_{n i}$ is written as $T_{i}$. It is implied that the unit normal vector to the surface on which the traction is defined is $\mathbf{n}$. With this change, equation (1.19) is simplified to

$$
\begin{equation*}
T_{i}=\sigma_{j i} n_{j} \tag{1.19a}
\end{equation*}
$$

### 1.2.4 Equilibrium Equations

If a body is in equilibrium, then the resultant force and moment on that body must be equal to zero.

### 1.2.4.1 Force Equilibrium

The resultant forces in the $x_{1}, x_{2}$, and $x_{3}$ directions are equated to zero to obtain the governing equilibrium equations. First, $x_{1}$ direction equilibrium is studied. Figure 1.7 shows all forces acting in the $x_{1}$ direction on an elemental volume.


FIGURE 1.7
Forces acting in the $x_{1}$ direction on an elemental volume.

Thus, the zero resultant force in the $x_{1}$ direction gives

$$
\begin{aligned}
& -\sigma_{11} d x_{2} d x_{3}+\left(\sigma_{11}+\frac{\partial \sigma_{11}}{\partial x_{1}} d x_{1}\right) d x_{2} d x_{3}-\sigma_{21} d x_{1} d x_{3}+\left(\sigma_{21}+\frac{\partial \sigma_{21}}{\partial x_{2}} d x_{2}\right) d x_{1} d x_{3} \\
& -\sigma_{31} d x_{2} d x_{1}+\left(\sigma_{31}+\frac{\partial \sigma_{31}}{\partial x_{3}} d x_{3}\right) d x_{1} d x_{2}+f_{1} d x_{1} d x_{2} d x_{3}=0
\end{aligned}
$$

or

$$
\left(\frac{\partial \sigma_{11}}{\partial x_{1}} d x_{1}\right) d x_{2} d x_{3}+\left(\frac{\partial \sigma_{21}}{\partial x_{2}} d x_{2}\right) d x_{1} d x_{3}+\left(\frac{\partial \sigma_{31}}{\partial x_{3}} d x_{3}\right) d x_{1} d x_{2}+f_{1} d x_{1} d x_{2} d x_{3}=0
$$

or

$$
\frac{\partial \sigma_{11}}{\partial x_{1}}+\frac{\partial \sigma_{21}}{\partial x_{2}}+\frac{\partial \sigma_{31}}{\partial x_{3}}+f_{1}=0
$$

or

$$
\begin{equation*}
\frac{\partial \sigma_{j 1}}{\partial x_{j}}+f_{1}=0 \tag{1.20}
\end{equation*}
$$

In equation (1.20) repeated index $j$ indicates summation.
Similarly, equilibrium in $x_{2}$ and $x_{3}$ directions gives

$$
\begin{align*}
& \frac{\partial \sigma_{j 2}}{\partial x_{j}}+f_{2}=0 \\
& \frac{\partial \sigma_{j 3}}{\partial x_{j}}+f_{3}=0 \tag{1.21}
\end{align*}
$$

The three equations in (1.20) and (1.21) can be combined in the following form:

$$
\begin{equation*}
\frac{\partial \sigma_{j i}}{\partial x_{j}}+f_{i}=\sigma_{j i, j}+f_{i}=0 \tag{1.22}
\end{equation*}
$$

The force equilibrium equations given in equation (1.22) are written in index notation, where the free index $i$ takes three values-1,2, and 3-and corresponds to three equilibrium equations, and the comma (,) indicates derivative.

### 1.2.4.2 Moment Equilibrium

Let us now compute the resultant moment in the $x_{3}$ direction (or, in other words, moment about the $x_{3}$ axis) for the elemental volume shown in Figure 1.8.
If we calculate the moment about an axis parallel to the $x_{3}$ axis and passing through the centroid of the elemental volume shown in Figure 1.8, then only four shear stresses shown on the four sides of the volume can produce moment. Body forces in $x_{1}$ and $x_{2}$ directions do not produce any moment because the resultant body force passes through the centroid of the volume. Since the resultant moment about this axis should be zero, one can write

$$
\begin{aligned}
& \left(\sigma_{12}+\frac{\partial \sigma_{12}}{\partial x_{1}} d x_{1}\right) d x_{2} d x_{3} \frac{d x_{1}}{2}+\left(\sigma_{12}\right) d x_{2} d x_{3} \frac{d x_{1}}{2}-\left(\sigma_{21}+\frac{\partial \sigma_{21}}{\partial x_{2}} d x_{2}\right) d x_{1} d x_{3} \frac{d x_{2}}{2} \\
& \quad-\left(\sigma_{12}\right) d x_{1} d x_{3} \frac{d x_{2}}{2}=0
\end{aligned}
$$



FIGURE 1.8
Forces on an element that may contribute to the moment in the $x_{3}$ direction.

Ignoring the higher order terms, one gets

$$
2\left(\sigma_{21}\right) d x_{2} d x_{3} \frac{d x_{1}}{2}-2\left(\sigma_{21}\right) d x_{1} d x_{3} \frac{d x_{2}}{2}=0
$$

or $\sigma_{12}=\sigma_{21}$.
Similarly, applying moment equilibrium about the other two axes, one can show that $\sigma_{13}=\sigma_{31}$ and $\sigma_{32}=\sigma_{23}$. In index notation,

$$
\begin{equation*}
\sigma_{i j}=\sigma_{j i} \tag{1.23}
\end{equation*}
$$

Thus, the stress tensor is symmetric. It should be noted here that if the body has internal body couple (or body moment per unit volume), then the stress tensor will not be symmetric.

Because of the symmetry of the stress tensor, equations (1.19a) and (1.22) can be written in the following form as well:

$$
\begin{align*}
T_{i} & =\sigma_{i j} n_{j}  \tag{1.24}\\
\sigma_{i j, j}+f_{i} & =0
\end{align*}
$$

### 1.2.5 Stress Transformation

Let us now investigate how the stress components in two Cartesian coordinate systems are related.

Figure 1.9 shows an inclined plane ABC whose normal is in the $x_{1^{\prime}}$ direction; thus, the $x_{2^{\prime}} x_{3^{\prime}}$ plane is parallel to the ABC plane. Traction $\mathbf{T}^{\left(1^{\prime}\right)}$ is acting on this plane. Three components of this traction in $x_{1^{\prime}}, x_{2^{\prime}}$ and $x_{3^{\prime}}$ directions are the three stress components $\sigma_{1^{\prime} \prime \prime}^{\prime \prime} \sigma_{1^{\prime} 2^{\prime}}$, and $\sigma_{1^{\prime} 3^{\prime \prime}}$ respectively. Note that the


FIGURE 1.9
Stress components in $x_{1^{\prime}} x_{2^{\prime}} x_{3^{\prime}}$ coordinate system.
first subscript indicates the plane on which the stress is acting and the second subscript gives the stress direction.

From equation (1.19) one can write

$$
\begin{equation*}
T_{1^{\prime} i}=\sigma_{j i} n_{j}^{\left(1^{\prime}\right)}=\sigma_{j i} \ell_{1^{\prime} j} \tag{1.25}
\end{equation*}
$$

where $n_{j}^{\left(1^{\prime}\right)}=\ell_{1^{\prime} j}$ is the $j$ th component of the unit normal vector on plane ABC or, in other words, the direction cosines of the $x_{1^{\prime}}$ axis.

Note that the dot product between $\mathbf{T}^{\left(1^{\prime}\right)}$ and the unit vector $\mathbf{n}^{\left(1^{1}\right)}$ gives the stress component $\sigma_{1^{\prime} 1^{\prime}}$; therefore,

$$
\begin{equation*}
\sigma_{1^{\prime} 1^{\prime}}=T_{1^{\prime} i} \ell_{1^{\prime} i}=\sigma_{j i} \ell_{1^{\prime} j} \ell_{1^{\prime} i} \tag{1.26}
\end{equation*}
$$

Similarly, the dot product between $\mathbf{T}^{\left(1^{1}\right)}$ and the unit vector $\mathbf{n}^{\left(\mathbf{2}^{2}\right)}$ gives $\sigma_{1^{\prime} 2^{\prime}}$ and the dot product between $\mathbf{T}^{\left(1^{1}\right)}$ and the unit vector $\mathbf{n}^{\left(3^{\prime}\right)}$ gives $\sigma_{1^{\prime} 3^{\prime}}$. Thus, we get

$$
\begin{align*}
& \sigma_{1^{\prime} 2^{\prime}}=T_{1^{\prime} i} \ell_{2^{\prime} i}=\sigma_{j i} \ell_{1^{\prime} j} \ell_{2^{\prime} i} \\
& \sigma_{1^{\prime} 3^{\prime}}=T_{1^{\prime} i} \ell_{3^{\prime} i}=\sigma_{j i} \ell_{1^{\prime} j} \ell_{3^{\prime} i} \tag{1.27}
\end{align*}
$$

Equations (1.26) and (1.27) can be written in index notation in the following form:

$$
\begin{equation*}
\sigma_{1^{\prime} m^{\prime}}=\ell_{1^{\prime} j} \sigma_{j i} \ell_{m^{\prime} i} \tag{1.28}
\end{equation*}
$$

In this equation, the free index $m^{\prime}$ can take values $1^{\prime}, 2^{\prime}$, or $3^{\prime}$.
Similarly, from the traction vector $\mathbf{T}^{\left(2^{\prime}\right)}$ on a plane whose normal is in the $x_{2^{\prime}}$ direction, one can show that

$$
\begin{equation*}
\sigma_{2^{\prime} m^{\prime}}=\ell_{2^{\prime} j} \sigma_{j i} \ell_{m^{\prime} i} \tag{1.29}
\end{equation*}
$$

From the traction vector $\mathbf{T}^{\left(3^{\prime}\right)}$ on the $x_{3^{\prime}}$ plane, one can derive

$$
\begin{equation*}
\sigma_{3^{\prime} m^{\prime}}=\ell_{3^{\prime} j} \sigma_{j i} \ell_{m^{\prime} i} \tag{1.30}
\end{equation*}
$$

Equations (1.28) to (1.30) can be combined to obtain the following equation in index notation:

$$
\sigma_{n^{\prime} m^{\prime}}=\ell_{n^{\prime} j} \sigma_{j i} \ell_{m^{\prime} i}
$$

Note that in the preceding equation, $i, j, m^{\prime}$, and $n^{\prime}$ are all dummy indices and can be interchanged to obtain

$$
\begin{equation*}
\sigma_{m^{\prime} n^{\prime}}=\ell_{m^{\prime} i} \sigma_{i j} \ell_{n^{\prime} j}=\ell_{m^{\prime} i} \ell_{n^{\prime} j} \sigma_{i j} \tag{1.31}
\end{equation*}
$$

### 1.2.5.1 Kronecker Delta Symbol ( $\delta_{i j}$ ) and Permutation Symbol ( $\varepsilon_{i j k}$ )

In index notation the Kronecker delta symbol $\left(\delta_{i j}\right)$ and permutation symbol ( $\varepsilon_{i j k}$, also known as the Levi-Civita symbol and alternating symbol) are often used. They are defined in the following manner:

$$
\begin{array}{ll}
\delta_{i j}=1 & \text { for } i=j \\
\delta_{i j}=0 & \text { for } i \neq j
\end{array}
$$

and
$\varepsilon_{i j k}=1$ for $i, j, k$ having values 1,2 , and 3 ; or 2,3 , and 1 ; or 3,1 , and 2 .
$\varepsilon_{i j k}=-1$ for $i, j, k$ having values 3,2 , and 1 ; or 1,3 , and 2 ; or 2,1 , and 3 .
$\varepsilon_{i j k}=0$ for $i, j, k$ not having three distinct values.

### 1.2.5.2 Examples of the Application of $\delta_{i j}$ and $\varepsilon_{i j k}$

Note that

$$
\begin{gathered}
\frac{\partial x_{i}}{\partial x_{j}}=\delta_{i j} ; \quad \mathbf{e}_{\mathbf{i}} \cdot \mathbf{e}_{\mathbf{j}}=\delta_{i j} \\
\operatorname{Det}\left|\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right|=\varepsilon_{i j k} a_{1 i} a_{2 j} a_{3 k} ; \quad \mathbf{b} \times \mathbf{c}=\varepsilon_{i j k} b_{j} c_{k} \mathbf{e}_{\mathbf{i}}
\end{gathered}
$$

where $\mathbf{e}_{\mathbf{i}}$ and $\mathbf{e}_{\mathrm{j}}$ are unit vectors in $x_{i}$ and $x_{j}$ directions, respectively, in the $x_{1} x_{2} x_{3}$ coordinate system. Also note that $\mathbf{b}$ and $\mathbf{c}$ are two vectors, while [ $a$ ] is a matrix.

One can prove that the following relation exists between these two symbols:

$$
\varepsilon_{i j k} \varepsilon_{i m n}=\delta_{j m} \delta_{k n}-\delta_{j n} \delta_{k m}
$$

## Example 1.3

Starting from the stress transformation law, prove that $\sigma_{m^{\prime} n^{\prime}} \sigma_{m^{\prime} n^{\prime}}=\sigma_{i j} \sigma_{i j}$ where $\sigma_{m^{\prime} n^{\prime}}$ and $\sigma_{i j}$ are stress tensors in two different Cartesian coordinate systems.

Solution

$$
\begin{aligned}
\sigma_{m^{\prime} n^{\prime}} \sigma_{m^{\prime} n^{\prime}} & =\left(\ell_{m^{\prime} i} \ell_{n^{\prime} j} \sigma_{i j}\right)\left(\ell_{m^{\prime} p} \ell_{n^{\prime} q} \sigma_{p q}\right)=\left(\ell_{m^{\prime} i} \ell_{n^{\prime} j}\right)\left(\ell_{m^{\prime} p} \ell_{n^{\prime} q}\right) \sigma_{i j} \sigma_{p q} \\
& \left(\ell_{m^{\prime} i} \ell_{m^{\prime} p}\right)\left(\ell_{n^{\prime} j} \ell_{n^{\prime} q}\right) \sigma_{i j} \sigma_{p q}=\delta_{i p} \delta_{j q} \sigma_{i j} \sigma_{p q}=\sigma_{i j} \sigma_{i j}
\end{aligned}
$$

### 1.2.6 Definition of Tensor

A Cartesian tensor of order (or rank) $r$ in $n$ dimensional space is a set of $n^{r}$ numbers (called the elements or components of tensor) that obey the following transformation law between two coordinate systems:

$$
\begin{equation*}
t_{m^{\prime} n^{\prime} p^{\prime} q^{\prime} \ldots . .}=\left(\ell_{m^{\prime} i} \ell_{n^{\prime} j} \ell_{p^{\prime} k} \ell_{q^{\prime} \ell} \ldots\right)\left(t_{i j k \ell \ldots}\right) \tag{1.32}
\end{equation*}
$$

where $t_{m^{\prime} n^{\prime} p^{\prime} q^{\prime} \ldots}$ and $t_{i j k . . . .}$ each has $r$ number of subscripts; $r$ number of direction cosines $\left(\ell_{m^{\prime} i} \ell_{n^{\prime} j} \ell_{p^{\prime} k} \ell_{q^{\prime} \ell} \ldots\right.$ ) are multiplied on the right-hand side. Comparing equation (1.31) with the definition of tensor transformation equation (1.32), one can conclude that the stress is a second-rank tensor.

### 1.2.7 Principal Stresses and Principal Planes

Planes on which the traction vectors are normal are called principal planes. Shear stress components on the principal planes are equal to zero. Normal stresses on the principal planes are called principal stresses.

In Figure 1.10, let $\mathbf{n}$ be the unit normal vector on the principal plane $A B C$ and $\lambda$ the principal stress value on this plane. Therefore, the traction vector on plane $A B C$ can be written as

$$
T_{i}=\lambda n_{i}
$$

Again, from equation (1.24),

$$
T_{i}=\sigma_{i j} n_{j}
$$

From the preceding two equations, one can write

$$
\begin{equation*}
\sigma_{i j} n_{j}-\lambda n_{i}=0 \tag{1.33}
\end{equation*}
$$



FIGURE 1.10
Principal stress $\lambda$ on the principal plane ABC.

