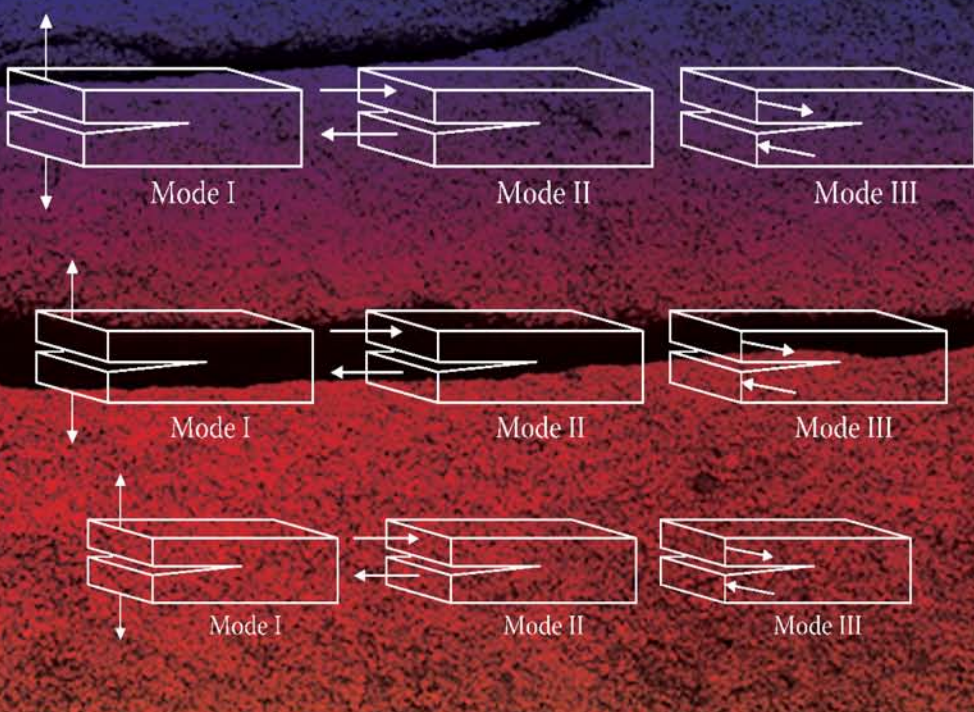
FUNDAMENTALS OF FRACTURE MECHANICS

TRIBIKRAM KUNDU



CRC Press
Taylor & Francis Group

FUNDAMENTALS OF FRACTURE MECHANICS

FUNDAMENTALS OF FRACTURE MECHANICS

TRIBIKRAM KUNDU

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-0-8493-8432-5 (Hardcover)

Library of Congress Cataloging-in-Publication Data

Kundu, T. (Tribikram)
Fundamentals of fracture mechanics / Tribikram Kundu.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-8493-8432-5 (alk. paper)
1. Fracture mechanics. I. Title.

TA409.K86 2008
620.1'126--dc22 2007038845

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Dedication

To my wife, Nupur, our daughters, Auni and Ina,

and our parents, Makhan Lal Kundu, Sandhya Rani Kundu,

Jyotirmoy Naha, and Rubi Naha

Contents

Preface

My students motivated me to write this book. Every time I teach the course on fracture mechanics my students love it and ask me to write a book on this subject, stating that my class notes are much more organized and easy to understand than the available textbooks. They say I should simply put together my class notes in the same order I teach so that any entry level graduate student or senior undergraduate student can learn fracture mechanics through self-study. Because of their encouragement and enthusiasm, I have undertaken this project.

When I teach this course I start my lectures reviewing the fundamentals of continuum mechanics and the theory of elasticity relevant to fracture mechanics. Chapter 1 of the book does this. Students lacking a continuum mechanics background should first go through this chapter, solve the exercise problems, and then start reading the other chapters. The materials in this book have been carefully selected and only the topics important enough to be covered in the first course on fracture mechanics have been included. Except for the last chapter, no advanced topics have been covered in this book. Therefore, instructors of elementary fracture mechanics courses should have a much easier time covering the entire book in a three-unit graduate level course; they will not have to spend too much time picking and choosing appropriate topics for the course from the vast knowledge presented in most fracture mechanics books available today.

A professor who has never taught fracture mechanics can easily adopt this book as the official textbook for his or her course and simply follow the book chapters and sections in the same order in which they are presented. A number of exercise problems that can be assigned as homework problems or test problems are also provided. At the end of the semester, if time permits, the instructor can cover some advanced topics presented in the last chapter or topics of his or her interest related to fracture mechanics.

From over 20 years of my teaching experience I can state with confidence that if the course is taught in this manner, the students will love it. My teaching evaluation score in fracture mechanics has always been very high and often it was perfect when I taught the course in this manner. Since many students of different backgrounds over the last two decades have loved the organization of the fracture mechanics course presented in this book, I am confident that any professor who follows this book closely will be liked by his or her students.

The book is titled *Fundamentals of Fracture Mechanics* because only the essential topics of fracture mechanics are covered here. Because I was motivated by my students, my main objective in writing this book has been to keep the materials and explanations very clear and simple for the benefit of students and first-time instructors. Almost all books on fracture mechanics available in the market today cover the majority of the topics presented in this book and often much more. These books are great as reference books but not necessarily as textbooks because the materials covered are not necessarily presented in the same order as most instructors present them in their lectures. Over half of the materials presented in any currently available fracture mechanics book is not covered in an introductory fracture mechanics course. For this reason, the course instructors always need to go through several fracture mechanics books' contents carefully and select appropriate topics to cover in their classes. It makes these books expensive and difficult for self-study. Often, instructors find that some important topics may be missing or explained in a complex manner in the fracture mechanics books currently available. For this reason, they are forced to follow several books in their course or provide supplementary class notes for clearer explanations of difficult topics. *Fundamentals of Fracture Mechanics* overcomes this shortcoming. Since it only covers the essential topics for an introductory fracture mechanics course, it is the right book for first-time learners, students, and instructors.

Tribikram Kundu

The Author

Tribikram Kundu, Ph.D., is a professor in the Department of Civil Engineering and Engineering Mechanics and the Aerospace and Mechanical Engineering Department at the University of Arizona, Tucson. He is the winner of the Humboldt Research Prize (senior scientist award) and Humboldt Fellowship from Germany. He has been an invited professor in France, Sweden, Denmark, Russia, and Switzerland. Dr. Kundu is the editor of 14 books and 3 research monographs, as well as author or coauthor of 2 textbooks and over 200 scientific papers, 100 of which have been published in refereed scientific journals; 3 have received "best paper" awards. Among his noteworthy recognitions are receipt of the President of India gold medal for ranking first in his graduating class in IIT Kharagpur and the regents' fellowship and outstanding MS graduate award from UCLA. He is a fellow of ASME, ASCE, and SPIE.

1

Fundamentals of the Theory of Elasticity

1.1 Introduction

It is necessary to have a good knowledge of the fundamentals of continuum mechanics and the theory of elasticity to understand fracture mechanics. This chapter is written with this in mind. The first part of the chapter (section 1.2) is devoted to the derivation of the basic equations of elasticity; in the second part (section 1.3), these basic equations are used to solve some classical boundary value problems of the theory of elasticity. It is very important to comprehend the first chapter fully before trying to understand the rest of the book.

1.2 Fundamentals of Continuum Mechanics and the Theory of Elasticity

Relations among the displacement, strain, and stress in an elastic body are derived in this section.

1.2.1 Deformation and Strain Tensor

Figure 1.1 shows the reference state R and the current deformed state D of a body in the Cartesian $x_1x_2x_3$ coordinate system. Deformation of the body and displacement of individual particles in the body are defined with respect to this reference state. As different points of the body move, due to applied force or change in temperature, the configuration of the body changes from the reference state to the current deformed state. After reaching equilibrium in one deformed state, if the applied force or temperature changes again, the deformed state also changes. The current deformed state of the body is the equilibrium position under current state of loads. Typically, the stress-free configuration of the body is considered as the reference state, but it is not necessary for the reference state to always be stress free. Any possible configuration of the body can be considered as the reference state. For simplicity, if it is not stated otherwise, the initial stress-free configuration of the body, before applying any external disturbance (force, temperature, etc.), will be considered as its reference state.

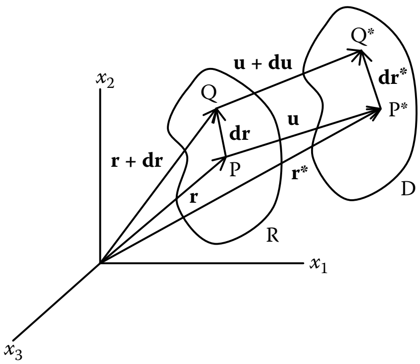


FIGURE 1.1
Deformation of a body: R is the reference state and D is the deformed state.

Consider two points P and Q in the reference state of the body. They move to P* and Q* positions after deformation. Displacement of points P and Q is denoted by vectors **u** and **u** + **du**, respectively. (Note: Here and in subsequent derivations, vector quantities will be denoted by boldface letters.) Position vectors of P, Q, P*, and Q* are **r**, **r** + **dr**, **r***, and **r*** + **dr***, respectively. Clearly, displacement and position vectors are related in the following manner:

$$\begin{aligned} \mathbf{r}^* &= \mathbf{r} + \mathbf{u} \\ \mathbf{r}^* + \mathbf{dr}^* &= \mathbf{r} + \mathbf{dr} + \mathbf{u} + \mathbf{du} \\ \therefore\ \mathbf{dr}^* &= \mathbf{dr} + \mathbf{du} \end{aligned} \tag{1.1}$$

In terms of the three Cartesian components, the preceding equation can be written as:

$$(dx_1^*\mathbf{e_1} + dx_2^*\mathbf{e_2} + dx_3^*\mathbf{e_3}) = (dx_1\mathbf{e_1} + dx_2\mathbf{e_2} + dx_3\mathbf{e_3}) + (du_1\mathbf{e_1} + du_2\mathbf{e_2} + du_3\mathbf{e_3}) \tag{1.2}$$

where $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ are unit vectors in x_1, x_2, and x_3 directions, respectively.

In index or tensorial notation, equation (1.2) can be written as

$$dx_i^* = dx_i + du_i \tag{1.3}$$

where the free index i can take values 1, 2, or 3.

Applying the chain rule, equation (1.3) can be written as

$$\begin{aligned} dx_i^* &= dx_i + \frac{\partial u_i}{\partial x_1}dx_1 + \frac{\partial u_i}{\partial x_2}dx_2 + \frac{\partial u_i}{\partial x_3}dx \\ \therefore\ dx_i^* &= dx_i + \sum_{j=1}^{3}\frac{\partial u_i}{\partial x_j}dx_j = dx_i + u_{i,j}dx_j \end{aligned} \tag{1.4}$$

In the preceding equation, the comma (,) means "derivative" and the summation convention (repeated dummy index means summation over 1, 2, and 3) has been adopted.

Equation (1.4) can also be written in matrix notation in the following form:

$$\begin{Bmatrix} dx_1^* \\ dx_2^* \\ dx_3^* \end{Bmatrix} = \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} \tag{1.5}$$

In short form, equation (1.5) can be written as

$$\{\mathbf{dr}^*\} = \{\mathbf{dr}\} + [\nabla \mathbf{u}]^{\mathrm{T}}\{\mathbf{dr}\} \tag{1.6}$$

If one defines

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.7a}$$

and

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{1.7b}$$

then equation (1.6) takes the following form:

$$\{\mathbf{dr}^*\} = \{\mathbf{dr}\} + [\varepsilon]\{\mathbf{dr}\} + [\omega]\{\mathbf{dr}\} \tag{1.7c}$$

1.2.1.1 Interpretation of ε_{ij} and ω_{ij} for Small Displacement Gradient

Consider the special case when $\mathbf{dr} = dx_1\mathbf{e_1}$. Then, after deformation, three components of $\mathbf{dr}^*$ can be computed from equation (1.5):

$$\begin{aligned} dx_1^* &= dx_1 + \frac{\partial u_1}{\partial x_1}dx_1 = (1+\varepsilon_{11})dx_1 \\ dx_2^* &= \frac{\partial u_2}{\partial x_1}dx_1 = (\varepsilon_{21}+\omega_{21})dx_1 \\ dx_3^* &= \frac{\partial u_3}{\partial x_1}dx_1 = (\varepsilon_{31}+\omega_{31})dx_1 \end{aligned} \tag{1.8}$$

In this case, the initial length of the element PQ is $dS = dx_1$, and the final length of the element P^*Q^* after deformation is

$$dS^* = \left[(dx_1^*)^2 + (dx_2^*)^2 + (dx_3^*)^2\right]^{\frac{1}{2}} = dx_1\left[(1+\varepsilon_{11})^2 + (\varepsilon_{21}+\omega_{21})^2 + (\varepsilon_{31}+\omega_{31})^2\right]^{\frac{1}{2}}$$
$$\approx dx_1\left[1+2\varepsilon_{11}\right]^{\frac{1}{2}} = dx_1(1+\varepsilon_{11}) \tag{1.9}$$

In equation (1.9) we have assumed that the displacement gradients $u_{i,j}$ are small. Hence, ε_{ij} and ω_{ij} are small. Therefore, the second-order terms involving ε_{ij} and ω_{ij} can be ignored.

From its definition, engineering normal strain (E_{11}) in x_1 direction can be written as

$$E_{11} = \frac{dS^* - dS}{dS} = \frac{dx_1(1+\varepsilon_{11}) - dx_1}{dx_1} = \varepsilon_{11} \tag{1.10}$$

Similarly one can show that ε_{22} and ε_{33} are engineering normal strains in x_2 and x_3 directions, respectively.

To interpret ε_{12} and ω_{12}, consider two mutually perpendicular elements PQ and PR in the reference state. In the deformed state these elements are moved to P^*Q^* and P^*R^* positions, respectively, as shown in Figure 1.2.

Let the vectors PQ and PR be $(\mathbf{dr})_{\mathbf{PQ}} = dx_1\mathbf{e_1}$ and $(\mathbf{dr})_{\mathbf{PR}} = dx_2\mathbf{e_2}$, respectively. Then, after deformation, three components of $(\mathbf{dr}^*)_{\mathbf{PQ}}$ and $(\mathbf{dr}^*)_{\mathbf{PR}}$ can be written in the forms of equations (1.11) and (1.12), respectively:

$$\begin{aligned}
(dx_1^*)_{PQ} &= dx_1 + \frac{\partial u_1}{\partial x_1}dx_1 = (1+\varepsilon_{11})dx_1 \\
(dx_2^*)_{PQ} &= \frac{\partial u_2}{\partial x_1}dx_1 = (\varepsilon_{21}+\omega_{21})dx_1 \\
(dx_3^*)_{PQ} &= \frac{\partial u_3}{\partial x_1}dx_1 = (\varepsilon_{31}+\omega_{31})dx_1
\end{aligned} \tag{1.11}$$

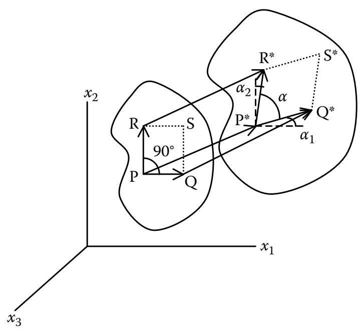


FIGURE 1.2
Two elements, PQ and PR, that are mutually perpendicular before deformation are no longer perpendicular after deformation.

$$(dx_1^*)_{PR} = \frac{\partial u_1}{\partial x_2} dx_2 = (\varepsilon_{12} + \omega_{12}) dx_1$$
$$(dx_2^*)_{PR} = \frac{\partial u_2}{\partial x_2} dx_2 = (1 + \varepsilon_{22}) dx_2 \quad (1.12)$$
$$(dx_3^*)_{PR} = \frac{\partial u_3}{\partial x_2} dx_2 = (\varepsilon_{32} + \omega_{32}) dx_1$$

Let α_1 be the angle between P*Q* and the horizontal axis, and α_2 the angle between P*R* and the vertical axis as shown in Figure 1.2. Note that $\alpha + \alpha_1 + \alpha_2 = 90°$. From equations (1.11) and (1.12), one can show that

$$\tan \alpha_1 = \frac{(\varepsilon_{21} + \omega_{21}) dx_1}{(1 + \varepsilon_{11}) dx_1} \approx \varepsilon_{21} + \omega_{21} = \varepsilon_{12} + \omega_{21}$$
$$\tan \alpha_2 = \frac{(\varepsilon_{12} + \omega_{12}) dx_2}{(1 + \varepsilon_{22}) dx_2} \approx \varepsilon_{12} - \omega_{21} \quad (1.13)$$

In the preceding equation, we have assumed a small displacement gradient and therefore $1 + \varepsilon_{ij} \approx 1$. For a small displacement gradient, $\tan \alpha_i \approx \alpha_i$ and one can write:

$$\alpha_1 = \varepsilon_{12} + \omega_{21}$$
$$\alpha_2 = \varepsilon_{12} - \omega_{21} \quad (1.14)$$
$$\therefore \varepsilon_{12} = \frac{1}{2}(\alpha_1 + \alpha_2) \quad \& \quad \omega_{21} = \frac{1}{2}(\alpha_1 - \alpha_2)$$

From equation (1.14) it is concluded that $2\varepsilon_{12}$ is the change in the angle between the elements PQ and PR after deformation. In other words, it is the engineering shear strain and ω_{21} is the rotation of the diagonal PS (see Figure 1.2) or the average rotation of the rectangular element PQSR about the x_3 axis after deformation.

In summary, ε_{ij} and ω_{ij} are strain tensor and rotation tensor, respectively, for small displacement gradients.

Example 1.1

Prove that the strain tensor satisfies the relation $\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} = \varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik}$.

This relation is known as the compatibility condition.

Solution

$$\text{Left-hand side} = \varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} = \frac{1}{2}(u_{i,jk\ell} + u_{j,ik\ell} + u_{k,\ell ij} + u_{\ell,kij})$$

$$\text{Right-hand side} = \varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik} = \frac{1}{2}(u_{i,kj\ell} + u_{k,ij\ell} + u_{j,\ell ik} + u_{\ell,jik})$$

Since the sequence of derivative should not make any difference, $u_{i,jk\ell} = u_{i,kj\ell}$; similarly, the other three terms in the two expressions can be shown as equal. Thus, the two sides of the equation are proved to be identical.

Example 1.2

Check if the following strain state is possible for an elasticity problem:

$$\varepsilon_{11} = k\left(x_1^2 + x_2^2\right), \qquad \varepsilon_{22} = k\left(x_2^2 + x_3^2\right), \qquad \varepsilon_{12} = kx_1x_2x_3, \qquad \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

Solution

From the compatibility condition, $\varepsilon_{ij,k\ell} + \varepsilon_{k\ell,ij} = \varepsilon_{ik,j\ell} + \varepsilon_{j\ell,ik}$, given in example 1.1, one can write

$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}$ by substituting $i = 1, j = 1, k = 2, \ell = 2$.

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2k + 0 = 2k$$

$$2\varepsilon_{12,12} = 2kx_3$$

Since the two sides of the compatibility equation are not equal, the given strain state is not a possible strain state.

1.2.2 Traction and Stress Tensor

Force per unit area on a surface is called traction. To define traction at a point P (see Figure 1.3), one needs to state on which surface, going through that point, the traction is defined. The traction value at point P changes if the orientation of the surface on which the traction is defined is changed.

Figure 1.3 shows a body in equilibrium under the action of some external forces. If it is cut into two halves by a plane going through point P, in general, to keep each half of the body in equilibrium, some force will exist at the cut plane. Force per unit area in the neighborhood of point P is defined as the traction at point P. If the cut plane is changed, then the traction at the same point will change. Therefore, to define traction at a point, its three components must be given and the plane on which it is defined must be identified. Thus, the traction can be denoted as $\mathbf{T}^{(\mathbf{n})}$, where the superscript $\mathbf{n}$ denotes the unit

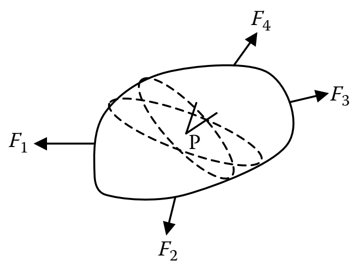


FIGURE 1.3
A body in equilibrium can be cut into two halves by an infinite number of planes going through a specific point P. Two such planes are shown in the figure.

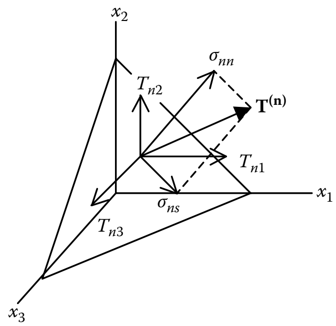


FIGURE 1.4
Traction $\mathbf{T}^{(\mathbf{n})}$ on an inclined plane can be decomposed into its three components, T_{ni}, or into two components: normal and shear stress components (σ_{nn} and σ_{ns}).

vector normal to the plane on which the traction is defined and where $\mathbf{T}^{(\mathbf{n})}$ has three components that correspond to the force per unit area in x_1, x_2, and x_3 directions, respectively.

Stress is similar to traction; both are defined as force per unit area. The only difference is that the stress components are always defined normal or parallel to a surface, while traction components are not necessarily normal or parallel to the surface. A traction $\mathbf{T}^{(\mathbf{n})}$ on an inclined plane is shown in Figure 1.4. Note that neither $\mathbf{T}^{(\mathbf{n})}$ nor its three components T_{ni} are necessarily normal or parallel to the inclined surface. However, its two components σ_{nn} and σ_{ns} are perpendicular and parallel to the inclined surface and are called normal and shear stress components, respectively.

Stress components are described by two subscripts. The first subscript indicates the plane (or normal to the plane) on which the stress component is defined and the second subscript indicates the direction of the force per unit area or stress value. Following this convention, different stress components in the $x_1x_2x_3$ coordinate system are defined in Figure 1.5.

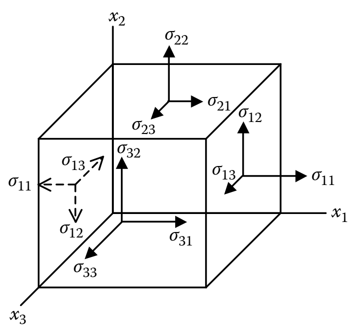


FIGURE 1.5
Different stress components in the $x_1x_2x_3$ coordinate system.

Note that on each of the six planes (i.e., the positive and negative x_1, x_2, and x_3 planes), three stress components (one normal and two shear stress components) are defined. If the outward normal to the plane is in the positive direction, then we call the plane a positive plane; otherwise, it is a negative plane. If the force direction is positive on a positive plane or negative on a negative plane, then the stress is positive. All stress components shown on positive x_1, x_2, and x_3 planes and negative x_1 plane in Figure 1.5 are positive stress components. Stress components on the other two negative planes are not shown to keep the Figure simple. Dashed arrows show three of the stress components on the negative x_1 plane while solid arrows show the stress components on positive planes. If the force direction and the plane direction have different signs, one positive and one negative, then the corresponding stress component is negative. Therefore, in Figure 1.5, if we change the direction of the arrow of any stress component, then that stress component becomes negative.

1.2.3 Traction–Stress Relation

Let us take a tetrahedron OABC from a continuum body in equilibrium (see Figure 1.6). Forces (per unit area) acting in the x_1 direction on the four surfaces of OABC are shown in Figure 1.6. From its equilibrium in the x_1 direction one can write

$$\sum F_1 = T_{n1}A - \sigma_{11}A_1 - \sigma_{21}A_2 - \sigma_{31}A_3 + f_1V = 0 \tag{1.15}$$

where A is the area of the surface ABC; A_1, A_2, and A_3 are the areas of the other three surfaces OBC, OAC, and OAB, respectively; and f_1 is the body force per unit volume in the x_1 direction.

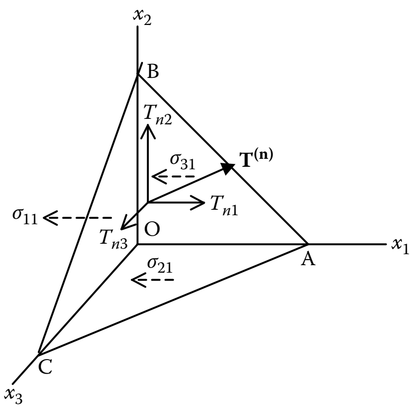


FIGURE 1.6
A tetrahedron showing traction components on plane ABC and x_1 direction stress components on planes AOC, BOC, and AOB.

If n_j is the jth component of the unit vector **n** that is normal to the plane ABC, then one can write $A_j = n_j A$ and $V = (Ah)/3$, where h is the height of the tetrahedron measured from the apex O. Thus, equation (1.15) is simplified to

$$T_{n1} - \sigma_{11}n_1 - \sigma_{21}n_2 - \sigma_{31}n_3 + f_1\frac{h}{3} = 0 \tag{1.16}$$

In the limiting case when the plane ABC passes through point O, the tetrahedron height h vanishes and equation (1.16) is simplified to

$$T_{n1} = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3 = \sigma_{j1}n_j \tag{1.17}$$

In this equation the summation convention (repeated index means summation) has been used.

Similarly, from the force equilibrium in x_2 and x_3 directions, one can write

$$\begin{aligned} T_{n2} &= \sigma_{j2}n_j \\ T_{n3} &= \sigma_{j3}n_j \end{aligned} \tag{1.18}$$

Combining equations (1.17) and (1.18), the traction–stress relation is obtained in index notation:

$$T_{ni} = \sigma_{ji}n_j \tag{1.19}$$

where the free index i takes values 1, 2, and 3 to generate three equations and the dummy index j takes values 1, 2, and 3 and is added in each equation.

For simplicity, the subscript n of T_{ni} is omitted and T_{ni} is written as T_i. It is implied that the unit normal vector to the surface on which the traction is defined is **n**. With this change, equation (1.19) is simplified to

$$T_i = \sigma_{ji}n_j \tag{1.19a}$$

1.2.4 Equilibrium Equations

If a body is in equilibrium, then the resultant force and moment on that body must be equal to zero.

1.2.4.1 Force Equilibrium

The resultant forces in the x_1, x_2, and x_3 directions are equated to zero to obtain the governing equilibrium equations. First, x_1 direction equilibrium is studied. Figure 1.7 shows all forces acting in the x_1 direction on an elemental volume.

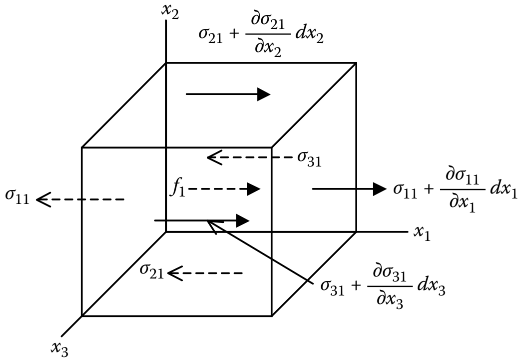


FIGURE 1.7
Forces acting in the x_1 direction on an elemental volume.

Thus, the zero resultant force in the x_1 direction gives

$$-\sigma_{11} dx_2 dx_3 + \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) dx_2 dx_3 - \sigma_{21} dx_1 dx_3 + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 \right) dx_1 dx_3$$

$$- \sigma_{31} dx_2 dx_1 + \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 + f_1 \, dx_1 dx_2 dx_3 = 0$$

or

$$\left(\frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) dx_2 dx_3 + \left(\frac{\partial \sigma_{21}}{\partial x_2} dx_2 \right) dx_1 dx_3 + \left(\frac{\partial \sigma_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 + f_1 \, dx_1 dx_2 dx_3 = 0$$

or

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0$$

or

$$\frac{\partial \sigma_{j1}}{\partial x_j} + f_1 = 0 \tag{1.20}$$

In equation (1.20) repeated index j indicates summation.

Similarly, equilibrium in x_2 and x_3 directions gives

$$\begin{aligned} \frac{\partial \sigma_{j2}}{\partial x_j} + f_2 = 0 \\ \frac{\partial \sigma_{j3}}{\partial x_j} + f_3 = 0 \end{aligned} \tag{1.21}$$

The three equations in (1.20) and (1.21) can be combined in the following form:

$$\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = \sigma_{ji,j} + f_i = 0 \tag{1.22}$$

The force equilibrium equations given in equation (1.22) are written in index notation, where the free index i takes three values—1, 2, and 3—and corresponds to three equilibrium equations, and the comma (,) indicates derivative.

1.2.4.2 Moment Equilibrium

Let us now compute the resultant moment in the x_3 direction (or, in other words, moment about the x_3 axis) for the elemental volume shown in Figure 1.8.

If we calculate the moment about an axis parallel to the x_3 axis and passing through the centroid of the elemental volume shown in Figure 1.8, then only four shear stresses shown on the four sides of the volume can produce moment. Body forces in x_1 and x_2 directions do not produce any moment because the resultant body force passes through the centroid of the volume. Since the resultant moment about this axis should be zero, one can write

$$\left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 dx_3 \frac{dx_1}{2} + (\sigma_{12}) dx_2 dx_3 \frac{dx_1}{2} - \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 \frac{dx_2}{2}$$
$$- (\sigma_{12}) dx_1 dx_3 \frac{dx_2}{2} = 0$$

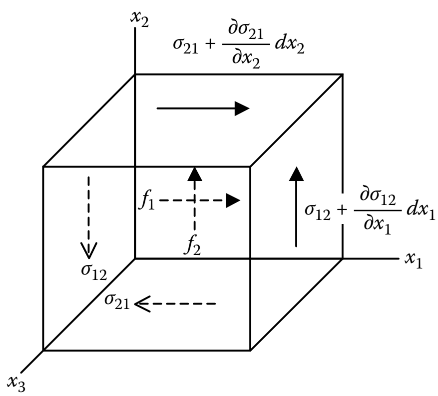


FIGURE 1.8
Forces on an element that may contribute to the moment in the x_3 direction.

Ignoring the higher order terms, one gets

$$2(\sigma_{21})dx_2dx_3\frac{dx_1}{2} - 2(\sigma_{21})dx_1dx_3\frac{dx_2}{2} = 0$$

or $\sigma_{12} = \sigma_{21}$.

Similarly, applying moment equilibrium about the other two axes, one can show that $\sigma_{13} = \sigma_{31}$ and $\sigma_{32} = \sigma_{23}$. In index notation,

$$\sigma_{ij} = \sigma_{ji} \tag{1.23}$$

Thus, the stress tensor is symmetric. It should be noted here that if the body has internal body couple (or body moment per unit volume), then the stress tensor will not be symmetric.

Because of the symmetry of the stress tensor, equations (1.19a) and (1.22) can be written in the following form as well:

$$\begin{aligned} T_i &= \sigma_{ij}n_j \\ \sigma_{ij,j} + f_i &= 0 \end{aligned} \tag{1.24}$$

1.2.5 Stress Transformation

Let us now investigate how the stress components in two Cartesian coordinate systems are related.

Figure 1.9 shows an inclined plane ABC whose normal is in the $x_{1'}$ direction; thus, the $x_{2'}x_{3'}$ plane is parallel to the ABC plane. Traction $\mathbf{T}^{(1')}$ is acting on this plane. Three components of this traction in $x_{1'}$, $x_{2'}$, and $x_{3'}$ directions are the three stress components $\sigma_{1'1'}$, $\sigma_{1'2'}$, and $\sigma_{1'3'}$, respectively. Note that the

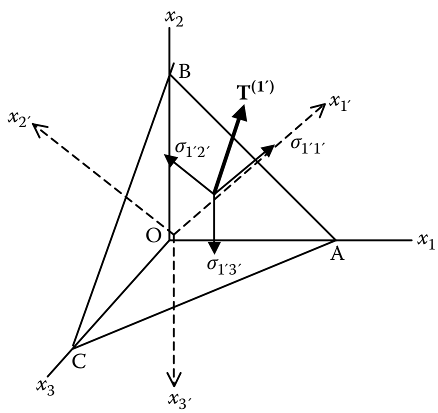


FIGURE 1.9
Stress components in $x_{1'}x_{2'}x_{3'}$ coordinate system.

first subscript indicates the plane on which the stress is acting and the second subscript gives the stress direction.

From equation (1.19) one can write

$$T_{1'i} = \sigma_{ji} n_j^{(1')} = \sigma_{ji} \ell_{1'j} \tag{1.25}$$

where $n_j^{(1')} = \ell_{1'j}$ is the jth component of the unit normal vector on plane ABC or, in other words, the direction cosines of the $x_{1'}$ axis.

Note that the dot product between $\mathbf{T}^{(1')}$ and the unit vector $\mathbf{n}^{(1')}$ gives the stress component $\sigma_{1'1'}$; therefore,

$$\sigma_{1'1'} = T_{1'i} \ell_{1'i} = \sigma_{ji} \ell_{1'j} \ell_{1'i} \tag{1.26}$$

Similarly, the dot product between $\mathbf{T}^{(1')}$ and the unit vector $\mathbf{n}^{(2')}$ gives $\sigma_{1'2'}$ and the dot product between $\mathbf{T}^{(1')}$ and the unit vector $\mathbf{n}^{(3')}$ gives $\sigma_{1'3'}$. Thus, we get

$$\begin{aligned} \sigma_{1'2'} &= T_{1'i} \ell_{2'i} = \sigma_{ji} \ell_{1'j} \ell_{2'i} \\ \sigma_{1'3'} &= T_{1'i} \ell_{3'i} = \sigma_{ji} \ell_{1'j} \ell_{3'i} \end{aligned} \tag{1.27}$$

Equations (1.26) and (1.27) can be written in index notation in the following form:

$$\sigma_{1'm'} = \ell_{1'j} \sigma_{ji} \ell_{m'i} \tag{1.28}$$

In this equation, the free index m' can take values 1′, 2′, or 3′.

Similarly, from the traction vector $\mathbf{T}^{(2')}$ on a plane whose normal is in the $x_{2'}$ direction, one can show that

$$\sigma_{2'm'} = \ell_{2'j} \sigma_{ji} \ell_{m'i} \tag{1.29}$$

From the traction vector $\mathbf{T}^{(3')}$ on the $x_{3'}$ plane, one can derive

$$\sigma_{3'm'} = \ell_{3'j} \sigma_{ji} \ell_{m'i} \tag{1.30}$$

Equations (1.28) to (1.30) can be combined to obtain the following equation in index notation:

$$\sigma_{n'm'} = \ell_{n'j} \sigma_{ji} \ell_{m'i}$$

Note that in the preceding equation, i, j, m', and n' are all dummy indices and can be interchanged to obtain

$$\sigma_{m'n'} = \ell_{m'i} \sigma_{ij} \ell_{n'j} = \ell_{m'i} \ell_{n'j} \sigma_{ij} \tag{1.31}$$

1.2.5.1 Kronecker Delta Symbol (δ_{ij}) and Permutation Symbol (ε_{ijk})

In index notation the Kronecker delta symbol (δ_{ij}) and permutation symbol (ε_{ijk}, also known as the Levi–Civita symbol and alternating symbol) are often used. They are defined in the following manner:

$$\delta_{ij} = 1 \quad \text{for} \quad i = j$$

$$\delta_{ij} = 0 \quad \text{for} \quad i \neq j$$

and

$\varepsilon_{ijk} = 1$ for i, j, k having values 1, 2, and 3; or 2, 3, and 1; or 3, 1, and 2.
$\varepsilon_{ijk} = -1$ for i, j, k having values 3, 2, and 1; or 1, 3, and 2; or 2, 1, and 3.
$\varepsilon_{ijk} = 0$ for i, j, k not having three distinct values.

1.2.5.2 Examples of the Application of δ_{ij} and ε_{ijk}

Note that

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}; \qquad \mathbf{e_i} \bullet \mathbf{e_j} = \delta_{ij}$$

$$Det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}; \qquad \mathbf{b} \times \mathbf{c} = \varepsilon_{ijk} b_j c_k \mathbf{e_i}$$

where $\mathbf{e_i}$ and $\mathbf{e_j}$ are unit vectors in x_i and x_j directions, respectively, in the $x_1x_2x_3$ coordinate system. Also note that $\mathbf{b}$ and $\mathbf{c}$ are two vectors, while $[a]$ is a matrix.

One can prove that the following relation exists between these two symbols:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Example 1.3
Starting from the stress transformation law, prove that $\sigma_{m'n'}\sigma_{m'n'} = \sigma_{ij}\sigma_{ij}$ where $\sigma_{m'n'}$ and σ_{ij} are stress tensors in two different Cartesian coordinate systems.

Solution

$$\sigma_{m'n'}\sigma_{m'n'} = (\ell_{m'i}\ell_{n'j}\sigma_{ij})(\ell_{m'p}\ell_{n'q}\sigma_{pq}) = (\ell_{m'i}\ell_{n'j})(\ell_{m'p}\ell_{n'q})\sigma_{ij}\sigma_{pq}$$

$$= (\ell_{m'i}\ell_{m'p})(\ell_{n'j}\ell_{n'q})\sigma_{ij}\sigma_{pq} = \delta_{ip}\delta_{jq}\sigma_{ij}\sigma_{pq} = \sigma_{ij}\sigma_{ij}$$

1.2.6 Definition of Tensor

A Cartesian tensor of order (or rank) r in n dimensional space is a set of n^r numbers (called the elements or components of tensor) that obey the following transformation law between two coordinate systems:

$$t_{m'n'p'q'.....} = (\ell_{m'i}\ell_{n'j}\ell_{p'k}\ell_{q'\ell}\ldots)(t_{ijk\ell\ldots}) \tag{1.32}$$

where $t_{m'n'p'q'\ldots}$ and $t_{ijk....}$ each has r number of subscripts; r number of direction cosines $(\ell_{m'i}\ell_{n'j}\ell_{p'k}\ell_{q'\ell}\ldots)$ are multiplied on the right-hand side. Comparing equation (1.31) with the definition of tensor transformation equation (1.32), one can conclude that the stress is a second-rank tensor.

1.2.7 Principal Stresses and Principal Planes

Planes on which the traction vectors are normal are called principal planes. Shear stress components on the principal planes are equal to zero. Normal stresses on the principal planes are called principal stresses.

In Figure 1.10, let **n** be the unit normal vector on the principal plane ABC and λ the principal stress value on this plane. Therefore, the traction vector on plane ABC can be written as

$$T_i = \lambda n_i$$

Again, from equation (1.24),

$$T_i = \sigma_{ij} n_j$$

From the preceding two equations, one can write

$$\sigma_{ij} n_j - \lambda n_i = 0 \tag{1.33}$$

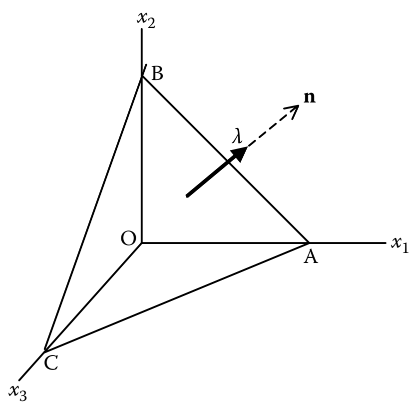


FIGURE 1.10
Principal stress λ on the principal plane ABC.