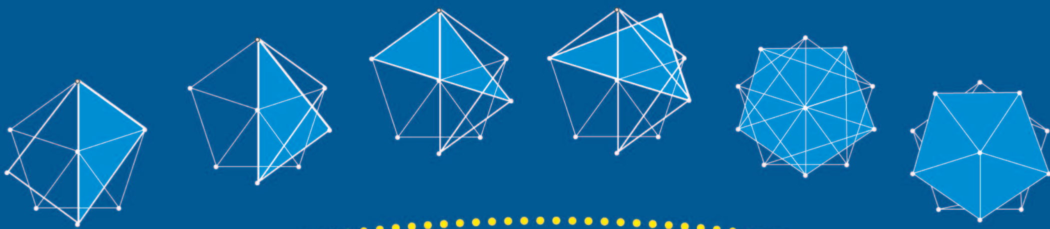
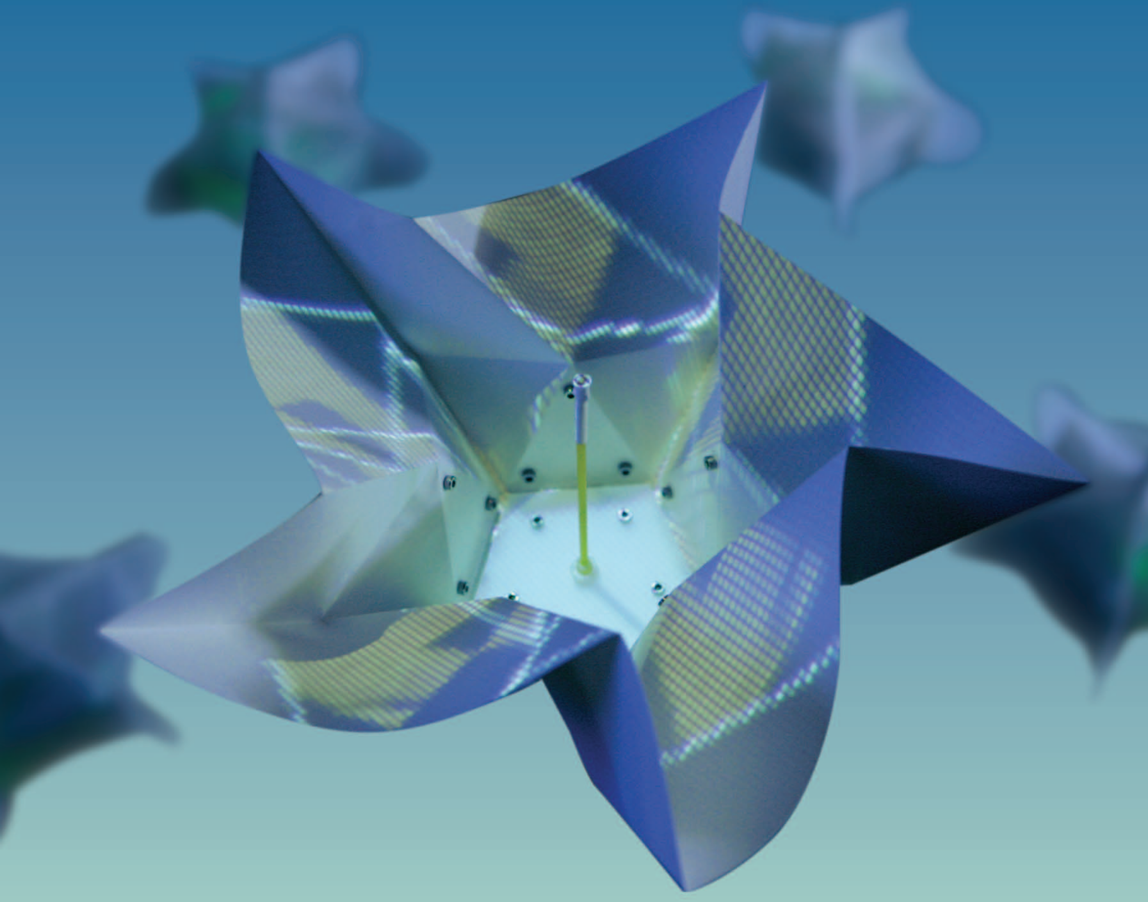


Origami^4



EDITED BY ROBERT J. LANG

Origami^4

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Fourth International Meeting of Origami Science, Mathematics, and Education

Robert J. Lang, editor

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Preface

The concepts of *origami* and *science* would seem to be about as far apart as you can get within human fields of endeavor: the former, an art, a craft, associated with a Japanese tradition hundreds of years old; the latter, a strict, rationalist way of knowing. But remarkably, both fields extend tendrils of influence into the other, exhibiting connections in manifold ways. And, in fact, they have done so for decades.

For upon closer examination, they are not as far apart as you might think, science and origami, or even science and art in general. While science is generally perceived among the public as the province of white-coated individuals following a rigid set of rules collectively known as "the scientific method," said scientific method is merely a discipline—a set of tools—that bring order to what is still a very human practice. Aesthetic terms like "elegance" pervade science; and while one may create and follow a double-blind protocol to evaluate a hypothesis or use advanced computational and mathematical tools to establish and explore a technology, the moment of scientific inspiration—that moment of "Aha!"—is widely known, if not widely advertised, as an art within the science. Many scientists, mathematicians, and technologists are as motivated by the order, beauty, and elegance within their field as any painter, writer, or sculptor. Scratch a successful scientist, and you will find an artist not far under the surface.

Conversely, the art of origami—folding uncut sheets of paper into beautiful objects—is deeply connected to the worlds of mathematics and science. The laws of origami—folding without cutting—would seem on their surface to be so restrictive as to prevent any significant variety of accomplishment. It is a testimony to the ingenuity of hundreds of origami artists that the opposite is true; there seems to be no limit on the range of artistic expression possible within origami. But there are absolute limits on the physical structures foldable with origami. Those limits are defined by the underlying mathematics of origami. By exploring, elucidating, and describing those mathematical laws, modern origami artists have found ways to push the art to undreamed-of heights, and to begin to develop computational tools that augment the capabilities of the human artist in order to more fully realize their artistic visions.

At the same time, these mathematical explorations have allowed origami, or more broadly, folded structures, to take on applications in the real world and bring real benefits to the world. Folded structures based on origami principles have found application in space flight, consumer electronics, health, and safety, to name just a few areas where origami has made an unexpected appearance.

These rich connections make origami an ideal vehicle to bridge the supposedly disparate worlds of math and science, and it should be no surprise that origami has found repeated application in education to form connections, to make mathematics accessible, and to provide concrete demonstration of the fact that mathematics is everywhere around us.

The connections between origami, mathematics, science, technology, and education have been a topic of considerable interest now for several decades. While many individuals have happened upon discrete connections among these fields during the twentieth century, the field began to take off when previously isolated individuals began to make further connections with each other through a series of conferences exploring the links between origami and "the outside world." The first such conference, the First International Meeting of Origami Science and Technology was held in Ferrara, Italy, in 1989, and was organized by Professor Humiaki Huzita at the University of Padova. This conference brought together researchers from all over the world, many meeting each other for the first time, and its published proceedings became almost immediately a standard reference for mathematical origami. (And now they are an extremely hard-to-find reference.)

This conference was so successful that a second conference, The Second International Meeting of Origami Science and Scientific Origami, was organized in Ohtsu, Japan, in 1994. It, too, produced a proceedings volume, which also became a key reference for this cross-disciplinary field. It was followed in 2001 by the Third International Meeting on Origami in Science, Mathematics, and Education, held in Monterey, California, whose proceedings were published as a book, *Origami³*, edited by Thomas Hull, and published by A K Peters, Ltd.

The success of these conferences—each year larger and with a more extensive program than the last—and their proceedings led to the Fourth International Meeting on Origami in Science, Mathematics, and Education

(4OSME), held in September, 2006, at the California Institute of Technology in Pasadena, California. The 4OSME brought together an unprecedented number of researchers presenting some 80 papers on fields ranging from mathematics, to technology, to educational uses of origami, to computer programs for the design of origami. Selected papers based on talks presented at that conference make up the book you hold in your hands.

It should be clear now that this book, and the conference that gave rise to it, owe their existence to those pioneering individuals who plumbed the fields of origami, math, science, and education. The contributors to those fields are innumerable, but I should like to acknowledge several people and organizations whose support was absolutely critical. First and foremost, the support of OrigamiUSA, which sponsored the conference, and of the California Institute of Technology, which provided facilities as well as financial support, was invaluable. The program committee, consisting of Tom Hull, Günter Rote, Ryda Rose, Koichi Tateishi, and Toshikazu Kawasaki, performed heroic duties in reviewing (and in many cases, recommending improvements to) both the conference papers and the works in this book. Tom, in particular, played a critical role in bringing this book together in many ways: advice, support, and through his extensive knowledge of origami-math. I must also express my thanks to an anonymous reviewer (you know who you are) who made extensive and helpful recommendations for several of the papers. Last, this book would not exist at all if not for the contributions of the authors, those who gave presentations at 4OSME, and who contributed to this book. My thanks to you all.

Robert J. Lang
General Chair, 4OSME
Editor, Origami⁴
Alamo, California, 2008

Part I

Origami in Design and Art

Paper Nautili: A Model for Three-Dimensional Planispiral Growth

Arle Lommel

The spiral forms of seashells have been of interest to many paper folders in recent years, with models such as the elegant intertwined flaps of Tomoko Fuse and Robert Lang's nautilus. This article describes a novel method for the construction of smoothly curved three-dimensional models of logarithmic spiral shell-like forms that approximate the curves of natural spiral shells.

This model differs from existing models in a number of regards. Rather than intertwining flaps, it is produced by repetition of a relatively simple folding sequence along the length of a tapered strip of folding medium in which a straight line is folded to a curved line, thereby causing the folding medium to buckle into three dimensions with a curve roughly catenary in form.

It should be emphasized that this model was designed initially through practical hands-on experimentation, not via the mathematical model presented herein, which is a post facto explanation of the results. As a result, even if any details of the mathematical model remain underspecified, the practical results demonstrate that the techniques described work well for producing actual models.

The natural basis chosen for this model was the shell of the chambered nautilus. Besides its traditional use as an image of mathematical perfection,

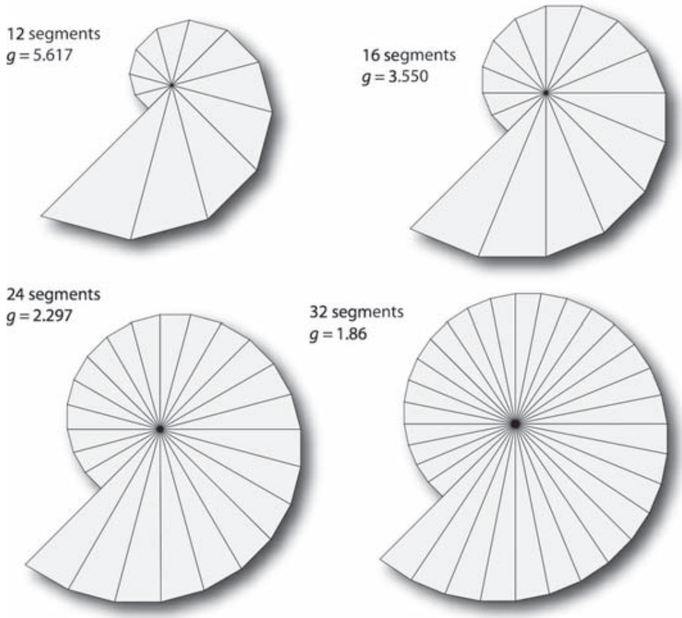


Figure 1. Various spirals produced by rotating similar right triangles about a radial axis.

the nautilus is planispiral (i.e., the spiral coils on a plane and is bilaterally symmetrical) and has a relatively simple catenary-type cross section when cut radially from the axis. It is thus a relatively simple shell form to model when compared to many other whorled shells found in nature.

Contrary to numerous published accounts, the chambered nautilus is not a so-called Golden Mean spiral; like the Golden Mean spiral it is a logarithmic (constant-slope) spiral, but a simple visual examination of both spirals shows that the nautilus has a much lower slope. The fact that so many sources cite it as a Golden Mean spiral demonstrates how powerful the belief in nature's mathematical basis can be, even in the face of manifest evidence to the contrary. One goal in producing this model was to generate a spiral that approximated the actual spiral of the nautilus rather than the idealized (but inaccurate) form that many scholars state that it has.

A useful starting point for designing this model is that any logarithmic spiral can be approximated as a series of similar triangles in which the hypotenuse of one triangle lies on (and is equal in length to) one leg of the previous triangle (see Figure 1). Each triangle differs in size from its neighbors by a fixed ratio. For purposes of this model, a series of right

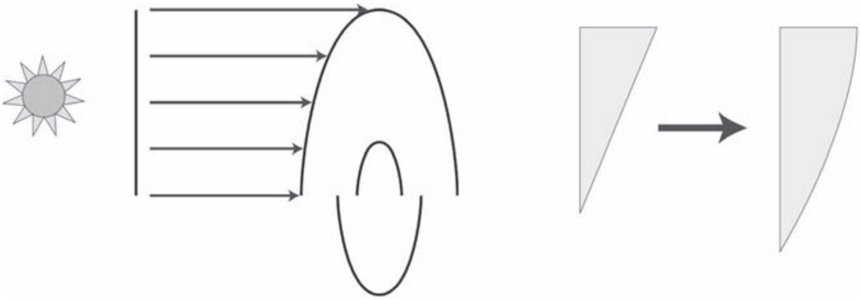


Figure 2. The shadow of right-triangle spiral segments on the curved form of the final model produces segment shapes that can cover the surface of the nautilus model.

triangles is particularly useful, because when right triangles are utilized to model a logarithmic spiral, a simple formula can be used to determine the growth rate per revolution (g) for any number of triangular segments per revolution (n):

$$g = \cos(2\pi/n)^{-n}.$$

This formula produces steeper slopes/growth rates for lower numbers of segments, as shown in Figure 1. As it turns out, measurements of actual nautilus shells yielded growth rates of roughly 3.5, for which the value $g =$ 3.55 of a 16-segment spiral model is a good approximation (a Golden Mean spiral, in contrast, has a g roughly equal to 6.9). Therefore, to simplify folding and design, this model of the nautilus adopts the 16-segment model. (It should be noted that *any* arbitrary logarithmic spiral can be produced in this manner, and I have produced 12- and 32-segment models in addition to the 16-segment model described here.)

Having established an appropriate two-dimensional model for the nautilus spiral's growth pattern, the problem of how to generate the three-dimensional structure, which includes roughly catenary radial cross sections, remains. However, this problem can be solved in a simple manner: the needed two-dimensional shapes can be conceived as the shadows of the right-angle triangles on the surface of the desired three-dimensional shape, as shown in Figure 2.

The shadows of the original right triangle sections show the same scaling factor with regard to adjacent segments as in the original two-dimensional model, allowing them to be arranged within an evenly tapered strip of folding medium, as shown in Figure 3.

It is important to note in Figure 3 that the curved segment BC (the shadow-distorted hypotenuse of an original ABC right triangle) is equal in length to the leg of BDE (the curve onto which each segment was projected

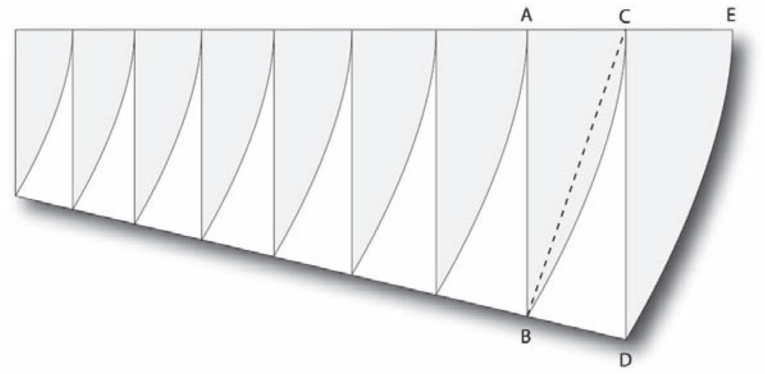


Figure 3. A sequence of distorted triangles that can cover the surface of a nautilus model arranged in a series.

was the same curve), and the straight-line segment BC (shown with a dotted line) is, consequently, shorter than CD. The ratio (r) of the length of the edge of any distorted triangle to the corresponding edge of its larger neighbor is defined as

$$r = g^{1/n}.$$

In the case of the 16-segment model, this yields a scale factor of 1.082. (This simple scaling factor aids in the production of templates for folding the model on a computer since each segment can be copied and scaled to yield the next segment.)

The fact that CD and CB are equal in length suggests a folding sequence that will yield the three-dimensional shape sought in this model. Segment CD is mountain folded and swung back to lie on CB, a process that is repeated on each segment of the model to leave only the gray shaded areas in Figure 3 visible. As the straight line CD is brought to lie on the curve BC, a curved valley fold forms equidistant between CD and BC. As this new fold is formed, the folding medium takes on the catenary-like shape onto which the original triangle sections were projected in Figure 2. Through the repetition of this process, the tips of the triangles (e.g., points B and D in Figure 3) are all brought to lie on the axis of the spiral, causing the overall spiral outline to form. One advantage of this model is that as points C and D are brought together, the valley fold (line CF in the crease pattern shown in Figure 4) automatically forms, similar to folding a straight angle bisector in conventional origami. Although it looks difficult, the folding is actually quite simple and automatic with a small amount of practice.

Figure 4 shows the resulting crease pattern, and Figure 5 shows a completed model made in this fashion from copper cloth.

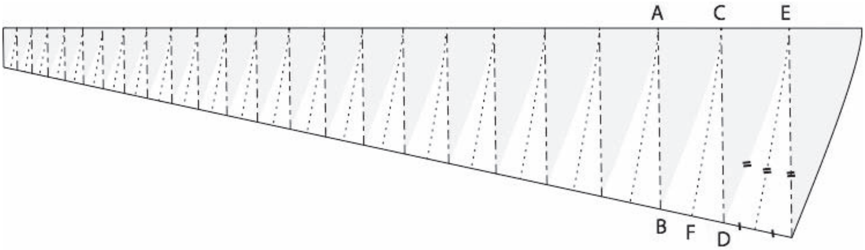


Figure 4. Crease pattern for a half nautilus shell. Gray areas will be visible from the outside of the completed model. Twenty four segments (1.5 revolutions) are shown.

Figure 5. Image of a completed half-shell model folded from copper cloth and chemically treated to variegate the surface.

As a practical matter, shells made following this model can be constructed by using a computer-drawing program (the author uses Adobe Illustrator) to generate a paper template used to place the mountain folds, which are precreased. This crease pattern can produce a half shell (as shown), or it can be reflected along its top edge to produce a bilaterally symmetrical crease pattern that yields a model of a complete shell. Either model is aesthetically pleasing, although only the half shell model affords a view of both the "inside" and the "outside" of the construction. In addition, the crease pattern can be extended to produce as many spirals as desired, although the folding is impractical below a certain size and adding extra segments on the big end can take up a large amount of folding medium for little additional spiral. If paper is used to fold these models, wet folding is helpful if the final model is to retain its shape, but in the author's experience, woven metal cloth is a superior folding medium for these models due to its malleability and receptiveness to hard creases.

In conclusion, this article has described a folded model that closely resembles an actual nautilus shell in its overall shape and spiral growth. It is constructed using a minimal set of repeated folds (and is thus conceptually elegant). The model is useful because it approximates the structure of an actual shell, rather than just its appearance, and does so in a gracefully curved form. In addition, the fold lines visible on the inside are evocative of the septa within a real nautilus shell, an unintended aesthetic bonus of the design. This novel technique for folding curves has proved capable of accurately modeling a variety of natural planispiral shells in an elegant and natural-seeming manner. To this point, the technique has been applied only to planispiral shells. The author has attempted to apply the technique to the more complex whorls of marine snails and other conically-spiral shells, but the results have not met expectations. Further research may enable the technique to be extended into these more complex shapes, but success is not yet certain.

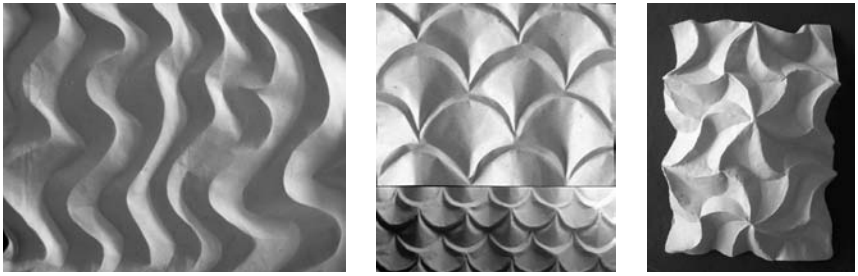
Curves and Flats

Saadya Sternberg

1 Background: Raising a Pattern, Keeping a Sheet Flat

What is the subject here? The aim is to gain control of this medium, mostly so as to be able to make those faces, my main proving ground. And the medium itself—is what? Clearly it involves *folding curves* (in, as it happens, rectangles of brown wrapping paper spray-glued to thick aluminum foil). Now curve folding, as is known, creates surfaces that won't lie flush to each other, that is, *open folds*; and open folds can be made voluntarily with straight lines too. So maybe our subject is best described as the *manipulation of open folds*, whether curved or not. And gaining control of this subject, taming it, means, for me—as in certain political theories—flattening it: being able to crush, squeeze, twist, bend the thing to the right or left .... In short, I want to be able to restore an *average flatness* to a surface *deformed by curved or open folds*, and then see whether and how such a textured or raised surface can be further manipulated.

But let's start at the beginning. Suppose you put curved folds of any kind in a flat sheet of paper. The paper will no longer lay flat. For that matter, you can easily enough use *straight folds* to create a surface that curves—a cone for example—by means of an angled crimp that originates in the interior of a sheet. But with non-flat surfaces made only via straight-line folds, you can always collapse the surface to a completely flat state (while retaining the initial folds) by adding a finite number of new straight-



(a) (b) (c)

Figure 1. (a) *Sand Curves.* (b) *Fish Scales.* (c) *Triangle Spirals.*

line folds. Curved surfaces made by *curved* folds cannot be collapsed flat while retaining those folds by *any* finite means, neither by adding straight folds nor by adding curved folds. (Both of these last conjectures seem to me eminently provable.)

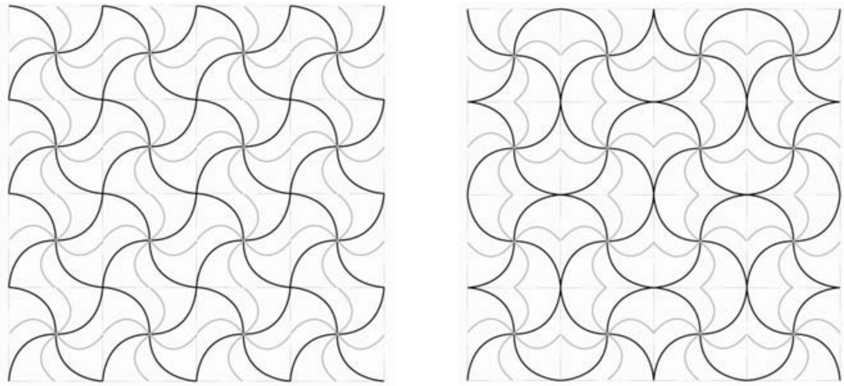
If real flatness is not to be had, there is still the next-best thing, *average* flatness. Here the surface has a raised texture of essentially the same height and depth throughout. The surface gives some of the appearance of flatness and shares some of its properties. This article mainly addresses some of the issues involved in making and manipulating surfaces of such a kind.

For a surface with a curved fold to be kept flat on average, a pattern of curves of roughly similar shape must typically be drawn on it. This can be done in one direction, with curves (for instance, waves) running parallel to each other. Can it be done in two? Clearly it can, for one instinctively flattens a cone shape (a surface created by a flat fold) by means of concentric circles. But another, less explored possibility is to divide a flat surface into a lattice of squares, triangles, or hexagons, and to place the identical curve pattern in each. This has the nice effect of shrinking the paper by the same amount in all places, so it is not forced to bend from the plane. And it not only maintains average flatness, but also yields a surface that is similar in outline to the one with which we started. (See the examples in Figure 1.)

However, this trick can't be done with every pattern, only with those that line up or *tessellate*—so that the left line in one tile's pattern turns into a right line in the tile next to it, and ditto for tops and bottoms.

2 Spiral Curvigami Tessellations

One ancient, well-studied pattern that tessellates very nicely is the vortex or spiral, so I want to spend a little time on it.



(a) (b)

Figure 2. (a) Regular square spirals. (b) Alternating spirals.

A vortex, whether in a bathtub or a galaxy, is nature's way of pulling material in a plane toward a center in the least objectionable manner, so it's an intuitive choice for shrinking a sheet of paper too. Liquid swirls were considered observationally by Leonardo da Vinci and patterns of ornamental spirals are to be found in the decorative art of many ancient cultures.

In origami too, spirals and vortex-like twist-folds have a distinguished pedigree, having been studied by, among others, S. Fujimoto, T. Kawasaki, Alex Bateman, Tomoko Fuse, Jeremy Shafer, Chris Palmer—indeed it sometimes seems by all the pioneers of the currently exploding field of *origami tessellations*. The spiral tessellations I'm introducing here are necessarily related to some of those more familiar ones in their underlying geometry, and they have other points in common too. But one difference is that the spirals here are drawn on the surface *as curves* and then folded *directly*—causing the paper to condense—rather than being created from straight folds of relatively free material in already condensed paper, folds that are then twisted into spirals. These spiral patterns are, for all that, one type of origami tessellation: they belong to the subset of tessellations that can be formed continuously with a lateral, bidirectional compression of a surface.

With a square and triangular grid, you can make spirals that curve in the same direction (Figure 2(a)) or you can alternate the direction (Figure 2(b)); the pattern will still line up. (With a hexagonal grid more thought is needed to achieve alternation.) Figure 1(c) is from a triangular grid with a unidirectional spiral; Figures 3(c) and 4 use a square grid. In

(a) (b) (c)

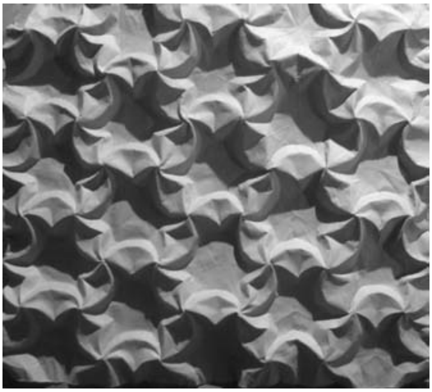
Figure 3. (a) *Hexagon Spirals.* (b) *Squeezed Hair.* (c) *Molly.*

these patterns the eye is naturally drawn to the shaded hollows between the ridges, but if you look at the vertices you'll see the pattern's spiral basis. Figure 5(a) is a fancy version of a spiral pattern based on a hexagonal grid; Figure 3(a) and (b) use simple spirals, also from hexagons.

Interestingly, an alternating spiral pattern compresses inward from the sides much less than a unidirectional pattern does. The degree of lateral compressibility is an important issue for any open-fold pattern, although it takes some practice to be able to recognize from a pattern drawing alone how well it will compress. I won't dwell on this subject here, but the issue of tangents, touched on below, bears on compressibility. It should be remembered that when a pattern contains curves it will not compress all the way, in the nice way that a Miura fold does. So the applicability of curving patterns for stents and such may be somewhat limited; but perhaps other uses can be found for them.

Note, too, that a regular division of the plane is not necessary for shrinking a sheet via spirals: any irregular polygonal lattice will do. Figure 6 shows a surface carved at random into irregular polygons, along with a (semiregular) spiral crease pattern for it. It is trivial to prove that any division into regular or irregular polygons will allow a spiral pattern to be created for it, and that the pattern will fold. It is less trivial to prove that a surface so divided and folded can always be made to lay flat—for the possible reason that this may not be true. In my own experiments, since the spiral within each polygon can be twisted with some independence from its neighbors, one always has a certain control over how flat or curved the overall surface will be. On the other hand, when the polygons are of a different size and the spirals in them are of a different height, the concept of average flatness loses some of its clarity.

Figure 4. *Ernestine.*



(a)

(b)

Figure 5. (a) *Fancy Hexagon Spirals.* (b) *Ben Gurion.*

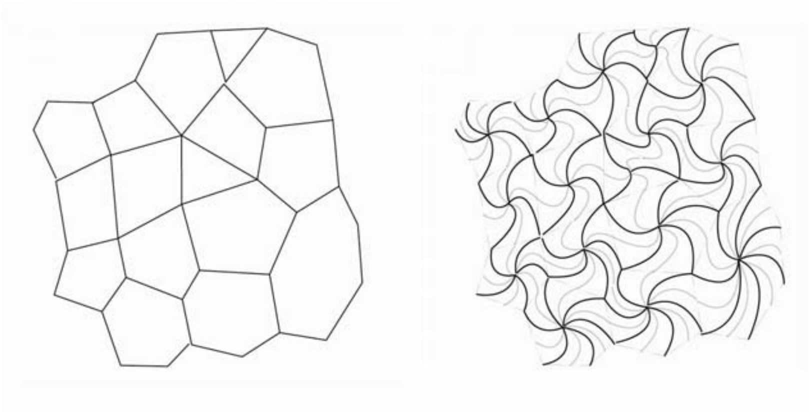


Figure 6. Spirals from an irregular grid.

As an autobiographical note, I came upon this subject of spiral tessellations only when seeking an elegant solution to a sculptural problem that was nagging at me: how to make from paper the dome of a person's head, which curves in two directions at once, as paper is loth to do. Many curve-based tessellations, while they can be kept flat, also introduce some bidirectional flexibility to a sheet of paper. Spiral ones happen fortuitously (see Figure 4) to look like hair.

Finally, to put this discussion of spirals back into perspective: spiral tessellations are just *one* kind of open-fold tessellation that will shrink a surface while preserving average flatness. There are many others (e.g., Figure 1(b)). Surfaces can also be shrunk without any tessellation at all using semiregular (Figure 1(a)) or random-crumple methods; and if edge proportions are allowed to change a great many other options are available. It seems that this field of compressive, flatness-preserving deformations of a sheet is still wide open for exploration in origami.

3 Folding Patterned Sheets

Let's move to our other main area of investigation. Once you have a surface with a raised pattern on it, what can you do with it? Specifically, can the usual origami manipulations done on smooth sheets be done on these textured ones too?

The answer, I'm afraid, is usually "no": most elaborate origami folding will typically be interfered with by the existence of a raised pattern. A counterexample among top-rank models is Roman Diaz' *Tiger's Head* (Fig-

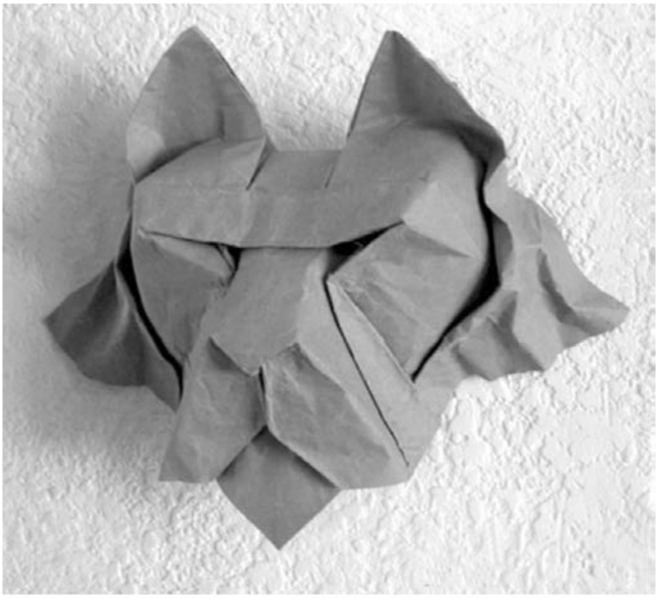


Figure 7. *Tiger's Head*, by Roman Diaz. (Folded by the author.)

ure 7: his design, my fold); but there the curves are put in at the final stage on the free flat edges that remain at that stage. Starting from the outset with a three-dimensional texture poses considerable difficulties for much origami. Having said that, folding a raised and especially a curved pattern around a corner line can create deep furrows and bulges that are visually quite arresting—enough by itself to make a fine model, as the beautiful 1976 Tower form by David Huffman, the great pioneer of curved folding, clearly demonstrates (Figure 8). Here, although the resultant shape has struck many people as wondrously complex, a crease pattern that folds to a similar form is actually quite simple (Figure 3; my reconstruction). I have tried absorbing some of its design principles in my own work (Figure 10).

The *Huffman Tower*, by the way, prompts a question that comes up more generally from various quarters when dealing with curved folds: is there any difference in principle between a curved fold and a straight one? Isn't a sine curve just a zigzag with the corners rounded off? In the case of the *Huffman Tower*, couldn't all the curvy lines have been replaced with straight segments, and the curving surfaces with flat ones? (And how about with my spirals?) This is not an insignificant question, and while the answer may be different in each separate instance it is always worth asking. There are some real differences between curved and straight folding (we await the full list ...) but the effect of curves can also be so hypnotic

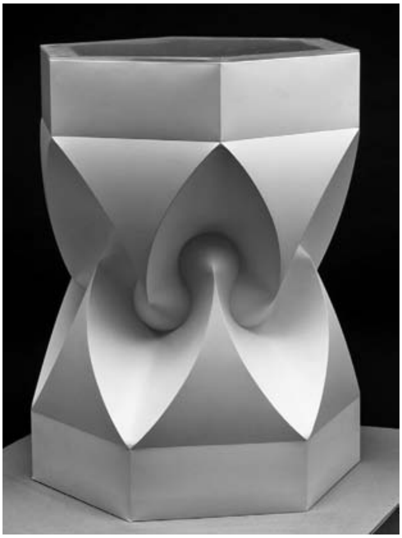


Figure 8. Mathematical paper folding, by David A. Huffman. (Courtesy of the Huffman family. Photo by Tony Grant.)

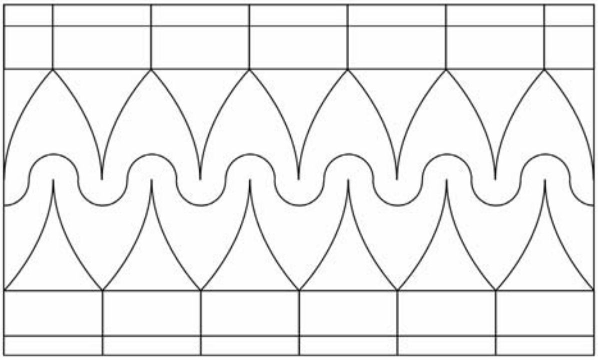


Figure 9. Reconstructed crease pattern for the *Huffman Tower*.

as to make us forget to check whether straight-line analogues exist. But let us leave that aside for now.

I want to consider what happens when a surface that is patterned in the way I've been describing is folded along a line—folded gradually anywhere from zero to 180 degrees. There are four different types of simple encounters of open folds (for now: mountain-valley pairs) with a corner line, and I'd like to show what happens in each.

Figure 10. *Jar of Muses.*

Figure 11 is an open-fold crease pattern, in which you are to imagine (or attempt) folding the more horizontal lines first into open mountain and valley folds, and then bending the pattern successively at each of the four vertical locations.

If you try bending the *straight-line* open-folds at A, the paper will resist. Eventually it will buckle, that is, it will form new fold-lines at awkward and unexpected locations. This is the corrugation effect, used for adding stability to flimsy sheet materials. Note that since the lines that intersect at A are all straight, there is nothing stopping you from folding them all the way into closed folds; A can then be folded without complaint.

At B, the horizontal open-fold lines, which are shown to be straight but may also be curved, meet line B from both sides *at an angle.* (Line B in fact will already be formed by having made the angled open-folds.) Bending the surface here can be done quite easily: the corrugation effect has disappeared. However, the result of such bending is that the height of the surface will compress along B, as the angles turn inward and trade some of their verticality for depth. If the open folds meeting B are straight lines, a 180° bend around B will close these folds completely.

At C, the horizontal lines are *arcs*; a hard fold along C itself encounters the same resistance as at A and for the same reasons. However, the *region* of C taken as a whole behaves just as the single line of B does; in fact it can be considered a stretched out version of B (one dimension stretching

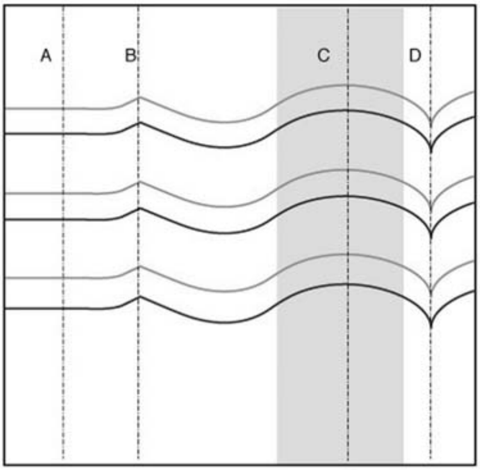


Figure 11. Encounters of curves with straight open folds.

into two!). Thus the entire region of C can be made into a corner that curves gradually, and if the corner is sharpened (edges bent back more), the furrows will deepen just as they did at B. The height will likewise shrink. But because they are curved, the folds will never shut completely. (On the other hand you are able to bend the surface back by more than 180°, indeed by more than 360°.)

At D, the open folds meet the line at a *tangent*: an angle of zero. Consequently there are no angles to rotate inward, and a fold here is not as disruptive to the vertical extension. It may be noted that this property of being able to meet a line at a tangent is one that curves possess and straight segments do not, so this is yet another answer to the question of what differences there are for folding purposes between curves and straights.

None of the above is earth-shaking mathematics, but it does account for many of the simpler cases of raised-pattern folding, so it needs to be stated. Fancier permutations (nonparallel mountains and valleys, mountain + mountain + valley open folds, open folds that meet curves, etc.) are of course possible too.

4 Concluding Thoughts

I think this is enough of a sketch to suggest some of the issues that come up when forming and manipulating curve patterns. I want to conclude with a few thoughts about method and the links and tensions here between art and science.

For experimental work, the ideal medium for curved folding is a foil-backed paper (preferably stiff foil, 50–100 microns thick) rather than paper on which the pattern has been plotted and scored. The reason is not aesthetic—aesthetics may in fact favor plain paper—but rather that foil-paper, which holds a curvy shape without springing, also allows you to erase a line with a fingernail and shift your curve at will. This helps avoid a trap one may fall into, especially if one takes an analytic rather than experimental approach to this field: the assumption that if a curve representing a particular function creates a nice effect, the effect is due to the function and no other curve can accomplish approximately the same thing. You can avoid such fixation by trying out other curves and straight-line variants—but that requires a comfortable medium for doing so. (This of course is not to say there are no specific curves that optimally solve well-defined problems, or that mathematics is not useful for finding them. But for most curved origami sculpture, at least in my experience, the details of a curve are not very determinative. Direction of curvature matters a great deal: degree and rate of curvature, usually less so.)

A similar fixation tends to happen with regular patterns, so these should always be tested against the most irregular version of the same pattern to see what in fact is doing the work.

Irregularity versus regularity, plotted and repeated patterns versus free-form and varying curves—all this raises another issue, this time a purely aesthetic one: the old, grand tension between mathematical optima and repeatability on the one side, and romantic and individual expression on the other. This is rather a large topic to broach just here: entire cultures are defined by where and how they come out on this continuum. I will say only this. Certainly in the animal world, the outline curve is a prime bearer of information about a living form's identity and emotional state; and in

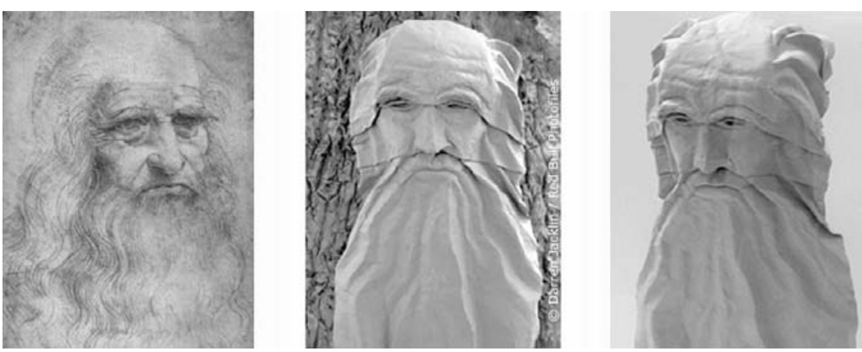


Figure 12. *Triptych of Leonardos.*

the handwriting and drawing of humans, the curve or the flourish is where personality is looked for—and found. It would be a shame if origami's inherent tendency for pattern and repetition should give rise in this new field to mainly a cold and crystalline form of model design, to the calculated rather than the expressive. Curved folds leave a great deal of freedom for the shaping of three-dimensional form: too much freedom, to many folders' tastes. But where there is freedom, there can also be—individuality.

The *Celes* Family of Modular Origami

Miyuki Kawamura

1 Genesis of *Celes*

Sometimes when square paper is cut from a larger sheet, long, slender paper strips remain. I wanted to make origami works with these paper strips and so I designed several models in 2001 and 2002. *Celes* [3], shown in Figure 1, is one of my modular works that is made with paper strips. The basic model is made with 30 strips in the proportions of 1 by 6, but other proportions can be used; 1 by 5 or longer strips are required.

The name *Celes* came from the word *celestial* because the model has 12 stars on the surface. *Celeste* might be the name in English, but the pronunciation of *Celes* is easier for me.

2 Variations of Symmetry

Polyhedral symmetry provides basic and important guidelines for the design and assembly of any modular work. There are basically three different kinds of symmetry, which dictate, among other things, the number of units needed for the structure. Phrases such as "assembled with 6, 12, or 30 modules" might be familiar to modular workers; these numbers, such as 6, 12, or 30, correspond to the number of edges in the underlying regular polyhedron. We can make two different types of models with 12 modules—those based on the cube and the octahedron. There is the same situation for 30-module models too, in which either the dodecahedron or icosahe-

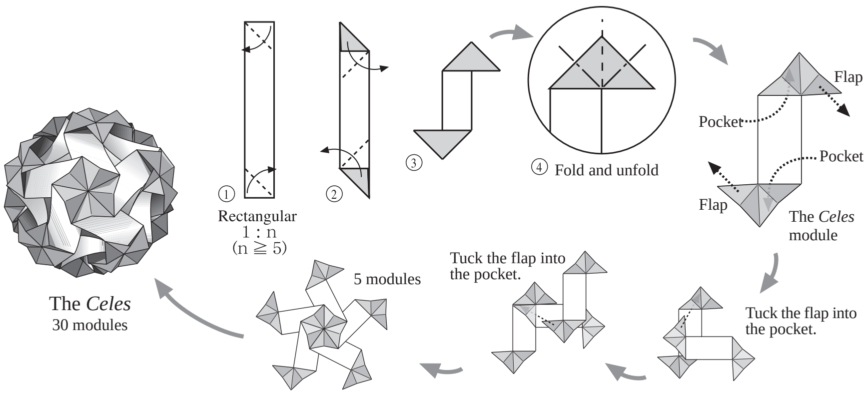


Figure 1. The *Celes* module.

dron is the underlying polyhedron. So, five different models corresponding to the five regular polyhedra can be made with one kind of module. It is possible to make five different models with the basic *Celes* modules, too, but one of the models made with 30 modules is not stable.

Generally, we can also make other, more complex models with larger numbers of modules. For example, polyhedra are possible using 24 units, 60 units, 90 units, and so on. These models correspond to the *semiregular* polyhedra; their symmetry is based, in turn, upon the symmetry of one of the five regular polyhedra. The symmetry of a prism is also available. We can design many variations of modular works by making use of different types of polyhedral symmetry.

3 Variation of Inside Out

The basic *Celes* modules can be assembled as a model turned inside out as well. It is very hard to complete this model because all of the connection parts are inside; the reader is encouraged to try.

4 Changing Angles of Connections

More exciting arrangements can be made by changing the angle of connection of the *Celes* module. The form of the connection of the basic module is a right triangle, as shown in Figure 2. The key angle inside the triangle is denoted by θ in Figure 2 and in the following discussion. This angle θ can be changed by redesigning the connection of module: specifically,

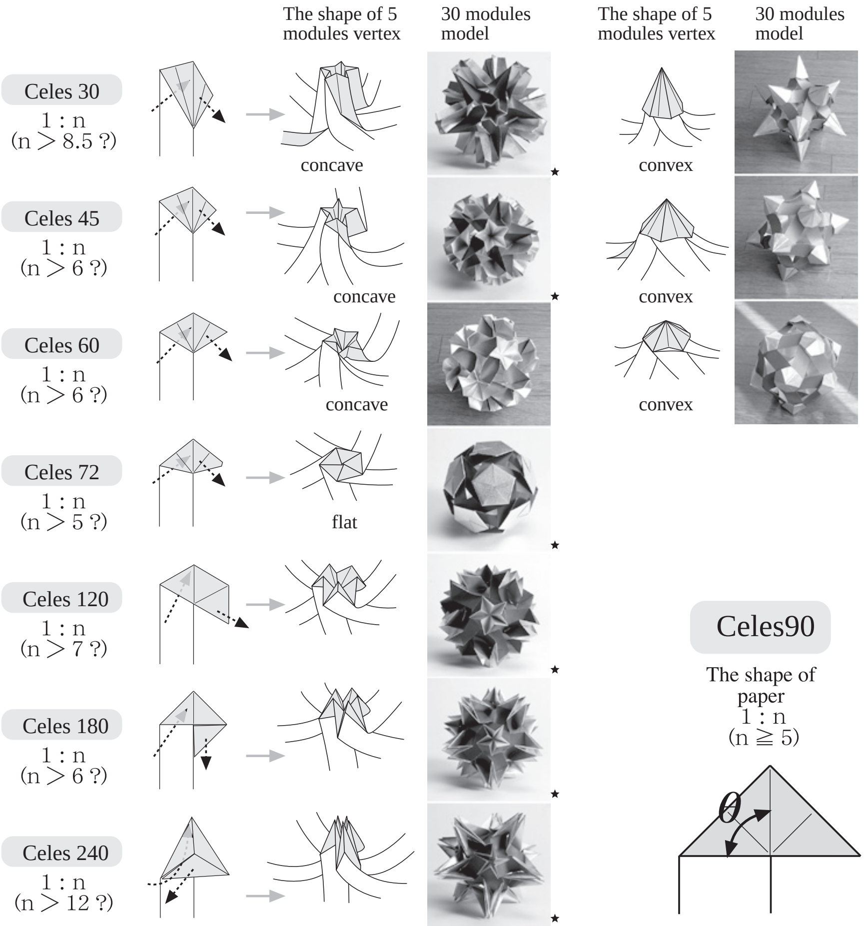


Figure 2. The *Celesθ* family.

by varying the angle at which each end of the strip is folded over. This defines a family of modules, parameterized by the angle θ, and so we call this family *Celesθ*. Individual members are named by replacing θ with the value of the angle; thus, the basic *Celes* module is called *Celes90*. Angle θ can be changed continuously from 0 to 360 degrees, so there are infinite variations of the *Celes* module.

To take just one example, in the complete model of the basic *Celes*90 model, the symmetry is the same as that of the icosahedron. Each star on the surface is made from the ends of five modules. With this symmetry, when the angle θ is smaller than 72 degrees, the curvature of a surface star is positive and we can make two different types of *Celes*. One has concave stars and the other one has convex stars. When θ is bigger than 72 degrees the star shape is wavy because the interior angles exceed 360 degrees, and when it is just 72 degrees the star is flat. So 72 degrees is the boundary between convex/concave and wavy stars.

When a star on the surface is made from four modules, the boundary angle is 90 degrees, and when a star is made by three modules, the boundary is 120 degrees. The relation between the shape of the star and the connection angle θ is the same as the relation between the form of a curved surface and its local curvature.

5 Bridge

Generally, many origami modules consist of two different and distinct functional regions. One region forms the connections between modules, e.g., pockets, flaps, and other assembly structures. The other part is not used in the connection between modules; instead, it extends from one connection region to another. That part is called the *bridge* [1, 2]. If the connection and the bridge are independent of one other, we can make the bridge any shape without influencing the connection. So there is some level of freedom in their arrangement.

In case of the basic *Celes* module, the two right triangles are the connection and the middle part is the bridge (Figure 3). We can fold the bridge into any shape: crane, flower, beetle, dragon, devil, etc., without affecting the connection. Because of this, there are innumerable variations of the bridge and it is difficult to describe all possible variations.

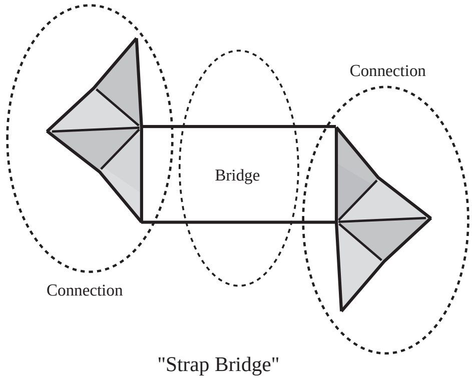


Figure 3. Bridge.

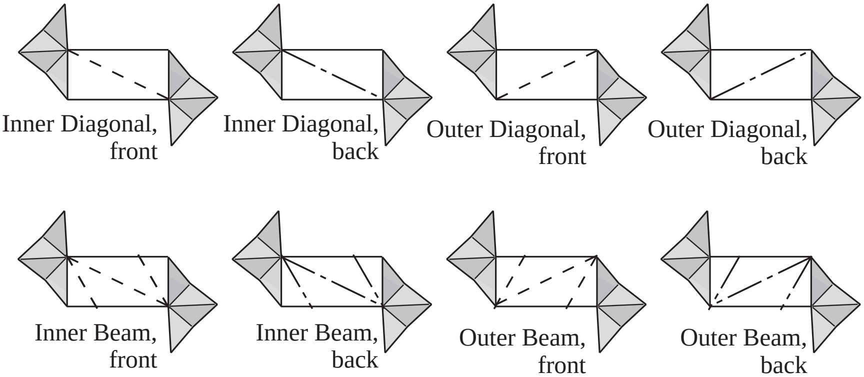


Figure 4. Diagonal and beam bridge.

However, we can begin with the simplest variations of the bridge. When the bridge has no crease line, we call it a *strap bridge*. If it has one crease line along the diagonal of the bridge, we call this the *diagonal bridge* (Figure 4). There are four possibilities for this crease.

As a second example, let's add further creases to the bridge. In this case, the bridge has three creases (Figure 4). This structure is named *beam bridge*. Fold along the diagonal line first, and then wrap each end around the raw edge of paper. As with the diagonal bridge, there are four variations; in each variation, all three creases are of the same type (mountain or valley).

6 Second Bridge

Each connection of the *Celes*90 module is made by two small right triangles. We redesign the module, split the two triangles and make a new bridge between these two (Figure 5). The new bridge is called the *second bridge*, and we rename the original bridge to be the *main bridge*. The complete module is called the *Celes spread module*. We can make the same treatments of the second bridge as on the main bridge, e.g., strap, diagonal, beam, and so forth.

Several examples are shown in Figure 6. All of these models are made from same length strips but the ratio of the lengths of the main bridge and the second bridge is different. The main bridge is shaped as a beam bridge (three diagonal creases) and the second bridges are shaped as strap bridges (no diagonal creases).

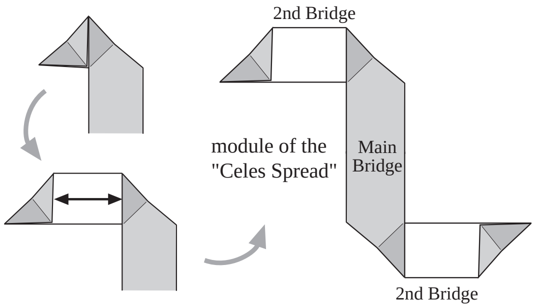


Figure 5. Second bridge.

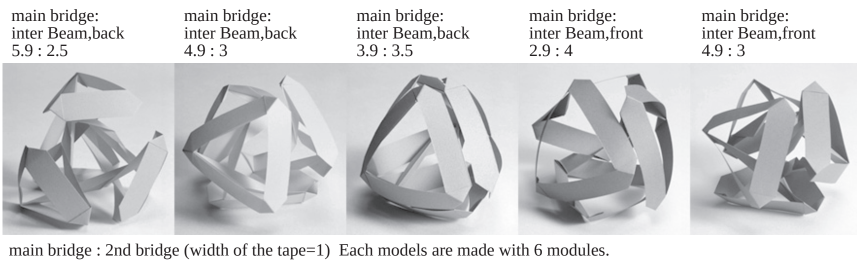


Figure 6. Main and second bridge.

7 Local Uniting Relation

The second bridge has a pocket or a flap on each end (Figure 7). There are two different ways to lay out the flap and pocket. Type 1 is called *basic* and Type 2 is called *twist*. The creases on the flap and the pocket can be independently chosen to be mountain or valley, giving eight kinds of module. One pocket has two slits, one on the front side and the other on the back side. When we choose two modules arbitrarily from the eight possibilities, the pattern to assemble is dictated by the choice of mountain or valley creases on the pocket and the flap, and so only one way of assembling the two is allowed. This property is called the *local uniting relation* of the module. Generally, many kinds of modules have this property, which strongly constrains the assembly and shapes of models made from the modules, and therefore dictates important characteristics of the modular works.

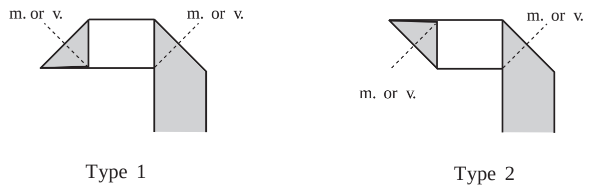


Figure 7. Two types of layout.

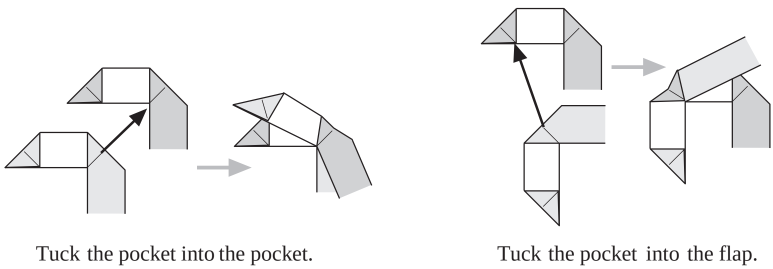


Figure 8. Flap and pocket.

8 Crease Pattern Formula

Here we generalize the module of the basic *Celes*90 and provide a compact notation for describing them. As noted earlier, the arrangement of the positions of the pockets and the flaps can be freely chosen. A flap has the same structure as a pocket, so we can tuck the pocket of one into the pocket of the other, or we can tuck the pocket of one into the flap of the other (Figure 8). Note that the number of pockets of the module need not be two and the shape of a module does not need to be symmetrical. (For that matter, a complete model does not need to be a closed polyhedral form.)

And so a model can be arbitrarily complex by repeatedly adding elements from the simple set of structures along the strip. Figure 9 shows an example of a generalized *Celes*90 module constructed according to this prescription. This module can, in fact, be assembled with copies of itself. The lower diagram in Figure 9 shows the crease pattern of this module.

Here is how we describe the module structure concisely:

- Between each crease, we give an integer that gives the length of the bridge as a multiple of the width of the strip. So, for example, in Figure 9, the numbers $1, 2, 4, 3, 1$ are the lengths of each bridge.

- We use brackets $[\ldots]$ to denote the two ends of the strip.

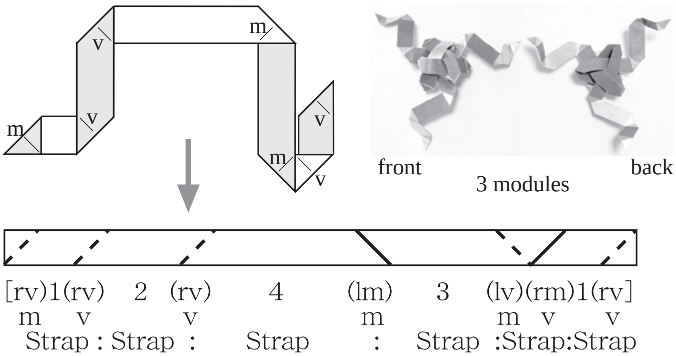


Figure 9. Crease pattern formula.

- We use the letters r and l to denote the slopes of the crease lines; r for a line rising from left to right, l for a line descending from left to right.
- We use the letters m and v to specify whether the fold is mountain or valley.

In general, there are four possible combination of r, l with m, v in each parenthetical pair, i.e., (rm), (rv), (lm), and (lv). Therefore, if a module has N pockets/flaps, the upper limit of the number of the kinds of shapes of the module is 4^N. However, this formula includes duplicates. For the example shown in Figure 9, there are four identical modules with different formulas:

$$[rv)1(rv)2(rv)4(lm)3(lv)0(rm)1(rv] \text{ (original module),}$$
$$[rv)1(rm)0(lv)3(lm)4(rv)2(rv)1(rv] \text{ (right-left reversal),}$$
$$[lm)1(lm)2(lm)4(rv)3(rm)0(lv)1(lm] \text{ (r-l and v-m reversal), and}$$
$$[lm)1(lv)0(rm)3(rv)4(lm)2(lm)1(lm] \text{ (right-left, r-l, and v-m reversal).}$$

With no other forms of duplication, the lower limit of the number of the kinds is 4^{N-1}. But we must also consider the number of forms that does not change with right-left reversal. This number changes with the parity of the module. So, the total number of different kinds of module with N pockets is as follows:

$$4^{N-1} \text{ if } N \text{ is odd,}$$
$$4^{N-1} - 2^{N-1} \text{ if } N \text{ is even.}$$

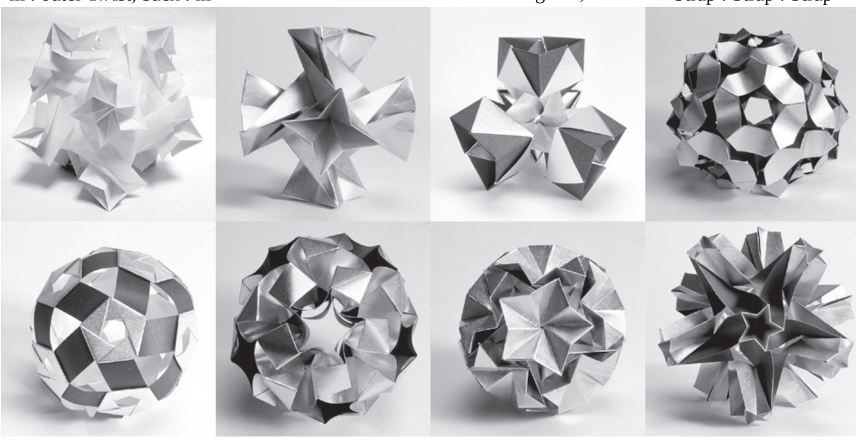
The m and v in the middle row of the three rows of symbols in Figure 9 indicate mountain or valley fold along the center line of each pocket.

"Whip"
30 modules
[rv)(lv)3(lv)(rv]
vmmv
m : outer Twist, back : m

"Beam 60"
12 modules

"Celes Beam"
6 modules
[lv)1(lv)7/3(lv)1(lv], mvvm,
inner Diagonal,back :
outer Beam, back :
inner Diagonal,back

"Celes Spread"
30 modules
[rv)1(lv)3.5(lv)1(rv]
vvvv
Strap : Strap : Strap



"Hole"
30 modules

"Cookie Maker"
30 modules
[rv)1(lv)(lv)1(rv], vmmv
Strap : Strap : Strap

"Celes Beam 90"
30 modules
[rv)(lv)1.5(lv)(rv], vvvv
m : outer Beam,front : m

"Celes 30"
30 modules

Figure 10. Variations of modular.

When there is no crease through the center line, it is indicated with a ".". The words in the third row indicate the type of each bridge, e.g., strap, diagonal, and beam. A shape of a module that belongs to the *Celes* family is uniquely described by this three-line formula, which we call the *crease pattern formula* for the module.

9 Variations of Modules and Assembly

One of the merits of using this formula is that it leads to automatic design of a module directly from its symbol. Figure 10 shows some models from the *Celes* family, along with the names that I have given them. The diagrams of *Whip* are published [4].

Since the bridge of a *Celes* module has only one layer, it is easy to change the form. A lot of interesting models that have beautiful curves are designed with long bridge modules. Some of these are shown in Figure 11 as well.

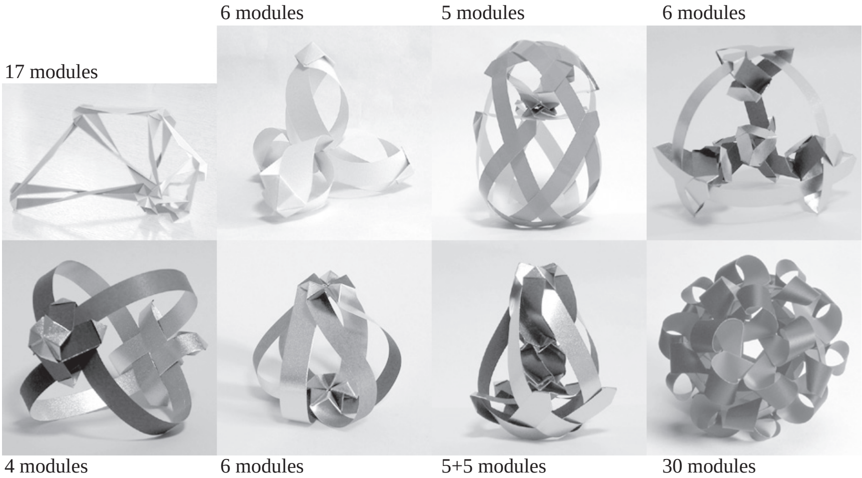


Figure 11. Variations of assembly.

10 Summary

The *Celes* module is very simple, and yet has great potential. In this module, the connection and bridge are separated clearly, so it is easy to construct arrangements of the module. The greatest feature of the *Celes* module is the flexibility to create pockets in arbitrary places within a module. Besides the work described here, many varieties have been made by many people. For example, Dr. Toshikazu Kawasaki has designed some kinds of the *Celes* family. Although the construction method of the *Celes* module is not yet common, it lends itself to a systematic approach for module design, and I expect that many new modular works will appear in the future.

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Fractal Crease Patterns

Ushio Ikegami

1 Redesigning the *Maekawa Pyramid*

Maekawa's pyramid model (Figure 1) is one of the infinite folding models he presented in [4]. By *infinite folding*, we mean that in the limit of infinite iterations, it produces an infinite number of branches in four directions from a finite square. Its crease pattern for any nth iteration consists of two kinds of generators. We can determine the foldability of infinite folding models (not flat foldability but the possibility of infinite iteration) by the existence of such finite generators and their relative arrangement within the crease pattern.

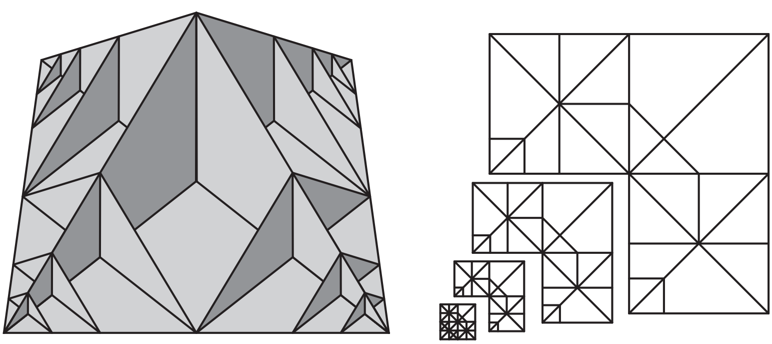

Figure 1. *Maekawa Pyramid*.

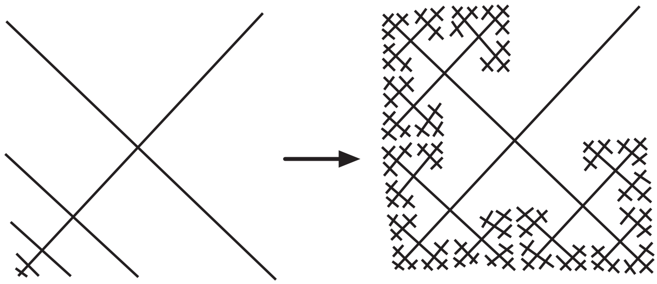

Figure 2. Sketch for the new design.

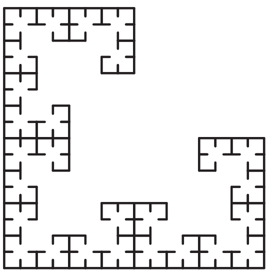

Figure 3. Pyramid curve.

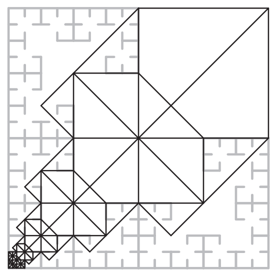

Figure 4. Overlap.

Now, in the *Maekawa Pyramid*, there are four main branches that, in the limit, produce an infinite number of secondary branches. However, each secondary branch doesn't branch any further after it comes off of a main branch. This raises the question: is it possible to fold an infinite number of branches from each secondary branch and subsequent branches as well? Figure 2 shows just such a branch pattern. It is much more complicated than the original pattern and its accumulation points form a curve shown in Figure 3. We will call this the *Pyramid curve*.

Let us use the *Maekawa Pyramid* itself for this new design. The accumulation points of the crease pattern form the same Pyramid curve as the accumulation points of the branch pattern. Furthermore, the curve overlaps the area that becomes the surface of the pyramid (Figure 4). The infinitely folded limit region cannot be made from a smooth surface. Thus, the *Maekawa Pyramid* itself cannot grow further; there is not enough paper. The crease pattern must be modified.

Thus, the Pyramid curve and the smooth surface must be separated within the crease pattern. And the individual contraction of generators shown in Figure 5 separates the curve and the surface because the contraction keeps the accumulation point fixed while the generators become smaller and smaller.

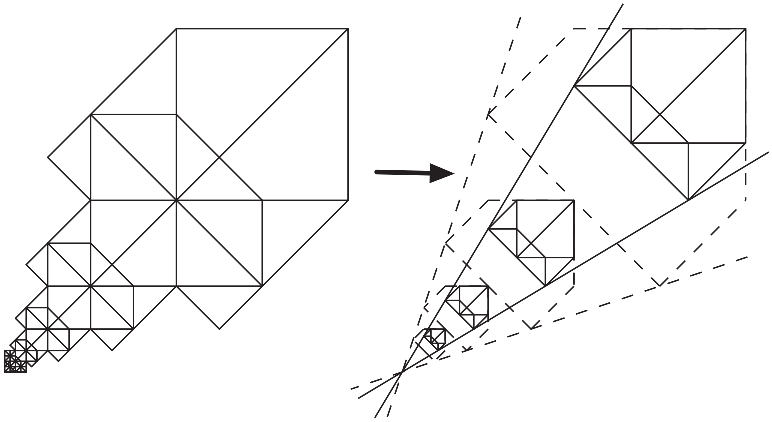

Figure 5. Contraction.

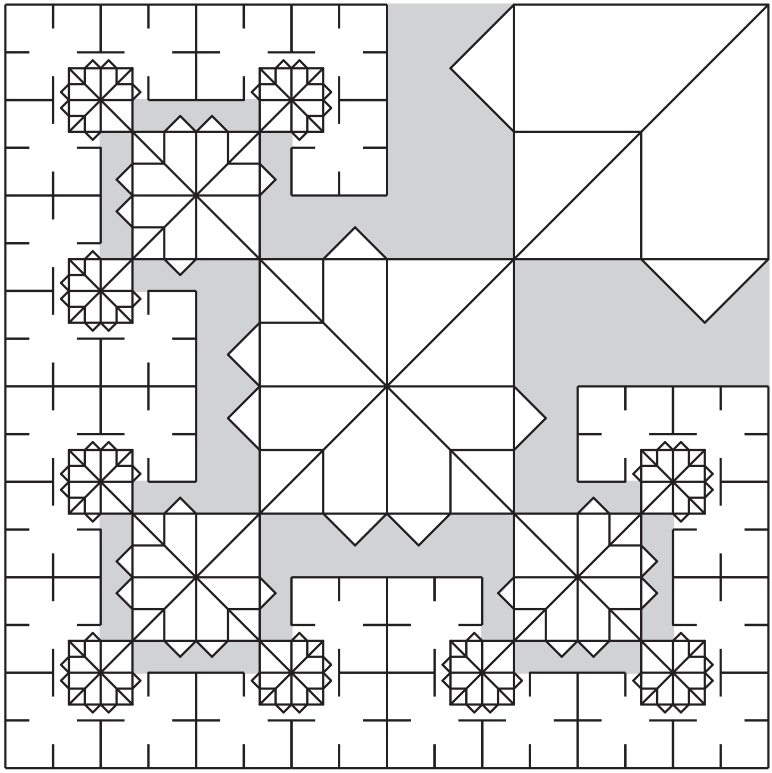

Figure 6. New composition.

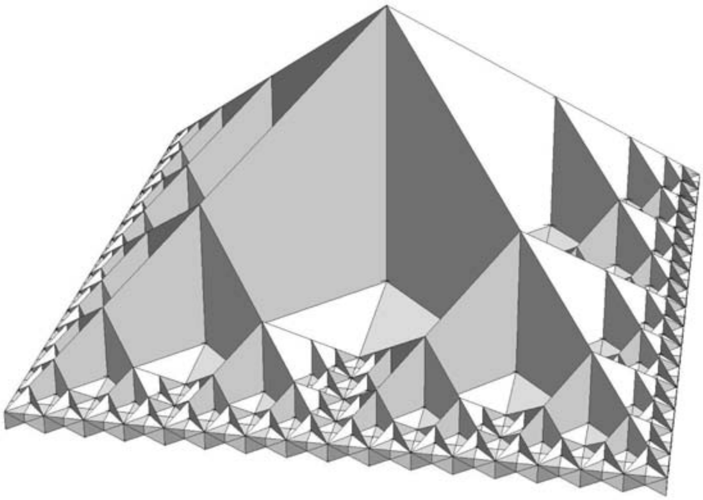

Figure 7. New pyramid.

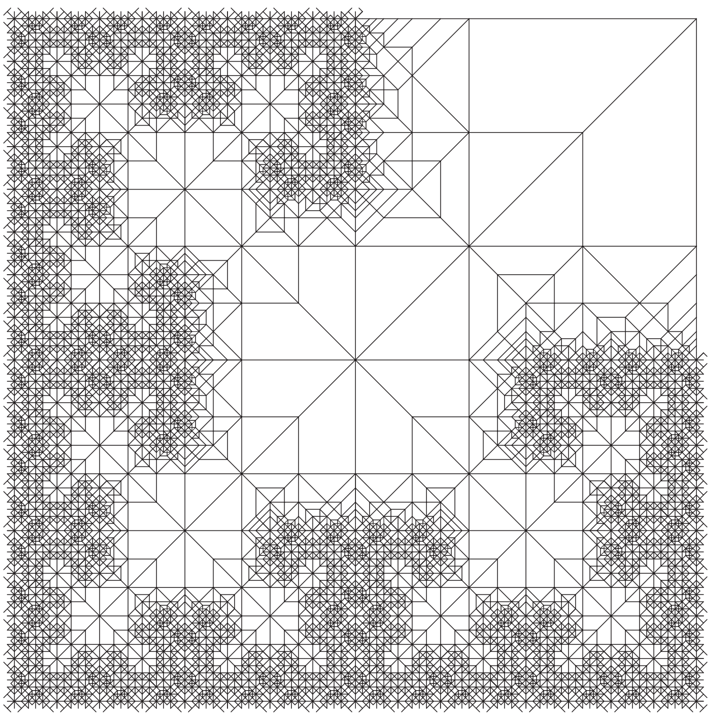

Figure 8. Crease pattern.

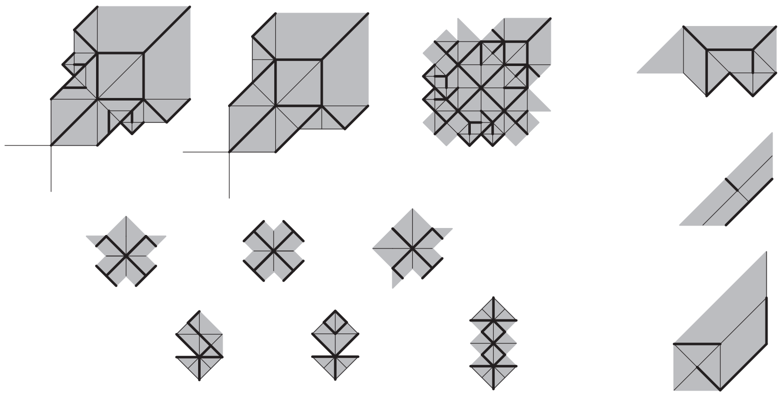

Figure 9. Crease-pattern generators.

Our next task is to fill up the space that was created by the contraction (and is colored gray in Figure 6) using a recursive crease pattern. I found such a crease pattern and resolved it into a finite set of generators. By combining all generators at the appropriate scales, the new pyramid may be completed as shown in Figure 7. Small pyramids protrude on the bottom of the folded structure; they follow the Pyramid curve. They can be folded flat underneath but I left them pointing downward to keep the crease pattern simple.

Figures 8 and 9 show the crease pattern at the fifth iteration and the generators and their representative tiles. Figure 10 illustrates their tiling pattern and thus establishes the foldability of this infinite folding model. My trial and the result of the work described here is also discussed in [2] and [3].

2 Hausdorff Dimension of the Pyramid Curve

The calculation of the Hausdorff dimension dim_H is generally difficult. But in this case, it is relatively easy and the Pyramid curve turns out to be a fractal set. Let C be a Pyramid curve of base length and height 1. C is self-similar, because there exist similarity transformations

$$f(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), \; g(x,y) = \left(1 - \frac{1}{2}x, \frac{1}{2}y\right), \; h(x,y) = \left(\frac{1}{2}x, 1 - \frac{1}{2}y\right)$$

such that

$$C = f(C) \cup g(C) \cup h(C).$$

Take an open set A as shown in Figure 11.

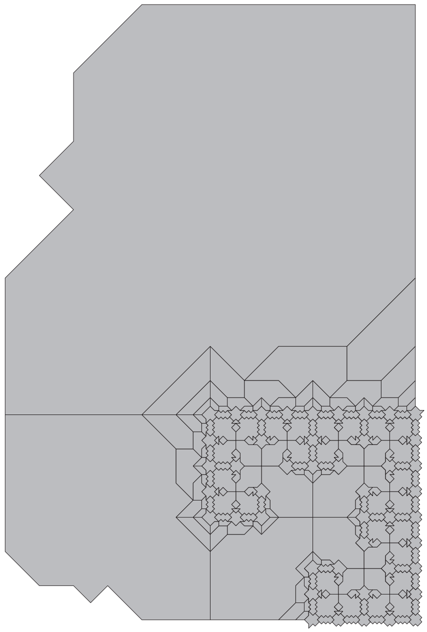

Figure 10. Tiling pattern.

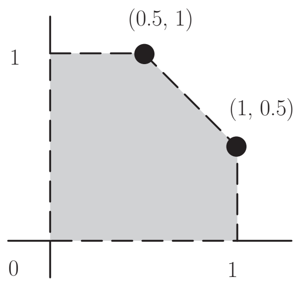

Figure 11. Set A.

Functions f, g, and h satisfy the open set conditions

$$f(A) \subset A,\ g(A) \subset A,\ h(A) \subset A,$$
$$f(A) \cap g(A) = \phi,\ f(A) \cap h(A) = \phi,\ g(A) \cap h(A) = \phi.$$

Hence, $dim_H(C)$ is equal to the similarity dimension of C $dim_S(C)$, which is the solution of $(1/2)^s + (1/2)^s + (1/2)^s = 1$.

Thus, $dim_H(C) = \log 3/\log 2 = 1.58\cdots$, which exceeds its topological dimension of 1. Therefore the Pyramid curve is fractal.

3 The Koch Curve as a Mountain Crease

The famous Koch curve K is defined as the limiting figure of a polygonal curve sequence $\{K_n\}$. Is it possible to use this curve as a flat-foldable crease pattern? (See Figure 12.)

For any given $n \in N$, place one of the curves K_n in the interior of paper as a set of mountain folds. It is obvious that this crease pattern by itself is not foldable. First of all, the Koch curve crease has its end points in the interior of paper. The real question is whether there is some additional crease pattern T_n such that the combination $T_n \cup K_n$ is foldable. As it turns

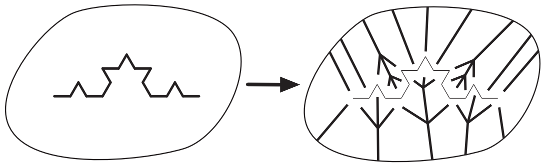

Figure 12. Is the Koch curve foldable?

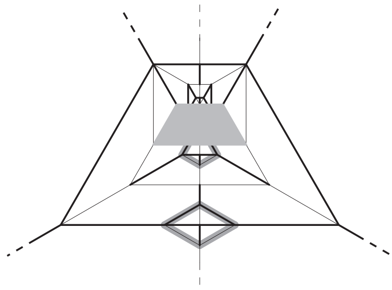

Figure 13. Entire view of the additional crease pattern.

out, there is; I was able to find a concrete example of $\{T_n\}_{n\in N}$, which is shown in Figures 13–17.

Let T be $\lim_{n\to\infty} T_n$. It has a set of accumulation points that correspond to K placed into the middle of it. Moreover, it is gained by open sink-folding at the tip of a single-vertex fold. In other words, the paper doesn't have to be bounded.

However, there is a problem. As you may notice, the highlighted zigzag crease in Figure 14 doesn't appear on the generator that covers the crease.

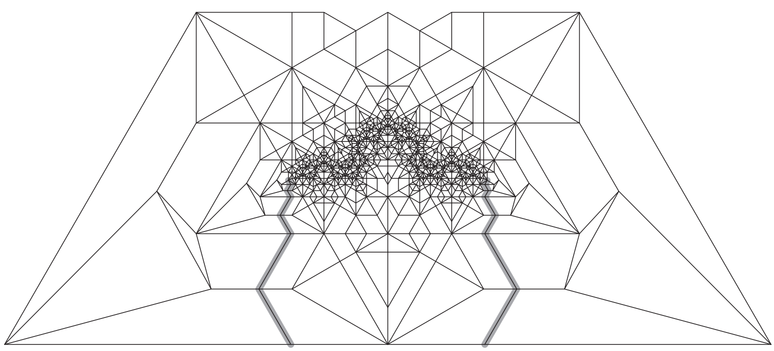

Figure 14. Blow-up of the center part.

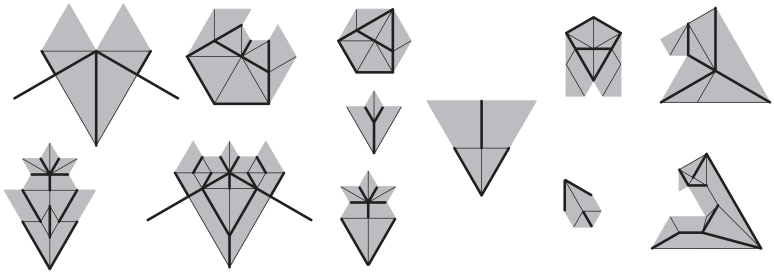

Figure 15. Generators for the center part.

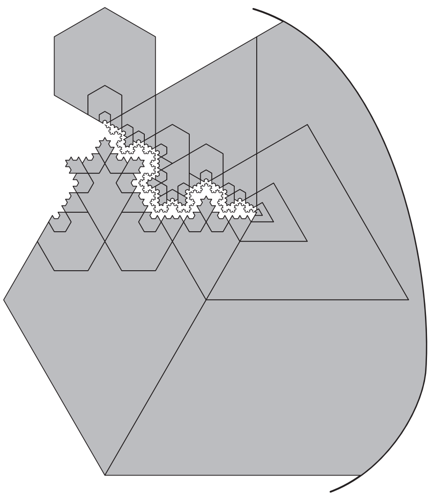

Figure 16. Tiling and Koch crease.

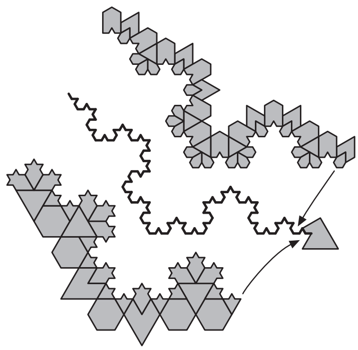

Figure 17. Detail.

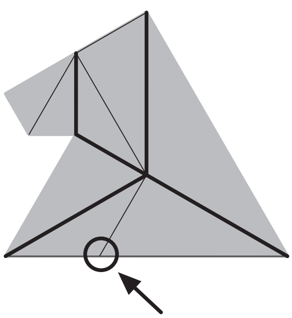

Figure 18. Starting point.

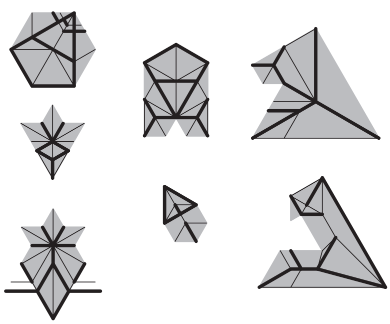

Figure 19. Altered generators.

This is because its location depends on n since its end point connects with the point indicated in Figure 18. Because this part of the pattern varies with the iteration order n, foldability at the limit $T \cup K$ is not yet established; we need to fix this crease on the generator somehow. Figure 19 shows altered generators with the zigzag crease now fixed upon them. In this case, a total of seven generators had to be converted. But by doing so, the foldability of the limit $T \cup K$ now becomes evident.

4 Creating a Snowflake Curve by Folding

We close with an open problem: is it possible to *create* a snowflake curve by folding? This was actually an earlier project for me than the two already described, but it is much more difficult—in fact, it is still open. So far, the trial crease patterns I have tried, including the one in [1], have

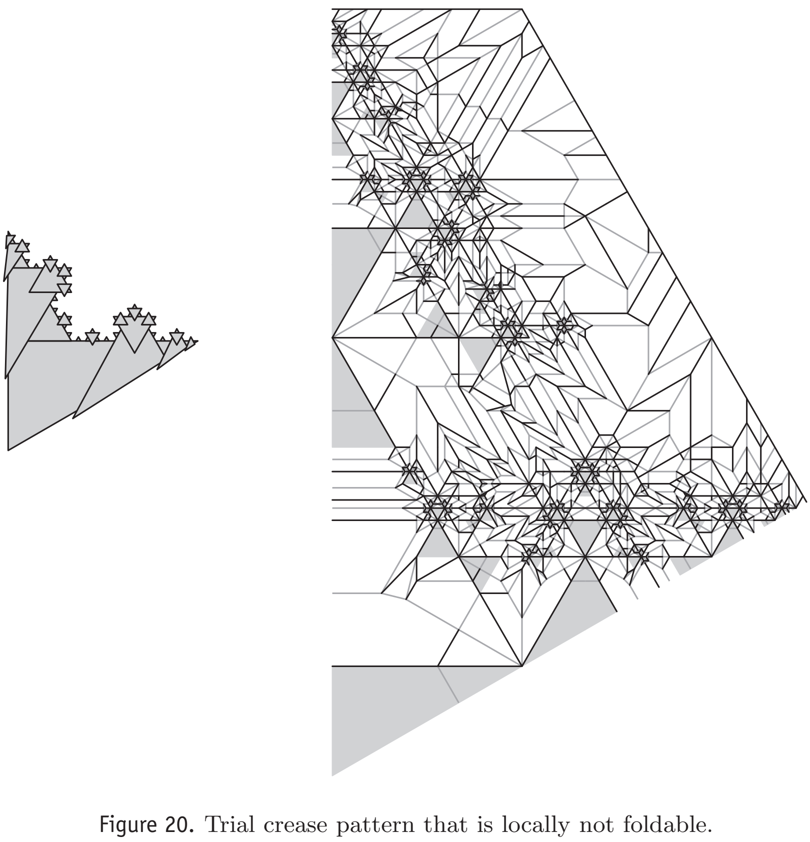

Figure 20. Trial crease pattern that is locally not foldable.

required irregular squash folding. This suggests that an infinite number of different types of generator may be required. As an inspiration to future investigators, I show one possible trial pattern in Figure 20.

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Constructing Regular n-gonal Twist Boxes

sarah-marie belcastro and Tamara Veenstra

1 Introduction

Among her one-piece boxes, Tomoko Fuse has a number of polygonal twist boxes [3, 4]. The crease patterns and folding sequences are structurally similar: divide the paper into $(n+1)$ths, fold across these $(n+1)$ths at some height h, fold some angle α emanating from each intersection of the vertical/horizontal folds, overlap the two ends of the paper, and collapse the twist.

Question. Can we generalize one of Fuse's constructions to create an n-gonal twist box for any n? That is, can we construct an n-gonal twist box from a 1×1 (or $1 \times m$) piece of paper by dividing the paper into vertical $(n+1)$ths, marking a horizontal height h, folding diagonals d in the resulting rectangles (formed by the height h fold line, the vertical folds, and the raw edge of the paper), making some folds to form the body of the box, overlapping the ends of the paper, and collapsing the twist? Better yet, can we find a formula for h in terms of n, so that the entire box construction is determined by n?

Answer. Yes! We will show how to construct this box for any n.

More precisely, we discuss the following mathematical considerations involved in proving that such a construction will work for all n. In order

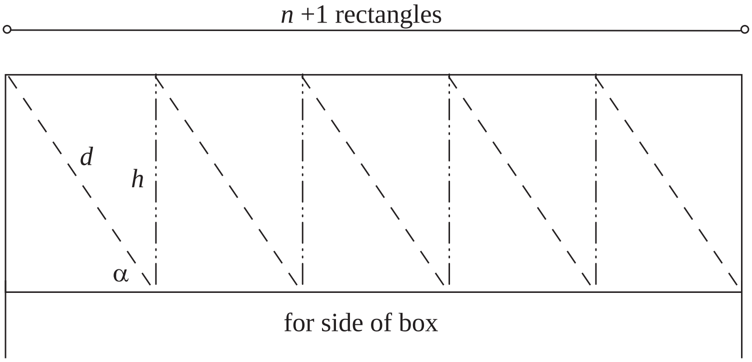

Figure 1. Part of the crease pattern to produce an n-gonal twist box.

for the twist to collapse so that the bottom of the box lies flat and has no hole in the center, the angle α formed by d and the vertical creases must be exact. The height h is determined by the angle α, which is in turn determined by n. We must also examine the paper between the twist center and the raw edge of the paper, and compare the length of d with the diameter of the box body to verify that the raw edges may always be contained within the body of the box. In constructing a folding sequence, we will need to determine a crease for either h or α; thus, we will consider which we can more easily and accurately find. Finally, we will examine the case of large n, give folding instructions for a 17-sided box, and look at the limiting (circular) case.

2 Determining α and h as a Function of n

We will first examine conditions on α in order to construct a regular n-gon. Each (interior) vertex of the rectangles in the crease pattern in Figure 1 has the same arrangement of angles. The sum of the angles around such a vertex before folding is $\pi = \frac{\pi}{2} + (\frac{\pi}{2} - \alpha) + \alpha$, and after folding it must be the interior angle of an n-gon, namely $\pi(n-2)/n$. Recall that the vertical creases h will be mountain folds and the diagonal creases d will be valley folds. The act of folding changes the sign of the angle between the mountain and valley folds, so we obtain

$$\frac{\pi(n-2)}{n} = \frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right) + \alpha = 2\alpha.$$

In other words, $\alpha = \pi(n-2)/(2n)$. Since α is half of the interior angle, the diagonal d bisects the interior polygon angle. This means it will cross

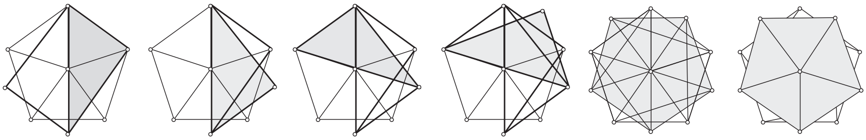

Figure 2. Intermediate folds for the twist of a pentagonal box.

through the polygon center, and thus our completed box will have neither holes nor paper intersections. A visual demonstration of this for $n = 5$ appears in Figure 2, where a sequence of theoretical partial-twist folds is given.

In general, for any n, the cumulative folded angle (for all vertices) is $\max(\frac{\pi}{2}, 2\alpha)$. If $n \geq 4$ then $\alpha \geq \frac{\pi}{4}$ and the cumulative folded angle is 2α. When $n = 3$, we have $\alpha = \frac{\pi}{6}$ so that the cumulative folded angle is $\frac{\pi}{2}$. Figure 3 shows the shape that is formed as a result of using our folding sequence in this case. While we can still construct a triangular box this way, there is some extra paper that must be tucked away.

Now that we have determined α in terms of n, we can construct h. We examine a triangle from the crease pattern for the twist as in Figure 4. The angle α is part of a right triangle with opposite side length h, adjacent side length s, and hypotenuse length d.

This shows that $h = s\tan(\alpha)$, and, given $1 \times m$ paper, $s = 1/(n+1)$. Thus, the formula for h in terms of n is

$$h = \frac{1}{n+1} \tan\left(\frac{\pi(n-2)}{2n}\right).$$

The height h is not particularly easy to approximate in general. In Section 4 we will discuss methods for constructing α and h.

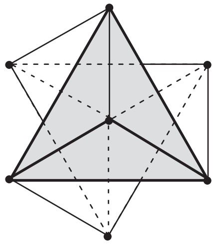

Figure 3. The twisted box when $n = 3$.

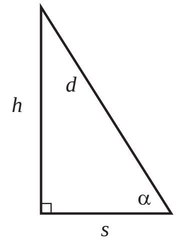

Figure 4. The basic triangle.

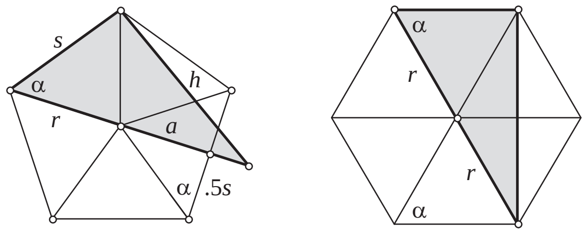

Figure 5. Pentagonal and hexagonal folds.

3 Differences for Even and Odd *n*

When folding the crease pattern from Figure 1, one sees that for $n \geq 4$ the raw edge of the twist will be a regular n-gon either coincident with the bottom of the box or rotated by $\frac{\pi}{n}$. Examples of the two cases are shown in Figure 5. To determine when each of these two cases will happen, we need to examine the placement of the diagonal d after completing the twist. Because d bisects the interior n-gon angle, the point where the diagonal intersects the raw edge of the paper lies at an opposing vertex of the polygon when n is even, and at the midpoint of an opposing edge when n is odd. We would like to compare the length d to the diameter of the n-gon, to see when the paper between the twist center and the raw edge will be contained within the boundary of the n-gon.

Let us consider our n-gon as inscribed in a circle. Radii of the circle partition the n-gon into n isosceles triangles with side lengths r and $s = 1/(n+1)$. Each isosceles triangle has altitude a. When n is even, we compare the length of d to $2r$, and when n is odd, we compare the length of d to $r + a$.

To calculate d we will use two similar right triangles, both with angle α, as in Figure 6. The larger triangle is part of the crease pattern and the smaller triangle is contained in an isosceles triangle of the n-gon. Comparing the hypotenuse and the side adjacent to the angle α, we have

$$\frac{d}{s} = \frac{r}{s/2},$$

so $d = 2r$. This computation may also be done using trigonometry, but the similar-triangles calculations are simpler.

We can now compare d to the length of the diameter of the n-gon. As $d = 2r$ is exactly the diameter of an even n-gon, we see that for even n the raw edge of the twist lines up perfectly with the bottom of the box. For odd n, the diameter of an n-gon is $r + a$. As r is the hypotenuse of the triangle and a is a leg of the triangle, $r + a < 2r$. Thus, the diagonal