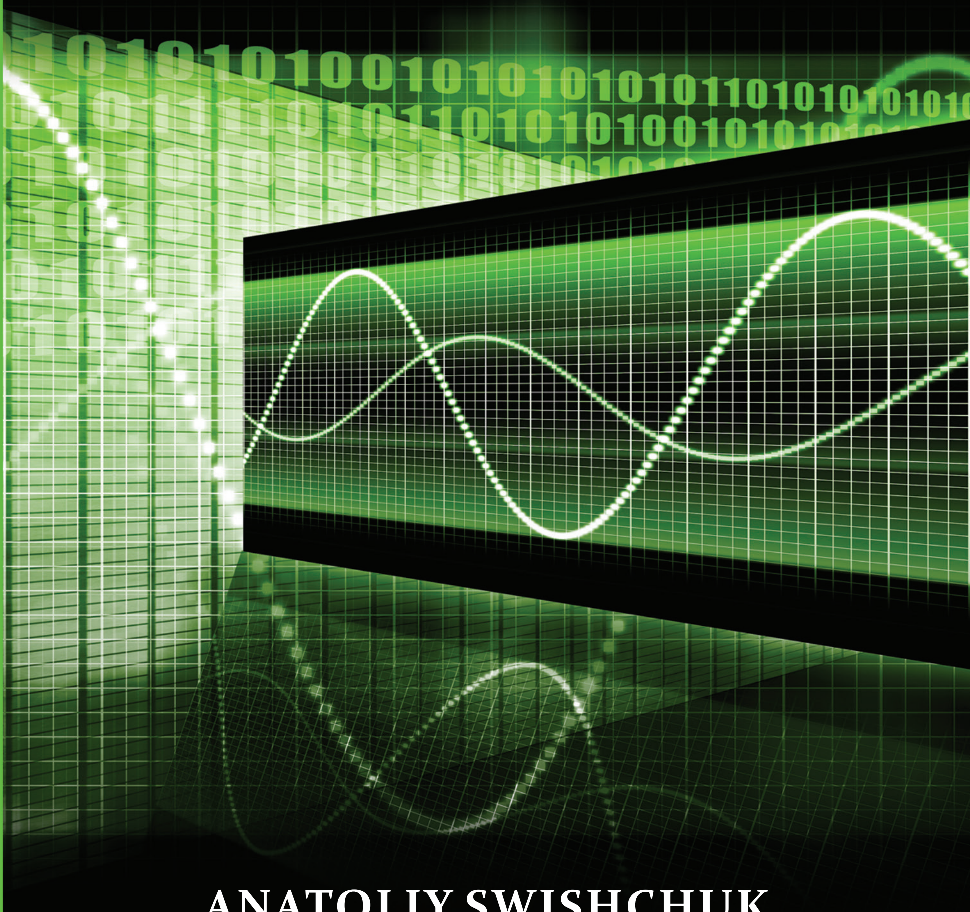
RANDOM DYNAMICAL SYSTEMS IN FINANCE



ANATOLIY SWISHCHUK

SHAFIQUL ISLAM

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

RANDOM DYNAMICAL SYSTEMS IN FINANCE

RANDOM DYNAMICAL SYSTEMS IN FINANCE

ANATOLIY SWISHCHUK

SHAFIQUL ISLAM

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20130321

International Standard Book Number-13: 978-1-4398-6719-8 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Contents

List of Figures

Preface

The theory and applications of random dynamical systems (RDS) are at the cutting edge of research in both mathematics and economics. There are many papers on RDS and also some books on RDS. As excellent examples we would like to mention *Random Dynamical Systems* by Ludwig Arnold (Springer, 2003) and *Random Dynamical Systems: Theory and Applications* by Rabi Bhattacharya and Mukul Majumdar (Cambridge, 2007).

Random dynamical systems have especially been studied in many contexts in economics, particularly in modeling long run evolution of economic systems subject to exogenous random shocks.

There are some papers on applications of RDS in economics, and a few papers on RDS in finance. However, there is no book containing any consideration of RDS in finance. Thus, this is the right time to publish a book on this topic.

Finance modeling with RDS is in its infancy. Our book is the first book that contains applications of random dynamical systems in finance.

In this way, the book is useful not only for researchers and academic people, but also for practitioners who work in the financial industry and for graduate students specializing in RDS and finance.

Anatoliy Swishchuk
Calgary, AB, Canada

Shafiqul Islam
Charlottetown, PEI, Canada

Acknowledgment

We would like to thank our colleagues and students for many useful discussions and unforgettable collaborations.

Anatoliy Swishchuk thanks Jianhong Wu, Anatoliy Ivanov, Yulia Mishura, Yuriy Kazmerchuk, Anna Kalemanova-Schlösser, Edson Alberto Coayla Teran, and LiFeng Zhang.

Shafiqul Islam wishes to give special recognition and appreciation to two individuals at Concordia University, Montreal, Canada. Pawel Góra and Abraham Boyarsky of Concordia University introduced and taught Shafiqul Islam the subject: random dynamical systems.

Anatoliy Swishchuk and Shafiqul Islam wish to recognize the patient encouragement and understanding of their families: wives Maryna Swishchuk and Monzu Ara Begum, and daughters Julia Swishchuk and Anika Tabassum, respectively. Without their dedication and support this book would remain just a dream. Anatoliy Swishchuk also thanks his son Victor Swishchuk for continuous support and inspiration.

We would also like to thank the anonymous referee for many remarks and comments. Our great appreciation and many thanks go to Matt Davison, who helped us to improve the book with many valuable and important comments, suggestions, and additions, especially on the matter of financial applications issues.

We both appreciate enormous help from David Grubbs (Chapman & Hall) and Shashi Kumar (Chapman & Hall) during preparation of this book.

Anatoliy Swishchuk
Calgary, AB, Canada

Shafiqul Islam
Charlottetown, PEI, Canada

Chapter 1

Introduction

This book is devoted to the study of random dynamical systems (RDS) and their applications in finance. The theory of RDS, developed by L. Arnold and co-workers, can be used to describe the asymptotic and qualitative behavior of systems of random and stochastic differential/difference equation in terms of stability, invariant manifolds, attractors, etc. Usually, a RDS consists of two parts: the first part is a model for the noise path, leading to a RDS, and the second part is the dynamics of a model.

In this book, we present many models of RDS and develop techniques in the RDS which can be implemented in finance.

Let us present just a few of many examples that can be used in finance or/and economics.

One of the examples of a model of RDS that can be used in finance is a geometric Markov renewal process (GMRP) for a stock price, which is defined as follows (see Chapter 6 for details):

$$S_t := S_0 \prod_{k=1}^{\nu(t)} (1+\rho(x_k)),\ t \in \mathbf{R}_+,$$

where function $\rho(x) > -1$ is continuous and bounded on phase space X of a Markov chain $x_n,\quad n \in \mathbf{Z}_+,\ \nu(t)$ is a counting process. This model is a generalization of Cox-Ross-Rubinstein binomial model for stock price (see [4], Chapter 6) and Aase's geometric compound Poisson process (see [1], Chapter 6).

The second example of a model of RDS that can be used in economics is a Ramsey (see [10], Chapter 13) stochastic model for capital that takes into account the delay and randomness in the production cycle (see Chapter 13 for details):

$$dK(t) = [AK(t-T) - u(K(t))C(t)]\,dt + \sigma(K(t-T))dw(t)$$

where K is the capital, C is the production rate, u is a control process, A is a positive constant, σ is a standard deviation of the "noise" $w(t)$. The "initial capital"

$$K(t) = \phi(t), \quad t \in [-T, 0],$$

is a continuous bounded positive function and depends not only on current t, but also on the past before t.

One more example is associated with a model for a stock price $S(t)$ that includes regime switching, delay, noise and Poisson jumps (see Chapter 12 for details):

$$\begin{aligned} dS(t) &= [a(r(t))S(t) + \mu(r(t))S(t-\tau)]dt + \sigma(r(t))S(t-\rho)dW(t) \\ &+ \int_{-1}^{\infty} yS(t)\nu(dy,dt). \end{aligned}$$

This model includes not only the current state of the stock price $S(t)$, but also, e.g., histories, $S(t-\tau)$ and $S(t-\rho)$, where ρ and τ are delayed parameters, and sudden shocks (Poisson jumps).

Dynamical systems are mathematical models of real-world problems and they provide a useful framework for analyzing various physical (see [7] and [9] in Chapter 3), engineering, social, and economic phenomena (see [37] in Chapter 3). A random dynamical system is a measure-theoretic formulation of a dynamical system with an element of randomness. A deterministic dynamical system is a system in which no randomness is involved in the development of future states of the system. The fundamental problem in the ergodic theory of deterministic dynamical systems is to describe the asymptotic behavior of trajectories defined by a deterministic dynamical system. In general, the long-time behavior of trajectories of a chaotic deterministic dynamical system is unpredictable (see [2] in Chapter 2). Therefore, it is natural to describe the behavior of the system as a whole by statistical means. In this approach, one attempts to describe the dynamics by proving the existence of an invariant measure and determining its ergodic properties (see [2] in Chapter 2). In particular, the existence of invariant measures which are absolutely continuous with respect to Lebesgue measure is very important from a physical point of view, because computer simulations of orbits of the system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure (see [18] in Chapter 3). The Birkhoff Ergodic Theorem (see [2] in Chapter 2) states that if $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and $\mu-$invariant and E is a measurable subset of X then the orbit of almost every point of X occurs in the set E with asymptotic frequency $\mu(E)$.

The Frobenius–Perron operator P_τ is the main tool for proving the existence of absolutely continuous invariant measures (acim) of a transformation τ. It is well known that f is the density of an acim μ under a transformation τ if and only if $P_\tau f = f$. In 1940, Ulam and von Neumann found examples of transformations having absolutely continuous invariant measures. In 1957, Rényi (see [35] in Chapter 3) defined a class of transformations that have an acim. Rényi's key idea of using distortion estimates has been used in the more general proofs of Adler and Flatto (see [2] in Chapter 3). In 1973, Lasota and Yorke (see [10] in Chapter 2) proved a general sufficient condition for the existence of an absolutely continuous invariant measure for piecewise expanding C^2 transformations. Their result was an important generalization of Rényi's (see [35] in Chapter 3) result using the theory of bounded variation and their essential observation was that, for piecewise expanding transformations, the Frobenius–Perron operator is a contraction. The bounded variation technique has been generalized in a number of directions (see [27] in Chapter 3). In Chapter 2 of this book, we briefly review deterministic dynamical systems, ergodic theory, the Frobenious–Perron operator, invariant measures, and stochastic perturbations. Many of these fundamental results in Chapter 2 of this book will be useful for Chapters 3–4. For more detailed results on the existence, properties, and approximations of invariant measures for deterministic dynamical systems, see the book by Boyarsky and Góra (see [2] in Chapter 2). The book by Ding and Zhou, (see [4] in Chapter 2) is another good reference for deterministic dynamical systems.

Random dynamical systems provide a useful framework for modeling and analyzing various physical, social, and economic phenomena (see [9], [37], and [38] in Chapter 3). A random dynamical system of special interest is a random map where the process switches from one map to another according to fixed probabilities (see [34] in Chapter 3) or, more generally, position dependent probabilities (see [3–6] and [16] in Chapter 3]. The existence and properties of invariant measures for random maps reflect their long-time behavior and play an important role in understanding their chaotic nature. Random maps have applications in the study of fractals (see [7] in Chapter 3), in modeling interference effects in quantum mechanics (see [9] in Chapter 3), in computing metric entropy (see [38] in Chapter 3), and in forecasting the financial markets (see [3] in Chapter 3). In 1984, Pelikan (see [34] in Chapter 3) proved sufficient conditions for the existence of acim for random maps with constant probabilities. Morita (see [32] in Chapter 3) proved a spectral decomposition theorem. In Chapter 3 of this book, we first present a general setup for a random dynamical system from Arnold's sense (see [1] in Chapter 3). Then we present skew product and random maps with constant probabilities. Some fundamental results on the properties of the Frobenius–Perron operator for random maps with constant probabilities are also presented in Chapter 3. We present necessary and sufficient conditions for the existence of absolutely continuous invariant measures for random maps. Moreover, we present two important properties of invariant measures for random maps with constant probabilities. At the end of Chapter 3, we present some applications of random maps in finance.

Position dependent random maps are more general random maps where the probabilities of choosing component maps are position dependent. Góra and Boyarsky (see [14] in Chapter 4) proved sufficient conditions for the existence of acim for random maps with position dependent probabilities. Bahsoun and Góra proved sufficient average expanding conditions for the existence of acim for position dependent random maps in one and higher dimensions (see [2] in Chapter 4), weakly convex and concave position dependent random maps (see [5] in Chapter 3). Bahsoun, Góra, and Boyarsky proved the sufficient condition for the existence of Markov switching random map with position dependent switching matrix (see [3] in chapter 4). In Chapter 4 of this book, we first present position dependent random maps and properties of the Frobenius–Perron operator. Then we present the existence of invariant measures for random maps, Markov switching random maps in one and higher dimensions.

Froyland (see [14] in Chapter 3) extended Ulam's method for a single transformation to random maps with constant probabilities (see [34] in Chapter 3). Góra and Boyarsky proved the convergence of Ulam's approximation for position dependent random maps (see [14] in Chapter 4). For Markov switching random maps, Froyland (see [14] in Chapter 3) considered the constant stochastic irreducible matrix W and proved the existence and convergence of Ulam's approximation of invariant measures. In Chapter 4 of this book, we also present numerical schemes for the approximation of invariant measures for position dependent random maps. Applications of position dependent random maps in finance are presented at the end of Chapter 4 of this book.

Chapter 5 is devoted to the study of random evolutions (REs). In mathematical language, a RE is a solution of stochastic operator integral equation in a Banach space. The operator coefficients of such equations depend on random parameters. The random evolution (RE), in physical language, is a model for a dynamical system whose state of evolution is subject to random variations. Such systems arise in many branches of science, e.g., random Hamiltonian and Shroedinger's equations with random potential in quantum mechanics, Maxwell's equation with a random reflective index in electrodynamics, transport equation, storage equation, etc. There are a lot of applications of REs in financial and insurance mathematics (see [11], Chapter 5). One of the recent applications of RE is associated with geometric Markov renewal processes which are regime-switching models for a stock price in financial mathematics, which will be studied intensively in the next chapters. Another recent application of RE is a semi-Markov risk process in insurance mathematics (see [11], Chapter 5). The REs are also examples of more general mathematical objects such as multiplicative operator functional (MOFs) (see [7, 10], Chapter 5), which are random dynamical systems in Banach space. The REs can be described by two objects: 1) operator dynamical system $V(t)$ and 2) random process $x(t)$. Depending on structure of $V(t)$ and properties of the stochastic process $x(t)$ we have different kinds of REs: continuous, discrete, Markov, semi-Markov, etc. In this chapter we deal with various problems for REs, including martingale property, asymptotical behavior of REs, such as averaging, merging, diffusion approximation, normal deviations, averaging, and diffusion approximation in reducible phase space for $x(t)$ rate of convergence for limit theorems for REs.

Chapters 6–9 deal with geometric Markov renewal processes (GMRP) as a special case of REs. We study approximation of GMRP in ergodic, merged, double averaged, diffusion, normal deviation, and Poisson cases. In all these cases we present applications of the obtained results to finance, including option pricing formulas.

In Chapter 6 we introduce the geometric Markov renewal processes as a model for a security market and study these processes in a series scheme. We consider its approximations in the form of averaged, merged, and double averaged geometric Markov renewal processes. Weak convergence analysis and rates of convergence of ergodic geometric Markov renewal processes, are presented. Martingale properties, infinitesimal operators of geometric Markov renewal processes are presented and a Markov renewal equation for expectation is derived. As an application, we consider the case of two ergodic classes. Moreover, we consider a generalized binomial model for a security market induced by a position dependent random map as a special case of a geometric Markov renewal process.

In Chapter 7 we study the geometric Markov renewal processes in a diffusion approximation scheme. Weak convergence analysis and rates of convergence of ergodic geometric Markov renewal processes in a diffusion scheme are presented. We present European call option pricing formulas in the case of ergodic, double averaged, and merged diffusion geometric Markov renewal processes.

Chapter 8 is devoted to the normal deviations of the geometric Markov renewal processes for ergodic averaging and double averaging schemes. Algorithms of averaging define the averaged systems (or models) which may be considered as the first approximation. Algorithms of diffusion under balance condition define diffusion models which may be considered as the second approximation. In this chapter we consider the algorithms of construction of the first and second approximation in the case when the balance condition is not fulfilled. Some applications in finance are presented; in particular, option pricing formulas in this case are derived.

In Chapter 9, we introduce the Poisson averaging scheme for the geometric Markov renewal processes. Financial applications in Poisson approximation schemes of the geometric Markov renewal processes are presented, including option pricing formulas.

Chapter 10 considers the stochastic stability of fractional (B,S)-security markets, that is, financial markets with a stochastic behavior that is caused by a random process with long-range dependence, fractional Brownian motion. Three financial models are considered. They arose as a result of different approaches to the definition of the stochastic integral with respect to fractional Brownian motion. The stochastic stability of fractional Brownian markets with jumps is also considered. In Appendix, we give some definitions of stability, Lyapunov indices, and some results on rates of convergence of fractional Brownian motion, which we use in our development of stochastic stability.

In Chapter 11, we study the stochastic stability of random dynamical systems arising in the interest rate theory. We introduce different definitions of stochastic stability. Then, the stochastic stability of interest rates for the Black-Scholes, Vasicek, Cox-Ingersoll-Ross models and their generalizations for the case of random jump changes are studied.

The subject of Chapter 12 is the stability of trivial solution of stochastic differential delay in Ito's equations with Markovian switchings and with Poisson bifurcations. Throughout the work stochastic analogue of second Lyapunov method is used.

Some applications in finance are considered as well.

RDS in the form of stochastic differential delay equations and their optimal control have received much attention in recent years. Delayed problems often appear in applications in physics, biology, engineering, and finance. Optimal controls of delayed RDS in finance in some specific and general settings are considered in Chapters 13 and 14, respectively.

Chapter 13 is devoted to the study of optimal control of random delayed dynamical systems and their applications. By using the Dynkin formula and solution of the Dirichlet-Poisson problem developed in Chapter 5, the Hamilton-Jacobi-Bellman (HJB) equation and the inverse HJB equation are derived. Application is given to a stochastic model in economics (stochastic Ramsey's model).

In Chapter 14 the problem of RDS arising in optimal control theory for vector stochastic differential delay equations (SDDEs) and its applications in mathematical finance and economics is studied. By using the Dynkin formula and solution of the Dirichlet-Poisson problem developed in Chapter 5, the Hamilton-Jacobi-Bellman (HJB) equation and the converse HJB equation are derived. Furthermore, applications are given to an optimal portfolio selection problem and a stochastic Ramsey model in economics.

The analogue of the Black-Scholes formula for vanilla call option price in conditions of (B,S)-securities market with delayed/past-dependent information is derived in Chapter 15. A special case of a continuous version of GARCH is considered. The results are compared with the results of the Black and Scholes (1973) formula.

All references are provided at the end of each chapter.

Thus, the book contains a variety of RDS which are used for approximations of financial models, studies of their stability and control, and presents many option pricing formulas for these models.

The book will be useful for researchers and academics who work in RDS and mathematical finance, and also for practitioners working in the financial industry. It will also be useful for graduate students specializing in the areas of RDS and mathematical finance.

Chapter 2

Deterministic Dynamical Systems and Stochastic Perturbations

2.1 Chapter overview

In this chapter we review deterministic dynamical systems and their invariant measures. Deterministic dynamical systems are special cases of random dynamical systems, and theories of deterministic dynamical systems play an important role for the study of random dynamical systems. The existence and properties of absolutely continuous invariant measures for deterministic dynamical systems reflect their long-time behavior and play an important role in understanding their chaotic nature. The Frobenius–Perron operator for deterministic dynamical systems is one of the key tools for the study of invariant measures for deterministic dynamical systems. In Chapter 3 and Chapter 4 we will see that the Frobenius–Perron operator for random dynamical systems is a combination of the Frobenius–Perron operator of the individual component systems which are deterministic dynamical systems. In this chapter we focus our special attention on the class of piecewise monotonic and expanding deterministic dynamical systems. Moreover, we present stochastic perturbations of deterministic dynamical systems. For the Frobenius–Perron operator and existence of invariant measures we closely follow [2, 4, 9, 10] and the references therein. For the stochastic perturbations we closely follow [7, 8, 9, 11] and the references therein.

2.2 Deterministic dynamical systems

Let $(X, \mathcal{B}, \mu)$ be a normalized measure space where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a measure such that $\mu(X) = 1$. Let ν be another measure on $(X, \mathcal{B})$. The measure μ is absolutely continuous with respect to ν if for any $A \in \mathcal{B}$ with $\nu(A) = 0$, we have $\mu(A) = 0$. Let $I = [a, b]$ be an interval of the real line $\mathbb{R}$. Throughout this chapter, we consider $X = I = [0, 1]$ and we denote by $V_I(\cdot)$ the standard one dimensional variation of a function on $[0, 1]$ and let $BV(I)$ be the space of functions of bounded variations on I equipped with the norm $\|\cdot\|_{BV} = V_I(\cdot) + \|\cdot\|_1$, where $\|\cdot\|_1$ denotes the L^1 norm on $L^1(I, \mathcal{B}, \mu)$.

Definition 2.1 *Let $\tau : I \to I$ be a transformation such that for any initial $x \in I$, the nth iteration of x under τ is defined by* $\tau^n(x) = \tau \circ \tau \circ \ldots \circ \tau(x)$ n times. *The transforma-*

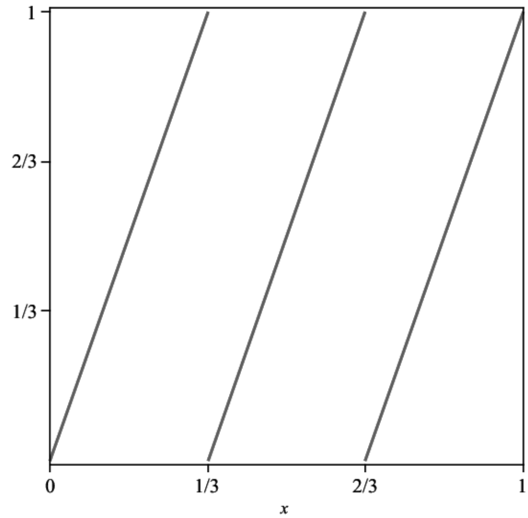


Figure 2.1 *The piecewise expanding map* τ.

tion $\tau : I \to I$ *is* **non-singular** *if for any* $A \in \mathcal{B}$ *with* $\mu(A) = 0$, *we have* $\mu(\tau^{-1}(A)) = 0$. *The transformation* τ **preserves the measure** μ *or* **the measure** μ **is** τ**-invariant** *if* $\mu(\tau^{-1}(A)) = \mu(A)$ *for all* $A \in \mathcal{B}$. *In this case the quadruple* $(X, \mathcal{B}, \mu, \tau)$ *is called a* **deterministic dynamical system**. *A family* $\mathcal{B}^*$ *of subsets of I is a* π**-system** *if and only if* $\mathcal{B}^*$ *is closed under intersections.*

The following Theorem (Theorem 3.1.1 in [2]) is useful for checking whether a transformation preserves a measure:

Theorem 2.2 *[2] Let* $(I, \mathcal{B}, \mu)$ *be a normalized measure space and* $\tau : I \to I$ *be a measurable transformation. Let* $\mathcal{B}^*$ *be a* $\pi-$ *system that generates* $\mathcal{B}$. *Then* μ *is* τ*-invariant if* $\mu(\tau^{-1}(A)) = \mu(A)$ *for any* $A \in \mathcal{B}^*$.

Example 2.1 *Consider the measure space* $([0,1], \mathcal{B}, \lambda)$, *where* $\mathcal{B}$ *is* σ*-algebra on* $[0,1]$ *and* λ *the Lebesgue measure on* $[0,1]$. *Let* $\tau : [0,1] \to [0,1]$ *be a map (see Figure 2.1) defined by*

$$\tau(x) = \begin{cases} 3x, & 0 \le x < \frac{1}{3}, \\ 3x - 1, & \frac{1}{3} \le x < \frac{2}{3}, \\ 3x - 2, & \frac{2}{3} \le x \le 1, \end{cases}$$

For any interval $[x,y] \subset [0,1]$, $\tau^{-1}([x,y]) = [\frac{x}{3}, \frac{y}{3}] \cup [\frac{x+1}{3}, \frac{y+1}{3}] \cup [\frac{x+2}{3}, \frac{y+2}{3}]$ *and*

$\lambda(\tau^{-1}([x,y])) = \lambda([\frac{x}{3},\frac{y}{3}] \cup [\frac{x+1}{3},\frac{y+1}{3}] \cup [\frac{x+2}{3},\frac{y+2}{3}]) = y-x = \lambda([x,y])$. *By Theorem 2.2, the transformation* τ *is* λ*-invariant. Thus,* $([0,1],\mathcal{B},\lambda,\tau)$ *is a deterministic dynamical system.*

2.2.1 Ergodicity and Birkhoff individual ergodic theorem

Let $\tau : [0,1] \to [0,1]$ be a measure preserving transformation and $x_0 \in [0,1]$. The Birkhoff ergodic theorem allows us to study the statistical behavior of orbit $\{x_0, x_1 = \tau(x_0), \ldots, x_n = \tau(x_{n-1}\}$. If τ is ergodic, then the Birkhoff ergodic theorem provides more specific information of the orbit. Let A be a measurable set of $[0,1]$ and χ_A be the characteristic function on A. For any $i \in \{0,1,\ldots,n\}$, $x_i = \tau^i(x_0) \in A$ if and only if $\chi_A(\tau^i(x_0)) = 1$.

Definition 2.3 *A measure-preserving transformation* $\tau : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ *is* **ergodic** *if for any* $B \in \mathcal{B}$ *such that* $\tau^{-1}B = B$*, we have* $\mu(B) = 0$ *or* $\mu(X \setminus B) = 0$*.*

Ergodicity of a measure preserving transformation $\tau : [0,1] \to [0,1]$ is an indecomposability property such that if τ has this indecomposability property then the study of τ cannot be split into separate parts. The following Theorem (Theorem 3.2.1 in [2], see also [4]) is useful for checking whether a transformation is ergodic:

Theorem 2.4 *[2] Let* $\tau : (I,\mathcal{B},\mu) \to (I,\mathcal{B},\mu)$ *be a measure preserving transformation. Then the following statements are equivalent:*

1. τ *is ergodic.*
2. *If* f *is measurable and* $(f \circ \tau)(x) = f(x)$ *almost everywhere, then* f *is constant almost everywhere.*
3. *If* $f \in L^2$ *and* $(f \circ \tau)(x) = f(x)$ *almost everywhere, then* f *is constant almost everywhere.*

Theorem 2.5 ***Birkhoff's ergodic theorem for deterministic dynamical systems*** *[2, 9]: Let* $\tau : (I,\mathcal{B},\mu) \to (I,\mathcal{B},\mu)$ *be* μ*-invariant and* $f \in L^1(I,\mathcal{B},\mu)$*. Then there exists a function* $f^* \in L^1(X,\mathcal{B},\mu)$ *such that for* $\mu-$*almost all* $x \in I$ *the limit of the time averages* $\frac{1}{n+1}\sum_{k=0}^{n} f(x_k)$ *exists and*

$$\frac{1}{n+1}\sum_{k=0}^{n} f(x_k) \to f^*, \tag{2.2.1}$$

$\mu-$ *almost everywhere. Moreover, if* τ *is ergodic and* $\mu(X) = 1$*, then* f^* *is constant* μ *a.e. and* $f^* = \int_X f d\mu$*.*

Application of the Birkoff ergodic theorem: Let $A \in \mathcal{B}$. Then $\sum_{k=0}^{n} \chi_A(x_k)$ is the number of points of the orbit $\{x_0, x_1 = \tau(x_0), \ldots, x_n = \tau(x_{n-1}\}$ in A and $\frac{1}{n+1}\sum_{k=0}^{n} \chi_A(x_k)$ is the relative frequency of the elements of $\{x_0, x_1 = \tau(x_0), \ldots, x_n =$

$\tau(x_{n-1}\}$. If we replace $f \in L^1$ by the characteristic function χ_A on the measurable set $A \subset [0,1]$ and if τ is ergodic and $\mu(I) = 1$ then by the Birkoff ergodic theorem 2.5,

$$\frac{1}{n+1}\sum_{k=0}^{n-1} \chi_A(\tau^k(x)) \to \mu(A), \tag{2.2.2}$$

$\mu-$ almost everywhere and thus the orbit of almost every point of I occurs in the set A with asymptotic frequency $\mu(A)$.

Example 2.2 *Consider the transformation τ in Example 2.1. τ preserves the Lebesgue measure λ and τ is λ-ergodic. Consider the measurable sets E_i of $[0,1]$ where $E_i = [\frac{i}{5}, \frac{i+1}{5}], i = 0,1,2,3$. Let x_0 be any initial point in $[0,1]$. By the Birkoff ergodic theorem 2.5*

$$\frac{1}{n+1}\sum_{k=0}^{n-1} \chi_{A_i}(\tau^k(x_0)) \to \lambda(E_i) = \frac{1}{3}, \tag{2.2.3}$$

2.2.2 *Stationary (invariant) measures and the Frobenius–Perron operator for deterministic dynamical systems*

Consider the measure space $(I, \mathcal{B}, \lambda)$ and let $\mathcal{M}(I) = \{m : m \text{ is a measure on I}\}$, that is, $\mathcal{M}(I)$ is the space of measures on $(I, \mathcal{B})$. Let $\tau : ([a,b], \mathcal{B}, \lambda) \to (I, \mathcal{B}, \lambda)$ be a piecewise monotonic non-singular transformation on the partition $\mathcal{P}$ of I where $\mathcal{P} = \{I_1, I_2, \ldots, I_N\}$ and $\tau_i = \tau|_{I_i}$. Let $\mu << \lambda$, that is, μ is absolutely continuous with respect to λ. The transformation τ induces an operator $\mathcal{O}$ on $\mathcal{M}(I)$ defined by

$$\mathcal{O}(\mu)(A) = \mu(\tau^{-1}(A)).$$

Non-singularity of τ implies that $\mathcal{O}(\mu) << \lambda$. Suppose that μ has a density $f \in \mathcal{D} = \{f \in L^1(I, \mathcal{B}, \mu) : f \geq 0 \text{ and } \parallel \text{f} \parallel_1 = 1\}$ with respect to λ. Then by the Radon-Nikodyn Theorem, $\mu(A) = \int_A f d\lambda$ for any measurable set $A \in \mathcal{B}$. Since μ has a density f, the induced measure $\mathcal{O}(\mu)$ also has a density $P_\tau f$. Thus,

$$\mathcal{O}(\mu)(A) = \int_A P_\tau f d\lambda = \mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f d\lambda.$$

Clearly, $P_\tau : L^1(I, \mathcal{B}, \lambda) \to L^1(I), \mathcal{B}, \lambda)$ is a linear operator. The above operator P_τ defined by

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda \tag{2.2.4}$$

is known as the Frobenius-Perron operator. Let $A = [0,x]$. Then

$$\int_0^x P_\tau f d\lambda = \int_{\tau^{-1}([0,x])} f d\lambda.$$

Differentiating on both sides of (2.2.4) with respect to P_τ we get

$$P_\tau f d\lambda = \frac{d}{dx}\int_{\tau^{-1}([0,x))} f d\lambda. \tag{2.2.5}$$

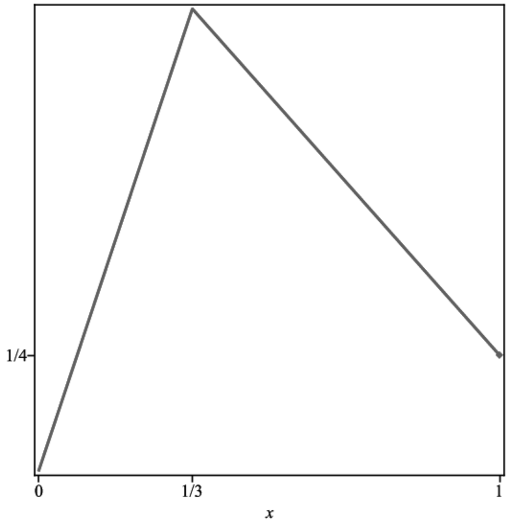


Figure 2.2 *The map τ.*

Example 2.3 *Let $\tau : [0,1] \to [0,1]$ be defined by*

$$\tau(x) = \begin{cases} 3x, & x \in [0, \frac{1}{3}] \\ -\frac{9}{8}x + \frac{11}{8}, & x \in [\frac{1}{3}, 1]. \end{cases}$$

See Figure 2.2. If $x >= \frac{1}{4}$, then $\tau^{-1}([0,x]) = [0, \frac{1}{3}x] \cup [\frac{11}{9} - \frac{8}{9}x, 1]$. If $0 \leq x < \frac{1}{3}$, then $\tau^{-1}([0,x]) = [0, \frac{1}{3}x]$. Therefore, Then $\tau^{-1}([0,x]) = [0, \frac{1}{3}x] \cup \{[\frac{11}{9} - \frac{8}{9}x, 1] \cap A\}$, where $A = [\frac{1}{3}, 1]$. For any $f \in L^1(0,1)$,

$$\begin{aligned} P_\tau f d\lambda &= \frac{d}{dx} \int_{\tau^{-1}([0,x])} f d\lambda \\ &= \frac{d}{dx} \int_{[0, \frac{1}{3}x] \cup \{[\frac{11}{9} - \frac{8}{9}x, 1] \cap A\}} f(x) d\lambda \\ &= \frac{d}{dx} \left[\int_0^{\frac{x}{3}} f(x) d\lambda + \int_{\frac{11}{9} - \frac{8}{9}x}^{1} f(x) \chi_A(x) d\lambda \right] \\ &= f(\frac{x}{3}) + \frac{8}{9} f(\frac{11}{9} - \frac{8}{9}x) \chi_J(x), \end{aligned}$$

where $J = \tau(A) = [\frac{1}{4}, 1]$.

Properties of the Frobenius–Perron operator operator P_τ [2, 9]: It is not difficult to show that the Frobenius–Perron operator operator P_τ of a transformation τ has the following useful properties:

1. Linearity: the Frobenius-Perron operator operator is a linear operator, that is
$$P_\tau(\alpha f+\beta g)=\alpha P_\tau f+\beta P_\tau g,$$
where α,β are real numbers and $f,g\in L^1$.
2. Positivity: Let $f\in L^1$ and assume $f\geq 0$. Then $P_\tau f\geq 0$.
3. Contraction Property: $P_\tau : L^1\to L^1$ is a contraction. It means that for any $f\in L^1$
$$\| P_\tau f \|_1 \leq \| f \|_1$$
4. Preservation of Integrals: P_τ preserves integrals, i.e., $\int_I f d\lambda=\int_I P_\tau f d\lambda$;
5. Composition Property: Let $\tau : I\to I$ and $\sigma : I\to$ I be non-singular, then
$$P_{\tau\cdot\sigma} f = P_\tau \cdot P_\sigma f$$
Moreover,
$$P_{\tau^n} f = P_\tau^n f$$
6. Adjoint Property: If $f\in L^1$ and $g\in L^\infty$, then
$$< P_\tau f, g >=< f, U_\tau g d\lambda >$$

For more details of the above properties see [2, 4, 9].

Definition 2.6 *A transformation* $\tau : [0,1]\to[0,1]$ *is piecewise monotonic if there exists a partition* $0=x_0<x_1<\cdots<x_n=1$ *and a constant* $r\geq 1$ *such that*

1. $|\tau'(x)|>0$ *for* $x\in(x_{i-1},x_i), i=1,2,\ldots,n.$
2. $\tau_{|(x_{i-1},x_i)}$ *is a r times continuously differentiable function which can be extended to a r times continuously differentiable function on the closed interval* $[x_{i-1},x_i], i=1,2,\ldots,n.$

A transformation $\tau : [0,1]\to[0,1]$ is piecewise expanding if τ is piecewise monotonic and $|\tau'(x)|>1$ for $x\in(x_{i-1},x_i), i=1,2,\ldots,n.$

Example 2.4 *The map* $\tau : [0,1]\to[0,1]$ *(see Figure 2.3) defined by*
$$\tau(x)=\begin{cases}\frac{x}{2}, & 0\leq x<\frac{1}{2},\\ 2x-1, & \frac{1}{2}\leq x\leq 1,\end{cases}$$
is piecewise monotonic and the tent map $\tau : [0,1]\to[0,1]$ *(see Figure 2.4) defined by*
$$\tau(x)=\begin{cases}2x, & 0\leq x\leq\frac{1}{2},\\ 2-2x, & \frac{1}{2}\leq x\leq 1,\end{cases}$$
is piecewise expanding.

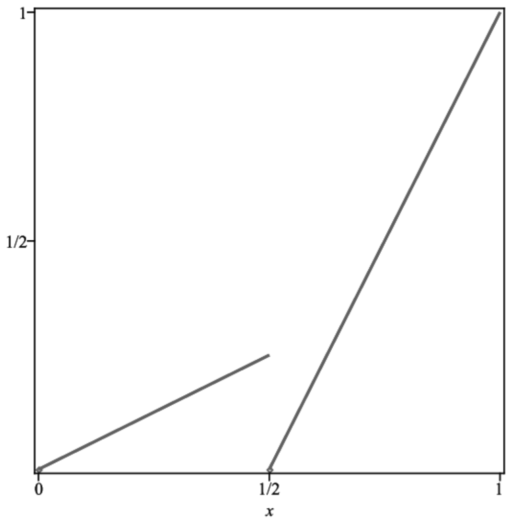


Figure 2.3 *The piecewise monotonic map* τ.

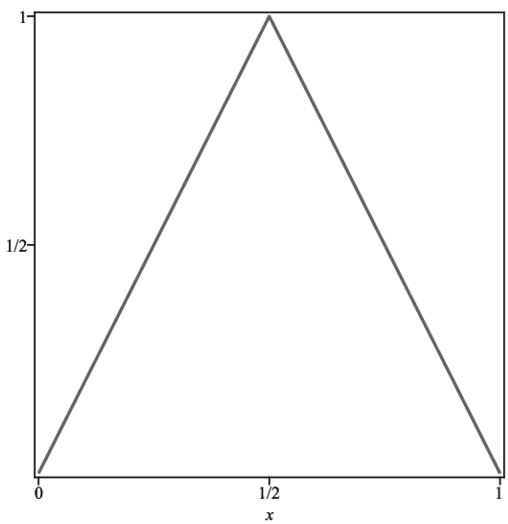


Figure 2.4 *The tent map* τ.

Representation of the Frobenius–Perron operator P_τ [2]: Let $\tau : [0,1] \to [0,1]$ be a piecewise monotonic transformation with respect to the partition $0 = x_0 < x_1 < \cdots < x_n = 1 = \{I_1, I_2, \ldots, I_n\}$. For each $1 \le i \le n$, let $D_i = \tau([x_{i-1}, i])$ and $g_i : D_i \to [x_{i-1}, x_i]$ is defined by $g_i(x) = \tau^{-1}_{|D_i}$. Piecewise monotonicity of τ implies that g_i exists for each $1 \le i \le n$. Let $A \in \mathcal{B}$. Then $\tau^{-1}(A) = \sum_{i=1}^n g_i(A \cap D_i)$ and $\{A \cap D_i\}_{1 \le i \le n}$ is a family of mutually disjoint sets. For $f \in L^1(I)$,

$$\begin{aligned}
\int_A (P_\tau(f))(x) d\lambda &= \int_{\tau^{-1}(A)} f d\lambda \\
&= \int_{\sum_{i=1}^n g_i(A \cap D_i)} f d\lambda \\
&= \sum_{i=1}^n \int_{g_i(A \cap D_i)} f d\lambda \\
&= \sum_{i=1}^n \int_{A \cap D_i} f(g_i(x)) |g_i'(x)| d\lambda \\
&= \sum_{i=1}^n \int_A f(g_i(x)) |g_i'(x)| \chi_{D_i}(x) d\lambda \\
&= \int_A \sum_{i=1}^n \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))} \chi_{\tau(x_{i-1}, x_i)}(x) d\lambda
\end{aligned}$$

$$(P_\tau(f))(x) = \sum_{i=1}^n \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))} \chi_{\tau(x_{i-1}, x_i)}(x), \tag{2.2.6}$$

for any measurable set A and $f \in L^1(I)$. Equation (2.2.6) is the representation of the Frobenius–Perron operator P_τ. Equation (2.2.6) can also be rewritten as

$$(P_\tau(f))(x) = \sum_{y \in \{\tau^{-1}(x)\}} \frac{f(y)}{\tau'(y)}, \tag{2.2.7}$$

Example 2.5 *Let* $\tau : [0,1] \to [0,1]$ *be the logistic map (see Figure 2.5) defined by* $\tau(x) = 4x(1-x)$. *It is not difficult to show that* τ *is piecewise monotonic with respect to the partition* $\{x_0, x_1, x_2\} = \{0, \frac{1}{2}, 1\}$.

$$\begin{aligned}
\tau_1^{-1}(x) &= \frac{1}{2} - \frac{1}{2}\sqrt{1-x},\ \tau_2^{-1}(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1-x}, \\
|\tau'(\tau_1^{-1}(x))| &= |\tau'(\tau_2^{-1}(x))| = 4\sqrt{1-x}.
\end{aligned}$$

Let $f \in L^1(I)$, *then by (2.2.6)*

$$\begin{aligned}
(P_\tau(f))(x) &= \sum_{i=1}^2 \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))} \chi_{\tau(x_{i-1}, x_i)}(x) \\
&= \frac{f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x})}{4\sqrt{1-x}} + \frac{f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})}{4\sqrt{1-x}} \\
&= \frac{1}{4\sqrt{1-x}} \left(f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}) \right).
\end{aligned}$$

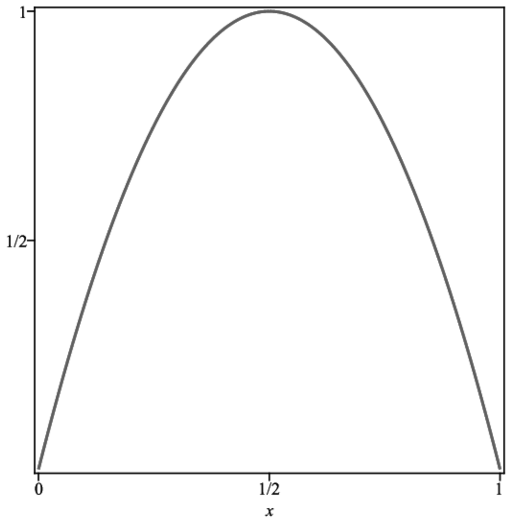


Figure 2.5 *The logistic map* $\tau = 4x(1-x)$.

Definition 2.7 *Let* $\mathcal{P} = \{I_1, I_2, \dots, I_n\}, I_i = (x_{i-1}, x_i), i = 1, 2, \dots, n$ *be a partition of* I, $\tau : I \to I$ *and* $\tau_i = \tau_{|_{I_i}}$. *For each* $i = 1, 2, \dots, n$ *if* τ_i *is a homeomorphism from* I_i *to a connected union of intervals of* $\mathcal{P}$ *then* τ *is called a Markov transformation. For each* $i = 1, 2, \dots, n$ *if* τ_i *is linear then* τ *is called a piecewise linear Markov transformation.*

Example 2.6 $\tau : [0,1] \to [0,1]$ *defined by*

$$\tau(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \le x \le 1, \end{cases}$$

is a piecewise Markov transformation on the partition $\mathcal{P} = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$.

The class of piecewise linear Markov transformations is a simple class of piecewise monotonic transformations and the matrix representation of the corresponding Frobenius–Perron operator can be calculated easily. In fact, it is a matrix which follows from the following theorem [2]:

Theorem 2.8 *(Theorem 9.2.1 in [2]) Let* $\tau : (I, \mathcal{B}, \lambda) \to (I, \mathcal{B}, \lambda)$ *be a piecewise linear Markov transformation with respect to the partition* $\{I_1, I_2, \dots, I_n\} = \{x_0, x_1, \dots, x_n\}$. *Then there exists a* $n \times n$ *matrix* M_τ *such that* $P_\tau f = f M_\tau^T$ *for every piecewise constant* $f = (f_1, f_2, \dots, f_n)$. *The matrix* $M_\tau = (m_{ij})$ *is defined by*

$$m_{ij} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}$$

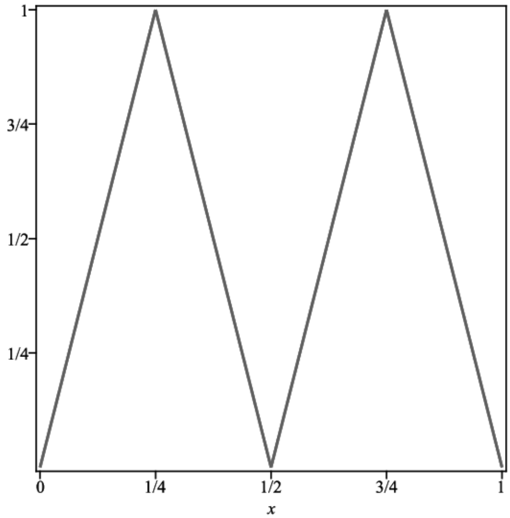


Figure 2.6 *The map τ^* which is the second iteration of the tent map in Figure 2.4.*

Example 2.7 *Let $\tau : [0,1] \to [0,1]$ be the tent map (see Figure 2.4)*

$$\tau(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2-2x, & \frac{1}{2} \le x \le 1, \end{cases}$$

and $\tau^* : [0,1] \to [0,1]$ (see Figure 2.6) is given by $\tau^*(x) = \tau^2(x)$. It can be easily checked that τ^* is a piecewise linear Markov on the partition $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. By the Theorem 2.8, the matrix representation of P_{τ^*} is M^*_τ where

$$M_{\tau^*} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Theorem 2.9 *[2] Let $\tau : I, \mathcal{B}, \lambda) \to (I, \mathcal{B}, \lambda)$ be a non-singular transformation. Then P_τ has a fixed point $f^* \in L^1, f^* \ge 0$ if and only if the measure $\mu = f^* \cdot \lambda$ defined by $\mu(A) = \int_A f^* d\lambda$ is $\tau-$invariant, that is, if and only if $\mu(\tau^{-1}(A) = \mu(A)$ for all measurable set A.*

Proof *Assume $\mu(\tau^{-1}(A)) = \mu(A)$ for any measurable set A. Then*

$$\int_{\tau^{-1}(A)} f^* d\lambda = \int_A f^* d\lambda$$

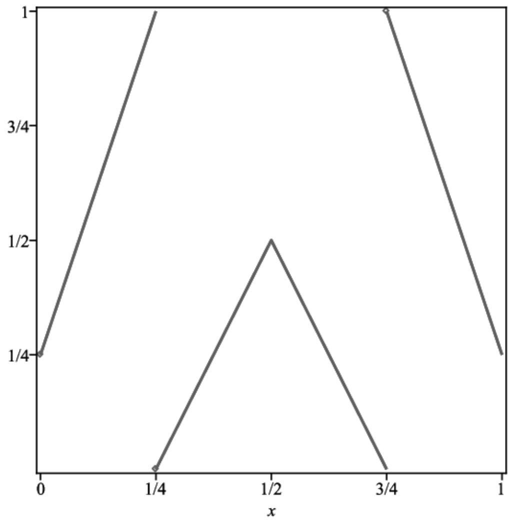


Figure 2.7 *The map τ in Example 2.8.*

and therefore

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda.$$

Since $A \in \mathcal{B}$ is arbitrary, $P_\tau f^* = f^*$ a.e.
Conversely, assume $P_\tau f^* = f^*$ a.e. Then

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda = \mu(A).$$

By definition,

$$\mu(A) = \int_A P_\tau f^* d\lambda = \int_{\tau^{-1}(A)} f^* d\lambda = \mu(\tau^{-1}(A)).$$

Example 2.8 *Let $\tau : [0,1] \to [0,1]$ be a piecewise linear Markov transformation on the partition $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ defined by*

$$\tau(x) = \begin{cases} 3x + \frac{1}{4}, & 0 \le x < \frac{1}{4}, \\ 2(x - \frac{1}{4}), & \frac{1}{4} \le x < \frac{1}{2}, \\ 2 - 2(x + \frac{1}{4}), & \frac{1}{2} \le x < \frac{3}{4}, \\ -3x + \frac{13}{4}, & \frac{3}{4} \le x \le 1. \end{cases}$$

It can be easily checked that τ is piecewise linear Markov on the partition