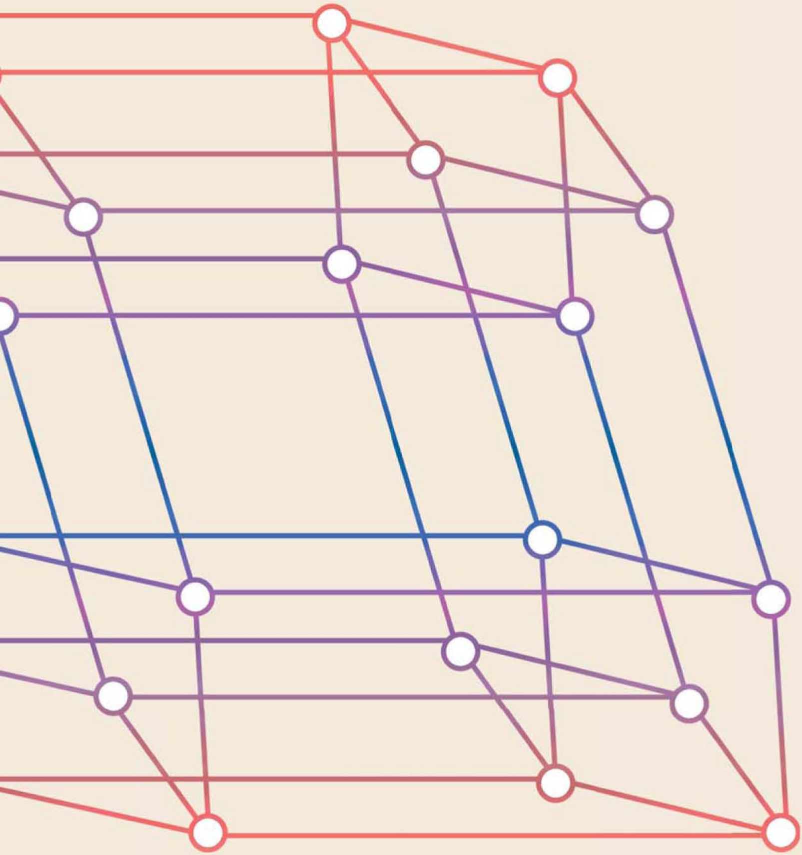

# Topics in Graph Theory

## Graphs and Their Cartesian Product

Wilfried Imrich • Sandi Klavžar • Douglas F. Rall

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*To our wives Gabi, Maja, and Naomi.*
*Without their love, patience, encouragement, support, and understanding, the chances of this book being published would have been infinitesimal at best.*

# Contents

# Preface

Graphs have become a convenient, practical, and efficient tool to model real-world problems. Their increasing utilization has become commonplace in the natural and social sciences, in computer science, and in engineering. The development of large-scale communication and computer networks as well as the efforts in biology to analyze the enormous amount of data arising from the human genome project are but two examples.

Not surprisingly, courses in graph theory have become part of the undergraduate curriculum of many applied sciences, computer science, and pure mathematics courses. Due to the complexity of the applications, many graduate programs in these areas now include a study of graph theory.

A multitude of excellent introductory and more advanced textbooks are on the market. In this book, we address a reader who has been exposed to a first course in graph theory, wishes to apply graph theory at a higher or more special level, and looks for a book that repeats the essentials in a new setting, with new perspectives and results. For this reader, we wish to communicate a working understanding of graph theory and general mathematical tools. The prerequisites are previous exposure to fundamental notions of graph theory, discrete mathematics, and algebra. Therefore, we will not strain the reader's patience with definitions of concepts such as equivalence relations or groups.

The context we chose for this task are graph products and their subgraphs. This includes Hamming graphs, prisms, and many other classes of graphs that are either graph products themselves or are closely related to them—often in surprising, unexpected ways.

This setting allows us to cover concepts with applications in many fields of mathematics and computer science. It includes problems from coding theory, frequency assignment, and mathematical chemistry, which are briefly treated to give the reader a flavor of the variety of the applications.

Many results in this book are recent in the sense that they first appeared in print around the time this book went to press. We have taken efforts to present them accurately and efficiently in a unified environment.

The book is divided into five parts. The first part is a short introduction to the Cartesian product—the main tool that is used throughout the remainder of the book. We convey the basic facts about the product, and apply them to Hamming graphs and Tower of Hanoi graphs, that is, to two classes of graphs that naturally appear.

Classic topics of graph theory are treated in Part II. Included are the fundamental notions of hamiltonicity, planarity, connectivity, and subgraphs. These standard concepts are introduced in most typical first courses in graph theory. We include several interesting results about these basic concepts, which were, somewhat surprisingly, only recently proved. Nonetheless, many challenging open problems still exist in these areas. For example, there is the unsettled conjecture by Rosenfeldt and Barnette that the prism over a 3-connected planar graph is hamiltonian and the determination of the crossing number of the so-called "torus graphs."

A large part of graph theory involves the computation of graphical invariants. The reason is that many applications in different fields reduce to such computations. It turns out that a variety of scheduling and optimization problems are actually coloring problems in graphs constructed from the constraints. In Part III, we therefore focus on several different graph coloring invariants, some standard and some more recently introduced. In a separate chapter we study the problem of determining the cardinality of a largest independent set in a graph. The remaining two chapters of Part III focus on the domination number of a graph with special emphasis on the famous conjecture of Vizing.

Distances in graphs represent another major area for applications. As an example of such an application we present the Wiener index, which is probably the most explored topological index in mathematical chemistry. In Part IV, we demonstrate that the Cartesian product is a natural environment for the standard shortest-path metric. The starting point for this is the fact that the distance function is additive on product graphs. The material in this part of the book culminates in the Graham-Winkler Theorem, asserting that every connected graph has a unique canonical, isometric embedding into a Cartesian product.

Mathematical structures can be properly understood only if one has a grasp of their symmetries. It also helps to know whether they can be constructed from smaller constituents. This approach is taken in Part V. It leads to the prime factorization of graphs and the description of their automorphism groups. These, in turn, simplify the investigation of algebraic properties of connected or disconnected graphs with respect to the Cartesian product. In particular, cancelation properties are derived and the unique $r^{\text{th}}$ root property is proved. Thereafter follows a chapter on the recent concept of the distinguishing number, which measures the effort needed to break all symmetries in a graph. The last chapter shows how the main result on the structure and the symmetries of Cartesian products lead to efficient factorization algorithms and the recognition of partial cubes.

Every chapter ends with a list of exercises. They are an integral part of the book because we are convinced that problem solving is not only at the core of mathematics, but is also essential for the comprehension and acquisition of mathematical proficiency. Checking one's mastery of ideas is crucial for strengthening self-confidence and self-reliance. Therefore some of the exercises are computational; others ask for the proof of a result in the chapter. The easier exercises let the reader check whether he or she grasps the concepts, but most of the exercises require an original idea, and a few demand a higher level of abstraction. Then there are problems whose solution requires the investigation of numerous cases. The idea for these problems is to find a way to minimize the effort and to solve some of the cases.

Hints and solutions to the exercises are collected at the end of the book.

We cordially thank Drago Bokal, Mietek Borowiecki, Boštjan Brešar, Ivan Gutman, Bert Hartnell, Iztok Peterin, and Simon Špacapan for invaluable comments, remarks and contributions to the manuscript. We are especially grateful to Amir Barghi, a graduate student at Dartmouth College, for a careful reading of the entire manuscript. His suggestions led to improvements in the presentation at numerous places in the text.

The manuscript was tested in courses at the University of Maribor, Slovenia; the Montanuniversität Leoben, Austria; and Furman University, Greenville, SC, United States. We wish to thank our students Matevž Črepnjak, Michael Hull, Marko Jakovac, Luka Komovec, Aneta Macura, Michał Mrzygłód, Mateusz Olejarka, Katja Prnaver, Jeannie Tanner, and Joseph Tenini for remarks that helped to make the text more accessible.

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W. Imrich, S. Klavžar, and D.F. Rall
Leoben, Maribor, Greenville
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# Part I

## Cartesian Products

# 1 The Cartesian Product

Throughout this book the Cartesian product will be the leading actor. With its help, the reader will develop a deeper understanding of graph theory. In addition, the reader will learn about important new concepts such as circular colorings, $L(2,1)$-labelings, prime factorizations, canonical metric embeddings, and distinguishing numbers.

In this chapter, we define the Cartesian product and introduce fibers and projections as important tools for further investigations. We also show that a product graph is connected if and only if its factors are connected. Along the way, we list several examples of Cartesian products. In particular, we observe that line graphs of complete bipartite graphs are products of complete graphs, and we show that these are the only products that are line graphs.

## 1.1 Definitions, Fibers, and Projections

Before we define the Cartesian product, we list some conventions to be used throughout the book. We write $g \in G$ instead of $g \in V(G)$ to indicate that $g$ is a vertex of $G$, and $|G| = |V(G)|$ for the number of vertices. An edge $\{u, v\}$ of a graph $G$ is denoted as $uv$. Sometimes, particularly when dealing with edges in products, we also write $[u, v]$.

The *Cartesian product* of two graphs $G$ and $H$, denoted $G \,\square\, H$, is a graph with vertex set

$$V(G \,\square\, H) = V(G) \times V(H), \tag{1.1}$$

that is, the set $\{(g, h) \mid g \in G, h \in H\}$.

The edge set of $G \,\square\, H$ consists of all pairs $[(g_1, h_1), (g_2, h_2)]$ of vertices with $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1, h_2] \in E(H)$.

For example, Figure 1.1 depicts $P_4 \,\square\, P_3$ (left) and $C_7 \,\square\, K_2$ (right). To the first example, we remark that Cartesian products $P_m \,\square\, P_n$ of two

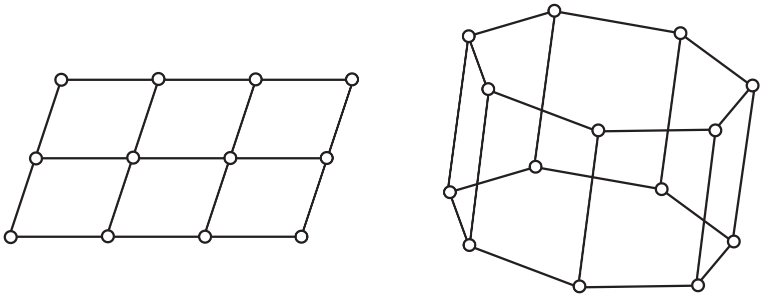

**Figure 1.1.** Cartesian products $P_4 \,\square\, P_3$ (left) and $C_7 \,\square\, K_2$ (right).

paths on $m$ and $n$ vertices are called *complete grid graphs*, and their subgraphs are known as *grid graphs*. Such graphs appear in many applications, for instance in the theory of communication networks.

Note that $K_2 \,\square\, K_2 = C_4$, that is, the Cartesian product of two edges is a square. This is the motivation for the introduction of the notation $\square$ for the Cartesian product.[1]

We can also define the edge set by the relation

$$E(G \,\square\, H) = (E(G) \times V(H)) \cup (V(G) \times E(H)), \tag{1.2}$$

where the edge $(e, h)$ of $G \,\square\, H$, with $e = [g_1, g_2] \in E(G)$, $h \in H$, has the endpoints $(g_1, h)$, $(g_2, h)$, and the edge $(g, f)$, with $g \in G$, $f = [h_1, h_2] \in E(H)$, has the endpoints $(g, h_1)$, $(g, h_2)$.

Since edges in simple graphs can be identified with their (unordered) sets of endpoints, the preceding two definitions of the edge set of $G \,\square\, H$ are equivalent.

Combining Equations (1.1) and (1.2), we obtain yet another, even more concise characterization of the Cartesian product of two graphs $G$ and $H$; see Gross and Yellen [49, p. 238]:

$$G \,\square\, H = (G \times V(H)) \cup (V(G) \times H).$$

Here

$$G \times V(H) = \bigcup_{h \in H} (G \times \{h\}),$$

and every $G \times \{h\}$ is a copy of $G$. We denote it by $G^h$ and call it a *G-fiber*.[2] Analogously, $V(G) \times H$ is the union of the *H-fibers* ${}^gH = \{g\} \times H$.

[1] Some authors use the term *box product* for the Cartesian product.

[2] In *Product Graphs* [66], fibers are referred to as *layers*.

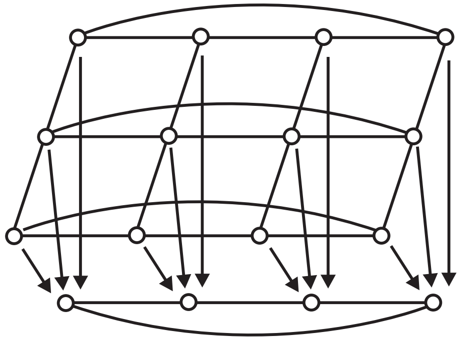

**Figure 1.2.** Projection $p_{C_4} : C_4 \,\Box\, P_3 \to C_4$.

For a given vertex $v = (g, h)$, we also write $G^v = G^h$, and respectively, ${}^vH = {}^gH$. Clearly $G^v$ can also be defined as the subgraph of $G \,\Box\, H$ induced by $\{(x, h) \mid x \in G\}$, and the mapping $\varphi : V(G^h) \to V(G)$ defined by

$$\varphi : (x, h) \mapsto x$$

is a bijection that preserves adjacency and nonadjacency. Such a bijection is called a *graph isomorphism*. One can say $G^h$ and $G$ are isomorphic, or in symbols, $G^h \cong G$.

Sometimes we also write $G = H$ for isomorphic graphs. Thus, $G = H$ may mean that $G$ and $H$ have the same vertex and edge sets or that $G$ and $H$ are isomorphic. For example, the phrase "$G$ is a $K_2$" or the equation "$G = K_2$" both mean that $G$ is isomorphic to $K_2$.

In contrast, for two $G$-fibers $G^v$ and $G^w$ of a product $G \,\Box\, H$, the statement $G^v = G^w$ expresses that these fibers are identical, whereas $G^v \cong {}^wH$ really means only that the two fibers are isomorphic because $G^v \neq {}^wH$; in fact, $|G^v \cap {}^wH| = 1$.

Note that $G \,\Box\, H \cong H \,\Box\, G$, and that $K_1 \,\Box\, G \cong G \,\Box\, K_1 \cong G$. In other words, Cartesian multiplication is commutative and $K_1$ is a unit (see Exercise 3).

For a product $G \,\Box\, H$, the *projection* $p_G : G \,\Box\, H \to G$ is defined by

$$p_G : (g, h) \mapsto g.$$

It is clear what we mean by $p_H$. See Figure 1.2 for the projection $p_{C_4}$ of $C_4 \,\Box\, P_3$ onto $C_4$.

Under the projections $p_G$ or $p_H$, the image of an edge is an edge or a single vertex. Such mappings are called *weak homomorphisms.* Clearly, the mapping $\varphi : V(G^h) \to V(G)$ defined above is the restriction of $p_G$ to $G^h$.

More generally, let $U$ be a subgraph of $G \,\Box\, H$. We follow common practice and define $p_G U$ as the subgraph of $G$ with the vertex set

$$\{p_G(v) \mid v \in U\}$$

and edge set

$$\{[p_G(u), p_G(v)] \mid [u, v] \in E(U),\ p_G(u) \neq p_G(v)\}.$$

## 1.2 Connectedness and More Examples

We continue with the following simple, yet fundamental observation about Cartesian products.

**Lemma 1.1.** *A Cartesian product $G \,\Box\, H$ is connected if and only if both factors are connected.*

*Proof:* Suppose $G \,\Box\, H$ is connected. We have to prove that both $G$ and $H$ are connected. Clearly, it suffices to prove it for $G$. Let $g$ and $g'$ be any two vertices of $G$, and let $h \in H$ be arbitrary. Then there is a path $P$ in $G \,\Box\, H$ from $(g, h)$ to $(g', h)$, and $p_G P \subseteq G$ contains a path from $g$ to $g'$.

Conversely, assume that $G$ and $H$ are connected. We have to show that there is a path between any two arbitrarily chosen vertices $(g, h)$ and $(g', h')$ of $G \,\Box\, H$. Let $R$ be a $g$,$g'$-path and $S$ an $h$,$h'$-path. Then

$$(R \times \{h\}) \cup (\{g'\} \times S)$$

is a $(g, h)$,$(g', h')$-path; see Figure 1.3. □

Before continuing with new concepts related to Cartesian products, we take a break with two examples: prisms and line graphs of complete bipartite graphs.

Prisms over graphs appear in many situations and are defined as follows. Let $G$ be a graph. Then the *prism over $G$* is the graph obtained from the disjoint union of graphs $G'$ and $G''$, both isomorphic to $G$, by joining any vertex of $G'$ with its isomorphic image in $G''$. An example of a prism is shown in Figure 1.4.

From our point of view, the prism over $G$ is the Cartesian product

$$G \,\Box\, K_2\,.$$

Let $G$ be a graph. Then the vertex set of the *line graph* $L(G)$ of $G$ consists of the edges of $G$. Two vertices of $L(G)$ are adjacent if the

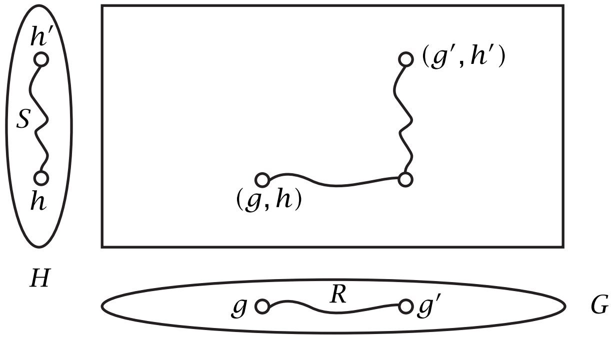


**Figure 1.3.** $G \,\Box\, H$ is connected provided $G$ and $H$ are connected.

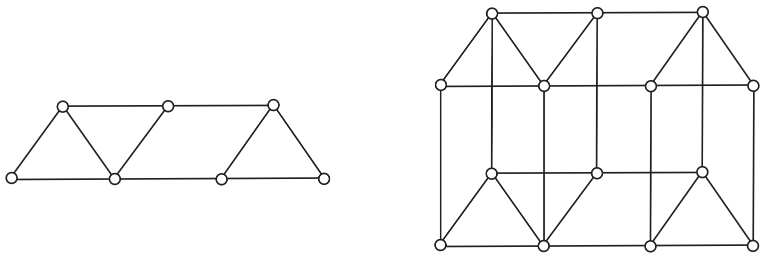

**Figure 1.4.** A graph (left) and the prism over it (right).

corresponding edges of $G$ are adjacent. Note that a vertex of degree $d$ in $G$ yields a complete subgraph $K_d$ in $L(G)$.

For instance, for any $n \geq 2$, $L(K_{1,n}) = K_n$ and $L(P_n) = P_{n-1}$.

Let $m, n \geq 2$ and consider the complete bipartite graph $K_{m,n}$ with the bipartition $\{u_1, u_2, \ldots, u_m\}$, $\{w_1, w_2, \ldots, w_n\}$. Then the vertex set of $L(K_{m,n})$ is

$$\{u_i w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\} .$$

Vertices $u_i w_j$ and $u_{i'} w_{j'}$ are adjacent in $L(K_{m,n})$ if and only if $i = i'$ and $j \neq j'$, or $j = j'$ and $i \neq i'$. This implies that

$$L(K_{m,n}) = K_m \,\Box\, K_n .$$

In fact, these are the only connected line graphs that are Cartesian products, as the following result of Palmer asserts.

**Proposition 1.2.** [97] *Let $X$ be a connected graph. Then $L(X)$ is a non-trivial Cartesian product if and only if $X = K_{m,n}$, $m, n \geq 2$.*