LECTURE NOTES IN LOGIC

NONSTANDARD METHODS AND APPLICATIONS IN MATHEMATICS

Edited by

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ASSOCIATION FOR SYMBOLIC LOGIC

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TEACHING

PREFACE

Nonstandard analysis is one of the the great achievements of modern applied mathematical logic. In addition to the important philosophical achievement of providing a sound mathematical basis for using infinitesimals in analysis, the methodology is now well established as a tool for both research and teaching, and has become a fruitful field of investigation in its own right. It has been used to discover and prove significant new standard theorems in such diverse areas as probability theory and stochastic analysis, functional analysis, fluid mechanics, dynamical systems and control theory, and recently there have been some striking and unexpected applications to additive number theory.

A conference on Nonstandard Methods and Applications in Mathematics (NS2002) was held in Pisa, Italy from June 12-16 2002. This was originally planned as a special section in the very successful first joint meeting of the American Mathematical Society and the Unione Matematica Italiana. In order to accommodate the large number of mathematicians interested in the field, a satellite conference, hosted by the Universitá di Pisa and held at the Domus Galilaeana, was added during the days preceding the main AMS/UMI meeting. A complete list of the registered participants appears later in this forward.

This volume is a byproduct of NS2002. Not a proceedings per se, it is a collection of peer-reviewed papers solicited from some of the participants with the aim of providing something more timely than a textbook, but less ephemeral than a conventional proceedings. To that end, the volume contains both survey papers on topics for which other surveys are either dated or nonexistent, and research articles on applications too recent to have received attention in older volumes.

One of the included papers, on an infinitesimal approach to calculus, deserves special mention. The use of infinitesimals in the teaching of calculus is of course not at all new, though they began to disappear from textbooks late in the 19th century due to concerns about their theoretical underpinnings. (Even today most instructors use infinitesimals in teaching *applications*, such as volumes of rotation, as they are more natural and compelling than Riemann sums in this context.) Any foundational concerns were of course completely dispelled by Abraham Robinson's work, and at least two calculus textbooks and several introductory analysis texts using infinitesimals have since appeared. By beginning the course with some basic rules for working in an extension of the real number system, such books make it possible to offer completely correct proofs to beginning students, proofs which better encapsulate mathematical intuition than do more conventional arguments.

A few months prior to NS2002, the organizers learned that an infinitesimal approach to calculus was being adopted by some high school teachers in Geneva, Switzerland. This was the first attempt we had heard of to use a modern infinitesimal approach at the high school level. Curious about the effort — which appeared to be independent of (and different than) the earlier approaches of Keisler et al — we asked one of the course's designers, Richard O'Donovan, to come to our meeting and report on their work. The paper here, *Nonstandard analysis at pre-university level: naive magnitude analysis* by O'Donovan and his colleague John Kimber describes their approach

We are grateful to the Istituto Nazionale di Alta Matematica, Gruppo Nazionale per le Strutture Algebriche, Geome-triche e le loro Applicazioni (INdAM-GNSAGA), and to the University of Pisa Interdepartmental Center for the Study of Complex Systems (CISSC), for the financial support which made NS2002 possible. We are also grateful to the Domus Galilaeana of Pisa for hosting part of the congress.

Thanks also to the ASL, in particular to C. Ward Henson and to Steffen Lempp, for their assistance at all stages of producing this volume.

The program comprised a total of thirty-three talks, including the following invited lectures:

- **N.J. Cutland** (Hull, UK): *Nonstandard techniques in stochastic fluid dynamics*
- **H. Osswald** (München, Germany): *Malliavin calculus on Banach space valued continuous functions*
- **F. Diener** (Nice, France): *Nonstandard tree model for financial mathematics: Beyond the continuous Black-Scholes approximation for vanilla and barrier options*
- **M. Wolff** (Tübingen, Germany): *Discrete approximation of spaces and operators*
- **V. Benci** (Pisa, Italia): *Numerosities of labelled sets: A new way of counting*
- **T. Nakamura** (Tsuda, Japan): *Construction of a path-space measure for the Ornstein-Uhlenbeck process by infinitesimal random walks*
- **A. Macintyre** (London, UK): *Ultraproducts of cohomology theories*
- **J.L. Bell** (Western Ontario, Canada): *Real lines in smooth infinitesimal analysis*

H.J. Keisler (Wisconsin, USA): *Products of Loeb spaces*
K. Hrbacek (CUNY, USA): *Nonstandard set theory*
P.A. Loeb (Illinois, USA): *Base operators in analysis and a generalization of monads*
R. Jin (Charleston, USA): *Nonstandard analysis and density problems: Introduction and recent developments*
D.A. Ross (Hawaii, USA): *Nonstandard measure constructions: Examples and problems*
S. Albeverio (Bonn, Germany): *No title*

The following is a complete list of registered participants in NS2002:

Eva Aigner, Petr Andreyev, David Ballard, Stefano Baratella, John Bell, Vieri Benci, Eric Benoit, Alessandro Berarducci, Josef Berger, Ouahiba Cherikh, Nigel Cutland, Francine Diener, Mauro Di Nasso, Antonino Drago, Ruggero Ferro, Marco Forti, Eberhard Gerlach, Guido Gherardi, Paolo Giordano, Karel Hrbacek, Chris Impens, Renling Jin, Vladimir Kanovei, Jerome Keisler, Giacomo Lenzi, Steven Leth, Peter Loeb, Angus Macintyre, Natalia Martins, Vladimir Molchanov, Mojtaba Moniri, Toru Nakamura, Vitor Neves, Siu-Ah Ng, Richard O'Donovan, Horst Osswald, Yves Peraire, Hans Ploss, Emiliano Rago, Giuseppe Randazzo, Hermann Render, Sergio Rodrigues, David Ross, Peter Schuster, Joao Teixeira, Hans Vernaeve, Guy Wallet, Manfred Wolff, Beate Zimmer.

The organizers note with sadness the death in May 2004 of our friend and colleague David Ballard. David was an invited participant in NS2002, and his work in the foundations of nonstandard set theory was intriguing and highly original.

The Editors
Nigel J. Cutland, *Hull*
Mauro Di Nasso, *Pisa*
David A. Ross, *Honolulu*

FOUNDATIONS

THE EIGHTFOLD PATH TO NONSTANDARD ANALYSIS

VIERI BENCI, MARCO FORTI, AND MAURO DI NASSO

Abstract. This paper consists of a quick introduction to the "hyper-methods" of nonstandard analysis, and of a review of eight different approaches to the subject, which have been recently elaborated by the authors.

Those who follow the noble Eightfold
Path are freed from the suffering and
are led ultimately to Enlightenment.

(Gautama Buddha)

Introduction. Since the original works [39, 40] by Abraham Robinson, many different presentations to the methods of nonstandard analysis have been proposed over the last forty years. The task of combining in a satisfactory manner rigorous theoretical foundations with an easily accessible exposition soon revealed very difficult to be accomplished. The first pioneering work in this direction was W.A.J. Luxemburg's lecture notes [36]. Based on a direct use of the ultrapower construction, those notes were very popular in the "nonstandard" community in the sixties. Also Robinson himself gave a contribution to the sake of simplification, by reformulating his initial type-theoretic approach in a more familiar set-theoretic framework. Precisely, in his joint work with E. Zakon [42], he introduced the *superstructure approach*, by now the most used foundational framework.

To the authors' knowledge, the first relevant contribution aimed to make the "hyper-methods" available even at a freshman level, is Keisler's book [33], which is a college textbook for a first course of elementary calculus. There, the principles of nonstandard analysis are presented axiomatically in a nice and elementary form (see the accompanying book [32] for the foundational aspects). Among the more recent works, there are the "gentle" introduction by W.C. Henson [26], R. Goldblatt's lectures on the hyperreals [25], and K.D. Stroyan's textbook [44].

2000 *Mathematics Subject Classification.* 26E35 Nonstandard analysis; 03E65 Other hypotheses and axioms.

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Lecture Notes in Logic, 25

Recently the authors investigated several different frameworks in algebra, topology, and set theory, that turn out to incorporate explicitly or implicitly the "hyper-methods". These approaches show that nonstandard extensions naturally arise in several quite different contexts of mathematics. An interesting phenomenon is that some of those approaches lead in a straightforward manner to ultrafilter properties that are independent of the axioms of Zermelo-Fraenkel set theory ZFC.

Contents. This article is divided into two parts. The first part consists of an introduction to the hyper-methods of nonstandard analysis, while the second one is an overview of eight different approaches to the subject recently elaborated by the authors. Most proofs are omitted, but precise references are given where the interested reader can find all details.

Part I contains two sections. The longest Section 1 is a soft introduction to the basics of nonstandard analysis, and will be used as a reference for the remaining sections of this article. The three fundamental "hyper-tools" are presented, namely the *star-map*, the *transfer principle*, and the *saturation property*, and several examples are given to illustrate their use in the practice. The material is intentionally presented in an elementary (and sometimes semi-formal) manner, so that it may also serve as a quick presentation of nonstandard analysis for newcomers. Section 2 is focused on the connections between the hyper-extensions of nonstandard analysis and ultrapowers. In particular, a useful characterization of the models of hyper-methods is presented in purely algebraic terms, by means of limit ultrapowers.

Each of the eight Sections 3–10 in Part II presents a different possible "path" to nonstandard analysis. The resulting eight approaches, although not strictly equivalent to each other, are all suitable for the practice, in that each of them explicitly or implicitly incorporates the fundamental "hyper-tools" introduced in Section 1.

Section 3 is about a modified version of the so-called *superstructure approach*, where a single superstructure is considered both as the standard and the nonstandard universe (see [3].) In Section 4, we present the purely algebraic approach introduced in [6, 7], which is based on the existence of a "special" ring homomorphism. Starting from such a homomorphism, we define in a direct manner a superstructure model of the hyper-methods, as defined in Section 3.

In Section 5, the axiomatic theory *ZFC of [17] is presented, that can be seen as an extension of the superstructure approach to the full generality of set theory. Section 6 is dedicated to the so-called *Alpha Theory*, an axiomatic presentation that postulates five elementary properties for an "ideal" (infinite) natural number α (see [4].) These axioms suffice for defining a star-map on the universal class of all mathematical objects.

Section 7 deals with *topological extensions*, a sort of "topological completions" of a given set X, introduced and studied in [9, 18]. These structures are topological spaces ${}^{\star}X$ where any function $f : X \to X$ has a continuous $\star$-extension, ${}^{*}f : {}^{*}X \to {}^{*}X$, and where the $\star$-extension ${}^{\star}A$ of a subset $A \subseteq X$ is simply its closure in ${}^{\star}X$. Hyper-extensions of nonstandard analysis, endowed with a natural topology, are characterized as those topological extensions that satisfy two simple additional properties. Moreover, several important features of nonstandard extensions, such as the enlarging and saturation properties, can be naturally described in this topological framework. Section 8, following [24], further simplifies the topological approach of the preceding section. By assuming that the $\star$-extensions of unary functions satisfy three simple "preservation properties" having a purely functional nature, one obtains all possible hyper-extensions of nonstandard analysis.

Section 9 deals with natural ring structures that can be given to suitable subspaces of $\beta\mathbb{Z}$, the Stone-Čech compactification of the integers $\mathbb{Z}$ (see [19].) Such rings turn out to be sets of hyperintegers with special properties that are independent of ZFC. In the final Section 10, we consider a new way of counting that has been proposed in [5] and which maintains the ancient principle that "the whole is larger than its parts". This counting procedure is suitable for those countable sets whose elements are "labelled" by natural numbers. We postulate that this procedure satisfies three natural "axioms of compatibility" with respect to inclusion, disjoint union, and Cartesian product. As a consequence, sums and products of numerosities can be defined, and the resulting semi-ring of numerosities becomes a special set of hypernatural numbers, whose existence is independent of ZFC.

Disclaimer. A disclaimer is in order. By no means the approaches presented here have been choosen because they are better than others, or because they provide an exhaustive picture of this field of research. Simply, this article surveys the authors' contributions to the subject over the last decade. In particular, throughout the paper we stick to the so-called *external* viewpoint of nonstandard methods, based on the existence of a star-map $*$ providing an hyper-extension ${}^{*}A$ for each standard object A. This is to be confronted with the *internal* approach of Nelson's IST [37], and other related nonstandard set theories where the *standard predicate* `st` is used in place of the star-map (cf. e.g. the recent book [30]; see also Hrbacek's article in this volume). Extensive treatments of nonstandard analysis based on the internal approach are given e.g. in the books [21, 22, 38].

Part I – The "Hyper-methods"

§1. What are the "hyper-methods"? Roughly, nonstandard analysis essentially consists of two fundamental tools: the *star-map* $*$ and the *transfer principle*. In most applications, a third fundamental tool is also considered, namely the *saturation property*.

There are several different frameworks where the methods of nonstandard analysis (the "hyper-methods") can be presented. The goal of this section is to introduce the basic notions in such a way that their formulations do not depend on the specific approach that one is adopting. Of course, there is a price we have to pay to reach this generality. Sometimes, the definitions as given here are not entirely formalized (at least from the point of view of a logician). However we are confident that they are still sufficiently clear and unambiguous to the point that some "practitioners" may find them suitable already. To reassure the suspicious reader, we anticipate that each of the eight Sections 3–10 consists of a specific approach where all notions presented here are given rigorous foundations.

Besides the fundamental tools and definitions, this section also contains the definition of internal element, sketchy proofs of the first consequences of the definitions, as well as a bunch of relevant examples. It is not a complete introduction (e.g. overspill and hyperfinite sets are not treated), but it may be used as a first reading for beginners interested in nonstandard analysis.

1.1. The basic definitions. In order to correctly formulate the fundamental tools of hyper-methods, we need the following

DEFINITION 1.1. A *universe* $\mathbb{U}$ is a nonempty collection of "mathematical objects" that is closed under subsets (i.e. $a \subseteq A \in \mathbb{U} \Rightarrow a \in \mathbb{U}$) and closed under the basic mathematical operations. Precisely, whenever $A, B \in \mathbb{U}$, we require that also the *union* $A \cup B$, the *intersection* $A \cap B$, the *set-difference* $A \setminus B$, the *ordered pair* (A, B), the *Cartesian product* $A \times B$, the *powerset* $\mathcal{P}(A) = \{a \mid a \subseteq A\}$, the *function-set* $B^A = \{f \mid f : A \to B\}$, all belong to $\mathbb{U}$.[1] A universe $\mathbb{U}$ is also assumed to contain (copies of) all sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \in \mathbb{U}$, and to be *transitive*, i.e. members of $\mathbb{U}$ belong to $\mathbb{U}$ (in formulæ: $a \in A \in \mathbb{U} \Rightarrow a \in \mathbb{U}$).

The notion of "mathematical object" includes all objects used in the ordinary practice of mathematics, namely: numbers, sets, functions, relations, ordered tuples, Cartesian products, *etc.* It is well-known that all these notions can be defined as sets and formalized in the foundational framework of Zermelo-Fraenkel axiomatic set theory ZFC.[2] For sake of simplicity, here we consider them as primitive concepts not necessarily reduced to sets.

Hyper-Tool # 1: STAR-MAP.

The star-map is a function $* : \mathbb{U} \to \mathbb{V}$ between two universes that associates to each object $A \in \mathbb{U}$ its hyper-extension (or nonstandard

[1] Clearly, here we implicitly assume that A and B are sets, otherwise these operation don't make sense. The only exception is the ordered pair, that makes sense for *all* mathematical objects A and B.

[2] E.g. in ZFC, an ordered pair (a, b) is defined as the *Kuratowski pair* $\{\{a\}, \{a, b\}\}$; an n-tuple is inductively defined by $(a_1, \dots, a_n, a_{n+1}) = ((a_1, \dots, a_n), a_{n+1})$; an n-place relation R on A is

extension) $^*A \in \mathbb{V}$. *It is assumed that* $^*n = n$ *for all natural numbers* $n \in \mathbb{N}$, *and that the* properness *condition* $^*\mathbb{N} \neq \mathbb{N}$ *holds.*

It is customary to call *standard* any object $A \in \mathbb{U}$ in the domain of the star-map, and *nonstandard* any object $B \in \mathbb{V}$ in the codomain. The adjective standard is also often used in the literature for hyper-extensions $^*A \in \mathbb{V}$.

We remark rightaway that one could directly consider a single universe $\mathbb{U} = \mathbb{V}$. Doing so, the traditional distinction between standard and nonstandard objects is overcome.[3] We point out that in all approaches appeared in the literature, the standard universe is taken to be large enough so as to include all mathematical objects under consideration.

We are now ready to introduce the second powerful tool of nonstandard methods. It states that the star-map preserves a large class of properties.

Hyper-Tool # 2: TRANSFER PRINCIPLE.

Let $P(a_1, \ldots, a_n)$ *be a property of the standard objects* $a_1, \ldots, a_n$ *expressed as an "elementary sentence". Then* $P(a_1, \ldots, a_n)$ *is true if and only if the same sentence is true about the corresponding hyper-extensions* $^*a_1, \ldots, ^*a_n$. *That is:*

$$P(a_1, \ldots, a_n) \Longleftrightarrow P(^*a_1, \ldots, ^*a_n)$$

The *transfer principle* (also known as *Leibniz principle*) is given a rigorous formulation by using the formalism of mathematical logic and, in particular, by appealing to the notion of *bounded quantifier formula* in the first-order language of set theory. Here we only give a semi-formal definition, and refer the reader to §4.4 of [12] for a rigorous treatment.

DEFINITION 1.2. We say that a property $P(x_1, \ldots, x_n)$ of the objects $x_1, \ldots, x_n$ is expressed as an *elementary sentence* if the following two conditions are fulfilled:

(1) Besides the usual logic connectives ("not", "and", "or", "if ... then", "if and only if") and quantifiers ("there exists", "for all"), only the basic notions of function, value of a function at a given point, relation,

identified with the set $R \subseteq A^n$ of n-tuples that satisfy it; a function $f : A \to B$ is identified with its graph $\{(a, b) \in A \times B \mid b = f(a)\}$; and so forth. As for numbers, complex numbers $\mathbb{C} = \mathbb{R} \times \mathbb{R}/\approx$ are defined as equivalence classes of ordered pairs of real numbers, and the real numbers $\mathbb{R}$ are defined as equivalence classes of suitable sets of rational numbers (namely, Dedekind cuts or Cauchy sequences). The rational numbers $\mathbb{Q}$ are a suitable quotient $\mathbb{Z} \times \mathbb{Z}/\approx$, and the integers $\mathbb{Z}$ are in turn a suitable quotient $\mathbb{N} \times \mathbb{N}/\approx$. The natural numbers of ZFC are defined as the set ω of von Neumann naturals: $0 = \emptyset$ and $n + 1 = n \cup \{n\}$ (so that each natural number $n = \{0, 1, \ldots, n-1\}$ is identified with the set of its predecessors.) We remark that these definitions are almost compulsory in order to obtain a set theoretic reductionist foundation, but certainly they are not needed in the ordinary development of analysis.

[3]This matter will be discussed in Section 3 (see Definition 3.3) and Section 5.

domain, codomain, ordered n-tuple, ith component of an ordered tuple, and membership $\in$, are involved.

(2) The scopes of all universal quantifiers $\forall$ ("for all") and existential quantifiers $\exists$ ("there exists") are "bounded" by some set.

A quantifier is *bounded* when it occurs in the form "for every $x \in X$" or "there exists $y \in Y$", for some specified sets X, Y. Thus, in order to correctly apply the *transfer principle*, one has to stick to the following rule.

Rule of the thumb. Whenever considering quantifiers: "$\forall x \dots$" or "$\exists y \dots$", we must always specify the range of the variables, i.e. we must specify sets X and Y and reformulate: "$\forall x \in X \dots$" and "$\exists y \in Y \dots$". In particular, all quantifications on subsets: "$\forall x \subseteq X \dots$" or "$\exists x \subseteq X \dots$", must be reformulated in the form "$\forall x \in \mathcal{P}(X) \dots$" and "$\exists x \in \mathcal{P}(X) \dots$" respectively, where $\mathcal{P}(X)$ is the powerset of X. Similarly, all quantifications on functions $f : A \to B$, must be bounded by B^A, the set of all functions from A to B.

We are now ready to give the

FUNDAMENTAL DEFINITION:

A model of hyper-methods (or a model of nonstandard analysis) is a triple $\langle * ; \mathbb{U} ; \mathbb{V} \rangle$ *where* $* : \mathbb{U} \to \mathbb{V}$ *is a star-map satisfying the transfer principle.*

1.2. Some applications of transfer. We now show a few simple applications of the *transfer principle*, aimed to clarify the crucial notion of elementary sentence.

EXAMPLE 1.3. By condition (1) of Definition 1.2, the following are all elementary sentences: "f is a function with domain A and codomain B"; "b is the value taken by f at the point a"; "R in an n-place relation on A"; "C is the Cartesian product of A and B". Thus by *transfer*, we get that "${}^*f : {}^*A \to {}^*B$ is a function with domain ${}^*A$ and codomain ${}^*B$"; "${}^*b = {}^*f({}^*a)$ is the value taken by ${}^*f$ at the point ${}^*a$", i.e. ${}^*(f(a)) = {}^*f({}^*a)$; "${}^*R$ is an n-place relation on ${}^*A$"; and "${}^*C = {}^*A \times {}^*B$ is the Cartesian product of ${}^*A$ and ${}^*B$".

EXAMPLE 1.4. The inclusion and all basic operations on sets are preserved under the star-map, with the only relevant exceptions of the powerset and the function-set (see Example 1.9 below). In fact the properties: "$A \subseteq B$"; "$C = A \cup B$"; "$C = A \cap B$"; and "$C = A \setminus B$" can all be formulated as elementary sentences. For instance, "$A \subseteq B$" means that "$\forall x \in A. x \in B$", *etc.* By *transfer* we obtain that "${}^*A \subseteq {}^*B$"; "${}^*C = {}^*A \cup {}^*B$"; "${}^*C = {}^*A \cap {}^*B$"; and "${}^*C = {}^*A \setminus {}^*B$".

EXAMPLE 1.5. Let $f : A \to B$ be any given standard function. Then the *images* $f(A') = \{f(a) \mid a \in A'\}$ of subsets $A' \subseteq A$, and the *preimages*

$f^{-1}(B') = \{a \in A \mid f(a) \in B'\}$ of subsets $B' \subseteq B$, are both preserved under the star-map, i.e. ${}^*(f(A')) = {}^*f({}^*A')$ and ${}^*(f^{-1}(B')) = {}^*f^{-1}({}^*B')$. In particular, ${}^*\text{Range}(f) = \text{Range}({}^*f)$, and so f is onto if and only if ${}^*f$ is. It is also easily shown that f is 1-1 if and only if ${}^*f$ is.

Two more relevant properties are the following: ${}^*\{a \in A \mid f(a) = g(a)\} = \{\alpha \in {}^*A \mid {}^*f(\alpha) = {}^*g(\alpha)\}$, and ${}^*\text{Graph}(f) = \text{Graph}({}^*f)$. All these properties are proved by direct applications of the *transfer principle*. E.g. the last equality is proved by transferring the elementary sentence:

$$\text{“}x \in \text{ Graph }(f) \text{ if and only if there exist } a \in A \text{ and } b \in B$$
$$\text{such that } b = f(a) \text{ and } x = (a,b)\text{”.}$$

EXAMPLE 1.6. Let A be a nonempty standard set, and consider the property: “$<$ is a linear ordering on A”. Notice first that $<$ is a binary relation, hence ${}^*{<}$ is a binary relation on ${}^*A$. By definition, $<$ is a linera ordering if and only if it satisfies the following three properties, that are expressed by means of bounded quantifiers.

$$\forall x \in A\ (x \not< x)$$
$$\forall x, y, z \in A\ (x < y \text{ and } y < z) \Rightarrow x < z$$
$$\forall x, y \in A\ (x < y \text{ or } y < x \text{ or } x = y)$$

Then we can apply the *transfer principle* and obtain that “${}^*{<}$ is a linear ordering on ${}^*A$”.

EXAMPLE 1.7. It directly follows from condition (1) of Definition 1.2 that the hyper-extension of an n-tuple of standard objects $A = (a_1, \dots, a_n)$ is ${}^*A = ({}^*a_1, \dots, {}^*a_n)$. Similarly, if $A = \{a_1, \dots, a_n\}$ is a finite set of standard objects, then its star-extension is ${}^*A = \{{}^*a_1, \dots, {}^*a_n\}$. This is proved by applying *transfer* to the following elementary sentence:

$$\text{“}a_1 \in A \text{ and } \dots \text{ and } a_n \in A \text{ and for all } x \in A,\ x = a_1 \text{ or } \dots \text{ or } x = a_n\text{”}$$

Notice that for every standard set A, $\{{}^*a \mid a \in A\} \subseteq {}^*A$ (apply *transfer* to all sentences “$a \in A$”). In the last example we have seen that the inclusion is actually an equality when A is finite. But this is never the case when A is infinite, as a consequence of the *properness condition* ${}^*\mathbb{N} \neq \mathbb{N}$.

PROPOSITION 1.8. *Let A be an infinite standard set A. Then the inclusion $\{{}^*a \mid a \in A\} \subset {}^*A$ is proper.*

PROOF. Fix a standard map $f : A \to \mathbb{N}$ which is onto. Then ${}^*f : {}^*A \to {}^*\mathbb{N}$ is onto as well. Now assume by contradiction that all elements in ${}^*A$ are of the form ${}^*a$ for some $a \in A$. Then:

$${}^*\mathbb{N} = \{{}^*f({}^*a) \mid a \in A\} = \{{}^*(f(a)) \mid a \in A\} = \{{}^*n \mid n \in \mathbb{N}\} = \mathbb{N},$$

against the properness condition ${}^*\mathbb{N} \neq \mathbb{N}$. ⊣

EXAMPLE 1.9. Let A and B be any standard sets. By transferring the sentences: "$\forall x \in \mathcal{P}(A)$, $\forall y \in x$, $y \in A$" and "$\forall f \in B^A$, f is a function with domain A and codomain B", it is proved that ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$, and ${}^*(B^A) \subseteq {}^*B^{{}^*A}$, respectively. Arguing similarly as in Example 1.7, one easily shows that these inclusions are equalities whenever both A and B are finite. In the infinite case, the inclusions are proper (cf. Proposition 1.25).

1.3. The basic sets of hypernumbers. Let us now concentrate on the hyper-extensions of sets of numbers.

DEFINITION 1.10. The elements of ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are called *hypernatural, hyperinteger, hyperrational, hyperreal*, and *hypercomplex* numbers, respectively.

Besides natural numbers, for convenience it is also customary to assume that ${}^*z = z$ for *all* numbers z. In this case, we have the inclusions $\mathbb{N} \subset {}^*\mathbb{N}$, $\mathbb{Z} \subset {}^*\mathbb{Z}$, $\mathbb{Q} \subset {}^*\mathbb{Q}$, $\mathbb{R} \subset {}^*\mathbb{R}$, and $\mathbb{C} \subset {}^*\mathbb{C}$ (the inclusions are proper by Proposition 1.8). Whenever confusion is unlikely, some asterisks will be omitted. For instance, we shall use the same symbols $+$ and $\cdot$ to denote both the sum and product operations on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and the corresponding operations defined on the hyper-extensions ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, ${}^*\mathbb{R}$, ${}^*\mathbb{C}$. Similarly for the ordering $\leq$.

In the next proposition we itemize the first properties of hypernumbers, all obtained as straightforward applications of the *transfer principle*.

PROPOSITION 1.11.

1. *${}^*\mathbb{Z}$ is a commutative ring, ${}^*\mathbb{Q}$ is an ordered field, ${}^*\mathbb{R}$ is a real-closed field, and ${}^*\mathbb{C}$ is an algebraically closed field;*[4]
2. *Every non-zero hypernatural number $\nu \in {}^*\mathbb{N}$ has a successor $\nu + 1$ and a predecessor $\nu - 1$;*[5]
3. *$(\mathbb{N}, \leq)$ is an initial segment of $({}^*\mathbb{N}, \leq)$, i.e. if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $\nu > n$ for all $n \in \mathbb{N}$;*
4. *For every positive $\xi \in {}^*\mathbb{R}$ there exists a unique $\nu \in {}^*\mathbb{N}$ such that $\nu \leq \xi < \nu + 1$. In particular, ${}^*\mathbb{N}$ is unbounded in ${}^*\mathbb{R}$;*
5. *The hyperrational numbers ${}^*\mathbb{Q}$, as well as the hyperirrational numbers ${}^*(\mathbb{R} \setminus \mathbb{Q}) = {}^*\mathbb{R} \setminus {}^*\mathbb{Q}$, are dense in ${}^*\mathbb{R}$;*[6]
6. *Let Z be any of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, and consider the open interval $(a, b) = \{x \in X \mid a < x < b\}$ determined by numbers $a < b$ in Z.*

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[4] Recall that an ordered field is *real-closed* if every positive element is a square, and every polynomial of odd degree has a root. A field is *algebraically closed* if all non-constant polynomials have a root.

[5] We say that ξ' is the *successor* of ξ (or ξ is the *predecessor* of ξ') if $\xi < \xi'$ and there exist no elements ζ such that $\xi < \zeta < \xi'$.

[6] I.e., for all $\xi < \zeta$ in ${}^*\mathbb{R}$, there exist $x \in {}^*\mathbb{Q}$ and $y \in {}^*\mathbb{R} \setminus {}^*\mathbb{Q}$ such that $\xi < x, y < \zeta$.

Then the hyper-extension $^*(a, b) = \{\xi \in {}^*\mathbb{Z} \mid a < \xi < b\}$. *Similar equalities also hold for intervals of the form* $[a, b)$, $(a, b]$, (a, b), $(-\infty, b]$ *and* $[a, +\infty)$.

As a consequence of property (3) above, the elements of $^*\mathbb{N} \setminus \mathbb{N}$ are called *infinite*. More generally:

DEFINITION 1.12. A hyperreal number $\xi \in {}^*\mathbb{R}$ is *infinite* if either $\xi > \nu$ or $\xi < -\nu$ for some $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Otherwise we say that ξ is *finite*. We call *infinitesimal* those hyperreal numbers $\varepsilon \in {}^*\mathbb{R}$ such that $-r < \varepsilon < r$ for all positive reals $r \in \mathbb{R}$. In this case we write $\varepsilon \sim 0$.

The following properties are easily seen:[7] $\varepsilon \neq 0$ is infinitesimal if and only if its reciprocal $1/\varepsilon$ is infinite; if ξ and ζ are finite, then also $\xi + \zeta$ and $\xi \cdot \zeta$ are finite; if $\varepsilon, \eta \sim 0$, then also $\varepsilon + \eta \sim 0$; if $\varepsilon \sim 0$ and ξ is finite, then $\varepsilon \cdot \xi \sim 0$; if ω is infinite and ξ is *not* infinitesimal, then $\omega \cdot \xi$ is infinite; if $\varepsilon \neq 0$ is infinitesimal but ξ is *not* infinitesimal, then ξ/ε is infinite; if ω is infinite and ξ is finite, then $\xi/\omega \sim 0$; *etc.*

Infinitesimal and infinite numbers can be seen as formalizations of the intuitive ideas of "small" number and "large" number, respectively. Also the idea of "closeness" can be formalized as follows.

DEFINITION 1.13. The hyperreal numbers ξ and ζ are *infinitely close* if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, $\sim$ is an equivalence relation. The completeness of the real numbers $\mathbb{R}$ yields the following result.

THEOREM 1.14 (Standard part). *For every finite* $\xi \in {}^*\mathbb{R}$, *there exists a unique real number* $r \in \mathbb{R}$ *(called the "standard part" of* ξ*) such that* $\xi \sim r$.

PROOF. The least upper bound $r = \sup\{a \in \mathbb{R} \mid a \leq \xi\}$ has the desired property. ⊣

The next interesting result shows that in a way the hyperrationals already "incorporate" the real numbers (see e.g. [45, Thm. 4.4.4] and [14, Ch.II, Thm. 2]).

THEOREM 1.15. *Let* $^*\mathbb{Q}_b$ *be the ring of finite hyperrationals, and let* $\mathfrak{I}$ *be the maximal ideal of its infinitesimals. Then* $\mathbb{R}$ *and* $^*\mathbb{Q}_b/\mathfrak{I}$ *are isomorphic as ordered fields.*

1.4. Correctly applying the transfer principle. From the examples presented so far, one might (wrongly) guess that applying the *transfer principle* merely consists in putting asterisks $*$ all over the place. It is not so, because — as we already pointed out — only elementary sentences can be transferred. We now give three relevant examples aimed to clarify this matter.

[7] In fact, they hold in any *non-archimedean* field (the archimedean property is defined in Example 1.18).

EXAMPLE 1.16. Recall the *well-ordering* property of $\mathbb{N}$:

"*Every nonempty subset of* $\mathbb{N}$ *has a least element*".

By applying the *transfer principle* to this formulation, we would get that "Every nonempty subset of ${}^*\mathbb{N}$ has a least element". But this is clearly false (e.g. the collection ${}^*\mathbb{N} \setminus \mathbb{N}$ of infinite hypernaturals has no least element, because if ν is infinite, then $\nu - 1$ is infinite as well). We reached a wrong conclusion because we transferred a sentence which is *not* elementary (the universal quantifier is not bounded). However, we can easily overcome this problem by reformulating the well-ordering property as the following elementary sentence: "*Every nonempty element of* $\mathcal{P}(\mathbb{N})$ *has a least element*", where $\mathcal{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$. (Notice that the property "X has a least element" is elementary, because it means: "there exists $x \in X$ such that for all $y \in X, x \leq y$".) We can now correctly apply the *transfer* and get: "*Every nonempty element of* ${}^*\mathcal{P}(\mathbb{N})$ *has a least element*", where it is intended that the ordering on ${}^*\mathbb{N}$ is the hyper-extension of the ordering on $\mathbb{N}$. The crucial point here is that ${}^*\mathcal{P}(\mathbb{N})$ is *properly included* in $\mathcal{P}({}^*\mathbb{N})$ (see Proposition 1.25 below).

EXAMPLE 1.17. Recall the *completeness property* of real numbers:

"*Every nonempty subset of* $\mathbb{R}$ *which is bounded above, has a l.u.b.*"

As in the previous example, if we directly apply *transfer* to this formulation, we reach a false conclusion, namely: "Every nonempty subset of ${}^*\mathbb{R}$ which is bounded above, has a l.u.b." (e.g. the set of infinitesimals is bounded above but has no least upper bound). Again, the problem is that the sentence above is *not* elementary because it contains a quantification over subsets. To fix the problem, we simply have to consider the powerset $\mathcal{P}(\mathbb{R})$ and reformulate: "*Every nonempty element of* $\mathcal{P}(\mathbb{R})$ *which is bounded above has a l.u.b.*". Thus, by the *transfer principle*, we have a least upper bound for each upper-bounded element of ${}^*\mathcal{P}(\mathbb{R})$ (which is a *proper* subset of $\mathcal{P}({}^*\mathbb{R})$, see Proposition 1.25 below).

As suggested by the last examples, restricting to elementary sentences is not a limitation, because virtually all mathematical properties can be equivalently rephrased in elementary terms.

Another delicate aspect that needs some caution, is the possibility of misreading a transferred sentence, once all asterisks $*$ have been put in the right place. A relevant example is given by the archimedean property.

EXAMPLE 1.18. The *archimedean property* of real numbers can be expressed in this elementary form:

"*For all positive* $x \in \mathbb{R}$, *there exists* $n \in \mathbb{N}$ *such that* $n \cdot x > 1$".

By *transfer*, we obtain: "*For all positive* $\xi \in {}^*\mathbb{R}$, *there exists* $\nu \in {}^*\mathbb{N}$ *such that* $\nu \cdot \xi > 1$". Notice that this sentence does *not* express the archimedean property of ${}^*\mathbb{R}$, because the element ν could be an *infinite* hypernatural.

Clearly, the hyperreal field ${}^*\mathbb{R}$ is not archimedean (in fact, an ordered field is non-archimedean if and only if it contains non-zero infinitesimals). In particular $\mathbb{R}$ and ${}^*\mathbb{R}$ are not isomorphic. We remark that this phenomenon of non-isomorphic mathematical structures that cannot be distinguished by any elementary sentence, is indeed the very essence of nonstandard analysis (and more generally, of model theory, a branch of mathematical logic).

1.5. Internal elements. We now introduce a fundamental class of objects in nonstandard analysis.

DEFINITION 1.19. An *internal* object is any element $\xi \in {}^*X$ belonging to some hyper-extension ${}^*X$. An element $\xi \in \mathbb{V}$ of the nonstandard universe is *external* if it is not internal.

Notice that all hyper-extensions ${}^*X$ are internal, because e.g. ${}^*X \in {}^*Y$, where $Y = \{X\}$ is the singleton of X. We remark that in most foundational approaches proposed in the literature, the collection of internal objects is assumed to be *transitive*, i.e. if $b \in B$ and B is internal, then b is internal as well.[8]

The following useful theorem is a straightforward consequence of the *transfer principle* and of the definition of internal object (see e.g. [12, Prop. 4.4.14]).

THEOREM 1.20 (Internal Definition Principle). *If $P(x, x_1, \dots, x_n)$ is an elementary sentence and $B, B_1, \dots, B_n$ are internal objects, then also the set $\{x \in B \mid P(x, B_1, \dots, B_n)\}$ is internal.*

By direct applications of this principle, the following is proved.

PROPOSITION 1.21.

1. *The collection $\mathcal{I}$ of internal sets is closed under union, intersection, set-difference, finite sets and tuples, finite Cartesian products, and under images and preimages of internal functions;*
2. *For every standard set A, ${}^*\mathcal{P}(A) = \mathcal{P}({}^*A) \cap \mathcal{I}$ is the set of all internal subsets of ${}^*A$;*
3. *For all standard sets A and B, ${}^*(B^A) = ({}^*B^{{}^*A}) \cap \mathcal{I}$ is the set of all internal functions from ${}^*A$ to ${}^*B$;*
4. *If $C, D \in \mathcal{I}$ are internal, then $\mathcal{P}(C) \cap \mathcal{I}$ (the set of all internal subsets of C) and $(D^C) \cap \mathcal{I}$ (the set of all internal functions from C to D) are internal.*

The notion of internal set is useful to correctly apply the *transfer principle*. In fact, any quantification on subsets or functions, can be transferred to a quantification on *internal* subsets or *internal* functions, respectively. For instance, let us go back to Examples 1.16 and 1.17. The *well-ordering* of $\mathbb{N}$ is transferred to: "Every nonempty *internal* subset of ${}^*\mathbb{N}$ has a least element". The *completeness* of $\mathbb{R}$ transfers to: "Every nonempty *internal* subset of ${}^*\mathbb{R}$ that is bounded above has a l.u.b.".

[8] The matter of transitivity of the class of internal sets gives rise to interesting considerations in the foundations of nonstandard set theories (cf. Hrbacek's remarks in Subsection 3.3 of [29].)

Another example is the following.

EXAMPLE 1.22. The well-ordering property of $\mathbb{N}$ implies that: "There is no decreasing function $f : \mathbb{N} \to \mathbb{N}$". Then, by *transfer*, "There is no *internal* decreasing function $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$".[9]

In general, we can state the following

Rule of the thumb. Properties about *subsets* or about *functions* of standard objects, transfer to the corresponding properties about *internal subsets* or *internal functions*, respectively.

We can use the above considerations to prove that certain objects are external.

EXAMPLE 1.23. The set ${}^*\mathbb{N} \setminus \mathbb{N}$ of the *infinite* hypernatural numbers is *external*, because it has no least element. Also $\mathbb{N}$ is external, otherwise the set-difference ${}^*\mathbb{N} \setminus \mathbb{N}$ would be internal.[10] The set of infinitesimal hyperreal numbers is another external collection, because it is bounded above but with no least upper bound.

An easy example of external function is the following.

EXAMPLE 1.24. Let $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$ be the function such that $g(n) = n$ if $n \in \mathbb{N}$, and $g(v) = 0$ if $v \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then g is external, otherwise its range $\mathbb{N}$ would be internal.

As a consequence of Proposition 1.21, the above Examples 1.23 and 1.24 show that ${}^*\mathcal{P}(\mathbb{N}) \neq \mathcal{P}({}^*\mathbb{N})$ and ${}^*(\mathbb{N}^{\mathbb{N}}) \neq {}^*\mathbb{N}^{{}^*\mathbb{N}}$. More generally, we have

PROPOSITION 1.25.

1. *Every infinite internal set has at least the size of the continuum, hence it cannot be countable. In particular, for every infinite standard set A, the inclusion ${}^*\mathcal{P}(A) \subset \mathcal{P}({}^*A)$ is proper;*
2. *If the standard set A is infinite and B contains at least two elements, then the inclusion ${}^*(B^A) \subset {}^*B^{{}^*A}$ is proper.*

We warn the reader that getting familiar with the distinction between internal and external objects is probably the hardest step in learning nonstandard analysis.

1.6. The saturation principle. The star-map and the *transfer principle* suffice to develop the basics of nonstandard analysis, but for more advanced applications a third tool is also necessary, namely:

Countable Saturation Principle: *Suppose $\{B_n\}_{n\in\mathbb{N}} \subseteq {}^*A$ is a countable family of internal sets with the "finite intersection property". Then the intersection $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$ is nonempty.*

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[9]We remark that there are models of hyper-methods where (external) decreasing functions $g : {}^*\mathbb{N} \to {}^*\mathbb{N}$ exist.

[10]Here $\mathbb{N} \subset {}^*\mathbb{N}$ is seen as an element of the nonstandard universe.

Recall that a family of sets $\mathcal{B}$ has the *finite intersection property* if $\bigcap_{i=1}^{n} B_i \neq \emptyset$ for all choices of finitely many $B_1, \ldots, B_n \in \mathcal{B}$. In several contexts, stronger saturation principles are considered where also families of larger size are allowed. Precisely, let κ be a given uncountable cardinal.

Fundamental Tool # 3: κ-SATURATION PROPERTY.

Suppose $\mathcal{B} \subseteq {}^*A$ is a family of internal subsets of some hyper-extension ${}^*A$, and suppose $|\mathcal{B}| < \kappa$. If $\mathcal{B}$ has the "finite intersection property", then $\bigcap \mathcal{B} \neq \emptyset$.

In this terminology, countable saturation is $\aleph_1$-saturation. The next example illustrates a relevant use of saturation.

EXAMPLE 1.26. Let (X, τ) be a Hausdorff topological space with *character* κ, hence each point $x \in X$ has a base of neighborhoods $\mathcal{N}_x$ of size at most κ. Clearly, the family of internal sets $\mathcal{B}_x = \{{}^*I \mid I \in \mathcal{N}_x\}$ has the finite intersection property. If we assume κ^+-saturation,[11] the intersection $\mu(x) = \bigcap_{I \in \mathcal{N}_x} {}^*I \neq \emptyset$. In the literature, $\mu(x)$ is called the *monad* of x. Notice that $\mu(x) \cap \mu(y) = \emptyset$ whenever $x \neq y$, since X is Hausdorff. Monads are the basic ingredient in applying the hyper-methods to topology, starting with the following characterizations (see e.g. [35, Ch.III]):

- $A \subseteq X$ is *open* if and only if for every $x \in A$, $\mu(a) \subseteq {}^*A$;
- $C \subseteq X$ is *closed* if and only if for every $x \notin C$, $\mu(x) \cap {}^*C = \emptyset$;
- $K \subseteq X$ is *compact* if and only if ${}^*K \subseteq \bigcup_{x \in K} \mu(x)$.

Sometimes in the literature, the following weakened version of saturation is considered, where only families of hyper-extensions are allowed.

DEFINITION 1.27 (*κ-enlarging property*). Suppose $\mathcal{F} \subseteq A$ is a family of subsets of some standard set A, and suppose that $|\mathcal{F}| < \kappa$. If $\mathcal{F}$ has the "finite intersection property", then $\bigcap_{F \in \mathcal{F}} {}^*F \neq \emptyset$.[12]

We remark that the κ^+-enlarging property suffices to prove the nonstandard characterizations for open, closed and compact sets.

§2. **Ultrapowers and hyper-extensions.** In this section we deal with the connections between ultrapowers and the hyper-extensions of nonstandard analysis. In particular, we will see that, up to isomorphisms, hyper-extensions are precisely suitable subsets of ultrapowers, namely the proper limit ultrapowers. This characterization theorem will be used in Part II of this article to show that the given definitions actually yield models of the hyper-methods.

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[11] κ^+ denotes the successor cardinal of κ. Thus $|\mathcal{B}| < \kappa^+$ is the same as $|\mathcal{B}| \leq \kappa$.

[12] We remark that the enlarging property is strictly weaker than saturation, in the sense that there are models of the hyper-methods where the κ-enlarging property holds but κ-saturation fails.

2.1. Ultrafilters and ultrapowers. Recall that a *filter* $\mathcal{F}$ on a set I is a nonempty family of subsets of I that is closed under intersections and supersets, i.e.

- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
- If $A \in \mathcal{F}$ and $B \supseteq A$, then also $B \in \mathcal{F}$.

A typical example of filter on a set I is the *Frechet filter* $\mathcal{F}r$ of cofinite subsets.

$$\mathcal{F}r = \{A \subseteq I \mid I \setminus A \text{ is finite}\}.$$

DEFINITION 2.1. An *ultrafilter* $\mathcal{U}$ on I is a filter that satisfies the additional property: $A \notin \mathcal{U} \Leftrightarrow I \setminus A \in \mathcal{U}$.

It is easily shown that ultrafilters on I are those non-trivial filters with are maximal with respect to inclusion.[13] As a consequence of the definition, if a finite union $A_1 \cup \cdots \cup A_n \in \mathcal{U}$ belongs to an ultrafilter, then at least one of the $A_i \in \mathcal{U}$.

First examples are the *principal* ultrafilters $\mathcal{U}_i = \{A \subseteq I \mid i \in A\}$, where i is a fixed element of I. Notice that an ultrafilter is non-principal if and only if it contains no finite sets (hence, if and only if it includes the Frechet filter). The existence of non-principal ultrafilters is proved by a straight application of Zorn's lemma.

Given an ultrafilter $\mathcal{U}$ on the set I, consider the following equivalence relation $\equiv_{\mathcal{U}}$ on functions with domain I:

$$f \equiv_{\mathcal{U}} g \Longleftrightarrow \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}.$$

The *ultrapower* of a set X modulo $\mathcal{U}$ is the quotient set:

$$X^I_{\mathcal{U}} = \{[f]_{\mathcal{U}} \mid f : I \to X\}$$

where we denoted by $[f]_{\mathcal{U}} = \{g \in X^I \mid f \equiv_{\mathcal{U}} g\}$ the equivalence class of f. When the ultrafilter $\mathcal{U}$ is clear from the context, we simply write $[f]$. X is canonically embedded into its ultrapower $X^I_{\mathcal{U}}$ by means of the *diagonal map* $d : x \mapsto [c_x]$, where $c_x : I \to X$ is the constant function with value x.

The ultrapower construction is commonly used to obtain models of hypermethods. Indeed, models of hyper-methods are fully characterized by means the generalized notion of limit ultrapower (see Theorem 2.10 below.)

Ultrafilters naturally arise in hyper-extensions.

DEFINITION 2.2. Let X be any standard set, and let $\alpha \in {}^*X$. The *ultrafilter generated* by $\alpha \in {}^*X$, is the following family of subsets of X:

$$\mathcal{U}_\alpha = \{A \subseteq X \mid \alpha \in {}^*A\}.$$

It is readily verified that $\mathcal{U}_\alpha$ is actually an ultrafilter on X. Moreover, $\mathcal{U}_\alpha$ is non-principal if and only if $\alpha \neq {}^*x$ for all $x \in X$.

[13]By the *trivial filter* on I we mean the collection $\mathcal{P}(I)$ of all subsets of I.