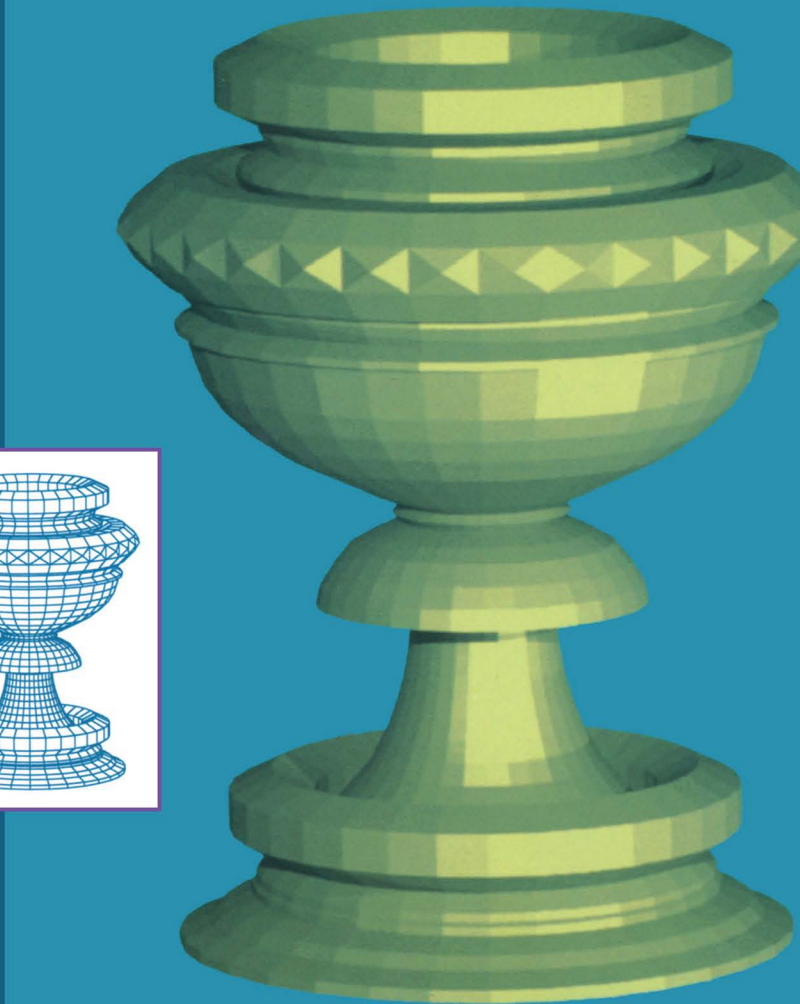
WOLFGANG BOEHM / HARTMUT PRAUTZSCH

Geometric Concepts for Geometric Design



Geometric Concepts for Geometric Design

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
AN A K PETERS BOOK

First published 1994 by A K Peters, Ltd.

Published 2018 by CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

ISBN-13: 978-1-56881-004-1 (hbk)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Library of Congress Cataloging-in-Publication Data

Boehm, Wolfgang, 1928-
Geometric Concepts for Geometric Design / Wolfgang Boehm, Hartmut Prautzsch.
p. cm.
Includes bibliographic references and index.
ISBN 1-56881-004-0
1. Geometry. 2. Geometry—Data processing. I. Prautzsch, Hartmut. II. Title.
QA445.B63 1993b
516—dc20 93-20666
CIP

About the cover: The Cover picture shows a computer generated shaded image of a chalice based on a drawing by Paolo Ucello (1397-1475). Ucello's hand drawing was the first extant complex geometrical form rendered according to the laws of perspective. (Perspective Study of a Chalice, Drawing, Gabinetto dei Disegni, Florence, ca 1430-1440.)

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Preface

This book addresses students, teachers and researchers in mathematics, computer science and engineering who are confronted with geometric problems, attracted by their beauty, and/or wish to get a deeper geometric background.

Its purpose is to give a solid foundation of geometric methods and their underlying principles. It may serve as an introduction to geometry as well as a practical guide to geometric design and modeling and to other applications of geometry.

The main idea of this book is to provide an imagination for what happens geometrically and to present tools for describing problems. A problem is often solved simply by finding the right description. The topics presented have been chosen from the many geometric problems the authors have confronted during their work in applied geometry and geometric design.

In writing this book we intended to disconnect geometric ideas and methods from special applications, in order to make these ideas clear and to allow the reader to apply the presented material to other problems of a geometric nature. Also, in many situations, a figure can say more than a thousand words. This old Chinese proverb ought to be a guideline in writing a text on geometry. Therefore, figures are crucial throughout this book, while diagrams are an integral part of Chapters 1 and 28.

This book owes its inception to lectures given by Boehm at Rensselaer Polytechnic Institute and the Technical University of Braunschweig several times between 1986 and 1990. This book has been partly written at Rensselaer, and we are greatly indebted to Harry McLaughlin who has been promoting Applied Geometry at Rensselaer and who together with Joe Ecker initiated their cooperation with the TU Braunschweig. Andreas Johannsen read the first and later drafts of the book very carefully, and we benefitted much from his helpful suggestions. We thank Dr. Michael Kaps and Wolfgang Völker for typing the manuscript; Daniel Bister for proof reading the mathematics and Jeannette Machnis for proofreading the English text; and Mrs. Diane McNulty for her judicial and committed assistance in the cooperation with Rensselaer.

Troy, in December 1992

Wolfgang Boehm
Hartmut Prautzsch

Notation

The following notation is used throughout this book:

Scalars	$\alpha, \beta, \ldots, a, b, \ldots$
Vectors, points, coordinate columns	$\mathbf{a}, \mathbf{b}, \ldots, \mathbf{p}, \mathbf{q}, \ldots$
Extended columns (by an additional coordinate)	$\mathtt{x}, \mathtt{y}, \ldots$
Differences between two points	$\triangle\mathbf{x}, \triangle\mathbf{y}, \ldots, \triangle\mathtt{x}, \triangle\mathtt{y} \ldots$
Matrices	$A, B, \ldots$
Augmented matrices	$\mathbb{A}, \mathbb{B}, \ldots$
Vector spaces	$\mathbf{V}, \mathbf{A}, \ldots$
Point spaces	$\mathcal{A}, \mathcal{P}, \ldots$
Orthogonal angles	
Parallelism	//

Bold type is used whenever a new term is introduced.

Remark: Each chapter starts with an abstract and a short bibliography for further information on the particular subject. The complete references are listed at the end of the book.

PART ONE

Some Linear Algebra

Many problems encountered in applied mathematics are linear or can be approximated by linear systems which are, in general, computationally tractable. The corresponding mathematical subdiscipline is called **linear algebra**. At the heart of linear algebra are techniques, such as Gaussian elimination and the Gauss-Jordan algorithm, for computing solutions of linear systems. The main tool of linear algebra is matrices which help to arrange coefficients and describe operations.

1 Linear Systems

Most finite linear systems can be described by matrices, a very useful shorthand notation which emphasizes the underlying linear structure and the interdependencies between the equations.

Literature: Atkinson, Boehm·Prautzsch, Conte·de Boor

1.1 Matrix Notation

A **linear system** is a set of equations of the form

$$\begin{aligned} a_{1,1}\,x_1 + \cdots + a_{1,n}\,x_n &= a_1 \\ \vdots \qquad\qquad\quad & \vdots \quad \vdots \\ a_{m,1}\,x_1 + \cdots + a_{m,n}\,x_n &= a_m \ , \end{aligned}$$

where the a's are given real numbers and the x's are unknowns. The array A of the coefficients $a_{i,k}$,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = [a_{i,k}] \ ,$$

is called an $m \times n$ **matrix**. The matrix A contains the element $a_{i,k}$ in its ith row and kth column. Similarly, the a_i can be written as an $m \times 1$ matrix or m **column**,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} .$$

Consequently one has

$$A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n] \ ,$$

where $\mathbf{a}_k$ represents the kth column of A. Note that a **scalar** a can be viewed as a 1×1 matrix.

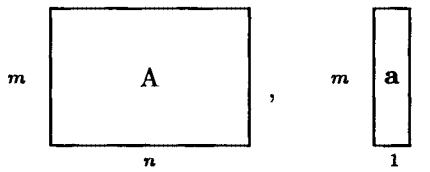
The $n \times m$ matrix $A^{\mathrm{t}} = [a^{\mathrm{t}}_{i,k}]$, defined by $a^{\mathrm{t}}_{i,k} = a_{k,i}$, is called the **transpose** of A, e.g., for A above

$$A^{\mathrm{t}} = \begin{bmatrix} a_{1,1} & \dots & a_{m,1} \\ \vdots & \ddots & \vdots \\ a_{1,n} & \dots & a_{m,n} \end{bmatrix} = [a_{k,i}] \ .$$

In particular, the transpose $\mathbf{a}^{\mathrm{t}}$ of an m column $\mathbf{a}$ forms an m **row**

$$\mathbf{a}^{\mathrm{t}} = [a_1 \ \dots \ a_m] \ .$$

Often it is helpful to visualize an $m \times n$ matrix A or an m column $\mathbf{a}$ in **block form**, i.e., as a rectangle of height m and width n or 1, respectively:



The matrix A is a **square matrix** if $m = n$, and it is **symmetric** if additionally $a_{i,k} = a_{k,i}$. A square matrix $[u_{i,k}]$ is called **upper triangular** if $u_{i,k} = 0$ for $i > k$. Similarly a square matrix $[l_{i,k}]$ is called **lower triangular** if $l_{i,k} = 0$ for $i < k$. The **Kronecker symbol**

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

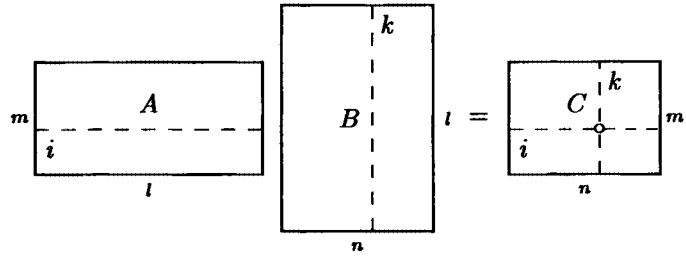
is used to define the **identity matrix** as the $n \times n$ square matrix $I = [\delta_{i,k}]$.

1.2 Matrix Multiplication

Let $A = [a_{i,j}]$ be an $m \times l$ matrix and $B = [b_{j,k}]$ an $l \times n$ matrix. The $m \times n$ matrix $C = [c_{i,k}]$ with the entries

$$c_{i,k} = \sum_{j=1}^{l} a_{i,j}\, b_{j,k}$$

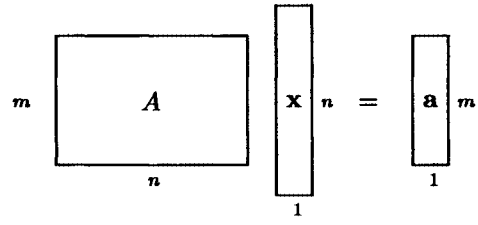
is called the **product** AB of A and B, in this order. Note that the width l of A has to match the height l of B. It is helpful to visualize the product $AB = C$ in block form, as introduced above:



The element $c_{i,k}$ is the **dot** or **scalar product** of the ith row of A with the kth column of B. This may be memorized as "**row times column**".
Using this product the linear system in Section 1.1 can be written more compactly as

$$A\,\mathbf{x} = \mathbf{a}\ ,$$

and visualized by blocks as



where $\mathbf{x}$ denotes the n column of the unknown x_i. Likewise the **scalar product** α of two m columns $\mathbf{a}$ and $\mathbf{b}$ can be written as

$$\alpha = \mathbf{a}^t\mathbf{b} = \mathbf{b}^t\mathbf{a}\ .$$

Note that the product $\mathbf{x}\alpha$ is defined as a matrix multiplication, but $A\alpha$ is not. It is convenient to define $A\alpha = \alpha A$ as the matrix of elements $a_{i,k}\alpha$, i.e., one has

$$A \cdot \alpha = [\mathbf{a}_1\alpha \ldots \mathbf{a}_n\alpha] = [a_{i,k}\alpha] \ .$$

In particular, one gets for $\alpha = 0$ the **null column** $\mathbf{o} = \mathbf{x}\,0$ and the **null matrix** $O = 0\,A$.

A square matrix A is said to be **non-singular** if its **inverse** A^{-1} defined by $A^{-1}A = AA^{-1} = I$ exists. Finally, a matrix B is said to be **orthonormal** if $B^tB = I$.

1.3 Gaussian Elimination

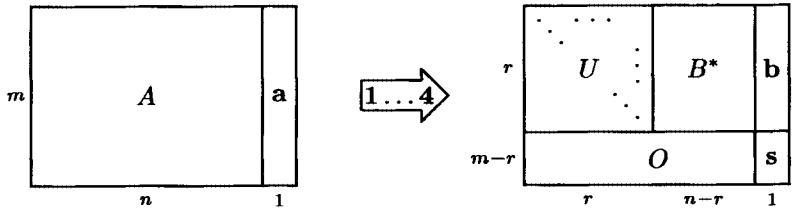
Linear systems are most frequently solved by **Gaussian elimination**. It is convenient to represent the linear system $A\mathbf{x} = \mathbf{a}$ by the **augmented matrix**

$$[A|\mathbf{a}] = [\mathbf{a}_1 \ldots \mathbf{a}_n|\mathbf{a}] \ .$$

Then the linear systems obtained by the following simple operations on $[A|\mathbf{a}]$ will have the same solutions:

1. exchanging two rows,
2. multiplying one row by a factor $\neq 0$,
3. adding one row to another,
4. exchanging two columns of A while simultaneously exchanging the corresponding unknowns in the column $\mathbf{x}$.

It was Gauss' idea to use these four simple operations to transform $[A|\mathbf{a}]$ into the matrix $[B|\mathbf{b}]$, where B is composed of an upper triangular, non-singular $r \times r$ matrix U, an $r \times n-r$ matrix matrix B^*, an $m-r \times n$ null matrix, an r column $\mathbf{b}$, and an $m-r$ column $\mathbf{s}$, as shown below.



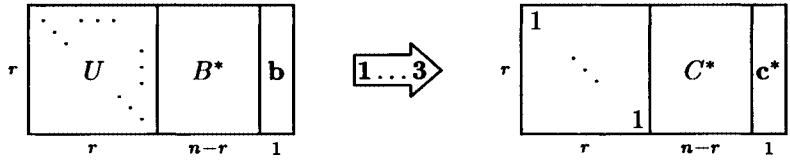
If $\mathbf{s} \neq \mathbf{o}$, there exists no solution. However, if $\mathbf{s} = \mathbf{o}$, there exists an $n-r$ parameter family of solutions which can easily be determined from the equivalent system $B\mathbf{x} = \mathbf{b}$ as follows. Assigning arbitrary values to $x_{r+1}, \ldots, x_n$ as parameters one can compute x_r **backward** from row r, then x_{r-1} from row $r-1$, ..., and finally x_1 from row 1.

Remark 1: The number r is called the **rank** of A, denoted by *rank* A. Note that $r \leq m$ and $r \leq n$.

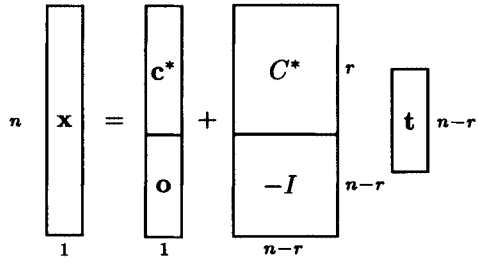
Remark 2: If $\mathbf{a} = \mathbf{o}$, the linear system $A\mathbf{x} = \mathbf{a}$ is called **homogeneous**. Then one also has $\mathbf{b} = \mathbf{o}$. The homogeneous system has a non-trivial solution if and only if $r < n$ as can be inferred from the equivalent system $B\mathbf{x} = O$. If $\mathbf{x}$ is a solution of a homogeneous system, then $\mathbf{x} \cdot \varrho$, where $\varrho \neq 0$, is also a solution.

1.4 Gauss-Jordan Algorithm

Gaussian elimination can further be used to construct an explicit representation for the set of all solutions of the linear system $A\mathbf{x} = \mathbf{a}$. With the aid of the operations **1**, **2**, **3**, one transforms the matrix $[U|B^*|\mathbf{b}]$ from above into $[I|C^*|\mathbf{c}^*]$ as illustrated in the following diagram.



The general solution of this system is depicted below where $-I$ denotes the negative $n-r \times n-r$ identity matrix and $\mathbf{t}$ an $n-r$ parameter column.

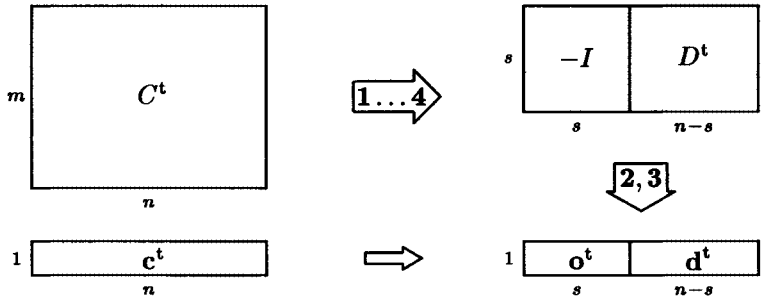


Note that for $r < n$ the representation of $\mathbf{x}$ depends on the sequence of operations performed during Gaussian elimination.

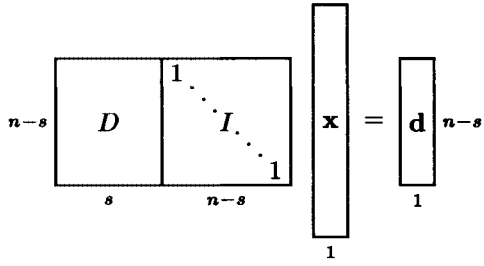
Remark 3: The construction can be reversed. Let $[C|\mathbf{c}]$ represent the set

$$\mathbf{x} = \mathbf{c} + C\mathbf{t}$$

of (given) solutions where $\mathbf{c}$ is some n column, C an $n \times m$ matrix, and $\mathbf{t}$ a column of m free parameters. The set represented by $[C|\mathbf{c}]$ does not change if the transposed matrix $[C|\mathbf{c}]^t$ is modified by Gaussian elimination. Using the operations $\mathbf{1}, \ldots, \mathbf{4}$, the matrix C^t can be transformed into an $s \times n$ matrix $[-I|D^t]$ provided $rank\, C = s$. This is illustrated below where the superfluous zero rows are discarded. Adding appropriate multiples of rows of $[-I|D^t]$ to $\mathbf{c}^t$ one obtains a row $[\mathbf{o}^t \mid \mathbf{d}^t]$ as illustrated below.



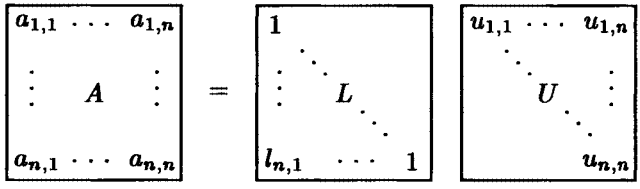
Now one easily obtains a linear system for which $\mathbf{x} = \mathbf{c} + C\mathbf{t}$ is a solution, namely



1.5 LU-Factorization

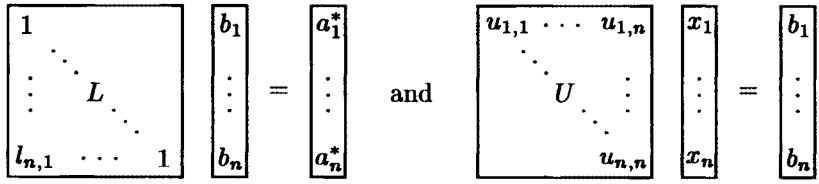
Often the matrix A of a linear system is square, i.e., $m = n$. Such a system is uniquely solvable if and only if A is non-singular or equivalently if $\mathit{rank}\, A = n$.

A non-singular matrix A can sometimes be factored into a lower-triangular matrix L, whose diagonal entries are all equal to 1, and an upper triangular matrix U, i.e., $A = LU$,



The entries of L and U can successively be computed by means of the matrix multiplication rule for $a_{1,1}, \ldots, a_{1,n}, a_{2,1}, \ldots, a_{2,n}, \ldots, a_{n,1}, \ldots, a_{n,n}$ in this order. At each step there is exactly one unknown $l_{i,k}$ or $u_{i,k}$ to be determined.

If a non-singular matrix A cannot be factored in this way, one can always rearrange the rows of A to obtain a matrix A^* which has an **LU-factorization**. In all cases one can start to compute L and U as if A were to be factored and interchange the rows of A during the computation whenever it becomes necessary to avoid dividing by zero. The LU-factorization is another organization of Gaussian elimination and can be used to solve a system $A\mathbf{x} = \mathbf{a}$. Let $[A^* | \mathbf{a}^*]$ be obtained from $[A|\mathbf{a}]$ by a row permutation such that an LU-factorization $A^* = LU$ exists. Solving the two triangular systems



by **forward** and **backward substitution** respectively, yields the solution for $A^*\mathbf{x} = \mathbf{a}^*$ and hence for $A\mathbf{x} = \mathbf{a}$.

Remark 4: The LU-factorization is useful for solving the system $A\mathbf{x} = \mathbf{a}$ repeatedly for a fixed coefficient matrix A and different right hand sides $\mathbf{a}$. In particular, if the right hand sides are the columns of the identity matrix I one obtains the inverse of A.

Remark 5: If A is symmetric and $\mathbf{x}^t A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then A is called positive definite, and a symmetric factorization $A = C^t C$, where C is an upper triangular matrix, is possible without row interchanges. This is called a **Cholesky factorization.**

1.6 Cramer's Rule

Let $A = [a_{i,k}]$ be a square $n \times n$ matrix and $A_{i,k}$ the submatrix obtained from A by deleting the ith row and kth column. Then the **determinant** of A, written $\det A$, is defined by the recursion

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{i,k} \det A_{i,k} \quad \text{and} \quad \det[a] = a,$$

for any scalar a. This definition does not depend on the choice of i and is called **Laplace expansion** along the ith row. The term $(-1)^{i+k} \det A_{i,k}$ is called the **cofactor** of $a_{i,k}$.

The determinant can be used to solve a non-homogeneous linear system $A\mathbf{x} = \mathbf{a}$ when A is some non-singular $n \times n$ matrix .

Let $A_k = [\mathbf{a}_1 \ldots \mathbf{a} \ldots \mathbf{a}_n]$ be obtained from $A = [\mathbf{a}_1 \ldots \mathbf{a}_k \ldots \mathbf{a}_n]$ by replacing the kth column with $\mathbf{a}$. Then **Cramer's rule**,

$$x_k = \frac{\det A_k}{\det A}, \quad k = 1, \ldots, n \ ,$$

gives the coordinates x_k of the solution. Note that $\det A \neq 0$ whenever A is non-singular.

In the case of a homogeneous system $A\mathbf{x} = \mathbf{o}$, where A is an $n\text{-}1 \times n$ matrix with $rank\, A = n - 1$, one can show that

$$x_k = \varrho \cdot (-1)^k \det A_k^*, \quad k = 1, \ldots, n \ ,$$

provides the solution of the system, where $\varrho \neq 0$ is a free parameter and A_k^* is obtained from A by deleting the kth column.

Remark 6: Cramer's rule is of practical use only for small n.

1.7 Notes and Problems

1 It is possible to improve the numerical stability of Gaussian elimination by row interchanges.

2 One of the numerically most stable algorithms used to solve linear systems is the so-called **Householder algorithm.**

3 Let A^* and I^* be obtained from the matrices A and I of equal height by the same row permutation. Then $P = I^*$ can be used to write down the permutation of A, i.e.,

$$A^* = PA \ .$$

P is called a permutation matrix. It is inverse to its transpose, i.e., $P^t P = I$.

4 Using a permutation matrix P (see Note 3), Gaussian elimination can be summarized as

$$L^{-1}P[A|\mathbf{a}] = U \ .$$

5 The LU-factorization of $A^* = PA$ can be used to compute $X = A^{-1}$ by solving $A^*X = P$ column by column.

6 Most elimination methods for solving linear systems are actually just different organizations of the Gaussian elimination process. They differ only in the ordering of the computation steps.

7 For two $n \times n$ matrices A and B one has $\det AB = \det A \cdot \det B$.

8 Matrix multiplication by hand is best organized by **Falk's scheme** as illustrated in Figure 1.1.

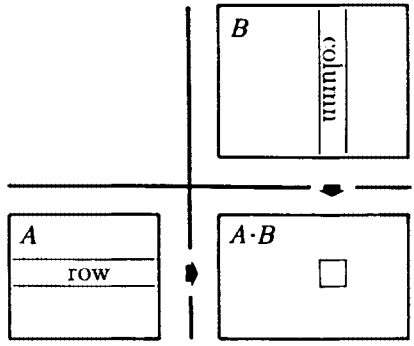


Figure 1.1: Falk's scheme.

9 Falk's scheme can also be used to procure an LU-factorization or a Cholesky factorization.

2 Linear Spaces

A **linear** or **vector space V over** $\mathbb{R}$ is a set which is closed under linear combinations with real coefficients. The elements of $\mathbf{V}$ are called **vectors**, the coefficients are **scalars**. A map from one linear space into another is called a **linear map** if it preserves linear combinations. The **standard vector space** is $\mathbb{R}^m$.

Literature: Greub, Strang, van der Waerden

2.1 Basis and Dimension

Let $\mathbf{o}$ denote the zero vector, then any r vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$ belonging to a vector space $\mathbf{V}$ are said to be **linearly dependent** if there exist scalars $x_1, \ldots, x_r$ not all of which are zero such that

$$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_r x_r = \mathbf{o} \ .$$

Otherwise $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are said to be **linearly independent**. On building the matrix $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_r]$, one has that $\mathbf{a}_1, \ldots, \mathbf{a}_r$ are linearly dependent if and only if $A\mathbf{x} = \mathbf{o}$ has a non-trivial solution.

The set of all linear combinations of the given $\mathbf{a}_i$ forms a linear space, called the **span** of the $\mathbf{a}_i$, or $span\,[\mathbf{a}_1 \ \ldots \ \mathbf{a}_r]$.

The space $\mathbf{A} = span\,[\mathbf{a}_1 \ \ldots \ \mathbf{a}_r]$ is called a **subspace** of $\mathbf{V}$. The **dimension** of $\mathbf{A}$, or $dim\,\mathbf{A}$, is defined as the maximum number of linearly independent vectors in $\mathbf{A}$. Occasionally, $n = dim\,\mathbf{A}$ is given as a superscript, $\mathbf{A}^n$

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be n linearly independent vectors of an n-dimensional linear space $\mathbf{V}$, and let $\mathbf{v}$ be some vector of $\mathbf{V}$. Then these $n+1$ vectors are linearly dependent, i.e.,

$$\mathbf{v} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n \ , \qquad x_i \in \mathbb{R} \ ,$$

in matrix notation $\mathbf{v} = A\mathbf{x}$. In this equation the factors x_i can be uniquely determined, otherwise the $\mathbf{a}_i$ would not be linearly independent. Hence $\mathbf{A}$ is non-singular. One says that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ form a **basis** of $\mathbf{V}$. The $\mathbf{a}_i x_i$ are called the **components** of $\mathbf{v}$, while the x_i are referred to as the **coordinates** of $\mathbf{v}$ with respect to the $\mathbf{a}_i$.

Remark 1: It is convenient to denote a vector by the vector of its coordinates $\mathbf{x} = [x_1 \ \ldots \ x_n]^t$. This convention is used throughout this book.

Remark 2: On choosing some fixed basis of $\mathbf{V}^n$ every vector of $\mathbf{V}^n$ corresponds to a unique element of $\mathbb{R}^n$, and every linear combination in $\mathbf{V}^n$ corresponds to the same linear combination in $\mathbb{R}^n$. Therefore it is sufficient to consider $\mathbb{R}^n$ instead of $\mathbf{V}^n$. In particular, the $\mathbf{a}_i$ from above may be viewed as elements of $\mathbb{R}^m$, $m \geq n$.

2.2 Change of Bases

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n$ denote two bases of a linear space $\mathbf{V}$. Then the $\mathbf{a}$'s can be expressed uniquely in terms of the $\mathbf{b}$'s,

$$\mathbf{a}_k = \mathbf{b}_1 c_{1,k} + \cdots + \mathbf{b}_n c_{n,k} \ .$$

Using matrix notation one gets

$$[\mathbf{a}_1 \ \ldots \ \mathbf{a}_n] = [\mathbf{b}_1 \ \ldots \ \mathbf{b}_n] \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix} ,$$

or more concisely $A = BC$. As a consequence one has $C = B^{-1}A$, i.e., $C = [c_{i,k}]$ is non-singular since it is the product of non-singular matrices. Let $\mathbf{v}$ be some arbitrary vector of $\mathbf{V}$ with the representations $\mathbf{v} = A\mathbf{x} = B\mathbf{y}$, i.e.,

$$\mathbf{v} = [\mathbf{a}_1 \quad \dots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} .$$

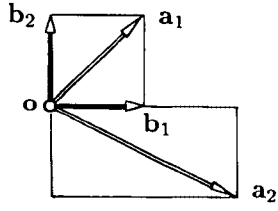
It then follows that $\mathbf{y} = C\mathbf{x}$, i.e.,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} .$$

Note that the $\mathbf{a}$'s are expressed in terms of the $\mathbf{b}$'s, but the y's are expressed in terms of the x's. Both transformations are called **contragredient** to each other.

The representation $\mathbf{a}_k = B\mathbf{c}_k$ has a simple but important **geometric meaning:**

> The column $\mathbf{c}_k$ of C represents the coordinates of the basis vector $\mathbf{a}_k$ with respect to the basis $\mathbf{b}_1, \dots, \mathbf{b}_n$.



Example 1:

For $[\mathbf{a}_1\, \mathbf{a}_2] = [\mathbf{b}_1\, \mathbf{b}_2] \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

one has $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2.3 Linear Maps

Of particular interest are maps which are compatible with the linear structure of linear spaces. Such maps must preserve linear combinations. Consider two linear spaces $\mathbf{A}$ and $\mathbf{B}$ with bases $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ respectively, and a map $\varphi : \mathbf{A} \to \mathbf{B}$ which preserves linear combinations, i.e.,

$$\varphi[\mathbf{a}\cdot\alpha + \mathbf{b}\cdot\beta] = \varphi\mathbf{a}\cdot\alpha + \varphi\mathbf{b}\cdot\beta$$

for all $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ and all $\alpha, \beta \in \mathbb{R}$. Such a map φ is called a **linear map**.

The images of the $\mathbf{a}_k$ can uniquely be expressed in terms of the $\mathbf{b}$'s,

$$\varphi\mathbf{a}_k = \mathbf{b}_1 c_{1,k} + \cdots + \mathbf{b}_m c_{m,k} \ ,$$

which may be written in matrix notation as

$$[\varphi\mathbf{a}_1 \ \ldots \ \varphi\mathbf{a}_n] = [\mathbf{b}_1 \ldots \mathbf{b}_m] \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & & \vdots \\ c_{m,1} & \cdots & c_{m,n} \end{bmatrix} \ ,$$

or in condensed form as $\varphi A = BC$. Let $\mathbf{a}$ be a vector of $\mathbf{A}$,

$$\mathbf{a} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n = A\mathbf{x} \ ,$$

and $\mathbf{b} = \varphi\mathbf{a}$ its image in $\mathbf{B}$,

$$\mathbf{b} = \mathbf{b}_1 y_1 + \cdots + \mathbf{b}_m y_m = B\mathbf{y} \ .$$

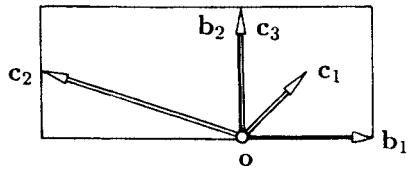
Then one has $\mathbf{y} = B^{-1}\mathbf{b}$ and $\mathbf{b} = \varphi A\mathbf{x} = BC\mathbf{x}$. This implies $\mathbf{y} = C\mathbf{x}$, i.e.,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & & \vdots \\ c_{m,1} & \cdots & c_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \ .$$

Note that the $\varphi\mathbf{a}$'s are expressed in terms of the $\mathbf{b}$'s via C, but the y's are expressed in terms of the x's, i.e., both transformations are **contragredient** to each other.

The representation $\varphi \mathbf{a}_k = B\mathbf{c}_k$ has a simple but important **geometric meaning**:

> The column $\mathbf{c}_k$ of C represents the coordinates of the image $\varphi \mathbf{a}_k$ of the basis vector $\mathbf{a}_k$ with respect to the basis $\mathbf{b}_1, \ldots, \mathbf{b}_m$.



Example 2: On inspecting the figure one obtains the matrix C,

$$[\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1/2 & -3/2 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} .$$

2.4 Kernel and Fibers

The images $\varphi \mathbf{a}_k$ span $\varphi \mathbf{A}$, the **image** of $\mathbf{A}$. The image of $\mathbf{A}$ is a subspace of $\mathbf{B}$ with $dim\ \varphi \mathbf{A} \leq dim\ \mathbf{A}$. These dimensions can be analyzed in more detail. There exists a subspace $\mathbf{K} \subset \mathbf{A}$, called the **kernel** of φ, $\mathbf{K} = kern\ \varphi$, which is the set of all vectors in $\mathbf{A}$ mapped into the null vector $\mathbf{o}$ of $\mathbf{B}$. The subspace $\mathbf{K}$ is represented by the solution of the homogeneous system

$$C\mathbf{x} = \mathbf{o} .$$

For any fixed vector $\mathbf{a}$ of $\mathbf{A}$ and all elements $\mathbf{k}$ of this kernel $\mathbf{K}$, the subset $\mathcal{F}_\mathbf{a}$ of $\mathbf{A}$ formed by all $\mathbf{a} + \mathbf{k}$ is called the **fiber** over $\mathbf{a}$. Evidently, φ maps all elements of $\mathcal{F}_\mathbf{a}$ into the same image $C\mathbf{a}$. Note that a fiber is a linear space only if $\mathbf{a} = \mathbf{o}$.

Using a basis of $\mathbf{A}$ which contains a basis of $\mathbf{K}$ one finds that

$$dim\ \varphi \mathbf{A} + dim\ \mathbf{K} = dim\ \mathbf{A} .$$

Example 3: In Example 2, $\mathbf{K}$ consists of all vectors $[3 \quad 1 \quad -2]^t \lambda$, with $\lambda \in \mathbb{R}$.

2.5 Point Spaces

One can see the world as a **space of points**. This point space is closely related to a linear space in a natural way. Two points are connected by a vector and a vector added to a point gives a point again. These relations are expressed by the notation

$$\mathbf{v} = \mathbf{p} - \mathbf{a} \qquad \text{and} \qquad \mathbf{p} = \mathbf{a} + \mathbf{v} \,,$$

where $\mathbf{a}$ and $\mathbf{p}$ are points and $\mathbf{v}$ is the vector pointing from $\mathbf{a}$ to $\mathbf{p}$. Let $\mathbf{v}$ be given with respect to a basis of $\mathbf{A}^n$, $\mathbf{v} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n$, then

$$\mathbf{p} = \mathbf{a} + \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n \,.$$

Let $\mathbf{a}$ be a fixed point, then every coordinate column $\mathbf{x} = [x_1 \ldots x_n]^t$ defines a point $\mathbf{p}$, with different $\mathbf{x}$'s generating different points.

Affine spaces: The set of points $\mathbf{p}$ corresponding to all $\mathbf{x} \in \mathbb{R}^n$ is called an **affine space** $\mathcal{A}$, while $span\,[\mathbf{a}_1 \ldots \mathbf{a}_n]$ is called the **underlying vector space** $\mathbf{A}$. One defines $dim\,\mathcal{A} = dim\,\mathbf{A}$. A point $\mathbf{a} \in \mathcal{A}$ together with a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbf{A}$ form an **affine system** in $\mathcal{A}$. The point $\mathbf{a}$ is referred to as the **origin** while the x_i are called the **affine coordinates** of $\mathbf{p}$ with respect to the frame $\mathbf{a}; \mathbf{a}_1, \ldots, \mathbf{a}_n$. Affine spaces are discussed in Part III.

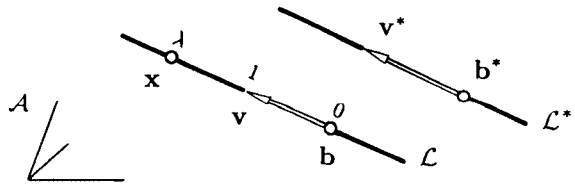


Figure 2.1: Parallelism and affine scale.

In most parts of this book, when points are viewed as vectors they will be denoted by their coordinate columns $\mathbf{x}$ with respect to some fixed frame. Note that $\mathbf{p}$ and $\mathbf{x}$ above denote the same point with respect to different systems.

An **affine subspace** $\mathcal{S}$ of $\mathcal{A}$ is defined by some point $\mathbf{b} \in \mathcal{A}$ and a subspace $\mathbf{S}$ of $\mathbf{A}$, i.e., $\mathcal{S} = \{\mathbf{b} + \mathbf{v} \,|\, \mathbf{v} \in \mathbf{S}\}$. In particular, a line $\mathcal{L}$ is a 1-dimensional subspace. It will be represented as

$$\mathbf{x} = \mathbf{b} + \mathbf{v}\lambda \ .$$

The parameter λ is called an **affine scale** on $\mathcal{L}$. It represents $\mathbf{x}$ with respect to the affine system $\mathbf{b}; \mathbf{v}$. Using this scale, the **ratio** of the point λ with respect to the points λ_0 and λ_1 is defined by

$$ratio\,(\lambda; \lambda_0, \lambda_1) = \frac{\lambda - \lambda_0}{\lambda_1 - \lambda} \ .$$

Note that this ratio depends on the ordering of the points, but not on the respective affine scale.

The line $\mathcal{L}$ is said to be **parallel** to a second line $\mathcal{L}^*$ given by

$$\mathbf{x} = \mathbf{b}^* + \mathbf{v}^*\mu$$

if $\mathbf{v} = \mathbf{v}^*\sigma$, $\sigma \neq 0$.

Euclidean spaces: If the basis vectors $\mathbf{a}_i$ of the underlying vector space $\mathbf{A}$ have length 1 and are pairwise perpendicular, then the corresponding affine system is called a **Cartesian system**. The x_i are called **Cartesian coordinates**, while the space $\mathcal{A}$ is called a **Euclidean space** and denoted by $\mathcal{E}$. In a Cartesian system the square of the distance between two points $\mathbf{x}$ and $\mathbf{x} + \mathbf{d}$ equals $\mathbf{d}^t\mathbf{d}$, and two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\mathbf{u}^t\mathbf{v} = 0$. Euclidean spaces are discussed in Part IV.

Projective spaces: Often it is easier to describe geometric properties if one introduces points at infinity — one point for each 1-dimensional subspace of $\mathbf{A}$. Then any two parallel lines meet in a point at infinity. These points are called **ideal points**, while the 1-dimensional subspaces of $\mathbf{A}$ are called **directions** of $\mathcal{A}$. The ideal points of $\mathcal{A}$ form the **ideal hyperplane** $\mathcal{A}_\infty$ of $\mathcal{A}$. The union $\mathcal{P} = \mathcal{A} \cup \mathcal{A}_\infty$ is called the **projective extension** of $\mathcal{A}$. It represents the prototype of a **projective space**. Projective spaces are discussed in Part V.

2.6 Notes and Problems

1 Although the elements of $\mathbb{R}^n$ can be interpreted as the elements of either an affine space or a linear space, the structures of these spaces are different.

2 The solution of a homogeneous linear system forms a linear space.

3 The solution of a non-homogeneous linear system forms an affine space.

4 More exactly, any r independent linear equations in n variables define an affine space of dimension $n - r$, provided that the corresponding linear system has a solution.

5 Any linear space is in a natural way an affine space, but not vice versa.

6 The set theoretical intersection of two subspaces $\mathbf{A}$ and $\mathbf{B}$ of a linear space is a linear space and is called the **intersection** $\mathbf{A} \sqcap \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$.

7 The set theoretical **union** of two subspaces $\mathbf{A}$ and $\mathbf{B}$ of a linear space is a linear space only if $\mathbf{A} \subset \mathbf{B}$ or $\mathbf{B} \subset \mathbf{A}$.

8 Let $\mathbf{A} = span\,[\mathbf{a}_1 \ldots \mathbf{a}_r]$ and $\mathbf{B} = span\,[\mathbf{a}_{r+1} \ldots \mathbf{a}_s]$, then $span\,[\mathbf{a}_1 \ldots \mathbf{a}_s]$ is called the **join** $\mathbf{A} \sqcup \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$.

9 Let $\mathbf{a}_1, \ldots, \mathbf{a}_r$ be linearly independent vectors of some n-dimensional linear space $\mathbf{A}$. They can be supplemented to a basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathbf{A}$.

10 Given r non-zero but linearly dependent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_r$, one can construct a basis of $span\,[\mathbf{a}_1 \ldots \mathbf{a}_r]$ by the Gauss-Jordan algorithm.

11 The set of all one-dimensional subspaces of a linear space $\mathbf{V}$ forms a **projective space** $\mathcal{P}$.

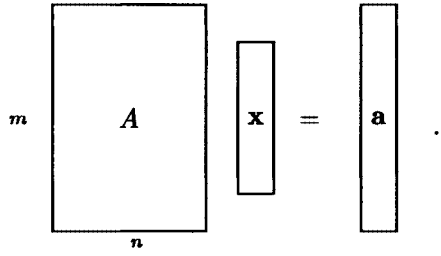
3 Least Squares

A linear system is **overdetermined** if the number of equations exceeds the number of unknowns. Since such a system has no solution in general, one usually seeks unknowns which "solve "the system best, approximatively. Frequently, one minimizes some Euclidean distance. This concept leads to the **method of least squares**.

Literature: Boehm·Prautzsch, Conte·de Boor, Wilkinson

3.1 Overdetermined Systems

Let A be a tall $m \times n$ matrix, i.e., $m > n$, with $rank\, A = n$, and let $A\mathbf{x} = \mathbf{a}$ be a linear system



Only if $\mathbf{a}$ is a linear combination of the columns $\mathbf{a}_k$ of A, is there a solution $\mathbf{x}$. But, in general, one has

$$\mathbf{r} = A\mathbf{x} - \mathbf{a} \neq \mathbf{o} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n .$$

The column $\mathbf{r}$ is called the **residual vector** associated with $\mathbf{x}$. It can be interpreted in $\mathbb{R}^m$ as the vector from the point $\mathbf{a}$ to the point $A\mathbf{x}$ as illustrated in Figure 3.1. An approximate solution $\mathbf{x}$ which minimizes $\mathbf{r} = \mathbf{r}(\mathbf{x})$ in some sense is all one can hope for. Minimizing $\mathbf{r}^t\mathbf{r}$ is rather a simple task.

In the Euclidean space $\mathcal{E}^m$ the length of $\mathbf{r}$ is minimal if $\mathbf{r}$ is orthogonal to the subspace $\mathcal{A}$ spanned by $\mathbf{o}; \mathbf{a}_1, \ldots, \mathbf{a}_n$, i.e., if

$$A^t\mathbf{r} = \mathbf{o} .$$

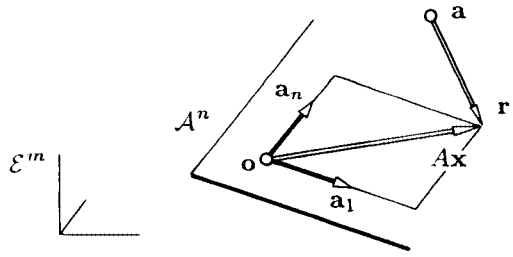


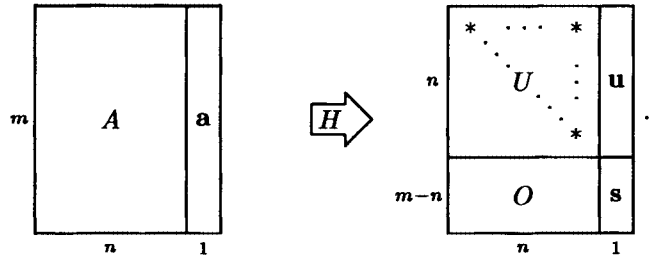
Figure 3.1: The residual vector.

Substituting $A\mathbf{x} - \mathbf{a}$ for $\mathbf{r}$ results in the so-called **Gaussian normal equations**,

$$A^tA\mathbf{x} = A^t\mathbf{a} .$$

The solution $\mathbf{x}$ represents the **foot** of the perpendicular from $\mathbf{a}$ onto $\mathcal{A}$ with respect to the affine system $\mathbf{o}; \mathbf{a}_1, \ldots, \mathbf{a}_n$. Note that A^tA is an $n \times n$ matrix and $A^t\mathbf{a}$ is an n column. Moreover, A^tA is symmetric and, if the $\mathbf{a}_i$ are linearly independent, also positive definite. In this case the normal equations can be solved via a symmetric factorization of A^tA, as mentioned in Remark 5 of Section 1.5.

Remark 1: In general, normal equations are poorly conditioned. Hence, it is advisable to use a numerically stable method such as Householder's. In **Householder's method**, the matrix $[A|\mathbf{a}]$ is multiplied by a sequence H of orthonormal transformations to obtain a matrix $[B|\mathbf{b}]$ such that B is composed of an upper triangular matrix U and a null matrix O,

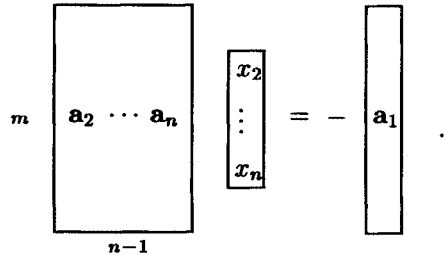


Since orthonormal transformations do not change the Euclidean length of a vector, $\mathbf{r} = A\mathbf{x} - \mathbf{a}$ and $H\mathbf{r} = B\mathbf{x} - \mathbf{b}$ have the same length, i.e., the solution of $U\mathbf{x} = \mathbf{u}$ minimizes $\mathbf{r}^t\mathbf{r}$, where $\mathbf{s}^t\mathbf{s}$ is the minimum value of $\mathbf{r}^t\mathbf{r}$.

Remark 2: The individual equations of $A\mathbf{x} = \mathbf{a}$ may be multiplied with arbitrary weights. This "scaling" changes the coordinates of the residual vector and, hence, influences the result. Thus one may distinguish equations corresponding to very accurate measurements. In this way, equations stemming from accurate measurements can become more influential than others.

3.2 Homogeneous Systems

The least squares method fails for homogeneous systems, i.e., if $\mathbf{a} = \mathbf{o}$, because $\mathbf{x} = \mathbf{o}$ solves the system. A simple way to avoid this problem is to add a constraint by setting one of the x_k's equal to 1. On constraining, e.g., x_1, one has to "solve" the overdetermined non-homogeneous system



Obviously, the "solution" depends on which coordinate x_k is constrained. Note that the corresponding $\mathbf{a}_k$ must be distinctly different from $\mathbf{o}$ to avoid numerical instabilities.

3.3 Constrained Least Squares

Sometimes the "solution" of an overdetermined system $A\mathbf{x} = \mathbf{a}$ is required to satisfy an additional system $B\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{o}$. One can compute the solution of the system $B\mathbf{x} = \mathbf{b}$ by the Gauss-Jordan algorithm and obtain an equivalent system

$$\mathbf{x} = \mathbf{c} + C\mathbf{t} \ .$$

These additional constraints are **hard constraints** which could be, for example, the boundary conditions of some initial problem. On substituting $\mathbf{c} + C\mathbf{y}$ for $\mathbf{x}$, the initial system $A\mathbf{x} = \mathbf{a}$ reduces to the overdetermined system

$$AC\mathbf{y} = \mathbf{a} - A\mathbf{c} \ .$$

If A is an $m \times n$ matrix and B is an $l \times n$ matrix, then AC is an $m \times n\text{-}l$ matrix. Note that $m > n > l$.

Geometrically, this procedure may be interpreted as the restriction of $\mathbf{x}$ to a subspace $\mathcal{C}$ of $\mathcal{A}$ and the introduction of new affine coordinates in $\mathcal{C}$ represented by $\mathbf{y}$. Note that this method works even if $\mathbf{a} = \mathbf{o}$.

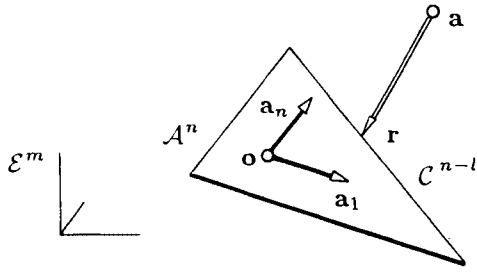


Figure 3.2: Constrained least squares.

Example 1: An example is discussed in Section 3.2 where $x_1 = 1$ represents the hard constraint, i.e.,

$$\mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} .$$