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Richard B. Melrose

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To my mother, Isla Louise Melrose

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PREFACE

This text is a somewhat expanded version of lecture notes written during, and directly after, a course at MIT in the Spring of 1991. Most of the participants had attended a course the preceding Fall on pseudodifferential operators on compact manifolds without boundary, including the Hodge theorem and the construction of the fundamental solution for the wave equation. Approximately this level of sophistication is assumed of the reader. The intention of the second course was to embed the Atiyah-Patodi-Singer index theorem in an analytic framework analogous to that provided by the theory of pseudodifferential operators for the Atiyah-Singer theorem. Since this treatment leads to a variety of current research topics it is presented here in the hope that it will be of use to a wider audience.

There are many people to thank. Foremost I am grateful to the members of the audience of the course for their tolerance and enthusiasm. I am especially grateful to Paolo Piazza for his comments during the course and also as a collaborator in work related to this subject. Others from whom I have learnt in this way are Xianzhe Dai, Charlie Epstein, Dan Freed, Rafe Mazzeo and Gerardo Mendoza. To the last of these I would like to take this opportunity to apologize for my part, whatever that was, in the somewhat mysterious non-appearance of the paper [64] on which a considerable part of Chapter 6, and indeed the general 'b-philosophy,' is based.

More generally I am happy to acknowledge the influence, through conversation, on my approach to this subject of Michael Atiyah, Ezra Getzler, Lars Hörmander, Werner Müller, Bob Seeley, Iz Singer and Michael Taylor. I am indebted to Antônio sá Barreto, Xianzhe Dai, Charlie Epstein and Maciej Zworski who tolerated my neglect of other projects during the process of writing, to Tanya Christiansen, Xianzhe Dai, Andrew Hassell, Lars Hörmander, Gerd Grubb, Rafe Mazzeo, Paolo Piazza and Lorenzo Ramero for comments on the manuscript and especially to Jillian Melrose for her forbearance. To Judy Romvos special thanks for turning my rather crude scrawlings into the original lecture notes.

Danville, Vermont. March 8, 1992

LIST OF FIGURES

Introduction and the proof

The Atiyah-Patodi-Singer index theorem (APS theorem) is used in this text as a pivot (or maybe an excuse) to discuss some aspects of geometry and analysis on manifolds with boundary. This volume does *not* contain a general treatment of index theorems even though they are amongst the most basic analytic-geometric results one can find. The power of such theorems in applications largely lies in their simplicity and generality. In particular the statement of the APS theorem is quite simple. In practice a great deal of effort, by many people, has gone into simplifying the proofs. This has lead to, and been accompanied by, a much wider understanding of the analytic framework in which they are centred. In fact, from an analytic perspective, index theorems can be thought of as much as testing grounds, for methods and concepts, as ends in themselves. The Atiyah-Singer theorem, which is the boundaryless precursor to the APS theorem, is intimately connected to the theory of pseudodifferential operators. This volume *is* intended to place the APS theorem in a similar context, the 'b' category and related calculus of b-pseudodifferential operators on a compact manifold with boundary.

The basic approach adopted here is to 'state' and 'prove' the APS theorem immediately, being necessarily superficial on a variety of points. The subsequent nine chapters consist largely in the fleshing out of this proof. Just as the initial discussion is brief, the later treatment is discursive and aims at considerably more than the proof of the index theorem alone. The proof given here is *direct* in two senses. The written proof itself is quite straightforward, given some conceptual background, and in particular the terms in the final formula come out directly in the course of the proof. The model here is Getzler's proof ([35]) of the Atiyah-Singer theorem for Dirac operators on a compact manifold without boundary.

The second sense in which the proof is direct is closely connected to the main thesis of this text. Namely that the APS theorem *is* the Atiyah-Singer theorem in the b-category, which is to say the category of compact manifolds with boundary with metrics having complete cylindrical ends. These metrics are called here (exact) b-metrics. This is by no means a radical position (since it is at least implicit in the original papers) but it is a position taken with some fervour. One consequence of this approach is the suggestion that there are other such theorems, especially on manifolds with corners. It is hoped that the context into which the APS theorem is placed will allow it to be readily understood and, perhaps more importantly, generalized. Of course extensions and generalizations already have been made, see in particular the work of Bismut and Cheeger [16], Cheeger [26], Moscovici and Stanton [69], Müller [70] and Stern [85]; see also Wu [91] and Getzler [36].

A review of the proof below, annotated with references to the intervening chapters to make it complete, can be found in §9.1. Towards the end of this Introduction there is an outline of the content of the later chapters.

1. The Atiyah-Singer index theorem.

Consider the Atiyah-Singer index theorem on a compact manifold without boundary. The version for Dirac operators is necessarily proved along the way to the APS theorem. It can be written in brief ([11])

$$\operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS}\,. \tag{In.1}$$

Here $\eth_E^+$ is a twisted Dirac operator, with coefficient bundle E, on the compact even-dimensional spin manifold X and AS is the Atiyah-Singer integrand. This is the volume part (form component of maximal degree) of the product of a characteristic class on X, the $\widehat{A}$ genus, and the Chern character of E :

$$\mathrm{AS} = \mathrm{Ev}_{\dim X}\left(\widehat{A}(X)\cdot \mathrm{Ch}(E)\right). \tag{In.2}$$

Here $\mathrm{Ev}_{\dim X}$ evaluates a form to the coefficient of the volume form of the manifold which it contains. A fundamental feature of (In.1) is that the left side is analytic in nature and the right side is topological, or geometric. One point in favour of Getzler's proof of the index formula is that it is *not* necessary to understand the properties of (In.2) independently, i.e. the theory of characteristic classes is not needed to derive the formula (although it certainly helps to understand it). The left side of (In.1) is, by definition,

$$\operatorname{ind}(\eth_E^+) = \dim \operatorname{null}(\eth_E^+) - \dim \operatorname{null}(\eth_E^-)$$

where $\eth_E^+$ and $\eth_E^-$ act on C^∞ sections of the appropriate bundles, $\eth_E^-$ being the adjoint of $\eth_E^+$, and the finite dimensionality of the null spaces follows by ellipticity. The direct proof of (In.1) simplifies the original proof of Atiyah and Singer ([11], [12], [72]) and the modifications by Patodi ([73]), Gilkey ([37]) and Atiyah, Bott and Patodi ([5]). Full treatments of proofs along these lines can be found in Berline, Getzler and Vergne [20], Freed [33], Hörmander [48], Roe [77] and Taylor [88].

2. The Atiyah-Patodi-Singer index theorem.

The APS theorem ([8]-[10]) is a generalization of (In.1) to manifolds with boundary. There are two, complementary, ways of thinking about a compact manifold with boundary. The most familiar way is to think of

X as half of a compact manifold *without* boundary by doubling across the boundary. The other approach is to think of the boundary as *at infinity* or at least impenetrable. The latter approach is the one adopted here, whereas in [8]-[10] both approaches are used and the celebrated Atiyah-Patodi-Singer boundary condition (see the discussion in §3 below) comes from the interplay between them. More precisely this means that

$$\text{(In.3)} \qquad \begin{gathered} \eth_E^+ \text{ is a twisted Dirac operator with} \\ \text{respect to an exact } b\text{-spin structure on } X^{2n}. \end{gathered}$$

The reason that X is assumed to be even-dimensional is that in case $\dim X$ is odd and $\partial X = \emptyset$ the index vanishes. There *is* an index theorem in the odd dimensional case but, for Dirac operators, it is relatively simple. For the moment, the notion of a spin structure is left undefined, as is the Dirac operator associated to it.

The exact b-metrics on a compact manifold with boundary are complete Riemann metrics on the interior which make the neighbourhood of the boundary into an asymptotically cylindrical end. More precisely, an exact b-metric is a Riemann metric which takes the form

$$\text{(In.4)} \qquad g = \left(\frac{dx}{x}\right)^2 + h$$

near ∂X, with h a smooth 2-cotensor which induces a Riemann metric on the boundary and $x \in \mathcal{C}^\infty(X)$ a defining function for the boundary. It is important to emphasize that this notion of a b-metric is taken seriously below. For example, the frame bundle of a metric of this type is a smooth principal bundle up to the boundary. An exact b-spin structure is simply a spin structure for an exact b-metric, i.e. a refinement of the frame bundle to a principal Spin bundle, where $\mathrm{Spin}(2n)$ is the non-trivial double cover of $\mathrm{SO}(2n)$.

One important property of an exact b-spin structure (which exists precisely when a spin structure exists) is that it induces a spin structure on the boundary. The corresponding Dirac operator on the boundary will be denoted $\eth_{0,E}$. A useful assumption, which will be removed later, is

$$\text{(In.5)} \qquad \eth_{0,E} \text{ is invertible.}$$

In fact $\eth_{0,E}$ is elliptic and self-adjoint so (In.5) just means that its null space reduces to $\{0\}$. As a consequence of (In.5), $\eth_E^+$ is Fredholm on its natural domain (the Sobolev space defined by the metric) and the APS theorem states that

$$\text{(In.6)} \qquad \operatorname{ind}(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}).$$

Here the Atiyah-Singer integrand, AS, is the same as before, manufactured from differential-geometric information in the spin structure and auxiliary bundle by local operations. On the other hand the η-invariant is a global object constructed from $\eth_{0,E}$, so fixed purely in terms of boundary data. In fact it is a spectral invariant of $\eth_{0,E}$. This decomposition into a 'local interior' and a 'global boundary' term is fundamental to the utility of the result.

3. Boundary conditions versus *b*-geometry.

The boundary of a compact manifold with boundary always has a collar neighbourhood, i.e. a neighbourhood of the form $[0,r)_x \times \partial X$, say for $r > 1$. An exact b-metric (In.4) is (or gives the manifold) a cylindrical end if, on the collar, h is simply the pull-back of a metric on ∂X. Often (although not so often here) the end is considered as unbounded in that $t = \log x$ is introduced as a variable, putting the boundary at $t = -\infty$. The manifold $X_1 = X \setminus ([0,1] \times \partial X)$ is diffeomorphic to X.

The spin bundles ${}^{\pm}S$ on X can be identified on the collar neighbourhood of the boundary, since they can be identified over the boundary, by an isomorphism with the spinor bundle of the boundary. Then (see §3.11) the Dirac operator becomes

$$\eth^+ = M_-^{-1} \cdot \left(x\frac{\partial}{\partial x} + \eth_0 \right) \cdot M_+, \tag{In.7}$$

where $M_\pm$ are the isomorphisms between the spinor bundles ${}^{\pm}S$ on the collar and S_0, the spinor bundle of ∂M and and $\eth_0$ is the Dirac operator on the boundary. The null space of $\eth^+$ acting on distributions on the collar can then be examined in terms of the eigen-decomposition for $\eth_0$, which is self-adjoint. Thus the solutions of $\eth^+ u_+ = 0$ are superpositions of the special solutions

$$u = x^{-z} M_+^{-1} v, \ \eth_0 v = zv.$$

This solution is square-integrable with respect to the metric if and only if $z < 0$.

The APS boundary condition for the Dirac operator restricted to the region $x \geq 1$, i.e. $t \geq 0$ is

$$Q_+ \left(M_+ u_{\restriction x=1} \right) = 0, \tag{In.8}$$

where Q_+ is the orthogonal projection onto the non-negative eigenspace of $\eth_0$. This projector is a pseudodifferential operator and $\eth^+$ with the boundary condition (In.8) can be considered as an elliptic boundary problem. In particular there is an associated Fredholm operator and the analysis can be carried out using the theory of elliptic boundary problems introduced by Calderón ([25]) and developed further by Seeley [81] (see also Boutet de Monvel [21] and Grubb [41]).

With the APS boundary condition, the appearance in (In.6) of the eta invariant is a little less striking, since it represents just a part of the information in the projection Q_+. The condition (In.8) reflects the square-integrability of an extension of the solution into $t < 0$, i.e. to the whole of the original manifold X. It will not be encountered below[1]. As already noted, the invertibility of the operator $\eth^+$, which in general for an exact b-metric is not quite as simple as (In.7), is attacked directly and its generalized inverse and the associated heat kernels are shown to be elements of the appropriate space of b-pseudodifferential operators.

4. Preliminaries to the proof.

Let ${}^{\pm}S$ be the two spinor bundles over the compact, even dimensional, exact b-spin manifold, X. The idea, used already in [5] and in a related manner by McKean and Singer [58] and dating back, in other contexts, at least to Minakshisundarum and Pleijel [68] is to consider the heat kernels

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-), \tag{In.9}$$

where the Dirac operator is

$$\eth_E^+ : C^\infty(X; {}^+S \otimes E) \longrightarrow C^\infty(X; {}^-S \otimes E)$$

and $\eth_E^-$ is its adjoint. Suppose for the moment that $\partial X = \emptyset$ and consider the Atiyah-Singer theorem. Both $\eth_E^-\eth_E^+$ and $\eth_E^+\eth_E^-$ are elliptic, self-adjoint and non-negative so the heat kernels (In.9) are, for $t > 0$, smoothing operators. The fact that 0 is an isolated spectral point of both means that

$$\begin{aligned} &\lim_{t\to\infty} \exp(-t\eth_E^-\eth_E^+) = \pi_{\operatorname{null}(\eth_E^+)} \\ &\lim_{t\to\infty} \exp(-t\eth_E^+\eth_E^-) = \pi_{\operatorname{null}(\eth_E^-)} \end{aligned} \tag{In.10}$$

where π_N is orthogonal projection onto the finite dimensional subspace $N \subset L^2(X; L)$, for the appropriate bundle L. The convergence in (In.10) is exponential, within smoothing operators. The trace functional, just the sum of the eigenvalues of a finite rank operator, extends continuously to smoothing operators, so from (In.10) it follows that

$$\lim_{t\to\infty} \operatorname{Tr}[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)] = \operatorname{ind}(\eth_E^+). \tag{In.11}$$

[1] The expunging of the APS boundary condition, in explicit form, is a 'feature' of this proof which is fundamental, although not universally welcomed.

The extension of the trace to smoothing operators is described by Lidskii's theorem. Namely smoothing operators are those with $\mathcal{C}^\infty$ Schwartz kernels ([79]), i.e. they can be written as integral operators

$$Ku(x) = \int_X K(x,x')u(x') \text{ with } K \in \mathcal{C}^\infty.$$

Then

$$\operatorname{Tr} K = \int_X K(x,x). \tag{In.12}$$

The single most important property of the trace functional is that it vanishes on commutators:

$$\operatorname{Tr}[K_1, K_2] = 0, \tag{In.13}$$

as follows readily from (In.12). This remains true if K_1 is a differential operator, provided K_2 is smoothing.

At least formally consider (since the operators act on different bundles)

$$\begin{aligned}&\frac{d}{dt}\left(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\right)\\ &= -\left(\eth_E^-\eth_E^+\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\eth_E^+\eth_E^-\right)\\ &= -[\eth_E^-, \eth_E^+\exp(-t\eth_E^-\eth_E^+)].\end{aligned} \tag{In.14}$$

Here the identity

$$\exp(-t\eth_E^+\eth_E^-)\eth_E^+ = \eth_E^+\exp(-t\eth_E^-\eth_E^+), \tag{In.15}$$

which follows from the uniqueness of solutions to the heat equation, has been used. With the trace taken in (In.14), (In.13) and (In.11) together give the remarkable identity of McKean and Singer

$$\operatorname{ind}(\eth_E^+) = \operatorname{Tr}\left[\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\right] \ \forall\, t > 0. \tag{In.16}$$

The formulæ (In.1) and (In.2) arise from a clear understanding of the behaviour of the heat kernels as $t \downarrow 0$, i.e. from the *local index theorem* (proved by Gilkey [37] and Patodi [72]):

$$\operatorname{AS}(x) = \lim_{t\downarrow 0} \operatorname{tr}\left(\exp(-t\eth_E^-\eth_E^+) - \exp(-t\eth_E^+\eth_E^-)\right)(x,x), \tag{In.17}$$

where the 'little' trace, tr, is just the trace functional on the bundles ${}^{\pm}S \otimes E$. The direct proof of (In.17) is Getzler's rescaling argument. The final formula (In.1) arises by applying (In.12) to (In.17).

So, to prove (In.6), in which the η-term should appear as a *defect*, it is natural to look at the heat kernels (In.9) when $\partial X \neq \emptyset$. The *fundamental problem* with the generalization of the proof outlined above is that, when $\partial X \neq \emptyset$ in the 'b-' setting,

$$\exp(-t\eth_E^-\eth_E^+),\ \exp(-t\eth_E^+\eth_E^-) \text{ are not trace class.}$$

Indeed the Atiyah-Patodi-Singer boundary condition was introduced to replace these by trace class operators. There is however a direct generalization of the statement that these exponentials are smoothing operators, which they still are in the interior. Namely there is a calculus of pseudodifferential operators ([61], [64], [47, §18.3], [66]), denoted here $\Psi_b^m(X; L_1, L_2)$ for any bundles L_1, L_2, which captures appropriate uniformity of the kernels up to the boundary. The assumption (In.3) implies that the Dirac operator is in the corresponding space of differential operators

$$\eth_E^+ \in \operatorname{Diff}_b^1(X; {}^+S \otimes E, {}^-S \otimes E) \subset \Psi_b^1(X; {}^+S \otimes E, {}^-S \otimes E);$$

it is elliptic. It follows from constructions essentially the same as in the standard case that

$$\begin{matrix} \exp(-t\eth_E^-\eth_E^+) \in \Psi_b^{-\infty}(X; {}^+S \otimes E) \\ \exp(-t\eth_E^+\eth_E^-) \in \Psi_b^{-\infty}(X; {}^-S \otimes E) \end{matrix} \quad \text{in } t > 0. \tag{In.18}$$

As already noted, these conditions do *not* mean that the operators are trace class. Despite this there is an *extension* of the trace functional to a linear functional

$$\text{b-Tr}_\nu \colon \Psi_b^{-\infty}(X; L) \longrightarrow \mathbb{C}.$$

This extension depends on ν, a trivialization of the normal bundle to ∂X, and is defined simply by regularization of (In.12), as in the work of Hadamard [43]. If $x \in C^\infty(X)$ is a defining function with $dx \cdot \nu = 1$, the b-trace is

$$\text{b-Tr}_\nu(K) = \lim_{\epsilon \downarrow 0} \left[\int_{x > \epsilon} \operatorname{tr} K(x, x) + \log \epsilon \cdot \widetilde{\operatorname{Tr}}(K) \right]. \tag{In.19}$$

The logarithmic term removed in (In.19) is precisely what is needed to regularize the integral and this fixes the coefficient $\widetilde{\operatorname{Tr}}(K)$.

The appearance of the defect in the index formula is directly related to the failure of (In.13) for the b-trace. There is an algebra homomorphism in the calculus

$$\Psi_b^m(X; L_1, L_2) \xrightarrow{(\ )_\partial} \Psi^m(\partial X; L_1, L_2), \tag{In.20}$$

where the image space consists of the pseudodifferential operators on the boundary acting on the restrictions of the bundles to the boundary. This map is defined by restriction:

$$A_\partial u = A\tilde{u}_{\restriction \partial X} \text{ if } \tilde{u} \in C^\infty(X; L_1),\ \tilde{u}_{\restriction \partial X} = u.$$

The homomorphism (In.20) can be extended by noting that the calculus is invariant under conjugation by complex powers of x (a boundary defining function)

$$\Psi_b^m(X; L) \ni A \longleftrightarrow x^{-i\lambda} A x^{i\lambda} \in \Psi_b^m(X; L).$$

Then

$$I(A, \lambda) = \left(x^{-i\lambda} A x^{i\lambda}\right)_\partial \in \Psi^m(\partial X; L) \tag{In.21}$$

is an entire analytic family of pseudodifferential operators, the *indicial family* of A. Moreover

$$K \in \Psi_b^{-\infty}(X; L) \text{ is trace class } \Longleftrightarrow I(K, \lambda) \equiv 0$$

and

$$I(K, \lambda) \equiv 0 \Longrightarrow \text{b-Tr}_\nu(K) = \text{Tr}(K).$$

The coefficient of the singular term in (In.19) is actually given by

$$\widetilde{\text{Tr}}(K) = \frac{1}{2\pi} \int_{\mathbb{R}} \text{Tr}\left(I(K, \lambda)\right) d\lambda.$$

The fundamental formula for the b-trace is:

$$\text{b-Tr}_\nu([A, B]) = \frac{i}{2\pi} \int \text{Tr}\big(\partial_\lambda I(A, \lambda) \circ I(B, \lambda)\big) d\lambda. \tag{In.22}$$

The integral on the right converges absolutely. This formula follows directly from (In.21), (In.19) and the definition of the b-calculus, i.e. it is elementary.

5. The proof.

First some bald facts, which need to be interpreted slightly but are 'true' enough. In terms of (In.21):

$$\begin{cases} I(\eth_E^+,\lambda) = M_-^{-1}(i\lambda+\eth_{0,E})M_+ \\ I(\eth_E^-,\lambda) = M_+^{-1}(-i\lambda+\eth_{0,E})M_- \\ I(\exp(-t\eth_E^-\eth_E^+),\lambda) = M_+^{-1}\exp(-t(\lambda^2+\eth_{0,E}^2))M_+. \end{cases} \tag{In.23}$$

Here $M_\pm$ are isomorphisms of ${}^\pm S$ restricted to ∂X and S_0, the spinor bundle over ∂X, with $M_+^{-1}M_-$ being Clifford multiplication by idx/x. The identity (In.14) still holds, so now taking b-traces it follows from (In.22) that

$$\begin{aligned} &\frac{d}{dt}\,\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^+\eth_E^-)\big) \\ &= \frac{1}{2\pi i}\int \operatorname{Tr}\left[\partial_\lambda I(\eth_E^-,\lambda)\circ I(\eth_E^+\exp(-t\eth_E^-\eth_E^+),\lambda)\right]d\lambda. \end{aligned} \tag{In.24}$$

Using the fact that I is a homomorphism and (In.23), the right side of (In.24) can be rewritten

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{Tr}\left[(i\lambda+\eth_{0,E})\exp(-t(\lambda^2+\eth_0^2))\right]d\lambda. \tag{In.25}$$

The λ integral can be carried out, replacing the integrand by its even part and changing variable to $t^{\frac12}\lambda$, to give

$$-\frac{1}{2\sqrt{\pi}}t^{-\frac12}\operatorname{Tr}\big(\eth_{0,E}\exp(-t\eth_{0,E}^2)\big). \tag{In.26}$$

As $t\to\infty$ (In.11) still holds in case $\partial X\neq\emptyset$, with Tr replaced by b-Tr_ν (the limit is independent of ν because the limiting operator is trace class). Similarly Getzler's scaling argument carries over to this setting to give (In.17), uniformly in x. Finally a similar scaling argument applies to (In.26) (as shown by Bismut and Freed [18], [19]). This allows (In.24) to be integrated over $(0,\infty)$ giving the limiting formula:

$$\begin{aligned} &\lim_{t\to\infty}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^-\eth_E^-)\big) \\ &\qquad -\lim_{t\to0}\text{b-Tr}_\nu\big(\exp(-t\eth_E^-\eth_E^+)-\exp(-t\eth_E^+\eth_E^-)\big) \\ &= -\frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-\frac12}\operatorname{Tr}\left(\eth_{0,E}\exp(-t\eth_{0,E}^2)\right)dt. \end{aligned} \tag{In.27}$$

This then gives (In.6) provided the eta invariant is *defined* to be

$$\eta(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\big(\eth_{0,E} \exp(-t\eth^2_{0,E})\big)dt, \tag{In.28}$$

where absolute convergence follows from the scaling argument. This is one of the 'standard formulæ' for the η-invariant, so the 'proof' is complete.

6. Weighting.

In this setting there is no explicit boundary condition on the Dirac operator $\eth^+_E$. Rather it is, precisely when (In.5) holds, a Fredholm operator on the Sobolev space fixed by the metric. In fact all exact b-metrics are quasi-isometric, so these Sobolev spaces are intrinsic to the compact manifold with boundary and are denoted $H^m_b(X; L)$, for sections of a vector bundle L.

To prove the APS theorem it is illuminating to embed the index problem in a one-parameter family. Namely the Sobolev spaces extend to weighted spaces, $x^s H^m_b(X; L)$, for $s \in \mathbb{R}$, where $x \in \mathcal{C}^\infty(X)$ is a defining function for the boundary. Then

$$\begin{gathered}\eth^+_E : x^s H^1_b(X; {}^+S \otimes E) \longrightarrow x^s H^0_b(X; {}^-S \otimes E)\\ \text{is Fredholm} \iff -s \text{ is not an eigenvalue of } \eth_{0,E}.\end{gathered} \tag{In.29}$$

The eigenvalues of $\eth_{0,E}$ form a discrete subset $\operatorname{spec}(\eth_{0,E}) \subset \mathbb{R}$, unbounded above and below, so $\operatorname{ind}_s(\eth^+_E)$ is defined for $-s \in \mathbb{R} \setminus \operatorname{spec}(\eth_{0,E})$. It is convenient to extend the definition of the index, even to the case that the operator is not Fredholm by setting

$$\widetilde{\operatorname{ind}}_s(\eth^+_E) = \lim_{\epsilon \downarrow 0} \frac{1}{2}\left[\operatorname{ind}_{s-\epsilon}(\eth^+_E) + \operatorname{ind}_{s+\epsilon}(\eth^+_E)\right]. \tag{In.30}$$

The parameter, s, can be absorbed into the operator by observing that the weighting factor, x^s, can be treated as 'rescaling' (in the sense of Chapter 8) of the coefficient bundle E to a bundle $E(s)$.

Thus the conjugated operator, $\eth^+_E(s) = x^{-s}\eth^+_E x^s$, is again the positive part of a (twisted) Dirac operator, however the new total Dirac operator is not self-adjoint; its negative part is $\eth^-_E(s) = x^{-s}\eth^-_E x^s$. All of the discussion above applies, provided $\eth^-_E$ is replaced throughout by $(\eth^+_E)^*$, except for the local index theorem, which no longer holds. The Atiyah-Singer integrand can still be defined as

$$\begin{gathered}\mathrm{AS}(s) = \text{The constant term as } t \downarrow 0 \text{ in}\\ \operatorname{tr}\big(\exp(-t\eth^-_E\eth^+_E) - \exp(-t\eth^+_E\eth^-_E)\big)(x, x)\end{gathered} \tag{In.31}$$

and similarly the modified eta invariant[2] is given by the regularized integral

$$\text{(In.32)}\quad \eta_s(\eth_{0,E}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E} + s] \exp(-t[\eth_{0,E} + s]^2\right) dt, \quad s \in \mathbb{R}.$$

The meaning here is the same as in (In.31): as $t \downarrow 0$ the integral

$$\frac{1}{\sqrt{\pi}} \int_\epsilon^\infty t^{-\frac{1}{2}} \operatorname{Tr}\left([\eth_{0,E} + s] \exp(-t[\eth_{0,E} + s]^2\right) dt$$

has an expansion in powers of ϵ and $\eta_s(\eth_{0,E})$ is the coefficient of the constant term. The APS theorem then can be written in weighted form

$$\text{(In.33)}\qquad \widetilde{\operatorname{ind}}_s(\eth_E^+) = \int_X \mathrm{AS}(s) - \frac{1}{2}\eta_s(\eth_{0,E}) \quad \forall\, s \in \mathbb{R}.$$

The proof outlined above only applies directly in the Fredholm case, when $s \notin -\operatorname{spec}(\eth_{0,E})$. However only the discussion as $t \to \infty$ needs to be modified to give (In.33) in general.

In fact $\widetilde{\operatorname{ind}}_s(\eth_E^+)$ is locally constant on the open set of Fredholm values of s. Its jump at a point $s \in -\operatorname{spec}(\eth_{0,E})$ can be computed using the relative index theorem discussed in §6.2 or from the direct analysis of $\eta_s(\eth_{0,E})$ in §8.14:

$$\widetilde{\operatorname{ind}}_s(\eth_{0,E}) = \begin{cases} \lim\limits_{r \downarrow s} \operatorname{ind}_r(\eth_{0,E}) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E} + s) \\ \lim\limits_{r \uparrow s} \operatorname{ind}_r(\eth_{0,E}) - \frac{1}{2}\dim\operatorname{null}(\eth_{0,E} + s). \end{cases}$$

This allows the value of $\widetilde{\operatorname{ind}}_0(\eth_E^+)$ to be computed explicitly as

$$\widetilde{\operatorname{ind}}_0(\eth_E^+) = \dim\operatorname{null}(\eth_E^+) - \dim\operatorname{null}_-(\eth_E^-) + \frac{1}{2}\dim\operatorname{null}(\eth_{0,E}).$$

Here $\operatorname{null}(\eth_E^+)$ is the null space of $\eth_E^+$ on $L^2(X; {}^+S \otimes E)$ and

$$\operatorname{null}_-(\eth_E^-) = \bigcap_{s<0} \left\{u \in x^s L^2(X; {}^-S \otimes E); \eth_E^- u = 0\right\}$$

[2] This is not to be confused with the eta function of $\eth_{0,E}$ which is discussed in Chapter 9.

is the 'extended L^2 null space.' This gives the familiar form of the index theorem in general, from [8], even when (In.5) is not valid:

$$\dim \operatorname{null}(\eth_E^+) - \dim \operatorname{null}_-(\eth_E^-) = \int_X \mathrm{AS} - \frac{1}{2}\left[\eta(\eth_{0,E}) + \dim \operatorname{null}(\eth_{0,E})\right].$$

The integer on the left, which is just the index on $x^s L^2(X; {}^+S \otimes E)$ for small $s > 0$, is sometimes called the extended L^2 index.

The absence of the local index theorem in the weighted case means that the form of the integrand, $\mathrm{AS}(s)$, in (In.33) cannot be so easily computed. However it is a polynomial in s. The removal of the non-constant terms allows the general formula to be recast as

$$\text{(In.34).} \qquad \widetilde{\operatorname{ind}}_s(\eth_E^+) = \int_X \mathrm{AS} - \frac{1}{2}\eta(\eth_{0,E}) - \widetilde{N}(\eth_{0,E}, s).$$

Here $\widetilde{N}(\eth_{0,E}, 0) = 0$ and for $s \neq 0$

$$\text{(In.35)} \quad \operatorname{sgn}(s) \times \widetilde{N}(\eth_{0,E}, s) = \text{Number of eigenvalues of } -\eth_{0,E} \text{ in } [0, s],$$

where eigenvalues are counted with their multiplicity and an eigenvalue at an endpoint of $[0, s]$ is counted with half its multiplicity.

7. Outline.

As already noted the remaining chapters are intended to place the proof outlined above on a firm basis and in context. First, in Chapter 1, the one-dimensional case, or rather analogue, of the theorem is discussed, although it is not proved. This discussion is not used later but serves to indicate the different ways of viewing a cylindrical end and introduces the power law behaviour of solutions, and fundamental solutions, which underlies the later analysis. Chapter 2 consists of a brief introduction to Riemannian geometry, the Levi-Civita connection and Riemann curvature tensor, presented in order that the extension to b-metrics should be straightforward. Again the point of view taken is that these metrics correspond in the category of compact manifolds with boundary to Riemann metrics in the boundaryless case. For example, they are fibre metrics on a vector bundle, the b-tangent bundle, which is *not* quite the ordinary tangent bundle but is a perfectly satisfactory replacement for it. The notion of a b-differential operator is introduced, as is the notion of ellipticity in this setting. In Chapter 3 the discussion is extended to examine the Clifford algebra and spin structures. It is shown that the Dirac operator associated to an exact b-spin structure is an elliptic b-differential operator.

The analytic part of the investigation begins, very geometrically, in Chapter 4, with a discussion of the b-stretched product of manifolds with boundary. This is the replacement for the ordinary product that it is convenient to use in the inversion of elliptic b-differential operators and it leads directly to the definition and basic properties of the small b-calculus. In particular the normal homomorphism underlying (In.23) and the product formula are then derived. As noted in the discussion of the composition formula, there is a more elegant, and general, approach using somewhat more differential-geometric machinery (see [63]). Some parts of this approach are introduced in the exercises but it is eschewed here in favour of a more elementary treatment. An effort is made to emphasize the structural properties of the b-calculus. One important feature is the b-trace functional. The commutator identity for this functional, (In.22), plays an important rôle in the proof in that it replaces otherwise cumbersome manipulations of the heat kernel on the cylindrical end, as carried out in [8] and in other versions of the theorem such as [85].

The small calculus of b-pseudodifferential operators reduces to the ordinary calculus of pseudodifferential operators when the compact manifold has no boundary. Philosophically there are two main uses for the ordinary calculus. It is used as an investigative tool (in microlocal analysis) and also to invert elliptic operators. The fact that the same space of operators serves both purposes, when $\partial X = \emptyset$, is somewhat fortuitous. For a manifold with boundary this is no longer the case and to invert elliptic b-differential operators it is necessary to enlarge the calculus. For this purpose, both the 'calculus with bounds' and the 'full calculus' are introduced in Chapter 5. Here the additional boundary terms which appear in the (generalized) inverse are described. The full calculus is applied to the examination of the mapping, and especially Fredholm, properties of elliptic b-differential operators. In Chapter 6 the calculus is further used to establish the relative index theorem and to describe the holomorphy properties of the resolvent family of a self-adjoint operator of second order. The boundary behaviour of the resolvent is also related to scattering theory. As an application of the relative index theorem, using an idea of Gromov and Shubin (see [40]), the Riemann-Roch theorem for surfaces is deduced. This chapter also contains a Hodge theoretic discussion (of course from the point of view of b-metrics) of the cohomology of a compact manifold with boundary.

In Chapter 7 the heat kernel of a second order operator is described. This is done in a manner consistent with the treatment of the b-calculus, i.e. using a blown-up 'heat space' to define the class of admissible kernels. This approach is very similar to the calculi described by Beals and Greiner in [14], by Taylor in [89] and more recently in [31]. Here the geometric structure is made explicit and this has the important consequence that the melding of the heat and b-calculi, to yield a detailed description of the heat kernel of a b-differential operator (such as $\eth_E^- \eth_E^+$) is straightforward. The discussion of the resolvent in Chapter 6 is used to analyze the long-time behaviour of the heat kernel.

Getzler's rescaling argument is formalized in Chapter 8 in the notion of the rescaling, at a boundary hypersurface, of a vector bundle. The weighting of the Sobolev spaces in (In.29) and the b-tangent bundle are both examples of this general procedure. The local index theorem then follows directly from this rescaling, the fundamental observation of Berezin and Patodi on the structure of the supertrace functional on the spin bundle, Lichnerowicz' formula for the difference between the Dirac and connection Laplacians and a generalization of Mehler's formula for the heat kernel of the harmonic oscillator, found by Getzler.

Finally in Chapter 9 the proof of the APS theorem outlined above is reviewed and completed by annotation with references to the intervening material. In fact the theorem is actually proved in the wider context of the Dirac operators on Hermitian Clifford modules (with graded unitary Clifford b-connections) on manifolds with exact b-metrics. The application to the signature formula given in [8] is then explained. It is also shown how the application of the b-calculus allows many of the standard analytic-geometric objects, such as the zeta function, the eta invariant and the Ray-Singer analytic torsion to be transferred to the b-category.

Chapter 1. Ordinary differential operators

The basic analytic tool developed below to carry out the proof of the APS theorem is the calculus of b-pseudodifferential operators. This allows the mapping, especially Fredholm, and spectral properties of $\eth_E^+$ to be readily understood. As motivation for the analytic part of the discussion the one-dimensional case will first be considered, although the result is not proved in detail. This case is 'easy' for many reasons, not least because the dimension is odd, which means there is no interior contribution to the index, and the boundary dimension is zero, so the boundary operator is a matrix, i.e. has finite rank. However, from an analytic point of view the one-dimensional cases serves as quite a good guide to the general case.

1.1. Operators and coordinates.

In one dimension there is *no* spin structure to be concerned about. The only connected one-dimensional compact manifold with non-trivial boundary is the interval $X = [0,1]$. In particular all bundles are trivial. Of course one should bear in mind that the boundary has two components. So consider a first-order linear differential operator acting on k functions

$$P = A(x)\frac{d}{dx} + B(x), \quad A, B \in \mathcal{C}^\infty([0,1]; M_{\mathbb{C}}(k)), \tag{1.1}$$

where $M_{\mathbb{C}}(k) \simeq \mathbb{C}^{k^2}$ is the algebra of $k \times k$ complex-valued matrices.

The operator should be *elliptic* in the interior, so $\det(A(x)) \neq 0$ for $x \in (0,1)$. It should also be of 'b' type at the end-points. This means that P should have regular-singular points at 0 and 1 :

$$A(x) = x(1-x)E(x), \quad E \in \mathcal{C}^\infty([0,1]; M_{\mathbb{C}}(k)), \ \det E \neq 0 \text{ on } [0,1]. \tag{1.2}$$

To analyze the *index* of P consider the adjoint, P^*, and set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim \operatorname{null}(P^*) \tag{1.3}$$

or find some space on which P is *Fredholm*:

a) $P : H_1 \longrightarrow H_2$ continuous, H_1, H_2 Hilbert spaces
b) $\operatorname{null}(P) \subset H_1$ finite dimensional
c) $\operatorname{range}(P) \subset H_2$ closed
d) $\operatorname{range}(P)^\perp \subset H_2$ finite dimensional

and then set

$$\operatorname{ind}(P) = \dim \operatorname{null}(P) - \dim(\operatorname{range}(P)^\perp). \tag{1.4}$$

If P is Fredholm these two definitions of the index are the same.

What is a reasonable space on which P can be expected to be Fredholm? Consider the simple case

$$P_c = x(1-x)\frac{d}{dx} + c, \quad c \in \mathbb{C}. \tag{1.5}$$

There are two transformations of the independent variable which yield even simpler operators. First

$$t = \frac{x}{1-x} : [0,\infty) \ni t \longmapsto x = \frac{t}{1+t} \in [0,1) \tag{1.6}$$

and then $s = \log t = \log x - \log(1-x)$, $(-\infty,\infty) \ni s \longmapsto t = e^s \in (0,\infty)$. Notice that

$$\frac{dt}{dx} = \frac{1}{(1-x)^2} = (1+t)^2, \ \frac{dt}{ds} = e^s = t,$$

so

$$x(1-x)\frac{d}{dx} = \frac{x}{1-x}\frac{d}{dt} = t\frac{d}{dt}$$
$$x(1-x)\frac{d}{dx} = \frac{d}{ds}.$$

Thus P_c in (1.5) becomes

$$P_c = t\frac{d}{dt} + c, \ \mathbb{R}^+\text{-invariant on } (0,\infty)$$
$$P_c = \frac{d}{ds} + c, \ \text{translation-invariant on } \mathbb{R}.$$

Certainly then P_c is easy to analyze. Acting on any reasonable class of functions, distributions or even hyperfunctions, P_c has at most a one-dimensional null space, given in the three coordinates systems by

$$P_c u = 0 \Longrightarrow u = \begin{cases} ax^{-c}(1-x)^c \\ at^{-c} & a \in \mathbb{C}. \\ ae^{-cs} \end{cases} \tag{1.7}$$

Then the only question is whether or not this solution is in the domain of P_c.

Suppose P_c^* is taken to be the adjoint with respect to Lebesgue measure, $|ds|$, on $\mathbb{R}$

$$P_c^* = -\frac{d}{ds} + \bar{c}.$$

Thus (1.7) applies to $P_c^* = -P_{-\bar{c}}$. It is reasonable to take the *domain* of P_c to be

$$H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}); \frac{d}{ds}u \in L^2(\mathbb{R})\}, \tag{1.8}$$

the standard Sobolev space.

EXERCISE 1.1. Check that if the domain of P_c^* is defined by

$$\mathrm{Dom}(P_c^*) = \{u \in L^2(\mathbb{R}); H^1(\mathbb{R}) \ni v \longmapsto \int u\overline{P_c v} ds$$
$$\text{extends by continuity to } L^2(\mathbb{R})\}$$

then $\mathrm{Dom}(P_c^*) = H^1(\mathbb{R})$.

1.2. Index.

Now notice that

$$\exp(-cs) \notin L^2(\mathbb{R}) \qquad \forall\, c \in \mathbb{C}, \tag{1.9}$$

since the exponential is *always* too large in one direction or the other (or both if $c \in i\mathbb{R}$). Thus, with domain (1.8) and definition (1.3), it is always the case that

$$\mathrm{ind}(P_c) = 0, \tag{1.10}$$

which is not too interesting!

The constant in P_c can be changed by conjugating by an exponential

$$e^{-as}\big(\frac{d}{ds} + c\big)e^{as}u = \big(\frac{d}{ds} + (a+c)\big)u. \tag{1.11}$$

Since the function e^{as} is real when a is, the adjoint changes to

$$e^{as}\big(-\frac{d}{ds} + \bar{c}\big)e^{-as} = \big(-\frac{d}{ds} + (\bar{c}+a)\big).$$

This corresponds to replacing the Sobolev space (1.8) by the exponentially weighted space

$$e^{as}H^1(\mathbb{R}) = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}); e^{-as}u \in H^1(\mathbb{R})\},$$

where $L^2_{\mathrm{loc}}(\mathbb{R})$ is the space of locally square-integrable functions on $\mathbb{R}$. Certainly (1.10) still holds on these spaces, but somehow not quite for the 'same' reason in that the part of infinity which causes (1.9) may have changed. This suggest that a less trivial result may follow by looking at spaces which are *weighted* differently at the *two* infinities, i.e. boundary points.

Recall that $H^1_{\mathrm{loc}}(0,1)$ is the space of locally square-integrable functions on $(0,1)$ with first derivative, in the distributional sense, also given by a locally square-integrable function.

DEFINITION 1.2. For $\alpha, \beta \in \mathbb{R}$ set

$$\begin{aligned}
&x^\alpha(1-x)^\beta H_b^1([0,1]) \\
&= \Big\{u \in H^1_{\mathrm{loc}}((0,1));\ \int_0^1 \Big|x^{-\alpha}(1-x)^{-\beta}u\Big|^2 \frac{dx}{x(1-x)} < \infty, \\
&\qquad \int_0^1 \Big|x^{-\alpha}(1-x)^{-\beta}\big(x(1-x)\frac{du}{dx}\big)\Big|^2 \frac{dx}{x(1-x)} < \infty\Big\}.
\end{aligned}$$

On passing to the variable s on $\mathbb{R}$, these conditions can be written in terms of $w \in H^1_{\mathrm{loc}}(\mathbb{R})$ as the requirements

$$\begin{aligned}
&\int_{-\infty}^0 \Big|e^{-\alpha s}w\Big|^2 ds, \quad \int_0^\infty \Big|e^{\beta s}w\Big|^2 ds < \infty \\
&\int_{-\infty}^0 \Big|e^{-\alpha s}\frac{dw}{ds}\Big|^2 ds, \quad \int_0^\infty \Big|e^{\beta s}\frac{dw}{ds}\Big|^2 ds < \infty.
\end{aligned} \tag{1.12}$$

Thus in terms of the variable s these spaces are exponentially weighted at infinity, with different weights.

EXERCISE 1.3. Check that in terms of the variable t the spaces in Definition 1.2 become

$$\begin{aligned}
&x^\alpha(1-x)^\beta H_b^1([0,1]) \longleftrightarrow \\
&\Big\{v \in H^1_{\mathrm{loc}}((0,\infty));\ \int_0^\infty \Big|\big(\frac{t}{1+t}\big)^{-\alpha}(1+t)^\beta v\Big|^2 \frac{dt}{t} < \infty, \\
&\qquad \int_0^\infty \Big|\big(\frac{t}{1+t}\big)^{-\alpha}(1+t)^\beta \big(t\frac{d}{dt}v\big)\Big|^2 \frac{dt}{t} < \infty\Big\}.
\end{aligned}$$

PROPOSITION 1.4. *The operator P_c in (1.5) is Fredholm as an operator*

$$P_c : x^\alpha(1-x)^\beta H_b^1([0,1]) \longrightarrow x^\alpha(1-x)^\beta L_b^2([0,1]) \tag{1.13}$$

if and only if

$$\alpha \neq -\operatorname{Re} c, \ \beta \neq \operatorname{Re} c; \tag{1.14}$$

its index is

$$\operatorname{ind}(P_c) = \begin{cases} 1 & \alpha < -\operatorname{Re} c, \ \beta < \operatorname{Re} c \\ 0 & \alpha > -\operatorname{Re} c, \ \beta < \operatorname{Re} c \text{ or } \alpha < -\operatorname{Re} c, \beta > \operatorname{Re} c \\ -1 & \alpha > -\operatorname{Re} c, \ \beta > \operatorname{Re} c. \end{cases} \tag{1.15}$$

PROOF: Certainly P_c is always a continuous linear map (1.13) as follows directly from the definition of these spaces. To study the Fredholm properties, it is enough to *invert* P and then see how the inverse is affected by the weighting. Using (1.11) the constant c can be removed since

$$\left(\frac{x}{1-x}\right)^{-c} : x^{\alpha}(1-x)^{\beta}H^1_b([0,1]) \longleftrightarrow x^{a-\operatorname{Re} c}(1-x)^{b+\operatorname{Re} c}H^1_b([0,1])$$

for any $c \in \mathbb{C}$ and

$$\left(\frac{x}{1-x}\right)^{c}P_c\left(\frac{x}{1-x}\right)^{-c} = x(1-x)\frac{d}{dx}. \tag{1.16}$$

This reduces the problem to the special case $c = 0$.

Now it is convenient to work in the translation-invariant picture, where $P = d/ds$. An inverse is given by integration, say from 0 :

$$Q'g(s) = \int_0^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g(s) = g. \tag{1.17}$$

Suppose $f \in x^{\alpha}(1-x)^{\beta}L^2_b([0,1])$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R})$ is its expression in terms of the coordinate s. From (1.12) it follows that

$$\beta < 0 \Longrightarrow \int_0^{\infty} \left|e^{s\beta}Q'g(s)\right|^2 ds < \infty$$

$$\alpha < 0 \Longrightarrow \int_{-\infty}^{0} \left|e^{-s\alpha}Q'g(s)\right|^2 ds < \infty.$$

Thus if Qf is $Q'g$ expressed in terms of the coordinate x,

$$\alpha, \beta < 0 \Longrightarrow Q : x^{\alpha}(1-x)^{\beta}L^2_b([0,1]) \longrightarrow x^{\alpha}(1-x)^{\beta}H^1_b([0,1]).$$

In this case P_c is surjective, so certainly Fredholm. From the definition, (1.4), of the index and the fact that the null space is spanned by

$$1 \in x^{\alpha}(1-x)^{\beta}H^1_b([0,1]) \text{ if } \alpha < 0,\ \beta < 0,$$

the validity of the first case in (1.15) follows.

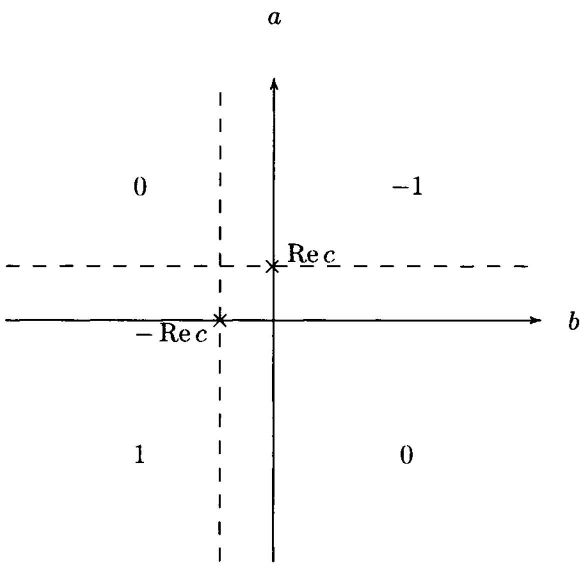

Figure 1. The index of P_c.

Suppose $\beta > 0$, then (1.17) does not give a solution correctly weighted at infinity, unless $\int_0^\infty g(s)ds = 0$. However taking instead

$$Q'g = \int_\infty^s g(r)dr \Longrightarrow \frac{d}{ds}Q'g = g, \tag{1.18}$$

since if $f \in x^\alpha(1-x)^\beta L_b^2$ then g *is* integrable near $s = +\infty$. In this case

$$\alpha < 0,\ \beta > 0 \Longrightarrow P \text{ is surjective, } \operatorname{ind}(P) = 0,$$

since 1 is no longer in $x^\alpha(1-x)^\beta \mathrm{L}_b^2$. The same argument applies if $\alpha > 0$, $\beta < 0$ by replacing x by $1-x$, and hence s by $-s$.

Finally if $\alpha > 0$, $\beta > 0$ then (1.18) is still correctly behaved near infinity but

$$Qf \in x^\alpha(1-x)^\beta L_b^2([0,1]),\ \alpha,\beta > 0 \text{ iff } \int_{-\infty}^{\infty} f(x)\frac{dx}{x(1-x)} = 0.$$

Certainly the constant solution is not in the domain so $\operatorname{ind}(P) = -1$.

This completes the proof of the proposition except for the part of (1.14) which states that P_c is not Fredholm as an operator (1.13) if

$$\alpha = -\operatorname{Re} c \text{ or } \beta = \operatorname{Re} c. \tag{1.19}$$

This is left as an exercise:

EXERCISE 1.5. Show that if $\alpha = \operatorname{Re} c$ or $\beta = -\operatorname{Re} c$ then P in (1.13) is not Fredholm because the range is *not* closed. [Hint: Find a sequence of functions in $L^2(\mathbb{R})$ of the form du_k/ds, $u_k \in H^1(\mathbb{R})$ such that $du_k/ds \longrightarrow 0$ in L^2 but $\|u_k\|_{L^2} \longrightarrow \infty$. Use this to show that there exists $f \in L^2(\mathbb{R})$ which is in the closure of the range but $f \neq du/ds$, for any $u \in H^1(\mathbb{R})$.]

1.3. General statement.

Proposition 1.4 can be interpreted informally as saying that the operator, P_c, is Fredholm *unless* there is an element of the null space (on distributions) which is *almost* in the domain, but is *not* in the domain. Notice also that in Figure 1 the index increases by 1 (the dimension of this null space) every time one of the lines in (1.19) is crossed downward. These two ideas will reappear in the higher dimensional setting below.

Now consider the extension of this result to the general case, (1.1) subject to (1.2). This is the prototype for the Dirac operator.

THEOREM 1.6. *The operator P in (1.1), subject to (1.2), is always a continuous linear operator*

$$P: \ x^\alpha(1-x)^\beta H^1_b([0,1];\mathbb{C}^k) \longrightarrow x^\alpha(1-x)^\beta L^2_b([0,1];\mathbb{C}^k), \ \alpha,\beta \in \mathbb{R},$$

which is Fredholm if and only if

$$\begin{aligned} &\alpha \neq \operatorname{Re}\lambda \text{ for any eigenvalue } \lambda \text{ of } -E(0)^{-1}B(0) \\ &\beta \neq \operatorname{Re}\lambda \text{ for any eigenvalue } \lambda \text{ of } E(1)^{-1}B(1) \end{aligned} \tag{1.20}$$

and then its index is

$$\operatorname{ind}(P) = -\frac{1}{2}(\eta^0_\alpha + \eta^1_\beta), \tag{1.21}$$

where if G^i_λ are, for $i = 0,1$, the eigenspaces with eigenvalue λ of the matrices $(-1)^i(E(i))^{-1}B(i)$,

$$\eta^i_r = \sum_{\operatorname{Re}\lambda > -r} \dim G^i_\lambda - \sum_{\operatorname{Re}\lambda < -r} \dim G^i_\lambda. \tag{1.22}$$

EXERCISE 1.7. Try to prove this result. It is not too hard, using standard results on solutions of ordinary differential equations, as in [28]. It is also illuminating to follow the lines of the proof outlined in the Introduction in this case.

Consider how the formula (1.21) reduces to (In.6). First note that the Atiyah-Singer integrand, AS, vanishes identically because the dimension is odd. It is worth noting the relationship between (1.22) and (In.28). From (In.23) it is reasonable to expect that

$$\eth_0 \longleftrightarrow E(0)^{-1}B(0).$$

To make this correspondence more exact, suppose that $\eth_0 = E(0)^{-1}B(0)$ is a self-adjoint matrix and $a = 0$. Then

$$\operatorname{tr} \eth_0 \exp(-t\eth_0^2) = \sum_{\text{eigenvalues}} \lambda e^{-t\lambda^2}, \tag{1.23}$$

where the eigenvalues are repeated with their multiplicity. By the assumption (In.5), or equivalently the condition (1.20), 0 should not be an eigenvalue. Then, inserting (1.23) into (In.28) gives

$$\begin{aligned}\eta(\eth_0) &= \frac{1}{\sqrt{\pi}} \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda) \int_0^\infty e^{-t|\lambda|^2} (t|\lambda|^2)^{-\frac{1}{2}} d(t|\lambda|^2) \\ &= \sum_{\text{eigenvalues}} \operatorname{sgn}(\lambda),\end{aligned}$$

since

$$\int_0^\infty e^{-s} \frac{ds}{s^{\frac{1}{2}}} = \sqrt{\pi}.$$

Thus $\eta(\eth_0) = \eta_0^0$ in terms of (1.22). This shows the relationship between (In.6) and (1.21). It also suggests that the eta invariant measures the *spectral asymmetry* of the operator, i.e. the difference between the number of positive and the number of negative eigenvalues.

EXERCISE 1.8. Check the relationship between (1.21) and (In.33).

1.4. Kernels.

To finish this look at the one-dimensional case, consider again the trivial case (1.5). For $a << 0$, $b >> 0$ the solution operator to P_c is obtained

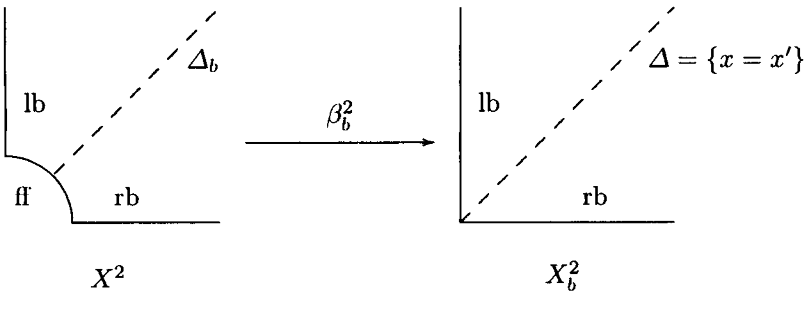

Figure 2. Blow-up of X^2, $X = [0, 1]$.

using the conjugation (1.16), the change of coordinates (1.6) and the integration formula (1.18). Consider this inverse in terms of the compact, x-representation. Then it can be written

$$Qf(x) = \int_1^x f(x') \frac{dx'}{x'(1-x')}.$$

Undoing (1.16) gives the inverse to P_c as

$$Q_c f(x) = \int_1^x \left(\frac{x}{1-x}\right)^c \left(\frac{x'}{1-x'}\right)^{-c} f(x') \frac{dx'}{x'(1-x')}.$$

This can be written as an integral operator

$$Q_c f(x) = \int_0^1 K(x, x') f(x') \frac{dx'}{x'(1-x')},$$

where the Schwartz kernel is

$$K_c(x, x') = -\frac{x^c}{(1-x)^c} \frac{(1-x')^c}{(x')^c} H(x' - x), \tag{1.24}$$

$H(t)$ being the Heaviside function.

Consider the structure of K_c. There are, in principle, five singular terms, with singularities at $x = 0, x = 1, x' = 0, x' = 1$ and $x = x'$ (except that those at $x' = 0$ and $x = 1$ happen to vanish). The singularities are simple power type, except that at the two corners $x = x' = 0$ and $x = x' = 1$

two singularities coincide and things look nasty. It is exactly such kernels that will be analyzed below. To do so, it is convenient to introduce *polar coordinates* around $x = x' = 0$ and $x = x' = 1$.

Actually, to carry out this process of *blowing up,* it is simpler to use the singular coordinates (near $x = x' = 0$)

$$r = x + x', \ \tau = \frac{x - x'}{x + x'}. \tag{1.25}$$

Then $x = \frac{1}{2}r(1 + \tau)$, $x' = \frac{1}{2}r(1 - \tau)$. Inserting this into (1.24) gives

$$K_c(x, x') = \frac{(1 + \tau)^c}{(1 - \tau)^c} \times H(-\tau) \times C^\infty \text{ near } x = x' = 0. \tag{1.26}$$

Notice what is accomplished by this maneuver. The kernel now has singularities at two separated surfaces, $\tau = 0$ and $\tau = -1$. This trick is the basis of the b-calculus.

EXERCISE 1.9. Write out the relationship between the coordinates r, τ in (1.25) and polar coordinates ρ, θ where $x = \rho \cos \theta$, $x' = \rho \sin \theta$ and $\theta \in [0, \frac{1}{4}\pi]$. Check that the map $\rho, \theta \longrightarrow (r, \tau)$ is a diffeomorphism from $[0, \infty) \times [0, \frac{1}{4}\pi]$ onto $[0, \infty) \times [-1, 1]$.

EXERCISE 1.10. Find formulæ similar to (1.26) for the Schwartz kernel of a generalized inverse to P_c in the other cases in (1.15).

Chapter 2. Exact b-geometry

Much of this chapter is geometric propaganda. It is intended to convince the reader that there is a 'category' of b-Riemann manifolds in which one can work systematically. This b-geometry can also be thought of as the geometry of manifolds with asymptotically cylindrical ends. There are other geometries which are similar to b-geometry (see [55], [54], [56], [32], [31] and [63] for a general discussion).

Following the definition and discussion of the most basic elements of (exact) b-geometry, the Levi-Civita connection is described *ab initio*. The notion of a b-connection is introduced and its relation to that of an ordinary connection is explained. Finally a brief description of characteristic classes is given. For the reader familiar with differential geometry, the main sections to peruse are §§2.2, 2.3, 2.4, 2.13, 2.16 and 2.17.

2.1. Manifolds.

It is assumed below that the reader is familiar with elementary global differential geometry, i.e. the concept of a manifold. However at various points later, the less familiar notion of a manifold with corners is encountered so, for the sake of clarity, definitions are given here. These have been selected for terseness rather than simplicity or accessibility!

A *topological manifold* of dimension N is a paracompact Hausdorff (connected unless otherwise noted) topological space, X, with the property that each point $p \in X$ is contained in an open set $O \subset X$ which is homeomorphic to $\mathbb{B}^N = \{x \in \mathbb{R}^N; |x| < 1\}$. These open sets, with their maps to $\mathbb{B}^N$, are called coordinate patches.

The algebra of all continuous functions, real-valued unless otherwise stated, is denoted $\mathcal{C}^0(X)$. A subalgebra $\mathcal{F} \subset \mathcal{C}^0(X)$ is said to be a $\mathcal{C}^\infty$ *subalgebra* if for any real-valued $g \in \mathcal{C}^\infty(\mathbb{R}^k)$, for any k, and any elements $f_1, \dots, f_k \in \mathcal{F}$ the continuous function $g(f_1, \dots, f_k) \in \mathcal{F}$. The subalgebra is said to be *local* if it contains each element $g \in \mathcal{C}^0(X)$ which has the property that for every set O_α in some covering of X by open sets there exists $g_\alpha \in \mathcal{F}$ with $g = g_\alpha$ on O_α.

A *manifold* (meaning here always an infinitely differentiable, shortened to $\mathcal{C}^\infty$, manifold) is a topological manifold with a real, local, $\mathcal{C}^\infty$ subalgebra $\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X)$ specified with the following property: X has a covering by open sets O_α, $\alpha \in A$, for each of which there are N elements $f_1^\alpha, \dots, f_N^\alpha \in \mathcal{C}^\infty(X)$ with $F^\alpha = (f_1^\alpha, \dots, f_N^\alpha)$ restricted to O_α making it a coordinate patch and $f \in \mathcal{C}^\infty(X)$ if and only if for each $\alpha \in A$ there exists $g_\alpha \in \mathcal{C}^\infty(\mathbb{B}^N)$ such that $f = g_\alpha \circ F^\alpha$ on O_α.

EXERCISE 2.1. Show that this definition is equivalent to the standard one involving covering by compatible infinitely differentiable coordinate