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FOURIER SERIES IN SEVERAL VARIABLES WITH APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Victor L. Shapiro

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Victor L. Shapiro
Department of Mathematics
University of California, Riverside
USA

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To Flo, my wife and dancing partner

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Preface

The primary purpose of this book is to show the great value that Fourier series methods provide in solving difficult problems in nonlinear partial differential equations. We illustrate these methods in three different cases.

Probably the most important of these three cases are the results that we present for the stationary Navier-Stokes equations. In particular, we show how to obtain the best possible results for periodic solutions of the stationary Navier-Stokes equations when the driving force is nonlinear. We also present the basic theorem for the distribution solutions of said equations. The ideas for this material come from a paper published by the author in the Journal of Differential Equations.

Also, we show how to obtain classical solutions to the stationary Navier-Stokes equations by applying the Calderon-Zygmund C^a-theory developed for multiple Fourier series earlier in the book. This technique using the Calderon-Zygmund C^a-theory does not appear to be in any other text dealing with this subject and is based on a paper that appeared in the Transactions of the AMS.

The second case we consider handles nonlinear reaction-diffusion systems and uses a technique involving multiple Fourier series to strongly improve on a theorem previously introduced by Brezis and Nirenberg. The idea for doing this comes from a recent (2009) paper published by the author in the Indiana University Math Journal. Reaction-diffusion systems are important in many areas of applied mathematics including mathematical biology. The main reason we were able to improve on the results of Brezis and Nirenberg is because the use of multiple Fourier series enables one to make sharper estimates and thus obtain a better compactness lemma. The second theorem we present in this area involves a conventional result involving weak solutions to the reaction-diffusion system.

The third case we consider is in the area of quasilinear elliptic partial differential equations and resonance theory. We deal with an elliptic operator of the form

$$\mathcal{Q}u = -\sum_{i,j=1}^{N} D_i[a^{ij}(x,u)D_ju] + \sum_{j=1}^{N} b^j(x,u,Du)D_ju$$

and establish a resonance result based on the work of Defigueredo and Gossez in a Journal of Differential Equations paper and on the work of the author in a Transactions of the AMS manuscript. The resonance result obtained is the best possible and is proved via a Galerkin type argument that illustrates once again the power of Fourier analysis in handling tough problems in nonlinear PDE. The second and third theorems that we present give necessary and sufficient conditions for the solution of certain other equations at a resonance involving the above operator $\mathcal{Q}u$.

Another aim of this book is to establish the connection between multiple Fourier series and number theory. We present an N-dimensional, $N \geq 2$, number theoretic result, which gives a necessary and sufficient condition that

$$C(\xi_1) \times \cdots \times C(\xi_N)$$

be a set of uniqueness for a class of distributions on the N-torus, T_N. The ideas behind this result come from a paper published in the Journal of Functional Analysis.

Here, $C(\xi_j)$ is the familiar symmetric Cantor set on $[-\pi, \pi]$ depending on the real number ξ_j where $0 < \xi_j < 1/2$. The condition is that each ξ_j^{-1} be an algebraic integer called a Pisot number. What is important about this result is that the considered class of distributions, labeled $\mathcal{A}(T_N)$, does not necessarily have Fourier coefficients that go to zero as the spherical norm $|m| = (m_1^2 + \cdots + m_N^2)^{1/2} \to \infty$ but as $\min(|m_1|, ..., |m_N|) \to \infty$. This gives rise to a wider class of distributions; when it appeared, it was the first result of this nature in the mathematical literature.

As a corollary to the result just mentioned, we have the following:

Let p and q be positive relatively-prime integers with $p < 2q$. Then a necessary and sufficient condition that

$$C(\frac{p_1}{q_1}) \times \cdots \times C(\frac{p_N}{q_N})$$

be a set of uniqueness for the class $\mathcal{A}(T_N)$ is that $p_j = 1$ for $j = 1, ..., N$.

An additional aim of this book is to present the periodic C^α-theory of Calderon and Zygmund. We deal with a Calderon-Zygmund kernel of spherical-harmonic type, called $K^*(x)$, and show that it has a principal-valued Fourier coefficient $\widehat{K^*}(m)$. We set $\widetilde{f} = f * K^*$ and show that the following very important theorem prevails:

$$f \in C^\alpha(T_N), 0 < \alpha < 1, \Rightarrow \widetilde{f} \in C^\alpha(T_N).$$

We also give an application of this theorem to a periodic boundary value problem involving the Laplace operator and later use it to obtain the regularity result mentioned above for the stationary Navier-Stokes equations.

Another aim of this book is to present the recent (2006) article in the Proceedings of the AMS, which extends Fatou's famous work on anti-derivatives and nontangential limits to higher dimensions. The big question answered is "How does an individual handle a concept that depends on the one-dimensional notion of the anti-derivative in dimension $N \geq 2$?" Our answer to the question is

> "Generalize the notion of the Lebesgue point set and show that the concepts are the same in one-dimension."

Chapter 1 of the book deals with four different summability methods used in the study of multiple Fourier series, namely the methods of (i) iterated Fejer, (ii) Bochner-Riesz, (iii) Abel, and (iv) Gauss-Weierstrass. The iterated Fejer method in §2 gives a global uniform approximation for continuous periodic functions as well as a global L^p approximation theorem. In §3, the classical Bochner theorem for pointwise Bochner-Riesz summability of multiple Fourier series is established. To understand the proof of this theorem, a knowledge of various Bessel identities and estimates is essential. This Bessel background material is presented in §1 and §2 of Appendix A.

Several Abel summability theorems, which are important in the study of harmonic functions including the nontangential result discussed above, are also presented in Chapter 1, §4. In §5 of Chapter 1, the Gauss-Weierstrass summability method, which is fundamental in the study of the heat equation, is developed; it includes a theorem necessary for a subsequent number theoretic result appearing later in the book.

Chapter 2 is devoted to the study of conjugate multiple Fourier series where the conjugacy is defined by means of periodic Calderon-Zygmund kernels that are of spherical harmonic type. In particular, the periodic Calderon-Zygmund kernel, $K^*(x)$, is defined, and it is proved that its principal-valued Fourier coefficient $\widehat{K^*}(m)$ exists. The conjugate function of f is designated by $\widetilde{f}$, and it is shown that if things are good, $\widehat{\widetilde{f}}\,(m) = \widehat{K^*}(m)\widehat{f}(m)$, which is similar to the one-dimensional situation. The main result established is the following: If $f \in C^\alpha\,(T_N)$, then $\widetilde{f} \in C^\alpha\,(T_N)$. This C^α- theorem is presented in complete detail in §4 of Chapter 2 and is based on a paper published by Calderon and Zygmund in the Studia Mathematica.

In §5 of Chapter 2, an application of this C^α- result to a periodic boundary value problem involves the Laplace operator. Also, a Tauberian convergence theorem for conjugate multiple Fourier series motivated by an interesting one-dimensional result of Hardy and Littlewood is given in §3 of Chapter 2. The Tauberian background material is developed in Appendix B.

Chapter 3 contains the details of the solution to *a one hundred year old problem*, namely

> Establish the two-dimensional analogue of Cantor's famous uniqueness theorem dealing with the convergence of one-dimensional trigonometric series.

The solution depends upon an elegant paper published by Roger Cooke in the Proceedings of the AMS establishing the two-dimensional Cantor-Lebesgue lemma joined with a manuscript of the author that appeared in the Annals of Mathematics.

Chapter 3 also contains the N-dimensional number theoretic theorem discussed above giving a necessary and sufficient condition that

$$C(\xi_1) \times \cdots \times C(\xi_N)$$

be a set of uniqueness for the class of distributions $\mathcal{A}(T_N)$ on the N-torus. In addition, Chapter 3 contains the recent (2004) article about fractal sets called generalized carpets that are not Cartesian product sets but are sets of uniqueness for a smaller class of distributions on the N-torus labeled $\mathcal{B}(T_N)$. These fractal results come from a paper published in the Proceedings of the AMS.

The analogous problem to Cantor's uniqueness theorem for a series of two-dimensional surface spherical harmonics on S_2 is still open and is presented in complete detail in Chapter 3, §2. This problem has been open now for 140 years. The background material in spherical harmonics, which plays an important role throughout this monograph, is presented in Appendix A, §3.

The material in Chapter 4 is motivated by Schoenberg's theorem involving positive definite functions on S_2 and surface spherical harmonics published in the Duke Journal of Math. It turns out that part of Schoenberg's theorem is highly useful in studying the kissing problem, $k(3)$, in discrete geometry, as Musin's 2006 result shows. Here, $k(3)$ is the largest number of white billiard balls that can simultaneously kiss (touch) a black billiard ball and represents a problem going back to Isaac Newton's time in 1694.

Chapter 4 presents Schoenberg's theorem on S_{N-1}, then on T_N, and finally on $S_{N_1-1} \times T_N$. The proof on $S_{N_1-1} \times T_N$ makes use of a number of different concepts that occur in this monograph.

Chapter 5 presents five theorems dealing with periodic solutions of nonlinear partial differential equations. As mentioned earlier, the methods employed illustrate the huge power of Fourier analysis in solving seemingly impenetrable problems in a nonlinear analysis. Chapter 5, §1 presents, in particular, periodic solutions in the space variables to a system of nonlinear reaction-diffusion equations of the form

$$\begin{cases} \frac{\partial u_j}{\partial t} - \Delta u_j = f_j(x, t, u_1, ..., u_J) & \text{in } T_N \times (0, T) \\ \\ u_j(x, 0) = 0 \end{cases}$$

$j = 1, ..., N$.

Two theorems are established with respect to this nonlinear parabolic system. The first theorem deals with one-sided conditions placed on the f_j, and the second deals with two-sided conditions on the f_j. As discussed above, the first theorem strongly improves (for periodic solutions) on a one-sided classical theorem previously established by Brezis and Nirenberg.

In §2 of Chapter 5, we deal with the equation

$$\mathcal{Q}u = f(x, u)$$

where $\mathcal{Q}u$ is the partial differential operator discussed above. We set

$$\mathcal{F}_{\pm}(x) = \limsup_{s\to\pm\infty} f(x,s)/s,$$

and show that if

$$\int_{T_N} \mathcal{F}_{+}(x)\,dx < 0 \text{ and } \int_{T_N} \mathcal{F}_{-}(x)\,dx < 0$$

and certain other conditions are met, then a distribution solution $u \in W^{1,2}(T_N)$ of $\mathcal{Q}u = f(x,u)$ exists. We also show that this is the best possible result.

In §2 of Chapter 5, we also handle the equation

$$\mathcal{Q}u = g(u) - h(x)$$

and define

$$\lim_{s\to\infty} g(s) = g(\infty) \text{ and } \lim_{s\to-\infty} g(s) = g(-\infty).$$

We show that if certain other assumptions are met, then the condition

$$(2\pi)^N g(\infty) < \int_{T_N} h(x)\,dx < (2\pi)^N g(-\infty)$$

is both necessary and sufficient that a distribution solution $u \in W^{1,2}(T_N)$ of $\mathcal{Q}u = g(u) - h(x)$ exists.

In §1 of Chapter 6, we handle the stationary Navier-Stokes equations with a nonlinear driving force:

$$-\nu\Delta\mathbf{v}(x) + (\mathbf{v}(x)\cdot\nabla)\,\mathbf{v}(x) + \nabla p(x) = \mathbf{f}(x,\mathbf{v}(x))$$

$$(\nabla\cdot\mathbf{v})(x) = 0$$

where ν is a positive constant, and $\mathbf{v}$ and $\mathbf{f}$ are vector-valued functions.

In particular, $\mathbf{f} = (f_1, ..., f_N) : T_N \times \mathbf{R}^N \to \mathbf{R}^N$. We set

$$E_j(\mathbf{f}) = \{x \in T_N : \limsup_{|s_j|\to\infty} f_j(x,\mathbf{s})/s_j < 0$$

uniformly for $s_k \in \mathbf{R}, k \neq j,\ k = 1, ..., N\}$

and show that if certain other assumptions are met, then

$$|E_j(\mathbf{f})| > 0 \quad \mathit{for} \quad j = 1, ..., N,$$

is a sufficient condition for the pair $(\mathbf{v}, p)$ to be a distribution solution of the stationary Navier-Stokes equations with $v_j \in W^{1,2}(T_N)$ and $p \in L^1(T_N)$. Here, $|E_j(\mathbf{f})|$ represents the Lebesgue measure of $E_j(\mathbf{f})$. We also demonstrate that this is the best possible result.

Another theorem that we establish in §1 of Chapter 6 handles the situation when

$$f_j(x,\mathbf{s}) = g_j(s_j) - h_j(x).$$

In particular, we prove that if certain other conditions are met, then

$$(2\pi)^N g_j(\infty) < \int_{T_N} h_j(x)\,dx < (2\pi)^N g_j(-\infty)$$

for $j = 1, ...N$, is both a necessary and sufficient condition that the pair $(\mathbf{v}, p)$ be a distribution solution of the stationary Navier-Stokes equations with $v_j \in W^{1,2}(T_N)$ and $p \in L^1(T_N)$.

In §2 of Chapter 6, we deal with the classical solutions of the stationary Navier-Stokes equations. The main tool for proving the theorem involved is the C^α-theory of Calderon and Zygmund established earlier in Chapter 2.

Given $\mathbf{f} \in [C(T_N)]^N$, we will say the pair $(\mathbf{v}, p)$ is a periodic classical solution of the stationary Navier-Stokes system provided:

$$\mathbf{v} \in \left[C^2(T_N)\right]^N \text{ and } p \in C^1(T_N)$$

and

$$-\nu\Delta\mathbf{v}(x) + (\mathbf{v}(x)\cdot\nabla)\mathbf{v}(x) + \nabla p(x) = \mathbf{f}(x) \qquad \forall x \in T_N$$

$$(\nabla\cdot\mathbf{v})(x) = 0 \qquad \forall x \in T_N.$$

To obtain the classical solutions of the Navier-Stokes system, we require slightly more for the driving force $\mathbf{f}$ than periodic continuity. In particular, we say $f_1 \in C^\alpha(T_N)$, $0 < \alpha < 1$, provided the following holds:

(*i*) $f_1 \in C(T_N)$;
(*ii*) $\exists\, c_1 > 0$ s. t. $|f_1(x) - f_1(y)| \le c_1 |x - y|^{\alpha} \qquad \forall x, y \in \mathbf{R}^N$.

Working in dimension $N = 2$ or 3, we show in §2 of Chapter 6 that if

$$f_j \in C^\alpha(T_N), 0 < \alpha < 1 \text{ for } j = 1, ...N,$$

then there is a pair $(\mathbf{v}, p)$ which is a periodic classical solution of the stationary Navier-Stokes system with $v_j \in C^{2+\alpha}(T_N)$ and $p \in C^{1+\alpha}(T_N)$.

I have lectured on the mathematics developed in this book at various mathematical seminars at the University of California, Riverside, where I have been a professor for the last 45 years. Also, I would like to thank my colleague James Stafney for the many discussions that we have had about spherical harmonics and related matters.

I had the good fortune to write my doctoral thesis with Antoni Zygmund at the University of Chicago. Also, I did post-doctoral work with Arne Beurling at the Institute for Advanced Study and with Salomon Bochner from Princeton University. My subsequent mathematical work was backed by Marston Morse from the Institute for Advanced Study. I am indebted to these four outstanding mathematicians.

Victor L. Shapiro
Riverside, California
January, 2010

Contents

CHAPTER 1

Summability of Multiple Fourier Series

1. Introduction

We shall operate in real N-dimensional Euclidean space, $\mathbf{R}^N$, $N \geq 1$, and use the following notation:

$$\begin{aligned} x &= (x_1, ..., \ x_N) \qquad y = (y_1, ..., \ y_N) \\ \alpha x + \beta y &= (\alpha x_1 + \beta y_1, ..., \ \alpha x_N + \beta y_N) \\ x \cdot y &= x_1 y_1 + ... + x_N y_N \ , \qquad |x| = (x \cdot x)^{\frac{1}{2}}. \end{aligned}$$

With T_N, the N-dimensional torus,

$$T_N = \{x : -\pi \leq x_j < \pi, \ j = 1, ..., N\} \, ,$$

we shall say $f \in L^p(T_N), 1 \leq p < \infty$, provided f is a real-valued (unless explicitly stated otherwise) Lebesgue measurable function defined on $\mathbf{R}^N$ of period 2π in each variable such that

$$\int_{T_N} |f|^p \, dx < \infty.$$

A similar definition prevails for $f \in L^\infty(T_N)$.

With m as an integral lattice point in $\mathbf{R}^N$ and Λ_N representing the set of all such points, we shall designate the series

$$\sum_{m \in \Lambda_N} \widehat{f}(m) e^{im \cdot x}$$

by $S[f]$ and call it the Fourier series of f where

$$\widehat{f}(m) = (2\pi)^{-N} \int_{T_N} e^{-im \cdot x} f(x) dx.$$

In this chapter, we study the relationship between f and its Fourier series $S[f]$.

To begin, we let $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2$ be the usual Laplace operator and observe that $\Delta e^{im \cdot x} = -|m|^2 e^{im \cdot x}$. Consequently, from an eigenvalue point of view, it is natural to ask, "In what manner does the series

$$\sum_{n=0}^{\infty} \left(\sum_{|m|^2 = n} \widehat{f}(m) e^{im \cdot x} \right) \tag{1.1}$$

approximate f?" Bearing in mind the classical counter-examples of both Fejer and Lebesgue concerning the convergence of one-dimensional Fourier series, [Zy1, Chapter 8], we see that the answer to the previous question should involve some spherical summability method of the series given in (1.1).

The two most natural methods involving spherical summability are those of Bochner-Riesz and Abel. In particular, we say that $S[f]$ is Bochner-Riesz summable of order α, henceforth designated by $(B-R,\alpha)$ to $f(x)$ if

$$\lim_{R\to\infty}\sum_{|m|\le R}\widehat{f}(m)e^{im\cdot x}(1-|m|^2/R^2)^\alpha = f(x). \tag{1.2}$$

Bochner-Riesz summability plays the same role for multiple Fourier series that Cesaro summability plays for one-dimensional Fourier series. In §3 of this chapter, we shall establish a fundamental result for Bochner-Riesz summability of Fourier series.

$S[f]$ is Abel summable to $f(x)$, this means that the

$$\lim_{t\to 0}\sum_{m\in\Lambda_N}\widehat{f}(m)e^{im\cdot x-|m|t} = f(x). \tag{1.3}$$

The reason for calling this method of summability Abel summability is motivated by the fact that the series

$$\sum_{m\in\Lambda_N}\widehat{f}(m)e^{im\cdot x-|m|t}$$

is harmonic in $\mathbf{R}_+^{N+1}$, i.e., in the variables (x,t) for $t>0$.

We shall discuss Abel summability in detail in §4 of this chapter. Also, in Chapter 2, we shall deal with the Abel summability of conjugate multiple Fourier series. But first, it turns out that we can get some very good global results connecting f and $S[f]$ by iterating well-known one-dimensional results involving the Fejer kernel, and we will now show this iteration.

2. Iterated Fejer Summability of Fourier Series

We leave $D_n(t)$ as the well-known one-dimensional Dirichlet kernel

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin(n+\frac{1}{2})t}{\sin(t/2)}, \tag{2.1}$$

and $K_n(t)$ as the well-known one-dimensional Fejer kernel [Ru1, p. 199],

$$K_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(t) = \frac{1}{n+1}\,\frac{1-\cos(n+1)t}{1-\cos t}. \tag{2.2}$$

We also observe from [Ru1, p. 199] that $K_n(t)$ has the following three properties:

$$
(2.3) \qquad
\begin{aligned}
&(a)\ K_n(t) \geq 0 \quad \forall t \in \mathbf{R}, \\
&(b)\ \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} K_n(t)dt = 1, \\
&(c)\ K_n(t) \leq \tfrac{1}{n+1}\ \tfrac{2}{1-\cos\delta} \quad \text{if } \ 0 < \delta \leq |t| \leq \pi.
\end{aligned}
$$

It follows from (2.1) and (2.2) that

$$
(2.4) \qquad K_n(t) = \sum_{j=-n}^{n} e^{ijt}(1 - \frac{|j|}{n+1}),
$$

and we shall refer to

$$
(2.5) \qquad K_n^{\Diamond}(x) = K_n(x_1)\cdots K_n(x_N)
$$

as the iterated N-dimensional Fejer kernel.

For $f \in L^1(T_N)$ with $S[f]$ as its Fourier series, we shall prove three global theorems involving $K_n^{\Diamond}(x)$ and the iterated Fejer summability of $S[f]$. In particular, we call $\sigma_n^{\Diamond}(f,x)$ the iterated Fejer partial sum of $S[f]$ where $m = (m_1, ..., m_N)$ and

$$
(2.6) \quad \sigma_n^{\Diamond}(f,x) = \sum_{m_1=-n}^{n} \cdots \sum_{m_N=-n}^{n} \widehat{f}(m)e^{im\cdot x}(1 - \frac{|m_1|}{n+1})\cdots(1 - \frac{|m_N|}{n+1}).
$$

With $f \in C(T_N)$ signifying that f is a real-valued continuous function defined on $\mathbf{R}^N$ of period 2π in each variable and with $B(x,r)$ designating the open N-ball with a center x and radius r, the first theorem we shall prove is the following:

Theorem 2.1. *Let $f \in C(T_N)$ and suppose $\sigma_n^{\Diamond}(f,x)$ is defined as in (2.6). Then*

$$
\lim_{n\to\infty} \sigma_n^{\Diamond}(f,x) = f(x) \quad \textit{uniformly for } x \in T_N.
$$

Proof of Theorem 2.1. We observe from (2.3)-(2.6) that

$$
(2.7) \qquad \sigma_n^{\Diamond}(f,x) - f(x) = (2\pi)^{-N} \int_{T_N} [f(x-y) - f(x)]K_n^{\Diamond}(y)dy.
$$

Let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that $|f(x-y) - f(x)| < \varepsilon$ for $y \in B(0,\delta)$ uniformly for $x \in T_N$. Now it is clear that $Cu(0, \frac{\delta}{N}) \subset B(0,\delta)$ where $Cu(0, \frac{\delta}{N})$ is the open N-cube with center 0 and a half-side δ/N. So

$$
(2.8) \quad |f(x-y) - f(x)| < \varepsilon \quad \text{for} \quad y \in Cu(0, \frac{\delta}{N}) \quad \text{uniformly for } \ x \in T_N.
$$

Designating $P_{1,\delta}^{+}$ as the rectangular parallelopiped

$$
P_{1,\delta}^{+} = \{x : \delta \leq x_1 \leq \pi,\ \text{-}\pi \leq x_j \leq \pi,\ j = 2, ..., N.\},
$$

we see from (2.3) that $\lim_{n\to\infty}\int_{P^{+}_{1,\delta/N}}\left|K_n^{\diamondsuit}(y)\right|dy=0$. Since $T_N\backslash Cu(0,\frac{\delta}{N})$ is covered by a finite number of parallelopipeds similar to $P^{+}_{1,\delta/N}$, we conclude that

$$\lim_{n\to\infty}\int_{T_N\backslash Cu(0,\frac{\delta}{N})}\left|K_n^{\diamondsuit}(y)\right|dy=0.$$

Since $f(x)$ is uniformly bounded on $\mathbf{R}^N$, we also see from this last limit that n_0 can be chosen so large that

$$(2.9)\quad (2\pi)^{-N}\int_{T_N\backslash Cu(0,\frac{\delta}{N})}|f(x-y)-f(x)|\left|K_n^{\diamondsuit}(y)\right|dy\leq\varepsilon\quad\text{for}\quad n\geq n_0$$

uniformly for $x\in T_N$.

Next, returning to (2.8), we obtain from (2.3)(b) that

$$\begin{aligned}\int_{Cu(0,\frac{\delta}{N})}|f(x-y)-f(x)|\left|K_n^{\diamondsuit}(y)\right|dy\quad&\leq\quad\varepsilon\int_{T_N}\left|K_n^{\diamondsuit}(y)\right|dy\\&\leq\quad\varepsilon\,(2\pi)^N\qquad\forall n\end{aligned}$$

uniformly for $x\in T_N$.

Hence, (2.7) and this last fact joined with (2.9) shows that

$$\left|\sigma_n^{\diamondsuit}(f,x)-f(x)\right|\leq 2\varepsilon\quad\text{ for }\ n\geq n_0\quad\text{uniformly for } x\in T_N,$$

which gives the conclusion to the theorem. ■

The second summability theorem that we obtain using the N-dimensional iterated Fejer kernel is the following:

Theorem 2.2. *Let* $f\in L^p(T_N)$, $1\leq p<\infty$ *and suppose* $\sigma_n^{\diamondsuit}(f,x)$ *is defined as in (2.6). Then*

$$\lim_{n\to\infty}\int_{T_N}\left|\sigma_n^{\diamondsuit}(f,x)-f(x)\right|^p dx=0.$$

Proof of Theorem 2.2. We prove this for the case $1<p<\infty$, with a similar proof prevailing for the case $p=1$. From (2.7) with $p^{-1}+p'^{-1}=1$, we see that

$$\left|\sigma_n^{\diamondsuit}(f,x)-f(x)\right|\leq(2\pi)^{-N}\int_{T_N}|f(x-y)-f(x)|\left|K_n^{\diamondsuit}(y)\right|^{p^{-1}+p'^{-1}}dy,$$

and hence from Holder's inequality and (2.3)(b) that

(2.10)

$$\int_{T_N}\left|\sigma_n^{\diamondsuit}(f,x)-f(x)\right|^p dx\leq\int_{T_N}\left|K_n^{\diamondsuit}(y)\right|[\int_{T_N}|f(x-y)-f(x)|^p\,dx]dy.$$

Now $f \in L^p(T_N)$ and is also periodic of period 2π in each variable. Therefore, it follows that given $\varepsilon > 0$, $\exists \delta > 0$,

$$\int_{T_N} |f(x-y) - f(x)|^p \, dx \leq \varepsilon(2\pi)^{-N} \quad \text{for} \quad y \in B(0,\delta).$$

Consequently, we obtain from (2.10) that

$$\int_{T_N} \left|\sigma_n^{\Diamond}(f,x) - f(x)\right|^p dx \leq \int_{T_N - B(0,\delta)} \left|K_n^{\Diamond}(y)\right| [\int_{T_N} |f(x-y) - f(x)|^p \, dx] dy + \varepsilon$$

But $f \in L^p(T_N)$ implies that the inner integral on the right-hand side of the above inequality is uniformly bounded. Therefore, since $Cu(0, \frac{\delta}{N}) \subset B(0,\delta)$, we infer from the limit above (2.9) and the above inequality that

$$\limsup_{n\to\infty} \int_{T_N} \left|\sigma_n^{\Diamond}(f,x) - f(x)\right|^p dx \leq \varepsilon.$$

Since ε >0 is arbitrary, this gives the conclusion to the theorem. ■

Theorem 2.2 has three important corollaries, the first of which is the following:

Corollary 2.3. $\left\{e^{im\cdot x}\right\}_{m\in\Lambda_N}$, *the trigonometric system, is a complete orthogonal system for* $L^1(T_N)$, *i.e., if f,g*$\in L^1(T_N)$ *and* $\widehat{f}(m) = \widehat{g}(m)$ *for every integral lattice point* m, *then f(x)=g(x) a.e. in* T_N.

Proof of Corollary 2.3. Since $f, g \in L^1(T_N)$ and $\widehat{f}(m) = \widehat{g}(m)$ for every integral lattice point m, it implies that $\sigma_n^{\Diamond}(f,x) = \sigma_n^{\Diamond}(g,x)$ $\forall x \in T_N$ and $\forall n$. Consequently, it follows from Theorem 2.2 that

$$\int_{T_N} |f(x) - g(x)| \, dx = 0,$$

which establishes the corollary. ■

The next corollary that we shall prove is called the Riemann-Lebesgue lemma and is the following:

Corollary 2.4 *If f*$\in L^1(T_N)$, *then* $\lim_{|m|\to\infty} \widehat{f}(m) = 0$.

Proof of Corollary 2.4. Let $\varepsilon > 0$ be given. Using Theorem 2.2, choose an n sufficiently large so that $\int_{T_N} \left|\sigma_n^{\Diamond}(f,x) - f(x)\right| dx < \varepsilon$. Then, it follows from the definition of $\widehat{f}(m)$ given above (1.1) that

$$\begin{aligned}\left|\widehat{f}(m)\right| &\leq (2\pi)^{-N}\{\int_{T_N}\left|\sigma_n^{\Diamond}(f,x)-f(x)\right|dx+\left|\int_{T_N}e^{-im\cdot x}\sigma_n^{\Diamond}(f,x)\;dx\right|\},\\ &\leq \varepsilon\;+(2\pi)^{-N}\left|\int_{T_N}e^{-im\cdot x}\sigma_n^{\Diamond}(f,x)\;dx\right|.\end{aligned}$$

Since $\sigma_n^{\Diamond}(f,x)$ is a fixed trigonometric polynomial, it follows that there is a positive number s_0 such that the integral in the second inequality is zero for $|m|\geq s_0$. We conclude that $\left|\widehat{f}(m)\right|\leq\varepsilon$ for $|m|\geq s_0$, which establishes the corollary. ■

The third corollary that we can obtain from Theorem 2.2 is called Parsevaal's theorem and is the following:

Corollary 2.5. *If* $f\in L^2(T_N)$, *then*

$$\lim_{n\to\infty}(2\pi)^N\sum_{m_1=-n}^{n}\cdots\sum_{m_N=-n}^{n}\left|\widehat{f}(m)\right|^2=\|f\|_{L^2}^2\,.$$

Proof of Corollary 2.5. From Theorem 2.2, we see that

$$\lim_{n\to\infty}\left\|\sigma_n^{\Diamond}\right\|_{L^2}^2=\|f\|_{L^2}^2\,.$$

Also, we have that $\left\{\|\sigma_n^{\Diamond}\|_{L^2}^2\right\}_{n=1}^{\infty}$ is an increasing sequence, and the proof follows easily from this last observation. ■

The third summability theorem that we get using the N-dimensional iterated Fejer kernel is the following:

Theorem 2.6. *Let* $f\in L^\infty(T_N)$ *and suppose* $\sigma_n^{\Diamond}(f,x)$ *is defined as in (2.6). Then* $\sigma_n^{\Diamond}(f,x)\to f(x)$ *in the weak** L^∞*-topology, i.e.,*

$$\lim_{n\to\infty}\int_{T_N}\sigma_n^{\Diamond}(f,x)\;h(x)dx=\int_{T_N}f(x)\;h(x)dx\quad\forall h\in L^1(T_N).$$

Proof of Theorem 2.6. Let h be a given function in $L^1(T_N)$. Then it follows from Theorem 2.2 that

$$\lim_{n\to\infty}\int_{T_N}\left|\sigma_n^{\Diamond}(h,x)-h(x)\right|dx=0. \tag{2.11}$$

Next, we set

$$I_n=(2\pi)^{-N}\int_{T_N}\sigma_n^{\Diamond}(f,x)h(x)dx,$$

and observe from (2.6) that

$$\begin{aligned} I_n &= \sum_{m_1=-n}^{n} \cdots \sum_{m_N=-n}^{n} \widehat{f}(m)\widehat{h}(-m)(1-\frac{|m_1|}{n+1})\cdots(1-\frac{|m_N|}{n+1}) \\ &= \sum_{m_1=-n}^{n} \cdots \sum_{m_N=-n}^{n} \widehat{f}(-m)\widehat{h}(m)(1-\frac{|m_1|}{n+1})\cdots(1-\frac{|m_N|}{n+1}). \end{aligned}$$

Consequently,

$$\int_{T_N} \sigma_n^{\diamondsuit}(f,x)h(x)dx = \int_{T_N} \sigma_n^{\diamondsuit}(h,x)f(x)dx.$$

But then

$$\int_{T_N} [\sigma_n^{\diamondsuit}(f,x) - f(x)]h(x)dx = \int_{T_N} [\sigma_n^{\diamondsuit}(h,x) - h(x)]f(x)dx.$$

Hence,

$$\left|\int_{T_N} [\sigma_n^{\diamondsuit}(f,x) - f(x)]h(x)dx\right| \leq \|f\|_{L^\infty(T_N)} \int_{T_N} \left|\sigma_n^{\diamondsuit}(h,x) - h(x)\right| dx,$$

and the conclusion to the theorem follows immediately from the limit in (2.11). ■

Exercises.

1. With $D_n(t) = \sum_{j=-n}^{n} e^{ijt}$, use the well-known formula for geometric progressions and prove that

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{\sin(t/2)}.$$

2. With $K_n(t) = \frac{1}{n+1}\sum_{j=0}^{n} D_j(t)$, use the familiar formula $1-\cos\phi = 2\sin^2(\phi/2)$ and prove that

$$K_n(t) = \frac{1}{n+1}\,\frac{1-\cos(n+1)t}{1-\cos t}.$$

3. Prove that $K_n(t)$ has the following properties:

(a) $K_n(t) \geq 0 \quad \forall t \in \mathbf{R}$,

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt = 1$,

(c) $K_n(t) \leq \frac{1}{n+1}\,\frac{2}{1-\cos\delta}$ if $0 < \delta \leq |t| \leq \pi$.

4. Complete the proof of Corollary 2.5.

3. Bochner-Riesz Summability of Fourier Series

As we observed in the introduction to this chapter, $\Delta e^{im\cdot x} = -|m|^2 e^{im\cdot x}$ where Δ is the usual Laplace operator. Hence, from an eigenvalue point of view, since the eigenfunctions with the same eigenvalue have their integral lattice points lying on spheres, it is a good idea to study multiple Fourier series using spherical techniques. One of the most effective spherical technique is the method of Bochner-Riesz summation, defined previously in (1.2). With $B(x, r)$ representing the open N-ball with center x and radius r, the first theorem for this method of summation that we shall prove is the following due to Bochner [Boc1]:

Theorem 3.1. *Let* $f \in L^1(T_N)$ *and set*

$$\sigma_R^\alpha(f, x) = \sum_{|m|\le R} \widehat{f}(m) e^{im\cdot x}(1 - |m|^2/R^2)^\alpha. \tag{3.1}$$

Suppose that $|B(0,\rho)|^{-1} \int_{B(0,\rho)} |f(x_0 + x) - f(x_0)|\, dx \to 0$ *as* $\rho \to 0$. *Then*

$$\lim_{R\to\infty} \sigma_R^\alpha(f, x_0) = f(x_0) \quad \text{for} \quad \alpha > (N-1)/2.$$

We refer to $\sigma_R^\alpha(f, x)$ on the left-hand side of (3.1) as the R-th Bochner-Riesz mean of order α. Also, $|B(0,\rho)|$ designates the volume of the N-ball of radius ρ, which we shall now compute.

In order to make this computation, we introduce the N-dimensional spherical coordinate notation

$$\begin{aligned}
x_1 &= r\cos\theta_1\\
x_2 &= r\sin\theta_1\cos\theta_2\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3\\
&\ \ \vdots\\
x_{N-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\phi\\
x_N &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\phi
\end{aligned}$$

where $0 \le r < \rho$, $0 \le \theta_j \le \pi$ for $j = 1, \ldots, N-2$, and $0 \le \phi < 2\pi$.

We label the Jacobian of this transformation, $\mathcal{J}_N(r, \theta_1, \ldots, \theta_{N-2}, \phi)$. For example,

$$\mathcal{J}_3(r, \theta_1, \phi) = r^2 \begin{vmatrix} \cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1\cos\phi & \cos\theta_1\cos\phi & -\sin\theta_1\sin\phi\\ \sin\theta_1\sin\phi & \cos\theta_1\sin\phi & \sin\theta_1\cos\phi \end{vmatrix},$$

and an easy computation shows that $\mathcal{J}_3(r, \theta_1, \phi) = r^2\sin\theta_1$.

In a similar manner, we see that $\mathcal{J}_4(r,\theta_1,\theta_2,\phi)/r^3$ is going to be the determinant of the following array:

$$\begin{array}{cccc}
\cos\theta_1 & -\sin\theta_1 & 0 & 0 \\
\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 & 0 \\
\sin\theta_1\sin\theta_2\cos\phi & \cos\theta_1\sin\theta_2\cos\phi & \sin\theta_1\cos\theta_2\cos\phi & -\sin\theta_1\sin\theta_2\sin\phi \\
\sin\theta_1\sin\theta_2\sin\phi & \cos\theta_1\sin\theta_2\sin\phi & \sin\theta_1\cos\theta_2\sin\phi & \sin\theta_1\sin\theta_2\cos\phi.
\end{array}$$

Expanding this determinant using the first row, we observe that

$$\begin{aligned}
\mathcal{J}_4(r,\theta_1,\theta_2,\phi)/r^3 &= \cos^2\theta_1\sin^2\theta_1\mathcal{J}_3(r,\theta_2,\phi)/r^2 + \sin^4\theta_1\mathcal{J}_3(r,\theta_2,\phi)/r^2 \\
&= \sin^2\theta_1\mathcal{J}_3(r,\theta_2,\phi)/r^2 \\
&= \sin^2\theta_1\sin\theta_2.
\end{aligned}$$

Hence, $\mathcal{J}_4(r,\theta_1,\theta_2,\phi) = r^3\sin^2\theta_1\sin\theta_2$.

Continuing in this manner, we compute $\mathcal{J}_N(r,\theta_1,\ldots,\theta_{N-2},\phi)$ using induction and obtain

$$\mathcal{J}_N(r,\theta_1,\ldots,\theta_{N-2},\phi) = r^{N-1}(\sin\theta_1)^{N-2}\cdots(\sin\theta_{N-3})^2(\sin\theta_{N-2}). \tag{3.2}$$

Now, is well known,

$$|B(0,\rho)| = \int_0^\rho\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\mathcal{J}_N(r,\theta_1,\ldots,\theta_{N-2},\phi)d\phi d\theta_1\cdots d\theta_{N-2}dr. \tag{3.3}$$

Also, it is easy to see that

$$|B(0,\rho)| = \int_0^\rho r^{N-1}|S_{N-1}|\,dr = \rho^N|S_{N-1}|/N,$$

where S_{N-1} is the unit (N-1)-sphere in $\mathbf{R}^N$ and $|S_{N-1}|$ is its $(N-1)$-dimensional volume.

In particular, we see from (3.2) and (3.3) that

$$\begin{aligned}
|S_{N-1}| &= 2\pi\int_0^\pi\cdots\int_0^\pi(\sin\theta_1)^{N-2}\cdots(\sin\theta_{N-2})d\theta_1\cdots d\theta_{N-2} \\
&= 2\pi\prod_{j=1}^{N-2}\int_0^\pi(\sin\theta)^j d\theta.
\end{aligned}$$

From [Ti1, p. 56], we obtain

$$\int_0^\pi(\sin\theta)^j d\theta = \Gamma(\frac{j+1}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{j+2}{2}).$$

Consequently, it follows from this last calculation that

$$|S_{N-1}| = 2\pi[\Gamma(\frac{1}{2})]^{N-2}/\Gamma(\frac{N}{2}) = 2(\pi)^{N/2}/\Gamma(\frac{N}{2}),$$

and therefore that

$$|B(0,\rho)| = \frac{2(\pi)^{N/2}}{N\Gamma(\frac{N}{2})}\rho^N. \tag{3.4}$$

$|S_{N-1}|$ can also be computed from the following observation:

$$\begin{aligned}\int_{\mathbf{R}^N} e^{-|x|^2} dx &= |S_{N-1}| \int_0^\infty r^{N-1} e^{-r^2} dr \\ &= |S_{N-1}| \, 2^{-1} \int_0^\infty s^{\frac{N}{2}-1} e^{-s} ds \\ &= |S_{N-1}| \, \Gamma(\frac{N}{2})/2.\end{aligned}$$

Since, is well known, $\int_0^\infty e^{-t^2} dt = \frac{\pi^{\frac{1}{2}}}{2}$, the left-hand side of the above equality is $(\pi)^{N/2}$, and we obtain the same value for $|S_{N-1}|$ as we did before.

In order to prove Theorem 3.1, we shall need some lemmas. The first of such lemmas is concerned with the Bochner-Riesz summability of Fourier integrals. In particular, if $g \in L^1(\mathbf{R}^N)$ and is complex-valued, we designate the Fourier transform of g by $\widehat{g}$ and define it in a manner analogous to the one used for the Fourier coefficients of a function in $L^1(T_N)$, namely,

$$\widehat{g}(y) = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{-iy\cdot x} g(x) dx.$$

The first lemma we prove is the following:

Lemma 3.2. *Let $g \in L^1(\mathbf{R}^N)$ and be complex-valued. Set*

$$\tau_R^\alpha(g,x) = \int_{B(0,R)} \widehat{g}(y) e^{ix\cdot y} (1 - |y|^2 / R^2)^\alpha \, dy. \tag{3.5}$$

Suppose that $|B(0,\rho)|^{-1} \int_{B(0,\rho)} |g(x_0 + x) - g(x_0)| \, dx \to 0$ as $\rho \to 0$. Then

$$\lim_{R\to\infty} \tau_R^\alpha(g, x_0) = g(x_0) \quad \textit{for} \quad \alpha > (N-1)/2.$$

Proof of Lemma 3.2. We will first prove a special case of the lemma, namely, when $g(x) = e^{-|x-x_0|^2}$. We start out by observing once again that $\int_0^\infty e^{-s^2} ds = \frac{\pi^{\frac{1}{2}}}{2}$, and from (1.12) in Appendix A that

$$\int_0^\infty e^{-s^2} \cos 2ts \; ds = \frac{\pi^{\frac{1}{2}}}{2} e^{-t^2}.$$

Hence, $\int_{-\infty}^\infty e^{-s^2} e^{-ist} ds = \pi^{\frac{1}{2}} e^{-\frac{t^2}{4}}$, and consequently

$$\underbrace{\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty}_{N} e^{-(x_1^2 + \cdots + x_N^2)} e^{-ix\cdot y} dx = \pi^{N/2} e^{-|y|^2/4} \quad \text{for y} \in \mathbf{R}^N. \tag{3.6}$$

On setting $2x = u$ and $y = 0$ in this last equation, we see that

$$\int_{\mathbf{R}^N} e^{-|u|^2/4} du = \pi^{N/2} 2^N. \tag{3.7}$$

We are now able to establish the lemma in the particular case when $g(x) = e^{-|x-x_0|^2}$. From (3.6), we obtain that

$$\begin{aligned}\widehat{g}(y) &= (2\pi)^{-N}\int_{\mathbf{R}^N} e^{-ix\cdot y}e^{-|x-x_0|^2}dx\\ &= (2\pi)^{-N}\int_{\mathbf{R}^N} e^{-iy\cdot(x+x_0)}e^{-|x|^2}dx\\ &= (2\pi)^{-N}e^{-iy\cdot x_0}\pi^{N/2}e^{-|y|^2/4}.\end{aligned}$$

Hence, $\widehat{g}(y) \in L^1(\mathbf{R}^N)$, and the equality in (3.7) together with this last value of $\widehat{g}(y)$ then implies that

$$\begin{aligned}\lim_{R\to\infty}\int_{B(0,R)} \widehat{g}(y)e^{iy\cdot x_0}(1-|y|^2/R^2)^\alpha\, dy &= (2\pi)^{-N}\pi^{N/2}\int_{\mathbf{R}^N} e^{-|y|^2/4}dy\\ &= (2\pi)^{-N}\pi^{N/2}\pi^{N/2}2^N\\ &= g(x_0).\end{aligned}$$

Therefore, the lemma is proved in the special case $g(x) = e^{-|x-x_0|^2}$.

From what we have just established, we can prove the lemma. Without loss of generality we can assume from the start that

$$g(x_0) = 0. \tag{3.8}$$

Otherwise, we could work with the function

$$h(x) = g(x) - g(x_0)e^{-|x-x_0|^2}.$$

In order to prove the lemma, we will need two estimates concerning Bessel functions that are established in Appendix A. The first estimate we need is

$$|J_\nu(t)| \le K_\nu t^\nu \text{ for } 0 < t \le 1 \text{ and } \nu > -\frac{1}{2}, \tag{3.9}$$

and the second is

$$|J_\nu(t)| \le K_\nu t^{-\frac{1}{2}} \text{ for } 1 \le t < \infty \text{ and } \nu > -1, \tag{3.10}$$

where K_ν is a positive constant. The estimates (3.9) and (3.10) correspond respectively to (2.1) and (2.2) in Appendix A.

Continuing with the proof of the lemma, we set

$$(2\pi)^N H_R^\alpha(x) = \int_{B(0,R)} e^{iy\cdot x}(1-|y|^2/R^2)^\alpha\, dy, \tag{3.11}$$

and observe from (3.5) and Fubini's theorem that

$$\tau_R^\alpha(g, x_0) = \int_{\mathbf{R}^N} g(x)H_R^\alpha(x-x_0)dx.$$

Hence,

$$\tau_R^\alpha(g, x_0) = \int_{\mathbf{R}^N} g(x+x_0)H_R^\alpha(x)dx. \tag{3.12}$$

In (1.11) in Appendix A, it is shown that

$$H_R^\alpha(x) = c(N,\alpha) J_{\frac{N}{2}+\alpha}(R|x|) R^{\frac{N}{2}-\alpha} / |x|^{\frac{N}{2}+\alpha}, \tag{3.13}$$

where $c(N,\alpha) = (2\pi)^{-N}\ \omega_{N-2} 2^\alpha \Gamma(\alpha+1) = 2^\alpha \Gamma(\alpha+1)/(2\pi)^{N/2}$. We therefore conclude from (3.12) and (3.13) that

$$\tau_R^\alpha(g, x_0) = c(N,\alpha) R^N \int_{\mathbf{R}^N} g(x+x_0) J_{\frac{N}{2}+\alpha}(R|x|)/(R|x|)^{\frac{N}{2}+\alpha} dx. \tag{3.14}$$

Next, we set

$$G(r) = \int_{B(0,r)} |g(x+x_0)|\, dx,$$

and observe from (3.8) and the hypothesis of the lemma that

$$\begin{aligned} &(i)\ G(r) = o(r^N) \quad \text{as } r \to 0, \\ &(ii)\ G(r) \text{ is uniformly bounded for } 0 < r < \infty, \\ &(iii)\ G(r) \text{ is absolutely continuous on } 0 < r < \infty, \\ &(iv)\ dG(r)/dr \geq 0 \text{ a.e. on } 0 < r < \infty. \end{aligned} \tag{3.15}$$

From the definition of $G(r)$ above, we see from (3.14) and (3.15)(iii) and (iv) that

$$|\tau_R^\alpha(g, x_0)| \leq c(N,\alpha) R^N \int_0^\infty \frac{dG(r)}{dr} \left| J_{\frac{N}{2}+\alpha}(Rr) \right| / (Rr)^{\frac{N}{2}+\alpha}\ dr. \tag{3.16}$$

Also, we see that the statements in (3.15) together with $\alpha > (N-1)/2$ imply that for any $\delta > 0$,

$$R^{N/2-(\alpha+\frac{1}{2})} \int_\delta^\infty r^{-(\frac{N}{2}+\alpha+\frac{1}{2})} dG(r)/dr\ dr = o(1) \text{ as } R \to \infty.$$

Hence, we obtain from (3.12) and (3.16) that

$$\limsup_{R\to\infty} |\tau_R^\alpha(g, x_0)| / c(N,\alpha) \leq R^N \int_0^\delta \frac{dG(r)}{dr} \frac{\left| J_{\frac{N}{2}+\alpha}(Rr) \right|}{(Rr)^{\frac{N}{2}+\alpha}}\ dr. \tag{3.17}$$

Next, given $\varepsilon > 0$ and using (3.15)(i), we choose δ, with $0 < \delta < 1$, so that

$$|G(r)| < \varepsilon r^N \qquad \text{for } 0 < r < \delta,$$

and observe after an integration by parts that

$$\limsup_{R\to\infty} R^{N/2-(\alpha+\frac{1}{2})} \int_{R^{-1}}^\delta r^{-(\frac{N}{2}+\alpha+\frac{1}{2})} dG(r)/dr\ dr \leq \varepsilon \frac{\alpha+\frac{1}{2}+\frac{N}{2}}{\alpha+\frac{1}{2}-\frac{N}{2}}.$$

So using (3.10) and this last computation, we obtain

$$\limsup_{R\to\infty} R^N \int_{R^{-1}}^\delta \frac{dG(r)}{dr} \frac{\left| J_{\frac{N}{2}+\alpha}(Rr) \right|}{(Rr)^{\frac{N}{2}+\alpha}}\ dr \leq \varepsilon K_{\frac{N}{2}+\alpha} \frac{\alpha+\frac{1}{2}+\frac{N}{2}}{\alpha+\frac{1}{2}-\frac{N}{2}}.$$

Also, using (3.9) and (3.15) (iii) and (iv), we see that

$$R^N \int_0^{R^{-1}} \frac{dG(r)}{dr} \left| J_{\frac{N}{2}+\alpha}(Rr) \right| / (Rr)^{\frac{N}{2}+\alpha} \, dr \leq \varepsilon K_{\frac{N}{2}+\alpha}$$

for R sufficiently large.

Hence, on writing the integral on the right-hand side of the inequality in (3.17) in the form $\int_0^\delta = \int_0^{R^{-1}} + \int_{R^{-1}}^\delta$, we see from these last two inequalities that

$$\limsup_{R\to\infty} \left|\tau_R^\alpha(g, x_0)\right| / c(N,\alpha) \leq \varepsilon K_{\frac{N}{2}+\alpha} \left(\frac{\alpha + \frac{1}{2} + \frac{N}{2}}{\alpha + \frac{1}{2} - \frac{N}{2}} + 1\right).$$

Since ε is an arbitrary positive number, we conclude that

$$\lim_{R\to\infty} \left|\tau_R^\alpha(g, x_0)\right| = 0,$$

which finishes the proof of the Lemma 3.2 because $g(x_0) = 0$. ■

The next lemma that we need for the proof of Theorem 3.1 is the following:

Lemma 3.3. *Let $S(x)$ be the trigonometric polynomial $\sum_{|m|\leq R_1} b_m e^{im\cdot x}$, i.e., $S(x){=}\sum_{m\in\Lambda_N} b_m e^{im\cdot x}$ where $b_m = 0$ for $|m| > R_1$. For $R{>}0$, set*

$$\sigma_R^\alpha(S, x) = \sum_{|m|\leq R} b_m e^{im\cdot x} (1 - |m|^2 / R^2)^\alpha.$$

Then for $\alpha > (N-1)/2$,

$$\sigma_R^\alpha(S, x) = c(N,\alpha) R^{N/2-\alpha} \int_{\mathbf{R}^N} S(y) \frac{J_{\frac{N}{2}+\alpha}(R\,|x-y|)}{|x-y|^{\frac{N}{2}+\alpha}} dy \tag{3.18}$$

where $c(N,\alpha)$ is the constant in (3.13).

Proof of Lemma 3.3. Define $\phi(t) = (1-t^2)^\alpha$, $0 \leq t \leq 1$, and $\phi(t) = 0$ for $t \geq 1$. Then since $S(x)$ is a finite linear combination of exponentials, it is clear that the lemma will follow if we can show that for fixed x and every $u \in \mathbf{R}^N$,

$$e^{iu\cdot x} \frac{\phi(|u|/R)}{c(N,\alpha)} = R^{N/2-\alpha} \int_{\mathbf{R}^N} e^{iu\cdot y} \frac{J_{\frac{N}{2}+\alpha}(R\,|x-y|)}{|x-y|^{\frac{N}{2}+\alpha}} dy. \tag{3.19}$$

Set $g(u) = e^{iu\cdot x}\phi(|u|/R)$. Then $g(u)$ is a continuous function which is also in $L^1(\mathbf{R}^N)$. If $\widehat{g}(y)$ is also in $L^1(\mathbf{R}^N)$, it follows from Lemma 3.2 and the Lebesgue dominated convergence theorem that

$$g(u) = \int_{\mathbf{R}^N} e^{iu\cdot y} \widehat{g}(y) dy.$$

For fixed x, (3.9) and (3.10) let

$$J_{\frac{N}{2}+\alpha}(R\,|x-y|)/\,|x-y|^{\frac{N}{2}+\alpha} \in L^1(\mathbf{R}^N) \text{ with respect to } y.$$

So (3.19) will be established if we show that

$$\frac{\widehat{g}(y)}{c(N,\alpha)} = R^{N/2-\alpha}\frac{J_{\frac{N}{2}+\alpha}(R\,|x-y|)}{|x-y|^{\frac{N}{2}+\alpha}}.$$

But from (3.11), we see that $\widehat{g}(y) = H_R^\alpha(x-y)$; this last fact follows from the equality in (3.13). ■

Proof of Theorem 3.1. We first observe from (3.9), (3.10), and (3.13) that there is a constant $K(\alpha, R)$ and an $\eta > 0$ such that for fixed R,

$$|H_R^\alpha(x)| \leq K(\alpha,R)/(1+|x|)^{N+\eta} \quad \text{for } x \in \mathbf{R}^N, \tag{3.20}$$

where $K(\alpha, R)$ is a constant depending on α and R. Consequently, the series

$$\sum_{m\in\Lambda_N} H_R^\alpha(x+2\pi m) = H_R^{*\alpha}(x)$$

is absolutely convergent, and furthermore

$$\lim_{R_1\to\infty} \sum_{|m|\leq R_1} H_R^\alpha(x+2\pi m) = H_R^{*\alpha}(x) \tag{3.21}$$

uniformly for x in a bounded domain.

Set $S^j(x) = \sigma_j^\Diamond(f,x)$, which is the trigonometric polynomial defined in (2.6). Then by (3.20), $S^j(y)H_R^\alpha(x-y) \in L^1(\mathbf{R}^N)$ with respect to y, and we obtain from Lemma 3.3 and (3.21) that for x in a bounded domain,

$$\begin{aligned}
\sigma_R^\alpha(S^j,x) &= \int_{\mathbf{R}^N} S^j(y)H_R^\alpha(x-y)dy \\
&= \lim_{R_1\to\infty} \sum_{|m|\leq R_1} \int_{T_N} S^j(y+2\pi m)H_R^\alpha(x-y-2\pi m)dy \\
&= \lim_{R_1\to\infty} \int_{T_N} S^j(y)\Big(\sum_{|m|\leq R_1} H_R^\alpha(x-y-2\pi m)\Big)dy \\
&= \int_{T_N} S^j(y)H_R^{*\alpha}(x-y)dy.
\end{aligned}$$

By Theorem 2.2, $S^j \to f$ in $L^1(T_N)$. Also, $H_R^{*\alpha} \in C(T_N)$. So from this last computation we can see by passing to the limit as $j \to \infty$, that

$$\sigma_R^\alpha(f,x) = \int_{T_N} f(y)H_R^{*\alpha}(x-y)dy. \tag{3.22}$$

But $f(y)$ is defined in $\mathbf{R}^N$ by periodicity of period 2π in each variable. So we see that

$$\int_{B(0,R_1+1)\setminus B(0,R_1)} |f(y)|\,dy = O(R_1^{N-1}) \text{ as } R_1 \to \infty. \tag{3.23}$$

This fact in conjunction with (3.20), implies that $f(y)H_R^\alpha(x-y) \in L^1(\mathbf{R}^N)$ with respect to y.

Hence, using (3.22), we can reverse the previous calculation and obtain

$$\sigma_R^\alpha(f,x_0) = \int_{\mathbf{R}^N} f(y)H_R^\alpha(x_0-y)dy = \int_{\mathbf{R}^N} f(x+x_0)H_R^\alpha(x)dx. \tag{3.24}$$

Since the theorem is obviously true if $f(x)$ is a constant function, we can prove the theorem, with no loss in generality, if we assume that $f(x_0)=0$. Therefore, from the hypothesis of the theorem,

$$\int_{B(0,r)} |f(x+x_0)|\,dx = o(r^N) \text{ as } r \to 0.$$

So using (3.15) and comparing (3.24) with (3.14), we see that locally the same proof will apply here as it was applied in the proof of Lemma 3.2. Consequently, to complete the proof of the theorem, we must to show that for fixed $\delta > 0$,

$$\lim_{R\to\infty} \int_{\mathbf{R}^N\setminus B(0,\delta)} f(x+x_0)H_R^\alpha(x)dx = 0. \tag{3.25}$$

Using (3.13) in conjunction with the estimate in (3.12), we see that

$$\left|\int_{\mathbf{R}^N\setminus B(0,\delta)} \frac{f(x+x_0)}{\lambda(N,\alpha)} H_R^\alpha(x)dx\right| \le R^{(N-1)/2-\alpha} \int_{\mathbf{R}^N\setminus B(0,\delta)} \frac{|f(x+x_0)|}{|x|^{\alpha+(N+1)/2}}dx$$

where $\lambda(N,\alpha) = c(N,\alpha)K_{\frac{N}{2}+\alpha}$ is a constant. Since $\alpha > (N-1)/2$, we see from (3.23) that the integral on the right-hand side of this last inequality is finite. Also we see that $(N-1)/2-\alpha$ is strictly negative. Consequently, the right-hand side of this last inequality is $o(1)$ as $R \to \infty$.

We conclude that the limit in (3.25) is indeed valid, and we complete the proof of Theorem 3.1. ■

$\alpha = (N-1)/2$ is called the critical index for Bochner-Riesz summability. What is very interesting about Theorem 3.1 is that it fails at the critical index for $N \ge 2$, even if $f=0$ in a neighborhood of x_0. Bochner has shown, in particular that with $0 < \delta < 1$,

$$\exists\ f \in L^1(T_N), N \ge 2, \text{with } f = 0 \text{ in } B(0,\delta) \tag{3.26}$$

such that
$\limsup_{R\to\infty} \left|\sigma_R^{(N-1)/2}(f,0)\right| = \infty$.

To see this ingenious counter-example, we refer the reader to [Boc, p. 193] or [Sh1, pp. 57-64].

It is clear from the Riemann-Lebesgue Lemma and the form of the Dirichlet kernel given in (2.1) that Bochner's counter-example itself does not hold when $N = 1$.

We close this section with the following corollary of Theorem 3.1:

Corollary 3.4. *Suppose* $f \in L^1(T_N)$. *Then for* $\alpha > (N-1)/2$,

$$\lim_{R\to\infty} \sigma_R^\alpha(x) = f(x) \qquad \textit{for a.e. } x \in T_N.$$

Proof of Corollary 3.4. Since almost every $x \in T_N$ is in the Lebesgue set of f (see page 22), Corollary 3.4 follows immediately from Theorem 3.1. ■

Exercises.

1. Show that Bochner's counter-example does indeed fail in dimension $N = 1$.

2. Find the third and fourth rows in the determinant corresponding to $\mathcal{J}_N(r, \theta_1, \dots, \theta_{N-2}, \phi)$ when $N = 5$.

3. By direct calculation, show that the following formula is true when $j = 3$:

$$\int_0^\pi (\sin\theta)^j d\theta = \Gamma(\frac{j+1}{2})\Gamma(\frac{1}{2})/\Gamma(\frac{j+2}{2}).$$

4. Given that $G(r)$ satisfies the conditions in (3.15) and that $\alpha > (N-1)/2$, $\delta > 0$ prove that

$$R^{N/2-(\alpha+\frac{1}{2})} \int_\delta^\infty r^{-(\frac{N}{2}+\alpha+\frac{1}{2})} dG(r)/dr \; dr = o(1) \text{ as } R \to \infty.$$

4. Abel Summability of Fourier Series

The Abel summability of Fourier series was defined in (1.3) of this chapter, and in this section, we shall prove three theorems regarding this method of summation. The first theorem we establish is an N-dimensional version of a well-known theorem in one dimension originally due to Fatou [Zy1, p. 100].

Theorem 4.1. *Let* $f \epsilon L^1(T_N)$, *and for* $t > 0$, *set*

$$A_t(f, x) = \sum_{m\in\Lambda_N} \widehat{f}(m) e^{im\cdot x - |m|t}.$$

Also, set

$$\beta^-(x) = \limsup_{r\to 0} \frac{\int_{B(x,r)} f(y)dy}{|B(x,r)|} \quad \textit{and} \quad \beta_-(x) = \liminf_{r\to 0} \frac{\int_{B(x,r)} f(y)dy}{|B(x,r)|} \ .$$

Then

$$\beta_-(x) \le \liminf_{t\to 0} A_t(f,x) \le \limsup_{t\to 0} A_t(f,x) \le \beta^-(x).$$

Of course, this theorem implies that in case $\beta^-(x) = \beta_-(x)$, then the Fourier series of f is Abel summable at x to this common value.

Proof of Theorem 4.1. To prove Theorem 4.1, we proceed in a manner similar to the proof given in Theorem 3.1. First, let $g\epsilon L^1(\mathbf{R}^N)$, and set

$$\mathcal{A}_t(g,x) = \int_{\mathbf{R}^N} \widehat{g}(y)e^{iy\cdot x-|y|t}dy \qquad \text{for } t>0, \tag{4.1}$$

where $\widehat{g}(y)$ is the Fourier transform of g and is defined above Lemma 3.2. Then, for $t>0$, by Fubini's theorem,

$$\mathcal{A}_t(g,x) = (2\pi)^{-N}\int_{\mathbf{R}^N} g(u)[\int_{\mathbf{R}^N} e^{iy\cdot(x-u)-|y|t}dy]du. \tag{4.2}$$

But, for $N\ge 2$,

$$\int_{\mathbf{R}^N} e^{iy\cdot(x-u)-|y|t}dy = |S_{N-2}|\int_0^\infty e^{-rt}r^{N-1}\int_0^\pi e^{i|x-u|r\cos\theta}(\sin\theta)^{N-2}d\theta.$$

Consequently,

$$\int_{\mathbf{R}^N} e^{iy\cdot(x-u)-|y|t}dy = \omega_{N-2}\int_0^\infty e^{-rt}r^{N-1}\frac{J_{(N-2)/2}(r\,|x-u|)}{(r\,|x-u|)^{(N-2)/2}}dr, \tag{4.3}$$

where we have made use of the integral identity in (1.5) in Appendix A and $\omega_{N-2} = (2\pi)^{N/2}$ is the constant defined below (1.11) in Appendix A.

For $N=1$, the equality in (4.3) continues to hold with $\omega_{-1} = (2\pi)^{1/2}$. This follows from a direct calculation that uses the well-known fact that

$$\cos t = (\pi/2)^{1/2}t^{1/2}J_{-1/2}(t) \quad \text{for } t>0.$$

Next, we use the integral identity (1.7) in Appendix A and conclude from the equality in (4.3) that

$$\int_{\mathbf{R}^N} e^{iy\cdot(x-u)-|y|t}dy = b_N t[t^2+|x-u|^2]^{-(N+1)/2}$$

where $b_N = (2)^{N/2}\Gamma(\frac{N+1}{2})\omega_{N-2}(\pi)^{-\frac{1}{2}}$.

This last equality, in conjunction with (4.2), establishes the useful fact that for $t>0$,

$$\mathcal{A}_t(g,x) = (2\pi)^{-N}b_N\int_{\mathbf{R}^N} g(y)t[t^2+|x-y|^2]^{-(N+1)/2}dy. \tag{4.4}$$

Next, we observe that the analog of Lemma 3.2 holds for $\mathcal{A}_t(g,x)$.

Also, we see that the analog of Lemma 3.3 holds, namely, if $S(x)$ is a trigonometric polynomial, then

$$\mathcal{A}_t(S,x) = (2\pi)^{-N} b_N \int_{\mathbf{R}^N} S(y) t[t^2 + |x-y|^2]^{-(N+1)/2} dy. \tag{4.4$'$}$$

To show that this is indeed the case, we need to only establish, as in the proof of Lemma 3.3, that

$$e^{iu \cdot x} e^{-|u|t}/b_N = (2\pi)^{-N} \int_{\mathbf{R}^N} e^{iu\cdot y} t[t^2 + |x-y|^2]^{-(N+1)/2} dy$$

for $u \in \mathbf{R}^N$ and $t > 0$. This equality will follow from the fact that the Fourier transform of $e^{iu\cdot x}e^{-|u|t}/b_N$ is

$$(2\pi)^{-N} t[t^2 + |x-y|^2]^{-(N+1)/2},$$

which is the statement three lines above (4.4) when u and y are interchanged.

Using the same technique that we used in the proof of Theorem 3.1 (i.e., see (3.25) through (3.27) in §3), to pass from Fourier integrals to Fourier series, we obtain from (4.4$'$) that for $f \in L^1(T_N)$,

$$A_t(f,x) = (2\pi)^{-N} b_N \int_{\mathbf{R}^N} f(x+y) t[t^2 + |y|^2]^{-(N+1)/2} dy. \tag{4.5}$$

To prove Theorem 4.1, it is sufficient to just establish the last inequality stated in the conclusion, namely,

$$\limsup_{t \to 0} A_t(f,x) \le \beta^-(x). \tag{4.6}$$

For then the first inequality follows from a consideration of $-f$.

If $\beta^-(x) = \infty$, (4.6) is established. So we need only consider the two cases: (i) $\beta^-(x)$ is finite, or (ii) $\beta^-(x) = -\infty$ in establishing (4.6). It is clear that the inequality in (4.6) will follow in both these cases if we show that the following holds for $\gamma \epsilon \mathbf{R}$:

$$\beta^-(x) < \gamma \Longrightarrow \limsup_{t \to 0} A_t(f,x) \le \gamma. \tag{4.7}$$

We now establish (4.7). To do this, first of all, we observe from (4.5) that $f(y)$ identically one implies that

$$(2\pi)^{-N} b_N t \int_{\mathbf{R}^N} [t^2 + |y|^2]^{-(N+1)/2} dy = 1 \quad \text{for} \quad t > 0. \tag{4.8}$$

Next, we set

$$f_{[r]}(x) = \frac{\int_{B(0,r)} f(x+y) dy}{|B(0,r)|}, \tag{4.9}$$

and use the hypothesis in (4.7) choose $\delta > 0$ so that

$$f_{[r]}(x) < \gamma \quad \text{for} \quad 0 < r < \delta. \tag{4.10}$$

Observing that $f(x+y)\,|y|^{-(N+1)}\ \epsilon L^1(\mathbf{R}^N\backslash B(0,\delta))$ with respect to y (because for fixed x, $f(x+y)\epsilon L^1(T_N)$ and is periodic of period 2π in each variable), we see that

$$\lim_{t\to 0} t\int_{\mathbf{R}^N\backslash B(0,\delta)} f(x+y)[t^2+|y|^2]^{-(N+1)/2}dy = 0.$$

Consequently, we obtain from (4.5) that

(4.11)

$$\limsup_{t\to 0} A_t(f,x) \leq (2\pi)^{-N}b_N \limsup_{t\to 0}\int_{B(0,\delta)} tf(x+y)[t^2+|y|^2]^{-(N+1)/2}dy.$$

From (4.9), we next observe that the integral on the right-hand side of this last inequality can be written as

$$t\int_0^\delta [t^2+r^2]^{-(N+1)/2}\frac{d[|B(0,r)|\, f_{[r]}(x)]}{dr}dr.$$

So we conclude from (4.9) and (4.10), after performing an integration by parts on this last integral, that

(4.12)

$$\limsup_{t\to 0} A_t(f,x) \leq \gamma(2\pi)^{-N}b_N \limsup_{t\to 0}(N+1)t\int_0^\delta r[t^2+r^2]^{-\frac{N+3}{2}}\,|B(0,r)|\,dr.$$

Likewise, after integrating by parts, we see from the identity in (4.8) that

$$(2\pi)^{-N}b_N \lim_{t\to 0}(N+1)t\int_0^\delta r[t^2+r^2]^{-\frac{N+3}{2}}\,|B(0,r)|\,dr = 1.$$

This last equality together with the inequality in (4.12) establishes the implication in (4.7) and concludes the proof to Theorem 4.1. ■

The next theorem that we establish involves the concept of nontangential Abel summability. With $x_0\epsilon\ \mathbf{R}^N$ and $\gamma > 0$, let $\mathcal{C}_\gamma(x_0)$ stand for the cone in $\mathbf{R}^{N+1}_+$ with vertex $(x_0,0)$ given as follows:

$$\mathcal{C}_\gamma(x_0) = \{(x,t) : t > 0 \text{ and } \frac{t}{|x-x_0|} \geq \gamma\ \}\ . \tag{4.13}$$

We say that the Fourier series of f, namely $S[f]$, is nontangentially Abel summable at x_0 to the limit l if for every $\gamma > 0$,

$$\lim_{(x,t)\to((x_0,0)} A_t(f,x) = l$$

where (x,t) tends to $(x_0,0)$ within the cone $\mathcal{C}_\gamma(x_0)$.

The nontangential Abel summability theorem that we shall present here is an improvement (for $N\geq 2$) over the usual one presented in books related to this subject (e.g., see [SW, p. 62]). In order to do this, we introduce the σ-set of f where $f\in L^1(T_N)$. We say x_0 is in the σ-set of f provided the following holds: $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $|x - x_0| < \delta$ and $r < \delta$ implies that

$$\left|\int_{B(x,r)} [f(y) - f(x_0)]dy\right| < \varepsilon(|x - x_0| + r)^N. \tag{4.14}$$

We prove the following theorem (see [Sh4]):

Theorem 4.2. *Let $f \epsilon L^1(T_N)$, and suppose that $x_0 \in \sigma$-set of f. Then $S[f]$ is nontangentially Abel summable at x_0 to $f(x_0)$.*

For $N = 1$, this result is the same as the result given in [Zy1, p. 61] which is evidently due to Fatou and states that if $F{=}\int f$ and F has a finite derivative equal to $f(x_0)$ (henceforth referred to as the Fatou condition at x_0), then nontangential Abel summability occurs at x_0. It is not difficult to show that for $N = 1$, $x_0 \in \sigma$-set of f if and only if the Fatou condition holds for f at x_0.

For $N \geq 2$, this result about $x_0 \in \sigma$-set of f has not appeared previously in any book and is due to the author (see [Sh 4]). The usual theorem proved is that if $x_0 \in$Lebesgue set of f, nontangential Abel summability occurs at x_0, [SW, p. 62]. After we prove the above theorem, we shall show $x_0 \in$ Lebesgue set of f implies that $x_0 \in \sigma$-set of f. Also, we shall give an example of an $f \in L^\infty(T_2)$ such that x_0 is not in the Lebesgue set of f, but x_0 is in the σ-set of f .

Proof of Theorem 4.2. To prove the theorem, it is easy to see from the start that we can assume that $x_0 = 0$. Therefore, to prove the theorem, we assume that $\gamma > 0$ and that $\{(x_n, t_n)\}_{n=1}^\infty \subset \mathcal{C}_\gamma(0)$ with $x_n \to 0$ and $t_n \to 0$. The proof will be complete when we show that

$$\lim_{n\to\infty} A_{t_n}(f, x_n) = f(0). \tag{4.15}$$

Given $\varepsilon > 0$, it is clear that the limit in (4.15) will follow if we show that

$$\limsup_{n\to\infty} \left|\frac{A_{t_n}(f, x_n) - f(0)}{(2\pi)^{-N} b_N}\right| \leq 2(N + 1)\varepsilon(\frac{1}{\gamma} + 1)^N. \tag{4.16}$$

It follows from (4.5) and (4.8) in the proof of Theorem 4.1 that

$$\frac{A_{t_n}(f, x_n) - f(0)}{(2\pi)^{-N} b_N} = \int_{\mathbf{R}^N} [f(x_n + y) - f(0)]t_n[t_n^2 + |y|^2]^{-(N+1)/2}dy$$

where $b_N = (2)^{N/2}\Gamma(\frac{N+1}{2})\omega_{N-2}(\pi)^{-\frac{1}{2}}$ and $\omega_{N-2} = (2\pi)^{N/2}$. Hence, the inequality in (4.16) will follow if we show that

(4.17)

$$\limsup_{n\to\infty} \left|\int_{\mathbf{R}^N} [f(x_n + y) - f(0)]t_n[t_n^2 + |y|^2]^{-(N+1)/2}dy\right| \leq 2(N{+}1)\varepsilon(\frac{1}{\gamma}{+}1)^N.$$