SERIES IN OPTICS AND OPTOELECTRONICS

LIGHT

The Physics of the Photon

Ole Keller

CRC Press
Taylor & Francis Group
A TAYLOR & FRANCIS BOOK

LIGHT

The Physics of the Photon

LIGHT

The Physics of the Photon

Ole Keller
Aalborg University, Denmark

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A TAYLOR & FRANCIS BOOK

In memory of my mother, Cecilie Marie Keller

Cover image: Courtesy of Esben Hanefelt Kristensen, based on a painting entitled "A Wordless Statement."

Taylor & Francis
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20140428

International Standard Book Number-13: 978-1-4398-4044-3 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Contents

Preface

I have often been asked *what is a photon?* In order to attempt to answer this question, as communicating human beings, above all we must learn how to use the word *is* in an unambiguous manner. The learning process takes us on a journey into deep philosophical questions, and many of us end up being bewildered before we finally are snowed under with philosophical thinking. In my understanding, the *is*-problem is like a Gordian knot. In physics we replace the word *is* by *characterizes*, although in everyday discussion among physicists we do not need to distinguish between *is* and *characterizes*, in general. So, I take the liberty to replace the original question with *what characterizes a photon?* If someone asks you who *is* this person you will "only" be able to answer by mentioning as many features, traits, etc., as you are aware of about the given person. In a sense, a good characterization of a *phenomenon* in physics means to look at the phenomenon from various perspectives (through different windows). In the case of the photon, we approach the original question *what is a photon* by looking at the phenomenon through as many windows as possible. Only in the never attainable limit, where the number (N) of windows [photon perspectives (PP)] approaches infinity, has one captured the photon phenomenon, at least in my understanding. Mathematically,

$$\text{Observational possibility} \equiv \sum_{i=1}^{N} (PP)_i$$
$$\Rightarrow \sum_{i=1}^{\infty} (PP)_i \equiv \text{The photon phenomenon.}$$

In this book I take a look at the photon phenomenon from a personal selection of a few perspectives. The insight obtained by looking through some of the windows may already be familiar to the reader.

Above I have made use of the word *phenomenon*, and replaced *photon* with *photon phenomenon*. The concept phenomenon was introduced in the physical literature by Niels Bohr, and the definition he first formulated publicly at a meeting in Warsaw in 1938, arranged by the International Institute of Intellectual Co-operation of the League of Nations. Niels Bohr, one of the monumental figures in the establishment of quantum mechanics, throughout his life, with ever-increasing force of the argument, emphasized that we must learn to use the words of the common language in an unambiguous manner, because after all, we as physicists essentially have only the common language when we discuss with each other what we have learned in our field of study. According to Bohr, *no elementary quantum phenomenon is a phenomenon until it is a registered (observed) phenomenon*. For Bohr quantum mechanics was a rational generalization of classical physics, and his definition of the phenomenon concept made it possible to unite the seemingly incompatible *particle* and *wave* aspects of the photon phenomenon, e.g., the single- and double-slit experiments with photons. Bohr's phenomenon concept, as well as another of his central points, viz., that the functioning of the measuring apparatus always must (and only can) be described in the language of classical physics, will be important for us to remember. To Bohr, every atomic phenomenon is closed in the sense that its observation is based on registrations obtained by means of suitable macroscopic devices (with irreversible functioning). Bohr considered the closure of fundamental significance not only in quantum physics but in the whole description of nature, and he often stressed in discussions that "reality" is a word in our language and that we must learn to use it correctly. Kalckar, in the 1967 book *Niels Bohr: His Life and Work as Seen by His Friends and Colleagues* (edited by S. Rozental) quoted Bohr for the following statement: *I am quite prepared to talk of the spiritual life of an electronic computer, to say that it is considering or that it is in a bad mood. What really matters is the unambiguous description of its behaviour, which is what we observe. The question as to whether the machine really feels, or whether it merely looks as though it did, is absolutely as meaningless as to ask whether light is "in reality" waves or particles. We must never forget that "reality" too is a human word just like "wave" or "consciousness." Our task is to learn to use these words correctly – that is, unambiguously and consistently.* It will be well to remember the fundamental (central) points of Niels Bohr throughout the reading of this book.

Notwithstanding the fact that field–matter interaction is needed for a *photon* to appear as a *photon phenomenon*, it is nevertheless indispensable to study the photon as a concept belonging to global vacuum (matter-free space). Although the photon of the vacuum is an abstraction of our mind, the photon concept must be firmly connected to the electromagnetic field concept in free space. The autonomy of the classical electromagnetic field in free space is solely connected with the vacuum speed of light (c): The classical electromagnetic field is an intermediary describing the delayed (with speed c) interaction between electrically charged particles in nonuniform motion. Although there is no room for accommodating the photon concept in the framework of classical electrodynamics, it is of value to investigate how far one may proceed toward the introduction of a classical light particle concept in a classical framework. The autonomy of the electromagnetic field increases in an essential manner with the introduction of the quantum of action (Planck's constant, h) in electrodynamics. The photon concept then flourishes, and the photon-free vacuum appears with its own autonomy. The modern era of the light particle (based on h) began when Einstein in 1905 concluded that *monochromatic* (*frequency*: ν) *radiation of low density (within the range of validity of Wien's radiation formula) behaves thermodynamically as though it consisted of a number of independent energy quanta of magnitude* $h\nu$.

In Part I, we prepare ourselves for photon physics by studying certain aspects of classical optics in a global vacuum on the basis of the free-space Maxwell equations. Since the photon in global vacuum (T-photon) is a transversely (T) polarized object belonging to the positive-frequency part of the electromagnetic spectrum, studies of transverse (longitudinal) vector fields, complex analytical signals, and the various polarization states of light are central. With an eye to the point-like Einstein light particle we also describe how the electromagnetic field can be resolved into a complete set of wave-packet modes. Because the massless photon necessarily is a relativistic object propagating with the vacuum speed of light, it is important to consider the fields of classical optics from the perspective of special relativity. Our brief account of optics in special relativity culminates with a demonstration of the manifest covariance of the Maxwell equations, and a discussion of the Lorentz transformation of the transverse and longitudinal parts of the electromagnetic field.

In Part II, we study light rays and geodesics, and we also present a brief account of the Maxwell theory in general relativity. In the framework of classical electrodynamics there is no hope for considering light as consisting of some sort of particles, in general. This is so because (wave) interference effects cannot occur in classical particle dynamics. In a corner of the classical field theory, known as geometrical optics, the wavelength (λ) of light plays no role; however, in the short wavelengths limit and here ($\lambda \to 0$) a geometrization of the field description in the form of light rays appears. The eikonal equation is the basic equation of geometrical optics. A classical particle moves along a trajectory, and in the framework of geometrical optics it makes sense to reflect on whether a kind of *approximate* light particle theory can be established in which the particle follows a trajectory (light ray) according to the possibilities inherent in the eikonal theory. From a somewhat different perspective a light ray appears as a geodetic line for particle motion. The equation for the geodetic line is obtained from a variational principle which also gives one Fermat's principle. Although it is not meaningless to consider a light ray as a particle trajectory, it is not possible to extend the formalism in such a manner that it describes the motion of a light particle which is spatially well-localized somewhere on the ray at a given time. The geodesic principle can be generalized to general relativity, and the "light particle" here propagates along null geodesics. On the basis of the principle of equivalence, the geodesic approach leads to the conclusion that the gravitational field may shift the frequency of a locally monochromatic light beam along the geodetic line. This so-called gravitational redshift can be understood from a somewhat different perspective that relates to quantum theory, viz., as a monochromatic photon in free fall in a gravitational field. It is possible to go beyond the geometrical optical approximation in general relativity, and establish an extension of the Maxwell–Lorentz theory to curved space-time. A beautiful reformulation of the basic theory allows one to present the Maxwell–Lorentz theory in general relativity in a form formally identical to that of macroscopic electrodynamics. Thus, the role of the metric tensor is reflected via effective permittivity and permeability tensors. In the quantum theory of the photon the scalar and vector potentials play a central role, and for this reason alone it is important that the possibilities for establishing a potential description of electrodynamics in curved space-time is presented to the reader.

In Part III, the theory of photon wave mechanics is discussed. The wave mechanical picture of light *partly* is based on a reinterpretation of the content of the free Maxwell equations. In this book, the properly normalized transverse part vector potential, a gauge-invariant quantity, is considered as the wave function of the free (transverse) photon. [Two transverse photon types having orthogonal polarizations (e.g., opposite helicities) are needed to establish the general theory.] Starting from the wave equation for the transverse vector potential $\mathbf{A}_T$, a Schrödinger-like (Hamiltonian) wave equation for the analytical signal, $\mathbf{A}_T^{(+)}$, emerges. In the framework of classical electrodynamics there is no room for the quantum of action, and only by *brute force* Planck's constant can be attached to photon wave mechanics based on the reinterpretation of the Maxwell theory. The division of the vector potential into transverse and longitudinal parts is not Lorentz invariant. This fact is not in itself a problem from the point of view that one finally *always* has to connect the abstract photon concept to the photon phenomenon. This concept relates to what an observer can measure, and a given inertial observer cannot be in two different inertial frames at a given instant of time. A Lorentz invariant photon wave mechanical theory can be established if one is willing to introduce (in addition to the two transverse photon types) a longitudinal and a scalar photon. In free space there is no net physical effect of these photons, often called virtual photons. In the rim (near-field) zone of matter (source/detector) this canceling does not occur. It is possible however to replace the longitudinal and scalar photons by two new ones, the so-called gauge and near-field photons. The gauge photon can be eliminated by a suitable gauge transformation *within* the Lorenz gauge, leaving us with the near-field photon. As the name indicates this photon plays an important role in near-field electrodynamics. Although the free photon in our present understanding (description) of the physical world is massless, it is interesting to reflect on the (hypothetical) situation where the transverse photon is endowed with a mass. The quantum mechanics of the massive photon is governed by the Proca equation and the Lorenz condition of the potentials, which usually is a subsidiary condition, *must* be satisfied. In cases where the photon's interaction with matter is dominated by the diamagnetic coupling (as in a BCS superconductor, for example) the transverse photon may acquire an effective mass. This circumstance, in a relativistic setting, leads to the conclusion that the interaction between a transverse photon and a relativistic spinless boson particle under certain conditions makes the photon massive, but still with the freedom of gauge invariance lost. Once the photon is made massive, it is possible to make a Lorentz transformation to the photon's rest frame. The new frame's velocity equals the light particle's group velocity in the original frame. Although the main emphasis in this book is devoted to a formalism in which photon wave mechanics is based on the transverse vector potential, alternatives exist. Starting with the Oppenheimer light quantum theory from 1930, I discuss the closely related *photon energy wave function* formalism in some detail. In this connection remarks on antiphotons are given.

In Part IV, we turn toward the field-quantized description of the electromagnetic field, paying particular attention to single-photon quantum optics in Minkowskian space. In textbooks the photon concept usually is connected with the elementary quantum excitations associated to monochromatic plane waves, yet sometimes to monochromatic multipole waves. A *single* photon may be emitted when an atom makes a stimulated downward transition from a stationary state $|a\rangle$ to a stationary state $|b\rangle$. From the Bohr relation $E_a - E_b = \hbar\omega$ it appears that the photon is monochromatic (angular frequency: ω). This result of the old quantum theory cannot be strictly correct in general since the decay time is finite. For single photon emission from a general many-body transition the same conclusion holds: *The photon is polychromatic.* To qualify as a polychromatic single-photon state the eigenvalue of the global number operator must be 1. First, I develop and discuss the polychromatic one-photon theory in Hilbert space. Next, I introduce a (new) T-photon "mean" position state in the state space in order to introduce a polychromatic single-photon wave function in direct space. Finally, I establish the dynamical (Schrödinger-like) wave equation for the photon. Our choice of T-photon wave function is based on a mean position state, $|\boldsymbol{\mathcal{R}}\rangle$, introduced via the action of the negative-frequency part of the local vector-potential operator $\hat{\mathbf{A}}_T^{(-)}(\mathbf{r},t)$ on the global photon vacuum, $|0\rangle$. Hence $|\boldsymbol{\mathcal{R}}\rangle(\mathbf{r},t) \equiv (2\epsilon_0 c/\hbar)^{1/2}\hat{\mathbf{A}}_T^{(-)}(\mathbf{r},t)|0\rangle$. This definition allows one to capture *all* observational photon phenomena, e.g., also those related to the Aharonov–Bohm effect. It is shown that it is possible to form a polychromatic (wave packet) basis for one-photon states. Atomic and field correlation matrices allow one to address the question: *How can a single-photon phenomenon manifest itself?* On the basis of a single-photon correlation matrix interference phenomena with single-photon wave packets are discussed.

In Part V, we concentrate on photon physics in the rim zone of matter, paying particular attention to photon emission processes. In the rim zone the "object" that ends up as a T-photon after the light source has stopped its activity is attached to matter. I have called the transverse part of the field state in the rim zone a *photon embryo.* As the T-embryo propagates outward from the source it gradually develops into a T-photon. Important insight into photon physics in the rim zone can be obtained in the covariant field formalism. Thus, in this formalism the coupling of the T-photon to its source is described as an interaction with longitudinal and scalar photons. I discuss basic aspects of the rim zone photon physics via studies of selected examples, viz., evanescent fields, photon tunneling, electric monopole dynamics, and Čerenkov shock waves. The chosen examples illustrate the first- and second-quantized versions of photon wave mechanics at work.

In Part VI, we take a closer look at the photon source domain, and the field propagators that in a convenient manner describe the photon field propagation in the vicinity of and far from the electronic source domain. The source domain of a transverse photon is identical to the domain occupied by the transverse part ($\mathbf{J}_T$) of the electronic current density ($\mathbf{J}$). The current density $\mathbf{J}$ is obtained via the relevant many-body (or single-particle) transition current densities. In most cases the related $\mathbf{J}_T$ is algebraically confined for atomic transitions [distance dependencies from the nucleus: r^{-3} (ED-transitions), r^{-4} (MD+EQ-transitions), etc.] as we illustrate by a nonrelativistic study of the hydrogen $1s \Leftrightarrow 2p_z$ transition. In a few exotic cases it turns out that $\mathbf{J}_T = \mathbf{J}$. When this happens the source domain of the T-photon becomes exponentially confined. Such so-called super-confined T-photon sources appear in what I denote as breathing mode transitions. The breathing mode current density originates in the diamagnetic part of the transition current density, a part that is needed for gauge invariance. I illustrate the breathing mode dynamics (and confinement) by a study of the $1s \Leftrightarrow 2s$ transition in hydrogen. *This transition is forbidden in all multipole orders if the diamagnetic part of the current density is neglected.* A pure spin-1/2 current density also may lead to exponential T-photon source confinement. Starting from the Gordon expression for the spin part of the relativistic spin-1/2 current density (an expression that I discuss in some detail) it is shown that the spin current density in the weakly relativistic limit is a transverse current density vector field. Part VI is closed with studies of massless photon propagators, such as the Huygens propagator, the transverse photon propagator, the Feynman photon propagator, and the longitudinal and scalar photon propagators. The close relation between the propagators and the related photon correlation matrices is emphasized, and the connection between T-photon time-ordered correlation events (based on the mean position state for transverse photons) and the transverse photon propagator is determined and discussed.

In Part VII, we study the photon vacuum and light quanta in Minkowskian space. In free space a *physical photon vacuum state*, $|0_{PHYS}\rangle$, is a state in which the number of transverse photons is zero. When an arbitrary transverse photon annihilation operator $\hat{\dagger}_T$ acts on the physical vacuum state one obtains

$$\hat{\dagger}_T|0_{PHYS}\rangle = 0,$$

a definition of the T-photon vacuum state. It is important to understand that $|0_{PHYS}\rangle$ is a state in "our physical world," whereas the zero on the right side of the relation above is "outside this world." In a sense one may say that the operator $\hat{\dagger}_T$ is the recipe for transferring one to the state of Nirvana, $0 = \textit{NIRVANA}$. From this state no operation $\hat{\mathcal{O}}$ can bring us back to the physical world. In Minkowskian space inertial observers have a privileged status. Although the physical photon vacuum state will be the same for *all* inertial observers, a Lorentz boost changes the number of scalar (S) and longitudinal (L) photons in $|0_{PHYS}\rangle$. In free space there is no net physical effect of these photon types, and a given allowed admixture of L- and S-photons can be removed by a suitable gauge transformation within the Lorenz gauge. An observer that accelerates through the Minkowskian vacuum will observe a spectrum of transverse photons. In the special case where the observer accelerates uniformly, with a magnitude of the four-acceleration equal to a, she/he will measure a thermal (Planck) spectrum of T-photons corresponding to an absolute temperature $T_0 = a/(2\pi k_B)$. The privileged status of inertial observers in special relativity makes the Minkowskian vacuum the "natural" choice for the "correct" physical vacuum. In general relativity inertial observers become free-falling observers, and in general detectors in different free falls will not agree on a definition of "physical vacuum." This fact raises deep unanswered questions concerning quantum electrodynamics in general relativity. If the photon vacuum in some sense is analogous to the ground state of an interacting many-body system, it is possible that the photon vacuum is degenerate (non-unique). Such a situation may lead one to a mass of the T-photon in vacuum, and to the presence of vacuum screening currents involving a real Higgs field. Although we have no experimental indication of the existence of a photon vacuum mass it is nevertheless of some interest to reflect on this topic. In a physical vacuum with spontaneous symmetry breaking the photon can acquire a vacuum mass without destroying the gauge invariance freedom.

A two-photon is not two photons, but a single entity one may call a *biphoton*. Thus, two-photon interference cannot be considered the interference of two photons. In Part VIII, we study the two-photon entanglement that is associated to the biphoton in space-time. In the wake of a brief account of the general formalism for quantum measurements bearing on only one part of a two-part physical system, we turn to a description of the formalism for two-photon wave mechanics. Afterward, the first- and second-order correlation matrices associated with two-photon wave packet correlations are discussed. The general theory is illustrated via a treatment of the photon wave mechanical picture of the correlated spontaneous photon cascade emission from a three-level atom. On the basis of the Weisskopf–Wigner theory for photon emission from a two-level atom I first determine the associated space-time photon wave function. My treatment extends previous studies by paying particular attention to the spontaneous emission in the atomic rim zone. In this atomic near-field zone one finds an interesting interplay between the spatial photon localization problem and the two-photon entanglement process.

Acknowledgments

On April 1, 2009, I was contacted by Dr. John Navas, senior acquisitions editor (physics) with Taylor & Francis, who invited me to discuss the idea of writing a theoretical book on "the nature of light." Since for many years the physics of the photon had been a subject of the greatest importance for me, it did not take me long to accept John's proposal. I started writing the manuscript in December 2010, and thus it has taken me three years to accomplish this book project. In particular, I want to acknowledge my former physics student, M.Sc. Dann S. Olesen, for the comprehensive work he has done converting my handwritten manuscript into a professional LaTeX version. A special thank you goes to Niels Maribo Bache, currently a physics student at Aalborg University, who in the final stage of the work has helped with the drawing of the figures.

About the author

Ole Keller is professor emeritus of theoretical physics, Aalborg University, Denmark. He earned his PhD degree in physics from the Danish Technical University in Copenhagen (1972), and the doctor of science degree from the University of Aarhus (1990). He is a fellow of the Optical Society of America. In recent years he has worked on theoretical research in fundamental photon physics, near-field quantum electrodynamics, mesoscopic physical optics, and magnetic monopole photon wave mechanics.

Part I

Classical optics in global vacuum

1

Heading for photon physics

Notwithstanding that it is possible to consider all electromagnetic fields as intermediaries, transmitting interactions between charged particles, it is fruitful to study the concepts connected to free fields, i.e., fields detached from the charges producing or absorbing them. To free electromagnetic fields, also referred to as radiation fields, one may associate many of the properties we are so familiar with for matter, e.g., energy, momentum and angular momentum. Even in the framework of classical physics, radiation fields take up a position almost on an equal footing with matter. In quantum physics, the autonomy of the radiation field is fully developed through the emergence of the photon concept. The photon, the elementary excitation of the electromagnetic quantum field, appears as a particle just as "fundamental" as the massive elementary particles attached to other quantum fields.

The photons referring to the quantized free electromagnetic field are called transverse photons because the electric field of the radiation field, from which these photons emerge, is a divergence-free vector field. A divergence-free electric field in direct space ($\mathbf{r}$-space) is perpendicular in the geometrical sense to the wave-vector ($\mathbf{q}$) direction in reciprocal space ($\mathbf{q}$-space), therefore the name transverse photon. Transverse photons are often referred to as physical (or real) photons, because of the (almost) autonomous status of the free field. However, it must be remembered that a physical photon is observable only when it interacts with matter (charged massive particles). In the photon-matter interaction photons are created or destroyed, so in a sense, one may say that a transverse photon manifests itself only during its birth or death process. After all, a real transverse photon is not very real left alone in free space. Perhaps, the only fingerprint left of a free photon is the fact that a number, the speed of light in vacuum, is attached to its "propagation" from source to detector (delayed interparticle interaction). The words "real photon" thus at best refers to the circumstance that one can establish an autonomous quantum theory of free electromagnetic fields. However, it must be remembered that a complete decoupling of the dynamics of charged massive particles and transverse photons is impossible. On top of the discussion above, we have learned from Niels Bohr that the words "real" and "reality" do not make much sense in physics unless they are attached to phenomena (in the Bohr sense) observable by human beings [207]. In the covariant theory of quantum electrodynamics (QED) so-called longitudinal and scalar photons are introduced. These types of photons are called virtual because they only play a physical role ("exist") during the time where a given field-matter interaction process takes place.

The virtual photons, which couple charged massive particles in so-called near-field contact, are active only in what I have named the rim (or Lorenz) zone of matter [122]. The rim zone is a vacuum domain in the sense that it is located outside what we refer to as a matter-filled region. In a quantum physical context "outside" means in a region where the (many-body) probability density distribution of the matter particle(s) effectively vanishes. The free transverse photon hence is an object related to those parts of vacuum regions that do not include rim zones.

By a certain reinterpretation of the Maxwell equations in free space, these appear as a first-quantized theory of the transverse photon, as we shall see later on. Because of this circumstance, it is important to study selected fundamental aspects of the classical electromagnetic field theory in free space.

We start our journey into classical electromagnetics (optics) from the Maxwell equations in free space, and the associated wave equations for the electric and magnetic fields (Sec. 2.1). The magnetic field, $\mathbf{B}(\mathbf{r}, t)$, is a divergence-free (transverse) field everywhere in space independent of whether we are inside or outside of matter. The electric field, $\mathbf{E}(\mathbf{r}, t)$, on the other hand, is a transverse field ($\mathbf{E}_T$) only in vacuum outside rim zones. In free space (global vacuum) both $\mathbf{E}$ and $\mathbf{B}$ are transverse fields. A vector field is called a transverse vector field provided it is divergence-free in *every* space point within its domain of definition. The magnetic field thus is a genuine transverse vector field because its divergence vanishes in all space points. In the presence of field-matter interactions the electric field possesses a rotational-free (longitudinal) part inside matter and in the rim zone. In Sec. 2.2, we show that up to a physically unimportant constant, a differentiable vector field is uniquely separable into divergence-free and rotational-free parts. Such a division for the electric field is of utmost importance in studies of field-matter interactions, in this book in particular in relation to photon physics.

In relativistic wave mechanics wave packets constructed by superposition of plane waves of positive frequencies relate to photons. Hence, it is important to introduce and discuss the complex analytical signal concept in classical optics. This is done in Sec. 2.3, where it is shown also that the real and imaginary parts of the analytical signal form a Hilbert transform pair (also called a conjugate pair). The analytic part of a signal is timely nonlocally related to the signal itself, a fact which is thought-provoking in a photon perspective.

The photon concept most often is introduced starting from an expansion of the transverse part of the electromagnetic into monochromatic plane waves (Sec. 2.4). The individual photon emerging from such an expansion attains energy $E = \hbar\omega$ and momentum $\mathbf{p} = \hbar\mathbf{q}$, where $\omega = c|\mathbf{q}|$ and $\mathbf{q}$ are the angular frequency and wave-vector of the given $(\omega, \mathbf{q})$-mode. Photons belonging to a selected monochromatic plane-wave mode appear in two different helicity eigenstates, which relate to right- and left-hand circular polarized field unit vectors. From these so-called positive- and negative-helicity states alternative sets of orthonormal base vectors can be constructed. In Sec. 2.5, we analyze the linear transformation connecting different sets of basis vectors and discuss the geometrical picture of the field polarization states.

Although it sometimes is claimed in the literature that the photons *are* synonymous with the quanta associated to monochromatic plane-wave modes, certainly this need not be the case. Thus, a wave packet composed of different $(\omega, \mathbf{q})$-modes may represent a single photon. Throughout this book we shall often consider a photon as a wave packet. As shown in Sec. 2.6, it is possible to expand a given transverse classical field after a set of orthonormalized wave-packet modes. Upon quantization, transverse wave-packet photons emerge. The wave-packet modes satisfy a completeness theorem in the subspace of transverse vector fields.

In global vacuum, the energy, the moment of energy, the momentum, and the angular momentum of the electromagnetic are conserved in time, as emphasized in Sec. 2.7. These conservation laws may be derived from Emmy Nöther's theorem [173] which provides us with a relationship between the symmetry (invariance) properties and conservation laws of a system [206]. The forms of the integrands appearing in the aforementioned quantities are not universal. Thus, these forms are valid for observers at rest in the inertial frame in which the fields are specified.

By means of the complex Riemann–Silberstein (RS) vectors it is possible to write the set of free Maxwell equations in compact form (Sec. 2.8). The two RS-vectors relate to states in which the electromagnetic field is composed of positive- and negative-helicity species. The dynamical equations for the positive-frequency parts of these vectors lead to photon wave mechanics based on the so-called energy wave function [16].

Since the photon necessarily is a relativistic object, it is important to consider the fields of classical optics in the perspective of the Special Theory of Relativity (Chapt. 3). A brief review of the Lorentz transformation and the proper time concept (Sec. 3.2), and the important four-vector and four-tensor formalism (Sec. 3.3) is given before turning the attention to the free-space electromagnetic field. The set of microscopic Maxwell–Lorentz equations, constituting the foundation of classical electrodynamics, is form-invariant under Lorentz transformations. This form-invariance, traditionally called covariance, necessarily also holds for the set of free Maxwell equations, and in Sec. 3.4 we rewrite this set in manifest covariant form. The virtual scalar and longitudinal photons appear in the wake of the covariant formalism. The Lorentz transformation of the electric and magnetic fields (Sec. 3.5) plays an important role in photon physics. The free-space electric and magnetic fields are transverse in all inertial frames, but a Lorentz transformation of the fields shows that $\mathbf{E}_T$ and $\mathbf{B}$ have no independent "existence." In the rim zone of matter, where the electric field has both transverse and longitudinal ($\mathbf{E}_L$) components, a Lorentz transformation will change $\mathbf{E}_L$ in a manner which involves the charge current density of the particle (system). The limitations on the localization of a transverse photon in space is linked to the spatial extension of the rim zone [123], and since this zone essentially is determined by the longitudinal field distribution, the spatial photon localization does not appear to be the same for different inertial observers.

2

Fundamentals of free electromagnetic fields

2.1 Maxwell equations and wave equations

Classical electromagnetics is summed up in the Maxwell–Lorentz equations [56, 57, 133], and in the absence of charges the electric and magnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, satisfy the equations

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \tag{2.1}$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = c^{-2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t), \tag{2.2}$$

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = 0, \tag{2.3}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{2.4}$$

in space ($\mathbf{r}$)-time (t). Eqs. (2.3) and (2.4) specify that both $\mathbf{E}$ and $\mathbf{B}$ are divergence-free (solenoidal) fields in matter-free regions of space. The magnetic field remains divergence-free in matter-filled domains, and this is so because our present theory is based on the fact that there is no experimental evidence for the existence of magnetic charges or monopoles. Since electric charges *do* exist, the electric field will not be divergence-free in matter-filled regions, and Eq. (2.3) thus must be modified in such regions. Whether a region can be characterized as matter-filled in the context of classical electromagnetics requires some remarks. In the macroscopic Maxwell theory matter is conceived as a continuum and the characterization complies with this. In the microscopic Maxwell–Lorentz theory all relevant charged particles (electrons, protons, ions) are treated as point-like entities. In consequence matter is present only in discrete points, and in these the charge density is infinite. In the covering theory of classical electrodynamics, named semiclassical electrodynamics [206], the dynamics of the charged elementary particles (electrons, etc.) is treated on the basis of quantum mechanics. Although we think of these particles as point-like entities, quantum theory does not allow one to determine (at a given time) a particle's position precisely. The probabilistic nature of quantum mechanics in a way leads us back to a continuum view of matter, yet in a quantum statistical sense to be described later on.

It appears from Eqs. (2.1) and (2.2) that the electric and magnetic fields are coupled, and a transformation of our description from one inertial frame to another shows that $\mathbf{E}$ and $\mathbf{B}$ have no independent existence, as we shall realize in Sec. 3.5.

It can be shown from Eqs. (2.1)-(2.4) that the electric and magnetic fields satisfy form-identical wave equations, viz.,

$$\Box\mathbf{E}(\mathbf{r}, t) = \mathbf{0}, \tag{2.5}$$

$$\Box\mathbf{B}(\mathbf{r}, t) = \mathbf{0}, \tag{2.6}$$

where

$$\Box = \nabla^2 - c^{-2}\frac{\partial^2}{\partial t^2} \tag{2.7}$$

is the d'Alembertian operator. The constant c is the speed of light, a universal quantity which is the same in all inertial systems. The wave equations in Eqs. (2.5) and (2.6) suggest the existence of electromagnetic waves that propagate through vacuum domains with speed c, a statement we shall put into the perspective of photon physics in Secs. 2.2-2.4. When we turn to the particle description of electrodynamics it will be seen that all photons propagate with speed c.

2.2 Transverse and longitudinal vector fields

In the following we take up a topic of utmost importance when we later discuss how photons are created and destroyed in space-time in their interaction with matter. The subject we touch upon here also is of relevance for the epistemology related to the photon concept (the photon measurement problem), and for a basic understanding of optical diffraction in regions near matter.

The electric and magnetic fields we deal with in classical electrodynamics always are generated by sources occupying a finite domain in space-time. In consequence, these fields vanish infinitely far away from their sources. In the Maxwell–Lorentz theory the fields are differentiable functions of the space coordinates except at the locations of the point-particles. Here the fields diverge. The fields are differential functions of time. In semiclassical electrodynamics, where the inherent probabilistic interpretation of quantum mechanics smears every singular behavior, the fields become differentiable everywhere in space-time.

Let $\mathbf{W}$ be a generic name for $\mathbf{E}$ and $\mathbf{B}$, and let us for brevity omit the reference to the time from the notation. Starting from the vector function

$$\mathbf{F}(\mathbf{r}) = \int_{-\infty}^{\infty} \frac{\mathbf{W}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', \tag{2.8}$$

which is a solution of the vectorial Poisson equation

$$\mathbf{W}(\mathbf{r}) = -\nabla^2 \mathbf{F}(\mathbf{r}) = \nabla \times (\nabla \times \mathbf{F}(\mathbf{r})) - \nabla\nabla \cdot \mathbf{F}(\mathbf{r}), \tag{2.9}$$

it is possible to prove that

$$\mathbf{W}(\mathbf{r}) = \nabla \times \int_{-\infty}^{\infty} \frac{\nabla' \times \mathbf{W}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r' - \nabla \int_{-\infty}^{\infty} \frac{\nabla' \cdot \mathbf{W}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', \tag{2.10}$$

because $\mathbf{W}(\mathbf{r})$ vanishes at infinity. The result in Eq. (2.10) shows that the vector field $\mathbf{W}(\mathbf{r})$ is uniquely separable into a divergence-free part,

$$\mathbf{W}_T(\mathbf{r}) = \nabla \times \int_{-\infty}^{\infty} \frac{\nabla' \times \mathbf{W}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', \tag{2.11}$$

and a rotational-free part,

$$\mathbf{W}_L(\mathbf{r}) = -\nabla \int_{-\infty}^{\infty} \frac{\nabla' \cdot \mathbf{W}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', . \tag{2.12}$$

This property,

$$\mathbf{W}(\mathbf{r}) = \mathbf{W}_T(\mathbf{r}) + \mathbf{W}_L(\mathbf{r}), \tag{2.13}$$

is called Helmholtz's theorem [163]. A field, $\mathbf{W}_T$, which is divergence-free in direct ($\mathbf{r}$) space is in reciprocal ($\mathbf{q}$) space perpendicular to $\mathbf{q}$, i.e., $\mathbf{q} \cdot \mathbf{W}_T(\mathbf{q}) = 0$, and a field, $\mathbf{W}_L$, which is rotational-free in $\mathbf{r}$-space is parallel to $\mathbf{q}$ in $\mathbf{q}$-space, that is $\mathbf{q} \times \mathbf{W}_L(\mathbf{q}) = \mathbf{0}$. This geometrical significance in reciprocal space is the reason that we shall use also the names transverse (with subscript T) and longitudinal (subscript L) for such fields in the remaining part of this book.

It is important to emphasize that a field, $\mathbf{W}(\mathbf{r})$, by definition, only qualifies as a transverse vector field if its divergence vanishes in every space point, i.e.,

$$\nabla \cdot \mathbf{W}(\mathbf{r}) = 0, \qquad \forall \mathbf{r}. \tag{2.14}$$

In accordance with this, a combination of Eqs. (2.12) and (2.14) gives $\mathbf{W}_L(\mathbf{r}) = \mathbf{0}$ for all $\mathbf{r}$, and thus $\mathbf{W}(\mathbf{r}) = \mathbf{W}_T(\mathbf{r})$. In analogy, a field $\mathbf{W}(\mathbf{r})$ is a longitudinal vector field only if

$$\nabla \times \mathbf{W}(\mathbf{r}) = \mathbf{0}, \qquad \forall \mathbf{r}, \tag{2.15}$$

in agreement with the fact that Eqs. (2.11) and (2.15) leads to $\mathbf{W}_T(\mathbf{r}) = \mathbf{0}$ for all $\mathbf{r}$, and hence $\mathbf{W}(\mathbf{r}) = \mathbf{W}_L(\mathbf{r})$.

Returning now to the free-space Maxwell equations given in (2.1)-(2.4), it appears that if no charges were present in the universe both the electric and magnetic field would be (genuine) transverse vector fields, i.e.,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_T(\mathbf{r}, t), \tag{2.16}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_T(\mathbf{r}, t). \tag{2.17}$$

Because of the absence of magnetic monopoles, the magnetic field will still qualify as a transverse vector field in the presence of matter. This circumstance makes it superfluous to add the subscript T to $\mathbf{B}$. The presence of a charge density distribution, $\rho(\mathbf{r}, t)$, changes Eq. (2.3) to $\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \rho(\mathbf{r}, t)/\epsilon_0$, where ϵ_0 is the vacuum permittivity. With $\mathbf{W}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t)$, Eqs. (2.11)-(2.13) then give

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_T(\mathbf{r}, t) - \frac{1}{4\pi\epsilon_0} \nabla \int_{-\infty}^{\infty} \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}^3 r'. \tag{2.18}$$

It appears from Eq. (2.18) that the electric field is not a transverse vector field when a charge density exists in a region of space. The $\mathbf{E}$-field has a longitudinal part $\mathbf{E}_L(\mathbf{r}, t)$ not only inside the charge distribution but also in the vacuum in a usually narrow zone surrounding matter; see Fig. 2.1. This zone, called the rim zone, is part of the source domain for transverse photons, as we shall understand later on. The rim zone concept plays a central role, e.g., in studies of evanescent fields (Chapt. 20), photon tunneling (Chapt. 21), the Čerenkov effect (Chapt. 22), and photon emission from atoms (Chapt. 24).

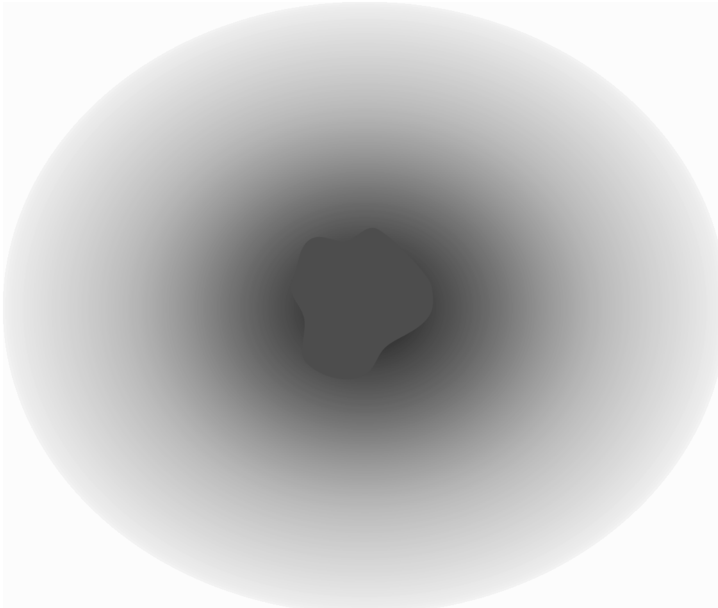

FIGURE 2.1
The black region indicates a domain in space where the quantum mechanical charge density of a system of (elementary) particles (atoms, molecules, a solid, etc.) is nonvanishing. Although transverse (T) photons can be generated by (certain) many-body (or single-body) transitions in the charge system, one cannot claim that such photons for sure are born *within* the charge distribution if one wants to uphold the criterion that T-photons propagate with the vacuum speed of light everywhere in space outside matter-filled domains. To maintain so-called Einsteinian causality one must admit that a T-photon in a quantum statistical sense also can be emitted (born) from every point within a larger so-called rim zone of matter. Schematically, the rim zone of the black charge density distribution is shown as a grey-toned domain. The fading out of the grey-toning away from the charge density region is meant to indicate that there is no sharp boundary between the rim zone domain and the surrounding vacuum.

2.3 Complex analytical signals

In free space relativistic wave equations have two main types of wave packet solutions, viz., those built from plane waves of positive frequencies, corresponding to particles, and those built from negative frequencies, relating to antiparticles [88, 209]. For photon physics it is therefore of interest to study the positive-frequency solutions to the free-space Maxwell equations.

Since the real and nonsingular vector field $\mathbf{W}(\mathbf{r}, t)[= \mathbf{E}(\mathbf{r}, t) or \mathbf{B}(\mathbf{r}, t)]$ always has finite support in time it may be represented as a Fourier integral

$$\mathbf{W}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{r}; \omega) e^{-i\omega t} \mathrm{d}\omega. \tag{2.19}$$

Below we are interested only in the time dependence of $\mathbf{W}$, and for brevity, we therefore leave out the reference to $\mathbf{r}$. Because $\mathbf{W}(t)$ is real, the (generally complex) Fourier amplitudes

$\mathbf{W}(\omega)$ obey the relation

$$\mathbf{W}(-\omega) = \mathbf{W}^*(\omega). \tag{2.20}$$

It appears from Eq. (2.20) that the negative frequency components ($\omega < 0$) do not contain any information not already carried by the positive frequency part of the spectrum. In a broader perspective this implies that the photon and antiphoton are identical.

The complex analytical signal [75, 155], denoted by $\mathbf{W}^{(+)}(\mathbf{r}, t)[\equiv \mathbf{W}^{(+)}(t)$ below] is obtained from the Fourier integral in Eq. (2.19) by suppressing the negative frequency components:

$$\mathbf{W}^{(+)}(t) = \frac{1}{2\pi} \int_0^\infty \mathbf{W}(\omega) e^{-i\omega t} \mathrm{d}\omega. \tag{2.21}$$

Since $\mathbf{W}(t)$ is real, the negative frequency part of Eq. (2.19),

$$\mathbf{W}^{(-)}(t) = \frac{1}{2\pi} \int_{-\infty}^0 \mathbf{W}(\omega) e^{-i\omega t} \mathrm{d}\omega, \tag{2.22}$$

is the complex conjugate of the analytical signal, i.e.,

$$\mathbf{W}^{(-)}(t) = (\mathbf{W}^{(+)}(t))^*, \tag{2.23}$$

as the reader may verify by a direct calculation involving use of Eq. (2.20). For what follows, it is useful to write the analytical signal as an integral over all ω:

$$\mathbf{W}^{(+)}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbf{W}^{(+)}(\omega) e^{-i\omega t} \mathrm{d}\omega. \tag{2.24}$$

where

$$\mathbf{W}^{(+)}(\omega) = \begin{cases} \mathbf{W}(\omega) & \text{for } \omega \geq 0, \\ \mathbf{0} & \text{for } \omega < 0. \end{cases} \tag{2.25}$$

The Fourier amplitude $\mathbf{W}^{(+)}(\omega)$ thus is given by an integral

$$\mathbf{W}^{(+)}(\omega) = \int_{-\infty}^\infty \mathbf{W}^{(+)}(t) e^{i\omega t} \mathrm{d}t. \tag{2.26}$$

which is zero for $\omega < 0$.

Let us next, albeit in a not quite rigorous manner, examine the possibility for extending the definition of the analytic signal given in Eq. (2.24) to complex valued arguments $\tau = t + is$. The demand that $\mathbf{W}^{(+)}(\omega)$ is zero for negative values of ω implies that $\mathbf{W}^{(+)}(\tau)$ is an analytic function in the lower half of the complex τ-plane. To see this, we consider the contour integral

$$\mathbf{I}(\omega) = \oint_C \mathbf{W}^{(+)}(\tau) e^{i\omega\tau} \mathrm{d}\tau, \tag{2.27}$$

and choose as the closed contour C a portion $-T < t < T$ of the real axis plus a semi-circle (of radius T) in the lower half-plane. It may be deduced that the integral along the semi-circle is zero in the limit $T \to \infty$ [155] for $\omega < 0$. In the limit, the integral along the real axis is just $\mathbf{W}^{(+)}(\omega)$. For $\omega < 0$, we thus must have

$$\mathbf{I}(\omega; T \to \infty) = \mathbf{W}^{(+)}(\omega) = \mathbf{0}. \tag{2.28}$$

If we require that $\mathbf{W}^{(+)}(\tau)$ is analytic in the lower half-plane, Cauchy's theorem ensures that $\mathbf{I}(\omega; T \to \infty)$ *is* zero for $\omega < 0$.

The analyticity of $\mathbf{W}^{(+)}(\tau)$ for $s \leq 0$ allows one to make use of Cauchy's integral formula, and thus obtain

$$\oint_C \frac{\mathbf{W}^{(+)}(\tau)}{\tau - \tau_0} \mathrm{d}\tau = \pi i \mathbf{W}^{(+)}(\tau_0) \tag{2.29}$$

in the case where τ_0 lies on the boundary curve. The integral must be interpreted as the Cauchy principal (P) value, and the curve C (located in the domain $s \leq 0$) is circulated in the counterclockwise sense. Let us now take for C the same contour as used in relation to Eq. (2.27), and let $\tau_0 = t$ be a point on the real axis. Since the contribution from the semi-circle again vanishes in the limit where the radius becomes infinite, we obtain

$$P \int_{-\infty}^{\infty} \frac{\mathbf{W}^{(+)}(t')}{t' - t} \mathrm{d}t' = -\pi i \mathbf{W}^{(+)}(t). \tag{2.30}$$

It appears from this integral identity that the real ($\mathcal{R}$) and imaginary ($\mathcal{I}$) parts of the complex analytic signal form a Hilbert transform pair, i.e.,

$$\mathcal{I}\mathbf{W}^{(+)}(\mathbf{r}, t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{R}\mathbf{W}^{(+)}(\mathbf{r}, t')}{t' - t} \mathrm{d}t' \tag{2.31}$$

$$\mathcal{R}\mathbf{W}^{(+)}(\mathbf{r}, t) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathcal{I}\mathbf{W}^{(+)}(\mathbf{r}, t')}{t' - t} \mathrm{d}t' \tag{2.32}$$

in a notation where the reference to the space coordinate has been reinserted. Utilizing that $\mathbf{W}(t) = 2\mathcal{R}\mathbf{W}^{(+)}(t)$, it is easy to show that the analytical part of a signal is related to the signal itself as follows:

$$\mathbf{W}^{(+)}(\mathbf{r}, t) = \frac{1}{2} \left(\mathbf{W}(\mathbf{r}, t) + \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{\mathbf{W}(\mathbf{r}, t')}{t' - t} \mathrm{d}t' \right). \tag{2.33}$$

The fact that the relation between $\mathbf{W}^{(+)}$ and $\mathbf{W}$ is nonlocal in time (yet local in space) is thought-provoking from the perspective of photon physics. Thus, as we shall see (Part III), it is possible in photon wave mechanics to associate the wave function of a transverse photon to a combination of the complex analytical fields $\mathbf{E}_T^{(+)}(\mathbf{r}, t)$ and $\mathbf{B}^{(+)}(\mathbf{r}, t)$ (Chapt. 13), or alternatively, to the positive-frequency part of the transverse vector potential $\mathbf{A}_T^{(+)}(\mathbf{r}, t)$ (Chapt. 10). For an electromagnetic field ($\mathbf{E}_T, \mathbf{B}$) of finite support in time, say from $t = 0$ to $t = T_0$, the associated photon wave function will be nonvanishing also outside the interval $(0|T_0)$. It must be remembered, however, that also a transverse antiphoton is associated to the given field. Together the photon and antiphoton have no net effect outside the $(0|T_0)$-interval.

Since $\mathbf{W}(\mathbf{r}, t)$ satisfies the wave equation $\Box \mathbf{W}(\mathbf{r}, t) = \mathbf{0}$, cf. Eqs. (2.5) and (2.6), it follows from the Fourier integral representation in Eq. (2.19) that the Fourier amplitude obeys the Helmholtz equation $[\nabla^2 + (\omega/c)^2]\mathbf{W}(\mathbf{r}; \omega) = \mathbf{0}$. An integration of this equation over all positive frequencies shows that the analytical part of the signal satisfies the same wave equation as the signal itself. The analytical parts of the free electric and magnetic fields hence obey wave equations of the usual form:

$$\Box \mathbf{E}_T^{(+)}(\mathbf{r}, t) = \mathbf{0}, \tag{2.34}$$

$$\Box \mathbf{B}^{(+)}(\mathbf{r}, t) = \mathbf{0}. \tag{2.35}$$

In Sec. 2.9, we shall see that the complex analytical signal also satisfies a certain type of integro-differential equation. Because this equation is of first-order in time, it is possible to determine the values of $\mathbf{W}^{(+)}(\mathbf{r}, t)$ for all $\mathbf{r}$ and t from a knowledge if $\mathbf{W}^{(+)}(\mathbf{r}, t_0)$ at any particular time t_0. This first-order equation in time helps us to obtain a unified view of the wave mechanics of massless and massive particles.

2.4 Monochromatic plane-wave expansion of the electromagnetic field

A Fourier integral representation of the spatial part of $\mathbf{W}(\mathbf{r},t)$, together with Eq. (2.19), lead to the monochromatic plane-wave expansion

$$\mathbf{W}(\mathbf{r},t) = (2\pi)^{-4} \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{q},\omega) e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \mathrm{d}^3 q \mathrm{d}\omega. \tag{2.36}$$

Since $\mathbf{W}(\mathbf{r},t)$ must satisfy the free-space wave equation $\Box \mathbf{W}(\mathbf{r},t) = \mathbf{0}$, the (angular) frequency (ω) and wave number ($q = |\mathbf{q}|$) are connected by the two-branch dispersion relation

$$\omega = \pm cq. \tag{2.37}$$

In the context of wave mechanics $\omega = +cq(>0)$ and $\omega = -cq(<0)$, upon multiplication by Planck's constant divided by 2π, give us the energy-momentum relation for plane-wave photons and antiphotons, respectively, as we shall see later on. The constraints in Eqs. (2.37), imply that the Fourier amplitude $\mathbf{W}(\mathbf{q},\omega)$ can be written in the form

$$\mathbf{W}(\mathbf{q},\omega) = 2\pi \left(\mathbf{W}(\mathbf{q},cq)\delta(\omega - cq) + \mathbf{W}(\mathbf{q},-cq)\delta(\omega + cq) \right), \tag{2.38}$$

where δ is the Dirac delta function. A combination of Eqs. (2.36) and (2.38) then splits $\mathbf{W}(\mathbf{r},t)$ into its positive- and negative-frequency parts:

$$\mathbf{W}(\mathbf{r},t) = \mathbf{W}^{(+)}(\mathbf{r},t) + \mathbf{W}^{(-)}(\mathbf{r},t), \tag{2.39}$$

where

$$\mathbf{W}^{(+)}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{q},cq) e^{i(\mathbf{q}\cdot\mathbf{r}-cqt)} \frac{\mathrm{d}^3 q}{(2\pi)^3}, \tag{2.40}$$

and

$$\mathbf{W}^{(-)}(\mathbf{r},t) = \int_{-\infty}^{\infty} \mathbf{W}(\mathbf{q},-cq) e^{i(\mathbf{q}\cdot\mathbf{r}+cqt)} \frac{\mathrm{d}^3 q}{(2\pi)^3}. \tag{2.41}$$

Because $(\mathbf{W}^{(-)}(\mathbf{r},t))^* = \mathbf{W}^{(+)}(\mathbf{r},t)$ the Fourier amplitudes satisfy the relation

$$\mathbf{W}(-\mathbf{q},-cq) = \mathbf{W}^*(\mathbf{q},cq), \tag{2.42}$$

as the reader may verify by complex conjugation of Eq. (2.41), followed by a variable change $\mathbf{q} \Rightarrow -\mathbf{q}$.

By inserting the expansion in Eq. (2.36) into the Maxwell equations in (2.1)-(2.4) it appears that the Fourier amplitudes satisfy the algebraic equations

$$\mathbf{q} \times \mathbf{E}_T(\mathbf{q},\omega) = \omega \mathbf{B}(\mathbf{q},\omega), \tag{2.43}$$
$$-c^2 \mathbf{q} \times \mathbf{B}(\mathbf{q},\omega) = \omega \mathbf{E}_T(\mathbf{q},\omega), \tag{2.44}$$
$$\mathbf{q} \cdot \mathbf{E}_T(\mathbf{q},\omega) = 0, \tag{2.45}$$
$$\mathbf{q} \cdot \mathbf{B}(\mathbf{q},\omega) = 0, \tag{2.46}$$

where $\omega = \pm cq$. Instead of $\mathbf{E}(\mathbf{q},\omega)$ we have written $\mathbf{E}_T(\mathbf{q},\omega)$ to emphasize that the electric field in a completely free space is a transverse vector field. In the photon wave mechanical description, to follow in Part III, the analytical parts of the fields play a prominent role. With the use of the dispersion relation $\omega = cq(> 0)$ inserted one obtains from Eqs. (2.43)-(2.46)

$$\boldsymbol{\kappa} \times \mathbf{E}_T(\mathbf{q}, cq) = c\mathbf{B}(\mathbf{q}, cq), \tag{2.47}$$

$$-c\boldsymbol{\kappa} \times \mathbf{B}(\mathbf{q}, cq) = \mathbf{E}_T(\mathbf{q}, cq), \tag{2.48}$$

$$\boldsymbol{\kappa} \cdot \mathbf{E}_T(\mathbf{q}, cq) = 0, \tag{2.49}$$

$$\boldsymbol{\kappa} \cdot \mathbf{B}(\mathbf{q}, cq) = 0, \tag{2.50}$$

where $\boldsymbol{\kappa} = \mathbf{q}/q$ is a unit vector in the direction of the wave vector $\mathbf{q}$. The algebraic set of equations satisfied by the negative-frequency $[\omega = -cq(< 0)]$ components of the fields $[\mathbf{E}_T(\mathbf{q}, -cq), \mathbf{B}(\mathbf{q}, -cq)]$ is readily obtained from Eqs. (2.47)-(2.50) utilizing the relation in Eq. (2.42).

2.5 Polarization of light

2.5.1 Transformation of base vectors

It appears from Eq. (2.45) that the electric field vector $\mathbf{E}_T(\mathbf{q}, \omega)$ always lies in a plane perpendicular to the wave vector $\mathbf{q}$. To characterize the state of the field we resolve the vector $\mathbf{E}_T(\mathbf{q}, \omega)$ into two orthogonal components by selecting a pair of generally complex orthonormal base vectors $\boldsymbol{\varepsilon}_1(\boldsymbol{\kappa})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{\kappa})$ that obey the following conditions:

$$\boldsymbol{\kappa} \cdot \boldsymbol{\varepsilon}_s(\boldsymbol{\kappa}) = 0, \qquad s = 1, 2 \tag{2.51}$$

$$\boldsymbol{\varepsilon}_s^*(\boldsymbol{\kappa}) \cdot \boldsymbol{\varepsilon}_{s'}(\boldsymbol{\kappa}) = \delta_{ss'}, \qquad s, s' = 1, 2 \tag{2.52}$$

where $\delta_{ss'}$ is the Kronecker symbol. Thus,

$$\mathbf{E}_T(\mathbf{q}, \omega) = \sum_{s=1,2} E_{T,s}(\mathbf{q}, \omega)\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa}). \tag{2.53}$$

The projections of $\mathbf{E}_T$ on the complex conjugates two basis vectors give the field components in the chosen basis. i.e.,

$$E_{T,s}(\mathbf{q}, \omega) = \boldsymbol{\varepsilon}_s^*(\boldsymbol{\kappa}) \cdot \mathbf{E}_T(\mathbf{q}, \omega). \tag{2.54}$$

The conditions in Eqs. (2.51) and (2.52) do not determine the basis vectors uniquely, and this is convenient because in a given application a particular set of basis vectors may be more useful than the others. Starting from a given set of OLD basis vectors, NEW sets can be constructed via a linear transformation

$$\begin{pmatrix} \boldsymbol{\varepsilon}_1^{NEW} \\ \boldsymbol{\varepsilon}_2^{NEW} \end{pmatrix} = \mathbf{T} \begin{pmatrix} \boldsymbol{\varepsilon}_1^{OLD} \\ \boldsymbol{\varepsilon}_2^{OLD} \end{pmatrix}, \tag{2.55}$$

where

$$\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.56}$$

is a 2×2 transformation matrix. In order that also the new set $(\boldsymbol{\varepsilon}_1^{NEW}, \boldsymbol{\varepsilon}_2^{NEW})$ satisfies the conditions in Eq. (2.52), the components of $\mathbf{T}$ must be related as follows:

$$aa^* + bb^* = cc^* + dd^* = 1, \tag{2.57}$$

$$a^*c + b^*d = 0, \tag{2.58}$$

as one readily may show. The constraints among the components are obeyed if the transformation matrix is unitary, i.e.,

$$\mathbf{T}^{-1} = \mathbf{T}^{\dagger}, \tag{2.59}$$

where

$$\mathbf{T}^{-1} = D^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{2.60}$$

and

$$\mathbf{T}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \tag{2.61}$$

are the inverse and Hermitian conjugate of $\mathbf{T}$, respectively. The quantity $D = ad - bc$ is the determinant of $\mathbf{T}$. Written in terms of the components unitarity is expressed by

$$a = d^* D, \quad b = -c^* D, \quad c = -b^* D, \quad d = a^* D. \tag{2.62}$$

The reader may verify to himself that these relations lead to

$$DD^* = 1, \tag{2.63}$$

and the fulfilment of Eqs. (2.57) and (2.58). It follows from Eq. (2.63) that the modulus of D is equal to one, and one may therefore write the determinant in the form $D = \exp(i\delta)$, where δ is a real phase parameter. By now, one may express the transformation matrix in the form

$$\mathbf{T} = \begin{pmatrix} a & b \\ -b^* e^{i\delta} & a^* e^{i\delta} \end{pmatrix}. \tag{2.64}$$

Remembering that $|a|^2 + |b|^2 = 1$, it appears that $\mathbf{T}$ contains four free parameters. One of these is δ. With a view to the geometrical analysis of the polarization states of light, given in Subsec. 2.5.2, it is useful to take

$$a = \left(1 + |\Delta|^2\right)^{-\frac{1}{2}} e^{i\alpha}, \tag{2.65}$$

$$b = \left(1 + |\Delta|^2\right)^{-\frac{1}{2}} \Delta, \tag{2.66}$$

with $\Delta = \Delta_R + i\Delta_I$. Expressed in terms of the four real free parameters δ, α, Δ_R, and Δ_I, the transformation is

$$\mathbf{T} = \frac{1}{(1 + |\Delta|^2)^{\frac{1}{2}}} \begin{pmatrix} e^{i\alpha} & \Delta \\ -\Delta^* e^{i\delta} & e^{i(\delta-\alpha)} \end{pmatrix}. \tag{2.67}$$

For what follows it is sufficient to employ only three free parameters, and it turns out to be convenient to make the choice $\alpha = 0$. Thus, the reduced transformation matrix

$$\mathbf{T}(\Delta, \delta) = \frac{1}{(1 + |\Delta|^2)^{\frac{1}{2}}} \begin{pmatrix} 1 & \Delta \\ -\Delta^* e^{i\delta} & e^{i\delta} \end{pmatrix} \tag{2.68}$$

will serve as the starting point for the subsequent study of the various polarization states of the electric field associated with a given monochromatic plane wave.

2.5.2 Geometrical picture of polarization states

The role of the phase factor $\exp(i\delta)$ becomes clear if one considers the vectorial product of the base vectors. Hence, one obtains from the unitary transformation in Eq. (2.55), with use of Eq. (2.68),

$$\boldsymbol{\varepsilon}_1^{NEW} \times \boldsymbol{\varepsilon}_2^{NEW} = e^{i\delta} \boldsymbol{\varepsilon}_1^{OLD} \times \boldsymbol{\varepsilon}_2^{OLD}. \tag{2.69}$$

With the choice $\delta = 0$, the vectorial product $\boldsymbol{\varepsilon}_1(\boldsymbol{\kappa}) \times \boldsymbol{\varepsilon}_2(\boldsymbol{\kappa})$ therefore is the same for all sets of basis vectors. The vector product always gives a vector parallel or antiparallel to $\boldsymbol{\kappa}$. With the choice

$$\boldsymbol{\varepsilon}_1(\boldsymbol{\kappa}) \times \boldsymbol{\varepsilon}_2(\boldsymbol{\kappa}) = \boldsymbol{\kappa}, \qquad \delta = 0, \tag{2.70}$$

the vectors $(\boldsymbol{\varepsilon}_1(\boldsymbol{\kappa}), \boldsymbol{\varepsilon}_2(\boldsymbol{\kappa}), \boldsymbol{\kappa})$ form a right-handed orthonormal set [in a generalized sense since $\boldsymbol{\varepsilon}_1(\boldsymbol{\kappa})$ and $\boldsymbol{\varepsilon}_2(\boldsymbol{\kappa})$ may be complex]. Below, we shall keep the phase factor in the analysis.

Classical electrodynamics is a deterministic theory. This implies that the end point of the electric field vector, at a fixed point in space, with increasing time describes a smooth curve. In general, the form of this curve is extremely complicated, and the curves are very different at the various points in space. For a monochromatic field, the curve is never more complicated than what results from a linear superposition of two ellipses. Below, we shall prove this assertion for a plane-wave field. The generalization of the proof from plane-wave fields to more complicated monochromatic fields is easy.

To examine the polarization state of the (transverse) electric field belonging to a given plane wave, one must analyze the expression

$$\mathcal{R}\left[\mathbf{E}_T(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\right] = \sum_{s=1,2} \mathcal{R}\left[E_{T,s}(\mathbf{q},\omega)\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\right]. \tag{2.71}$$

If one writes the complex amplitude $E_{T,s}(\mathbf{q},\omega)$ in the polar form $E_{T,s}(\mathbf{q},\omega) = |E_{T,s}(\mathbf{q},\omega)| \exp[i\phi_s(\mathbf{q},\omega)]$, one obtains

$$\mathcal{R}\left[\mathbf{E}_T(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\right] = \sum_{s=1,2} |E_{T,s}(\mathbf{q},\omega)|\mathcal{R}\left[\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t+\phi_s(\mathbf{q},\omega))}\right], \tag{2.72}$$

a form which is convenient for the subsequent analysis. Let us assume now that the old basis vectors are real (superscript R): $(\boldsymbol{\varepsilon}_1^{OLD}, \boldsymbol{\varepsilon}_2^{OLD}) = (\boldsymbol{\varepsilon}_1^R, \boldsymbol{\varepsilon}_2^R)$. From Eqs. (2.55) and (2.68), we then find that the new basis vectors $(\boldsymbol{\varepsilon}_1^{NEW}, \boldsymbol{\varepsilon}_2^{NEW}) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)$ are given by

$$\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa}) = (\mathbf{p}_s + i\mathbf{q}_s)\exp(i\delta\delta_{s2}), \qquad s = 1, 2, \tag{2.73}$$

where

$$\mathbf{p}_1 = K(\boldsymbol{\varepsilon}_1^R + \Delta_R \boldsymbol{\varepsilon}_2^R), \tag{2.74}$$

$$\mathbf{q}_1 = K\Delta_I \boldsymbol{\varepsilon}_2^R, \tag{2.75}$$

and

$$\mathbf{p}_2 = K(-\Delta_R \boldsymbol{\varepsilon}_1^R + \boldsymbol{\varepsilon}_2^R), \tag{2.76}$$

$$\mathbf{q}_2 = K\Delta_I \boldsymbol{\varepsilon}_1^R, \tag{2.77}$$

with $K = (1 + |\Delta|^2)^{-1/2}$. The four vectors $(\mathbf{p}_s, \mathbf{q}_s)[s = 1, 2]$ are real, and their geometrical significance will soon become clear. With the abbreviation

$$\Phi_s = \mathbf{q}\cdot\mathbf{r} + \phi_s(\mathbf{q},\omega) + \delta\delta_{s2}, \tag{2.78}$$

we obtain by combining Eqs. (2.72) and (2.73)

$$\mathcal{R}\left[\mathbf{E}_T(\mathbf{q},\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\right] = \sum_{s=1,2} |E_{T,s}| \left[\mathbf{p}_s \cos(\omega t - \Phi_s) + \mathbf{q}_s \sin(\omega t - \Phi_s)\right]. \tag{2.79}$$

As a function of time the expression in the bracket describes (in general) for the given s an ellipse, and $\mathbf{p}_s$ and $\mathbf{q}_s$ are a pair of so-called conjugate semi-diameters for the ellipse. The state of polarization of the field associated with the monochromatic plane wave $(\mathbf{q}, \omega)$, thus appears as a superposition of two elliptical polarization states with weights $|E_{T,s}|$, $s = 1, 2$. Since

$$\mathbf{p}_1 \times \mathbf{q}_1 \cdot \boldsymbol{\kappa} = -\mathbf{p}_2 \times \mathbf{q}_2 \cdot \boldsymbol{\kappa} = K^2 \Delta_I, \tag{2.80}$$

the end points of the two vectors describing the ellipses traverse these in opposite directions. For $\Delta_I > 0$, we say that the polarization is right-handed for $s = 1$, and left-handed for $s = 2$ (see Fig. 2.2). For $\Delta_I < 0$, the classification is opposite. The corresponding conjugate semi-diameters for the two ellipses are orthogonal, i.e., $\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{q}_1 \cdot \mathbf{q}_2 = 0$. When $\Delta_I = 0$,

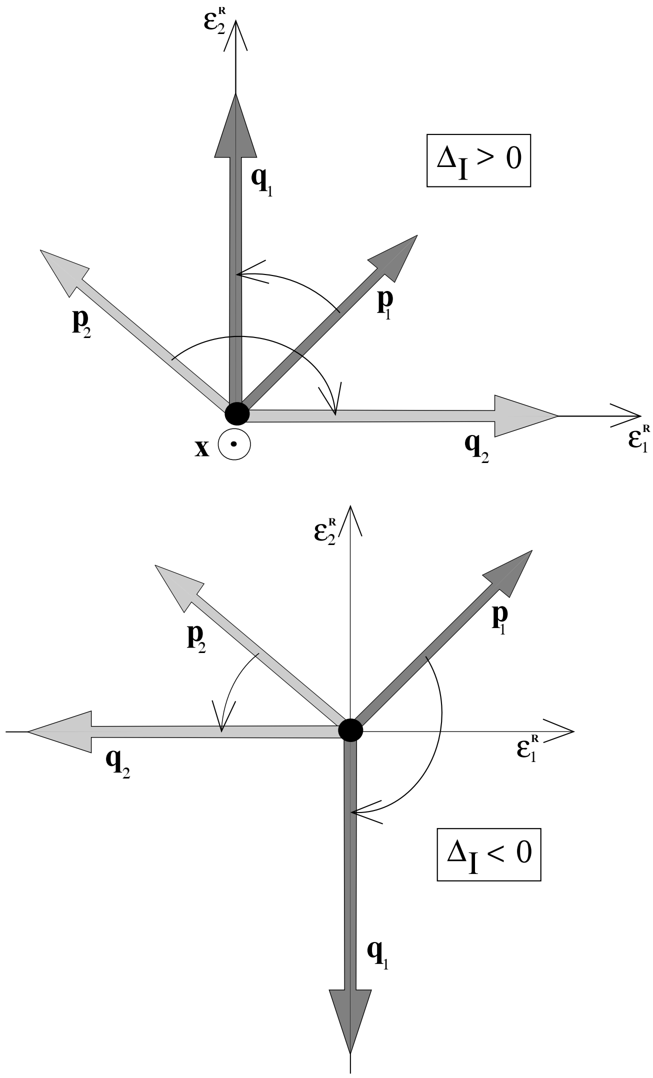

FIGURE 2.2
Sketch of the vector sets $(\mathbf{p}_s, \mathbf{q}_s)$, $s = 1, 2$, related to a general transformation from an OLD set of real and orthonormalized polarization basis vectors, $(\boldsymbol{\varepsilon}_1^R, \boldsymbol{\varepsilon}_2^R)$, to a NEW set of generally complex polarization vectors [see Eqs. (2.73)–(2.77)]. For $\Delta_I > 0$, the polarization is right-handed (with respect to the wave-vector direction $\boldsymbol{\kappa}$) for $s = 1$, and left-handed for $s = 2$. For $\Delta_I < 0$, the classification is the opposite. The real quantity Δ_I is one of three parameters characterizing the employed transformation matrix $\mathbf{T}$ (Eq. (2.68)).

we have $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{0}$, and the two fundamental states now are linearly polarized. In relation to photon physics, the states characterized by $\Delta_R = 0$ and $\Delta_I = +1$ [and hence $K = 1/\sqrt{2}$] are of special importance. Since, now $\mathbf{p}_1 = \mathbf{q}_2 = \boldsymbol{\varepsilon}_1^R/\sqrt{2}$ and $\mathbf{p}_2 = \mathbf{q}_1 = \boldsymbol{\varepsilon}_2^R/\sqrt{2}$, the fundamental states are circularly polarized: right-handed for $s = 1$ and left-handed for $s = 2$. If one makes the choice $\Delta_I = -1$, instead, the two circles are traversed in the opposite sense as before.

The reader may convince herself that an expression of the form $\mathcal{R}\left[(\mathbf{p} + i\mathbf{q})\exp(i\omega t)\right]$ describes an ellipse: set $\mathbf{p} + i\mathbf{q} = (\mathbf{a} + i\mathbf{b})\exp(i\eta)$, and choose the real phase parameter η so that the real vectors $\mathbf{a}$ and $\mathbf{b}$ become mutually orthogonal. With $\tan(2\eta) = 2\mathbf{p}\cdot\mathbf{q}/(p^2 - q^2)$ we find $\mathbf{a} \perp \mathbf{b}$. The connection $\mathcal{R}\left[(\mathbf{p} + i\mathbf{q})\exp(-i\omega t)\right] = \mathbf{a}\cos(\omega t - \eta) + \mathbf{b}\sin(\omega t - \eta)$ in turn evidently shows that the original expression represents an ellipse, with semi-axes $|\mathbf{a}|$ and $|\mathbf{b}|$.

2.6 Wave packets as field modes

In Sec. 2.4, we made an expansion of the electromagnetic field in monochromatic plane-wave modes, and in Sec. 2.5 an analysis of the polarization states associated with the individual modes was undertaken. Notwithstanding the extreme importance of the monochromatic plane-wave expansion in both classical and quantum optics, not least for technical mathematical reasons, it is from a conceptual point of view interesting to investigate the possibility for expanding the classical free field in wave-packet modes. If we follow Einstein's original idea [60] that light might consist of quanta of energy with a point-like structure, it is natural, if one starts from classical optics, to seek to localize the electromagnetic field in narrow wave packets in space-time.

Without loss of generality, let us focus the attention on the positive-frequency part of the transverse electric field, i.e., $\mathbf{E}_T^{(+)}(\mathbf{r}, t)$. In Sec. 2.4 the field was expanded over infinite space. In the following we replace this continuous mode representation by an expansion over a finite cubic volume, $V = L^3$. Here, this is done for mathematical simplicity but we note that the expansion over a finite volume, not necessarily cubic, is of physical importance in studies of for instance atom-field interaction in cavities [211, 158] and field propagation in confined structures [225, 101, 91]. In going from infinite-space to finite-space mode expansion, Fourier integrals involving a continuum of wave vectors $\mathbf{q}$, are replaced by Fourier series (discrete sums) with discrete wave vectors $\mathbf{q}_\alpha$, only. In expressions involving *bilinear* combinations of fields (e.g., field energy, momentum, and angular momentum) the correspondence is as follows:

$$\int_{-\infty}^{\infty} (\cdots)\frac{\mathrm{d}^3 q}{(2\pi)^3} \Leftrightarrow \frac{1}{L^3}\sum_{\mathbf{q}_\alpha}(\cdots). \tag{2.81}$$

If, for brevity , we let the index i (or j) stand for the combination $(\mathbf{q}_\alpha, s_\beta)$ of wave vector and polarization $(\beta = 1, 2)$ indices, the correspondence for the positive-frequency electric field takes the form

$$\begin{aligned}\mathbf{E}_T^{(+)}(\mathbf{r}, t) &= \sum_s \int_{-\infty}^{\infty} E_{T,s}(\mathbf{q}, cq)\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})e^{i(\mathbf{q}\cdot\mathbf{r} - cqt)}\frac{\mathrm{d}^3 q}{(2\pi)^3} \\ &\Leftrightarrow \frac{1}{L^{\frac{3}{2}}}\sum_i E_{T,i}\boldsymbol{\varepsilon}_i e^{i(\mathbf{q}_i\cdot\mathbf{r} - \omega_i t)},\end{aligned} \tag{2.82}$$

with $\omega_i = c|\mathbf{q}_i|$. The discrete ($E_{T,i}$) and continuous ($E_{T,s}(\mathbf{q}, cq)$) amplitudes do not have the same dimension, but the precise relation between them is not needed in the treatment below.

Let us thus consider wave packets formed by linear combination of monochromatic plane-wave modes, viz.,

$$\mathbf{w}_m^{(+)}(\mathbf{r}, t) = \frac{1}{L^{\frac{3}{2}}} \sum_j t_{mj} \boldsymbol{\varepsilon}_j e^{i(\mathbf{q}_j \cdot \mathbf{r} - \omega_j t)}, \tag{2.83}$$

where the coefficients t_{mj} are elements of a certain type of square matrix, as we shall see soon. To each m belongs a wave-packet mode composed of only positive-frequency plane waves. With an appropriate $\mathbf{t}$-matrix, the general field $\mathbf{E}_T^{(+)}(\mathbf{r}, t)$ can be resolved in terms of a complete orthonormal set of $\mathbf{w}_m^{(+)}(\mathbf{r}, t)$-modes. Utilizing Eq. (2.83) one obtains

$$\begin{aligned}
&\int_V [\mathbf{w}_{\mathbf{m}}^{(+)}(\mathbf{r}, \mathbf{t})]^* \cdot \mathbf{w}_n^{(+)}(\mathbf{r}, t) \mathrm{d}^3 r \\
&= \sum_{i,j} t_{mi}^* t_{nj} \boldsymbol{\varepsilon}_i^* \cdot \boldsymbol{\varepsilon}_j e^{i(\omega_i - \omega_j)t} \left[\frac{1}{L^3} \int_V e^{i(\mathbf{q}_j - \mathbf{q}_i) \cdot \mathbf{r}} \mathrm{d}^3 r \right] \\
&= \sum_{i,j} t_{mi}^* t_{nj} \boldsymbol{\varepsilon}_i^* \cdot \boldsymbol{\varepsilon}_j \delta_{\mathbf{q}_i \mathbf{q}_j}.
\end{aligned} \tag{2.84}$$

The presence of the Kronecker delta $\delta_{\mathbf{q}_i \mathbf{q}_j}$ implies that the base vectors $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}_j$ belong to the same wave-vector direction. Use of the orthonormality condition in Eq. (2.52) then leads to the result

$$\int_V [\mathbf{w}_{\mathbf{m}}^{(+)}(\mathbf{r}, \mathbf{t})]^* \cdot \mathbf{w}_n^{(+)}(\mathbf{r}, t) \mathrm{d}^3 r = \delta_{mn} \tag{2.85}$$

provided the $\mathbf{t}$-matrix satisfies the condition

$$\sum_i t_{mi}^* t_{ni} = \delta_{mn}. \tag{2.86}$$

If Eq. (2.85) is obeyed the wave-packet modes are orthonormalized. The condition in Eq. (2.86) will be met if $\mathbf{t}$ is a unitary matrix:

$$\mathbf{t}\mathbf{t}^\dagger = \mathbf{U} = \mathbf{t}^\dagger \mathbf{t}, \tag{2.87}$$

or in component form

$$\sum_m t_{im} t_{jm}^* = \delta_{ij} = \sum_m t_{mi}^* t_{mj}, \tag{2.88}$$

since per definition $t_{mj}^\dagger = t_{jm}^*$ ($t_{im}^\dagger = t_{mi}^*$). [The first member of Eq. (2.88) leads to Eq. (2.86) upon the following renaming of indices: $i \to n$, $j \to m$, $m \to i$.] The second member of Eq. (2.88) allows one to rewrite the expression for the positive-frequency electric field in the form

$$\begin{aligned}
\mathbf{E}_T^{(+)}(\mathbf{r}, t) &= L^{-\frac{3}{2}} \sum_{i,j} E_{T,i} \delta_{ij} \boldsymbol{\varepsilon}_j e^{i(\mathbf{q}_j \cdot \mathbf{r} - \omega_j t)} \\
&= L^{-\frac{3}{2}} \sum_{i,j,m} E_{T,i} t_{mi}^* t_{mj} \boldsymbol{\varepsilon}_j e^{i(\mathbf{q}_j \cdot \mathbf{r} - \omega_j t)}.
\end{aligned} \tag{2.89}$$

By means of Eq. (2.83) one finally obtains

$$\mathbf{E}_T^{(+)}(\mathbf{r},t) = \sum_m \mathcal{E}_{T,m}\mathbf{w}_m^{(+)}(\mathbf{r},t), \tag{2.90}$$

where

$$\mathcal{E}_{T,m} = \sum_i E_{T,i}t^*_{mi}. \tag{2.91}$$

Eq. (2.90) shows that an arbitrary transverse electric field, $\mathbf{E}_T^{(+)}(\mathbf{r},t)$, can be expanded after an orthonormalized set of wave-packet modes, $\mathbf{w}_m^{(+)}(\mathbf{r},t)$, the expansion coefficient being $\mathcal{E}_{T,m}$ [Eq. (2.91)] for the mth mode. Such an expansion is important in photon physics, because it enables one to associate a photon concept also with non-monochromatic localized field distributions.

The wave-packet modes satisfy a completeness (closure) theorem in the subspace of transverse vector fields. To prove this, one inserts the plane-wave expansion of $\mathbf{w}_m^{(+)}$ [Eq. (2.83)] in the relevant sum of dyadic products. Hence,

$$\sum_m [\mathbf{w}_m^{(+)}(\mathbf{r}',t)]^*\mathbf{w}_n(\mathbf{r},t) = \frac{1}{L^3}\sum_{m,i,j} t^*_{mi}t_{mj}\boldsymbol{\varepsilon}^*_i\boldsymbol{\varepsilon}_j e^{i(\mathbf{q}_j\cdot\mathbf{r}-\omega_j t)}e^{-i(\mathbf{q}_i\cdot\mathbf{r}'-\omega_i t)}. \tag{2.92}$$

The sum over m gives δ_{ij}, since $\mathbf{t}$ is unitary ([Eq. (2.88)], last member), and the presence of the Kronecker delta immediately reduces the remaining double summation to a single summation. Therefore,

$$\begin{aligned}\sum_m [\mathbf{w}_m^{(+)}(\mathbf{r}',t)]^*\mathbf{w}_m^{(+)}(\mathbf{r},t) &= \frac{1}{L^3}\sum_i \boldsymbol{\varepsilon}^*_i\boldsymbol{\varepsilon}_i e^{i\mathbf{q}_i\cdot(\mathbf{r}-\mathbf{r}')}\\ \Leftrightarrow \int_{-\infty}^{\infty}&\left[\sum_s \boldsymbol{\varepsilon}^*_s(\boldsymbol{\kappa})\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})\right]e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\frac{\mathrm{d}^3q}{(2\pi)^3}.\end{aligned} \tag{2.93}$$

If one multiplies Eq. (2.52) (with the factors in the scalar product interchanged) by $\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})$, and thereafter makes a summation over s, one obtains

$$\boldsymbol{\varepsilon}_{s'}(\boldsymbol{\kappa})\cdot\left[\sum_s \boldsymbol{\varepsilon}^*_s(\boldsymbol{\kappa})\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa})\right] = \sum_s \delta_{ss'}\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa}) = \boldsymbol{\varepsilon}_{s'}(\boldsymbol{\kappa}). \tag{2.94}$$

The sum in the square bracket hence must be the 2×2 unit tensor in the subspace of the transverse vector fields. If $\mathbf{U}$ denotes the unit tensor (3×3) in the full vector field space, we have the dyadic relation

$$\sum_s \boldsymbol{\varepsilon}^*_s(\boldsymbol{\kappa})\boldsymbol{\varepsilon}_s(\boldsymbol{\kappa}) = \mathbf{U} - \boldsymbol{\kappa}\boldsymbol{\kappa}. \tag{2.95}$$

When Eq. (2.95) is inserted into the integral expression in Eq. (2.93), one obtains a completeness theorem of the form

$$\sum_m [\mathbf{w}_m^{(+)}(\mathbf{r}',t)]^*\mathbf{w}_m^{(+)}(\mathbf{r},t) = \delta_T(\mathbf{r}-\mathbf{r}'), \tag{2.96}$$

where $\delta_T(\mathbf{r}-\mathbf{r}')$ is the transverse delta function, a dyadic quantity, which in the description involving a continuum of wave vectors is given by

$$\delta_T(\mathbf{r}-\mathbf{r}') = \int_{-\infty}^{\infty}(\mathbf{U}-\boldsymbol{\kappa}\boldsymbol{\kappa})e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\frac{\mathrm{d}^3q}{(2\pi)^3}. \tag{2.97}$$

The transverse delta function is *not* zero for $\mathbf{r} \neq \mathbf{r}'$, but decays as $|\mathbf{r}-\mathbf{r}'|^{-3}$. The fact that the left-hand side of Eq. (2.96) is appreciably different from zero for space points with so-called near-field separation has important consequences for the spatial localization problem for transverse photons, as we shall see in later chapters. Since $\delta_T(\mathbf{r}-\mathbf{r}')$ formally is the Fourier transform of a function, $\mathbf{U}-\boldsymbol{\kappa\kappa}$, which does not tend to zero for $|\mathbf{q}| \to \infty$, $\delta_T(\mathbf{r}-\mathbf{r}')$ has a singularity at $\mathbf{r}' = \mathbf{r}$ which one must regularize by a procedure relating in the proper manner to the physics over short distances.

The photon localization problem relates to the impossibility of creating an electromagnetic field which is different from zero only in a single space point (at a given time). A field with only delta function support is physically untenable for several reasons, as we shall discuss later on. From a mathematical point of view, all purely transverse electric fields, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_T(\mathbf{r},t)$, $\forall\mathbf{r}$, must obey the identity

$$\mathbf{E}_T(\mathbf{r},t) = \nabla \times \int_{-\infty}^{\infty} \frac{\nabla' \times \mathbf{E}_T(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathrm{d}^3 r', \tag{2.98}$$

cf. the Helmholtz theorem (in particular Eq. (2.11)). The reader may readily convince herself that a postulated field of the form $\mathbf{E}_T(\mathbf{r},t) = \mathbf{A}(t)\delta(\mathbf{r})$ cannot satisfy Eq. (2.98). Furthermore, such a field is not a genuine transverse vector field because its divergence does not vanish at $\mathbf{r} = \mathbf{0}$ (the subscript T on the postulated field thus is misleading!).

2.7 Conservation of energy, moment of energy, momentum, and angular momentum

For a free electromagnetic field the total energy

$$H_T = \frac{\epsilon_0}{2} \int_{-\infty}^{\infty} \left[\mathbf{E}_T(\mathbf{r},t) \cdot \mathbf{E}_T(\mathbf{r},t) + c^2 \mathbf{B}(\mathbf{r},t) \cdot \mathbf{B}(\mathbf{r},t)\right] \mathrm{d}^3 r, \tag{2.99}$$

the total moment of energy

$$\mathbf{K}_T = \frac{\epsilon_0}{2} \int_{-\infty}^{\infty} \mathbf{r} \left[\mathbf{E}_T(\mathbf{r},t) \cdot \mathbf{E}_T(\mathbf{r},t) + c^2 \mathbf{B}(\mathbf{r},t) \cdot \mathbf{B}(\mathbf{r},t)\right] \mathrm{d}^3 r, \tag{2.100}$$

the total momentum

$$\mathbf{P}_T = \epsilon_0 \int_{-\infty}^{\infty} \mathbf{E}_T(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t) \mathrm{d}^3 r, \tag{2.101}$$

and the total angular momentum

$$\mathbf{J}_T = \epsilon_0 \int_{-\infty}^{\infty} \mathbf{r} \times \left[\mathbf{E}_T(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)\right] \mathrm{d}^3 r, \tag{2.102}$$

all are constants, that is, independent of time. In order to prove that

$$\frac{\mathrm{d}H_T}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\mathbf{P}_T}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{J}_T}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{K}_T}{\mathrm{d}t} = \mathbf{0}, \tag{2.103}$$

one first differentiates the various integrands of Eq. (2.99)-(2.102) with respect to t, and thereafter the time derivatives $\partial\mathbf{E}_T(\mathbf{r},t)/\partial t$ and $\partial\mathbf{B}(\mathbf{r},t)/\partial t$ are eliminated by means of the Maxwell equations given in Eqs. (2.1) and (2.2). Finally, certain integrals, viz., those containing integrands of the types $\nabla \cdot \mathbf{f}$ and ∇f, are transformed into surface integrals. Since all free electromagnetic fields are generated (emitted) by sources with finite support in space-time, the detached fields are for all finite times contained in a certain (finite) volume of space. If the surfaces mentioned above are placed outside this volume, the surface integrals vanish, and the results in Eqs. (2.103) follow.

In relativistic quantum theory, the ten generators of the Poincaré group are identified with the operators $\hat{H}_T$, $\hat{\mathbf{K}}_T$, $\hat{\mathbf{P}}_T$, and $\hat{\mathbf{J}}_T$ associated with the classical quantities in Eq. (2.99)-(2.102). These operators generate infinitesimal time ($\hat{H}_T$)- and space ($\hat{\mathbf{P}}_T$)- translations, rotations ($\hat{\mathbf{J}}_T$), and boosts ($\hat{\mathbf{K}}_T$). Physical states are labelled by the eigenvalues of those operators which are conserved, i.e., that commute with the energy operator. Since the commutator $[\mathbf{K}_T, H_T] \neq \mathbf{0}$, the eigenvalues of the boost operator are not used to label a free photon state [209, 242].

The quantities $\mathbf{W}_T(\mathbf{r},t) \equiv (\epsilon_0/2)[\mathbf{E}_T^2(\mathbf{r},t) + c^2\mathbf{B}^2(\mathbf{r},t)]$ and $c^2\mathbf{P}_T(\mathbf{r},t) = \mathbf{S}_T(\mathbf{r},t) \equiv \mu_0^{-1}\mathbf{E}_T(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)$ are known as the electromagnetic energy density and the Poynting vector (or energy flux density). It may be shown, that these designations are meaningful only relative to an observer at rest in the frame in which the fields are specified [206, 101].

2.8 Riemann–Silberstein formalism

The information contained in the set of free-space Maxwell equations, given in Eqs. (2.1)-(2.4), can be written in compact form by introduction of the complex so-called Riemann–Silberstein vector

$$\mathbf{F}_+(\mathbf{r},t) = \sqrt{\frac{\epsilon_0}{2}}\left(\mathbf{E}_T(\mathbf{r},t) + ic\mathbf{B}(\mathbf{r},t)\right). \tag{2.104}$$

Thus, by multiplying Eq. (2.2) by ic, and adding hereafter the resulting equation and Eq. (2.1), it appears that $\mathbf{F}_+(\mathbf{r},t)$ satisfies the dynamical equation

$$\frac{i}{c}\frac{\partial}{\partial t}\mathbf{F}_+(\mathbf{r},t) = \nabla \times \mathbf{F}_+(\mathbf{r},t). \tag{2.105}$$

By taking the divergence of this equation, and interchanging the $\partial/\partial t$- and ∇-operators it follows that $\nabla \cdot \mathbf{F}_+(\mathbf{r},t)$ must equal a time independent constant, which possibly may be space dependent. However, the combination of the Maxwell Eqs. (2.3) and (2.4) shows that $\mathbf{F}_+(\mathbf{r},t)$ is a divergence-free complex vector field, i.e.,

$$\nabla \cdot \mathbf{F}_+(\mathbf{r},t) = 0. \tag{2.106}$$

Together, Eqs. (2.105) and (2.106) represent a complex version of the free-space Maxwell equations. Applications in the framework of classical electrodynamics of this so-called Riemann–Silberstein version, which seems to date back to Riemann [240], were given a hundred years ago first by Silberstein [219, 220, 221], and shortly afterward by Bateman [11]. In the presence of charges, the electric field $\mathbf{E}(\mathbf{r},t)$ is no longer a transverse vector field, and in the original Riemann–Silberstein description of classical electrodynamics $\mathbf{E}(\mathbf{r},t)$ enters the definition of $\mathbf{F}_+(\mathbf{r},t)$ instead of $\mathbf{E}_T$.

In recent time $\mathbf{F}_+(\mathbf{r},t)$, and its complex conjugate

$$\mathbf{F}_-(\mathbf{r},t) = \sqrt{\frac{\epsilon_0}{2}}\left(\mathbf{E}_T(\mathbf{r},t) - ic\mathbf{B}(\mathbf{r},t)\right), \tag{2.107}$$

have played an important role in photon wave mechanics, because the positive-frequency parts of these vectors relate to the so-called energy wave function of the photon [see Chapt. 13]. Because only the transverse field dynamics is quantized in this approach it is useful here to use $\mathbf{E}_T(\mathbf{r},t)$ in the definition of $\mathbf{F}_+(\mathbf{r},t)$ and $\mathbf{F}_-(\mathbf{r},t)$. It is easy to show that $\mathbf{F}_-(\mathbf{r},t)$ satisfy the dynamical equation

$$\frac{i}{c}\frac{\partial}{\partial t}\mathbf{F}_-(\mathbf{r},t) = -\nabla\times\mathbf{F}_-(\mathbf{r},t), \tag{2.108}$$

and, of course

$$\nabla\cdot\mathbf{F}_-(\mathbf{r},t) = 0. \tag{2.109}$$

It is convenient for later use in photon wave mechanics and quantum optics, to write the equations for $\mathbf{F}_+(\mathbf{r},t)$ and $\mathbf{F}_-(\mathbf{r},t)$ in the compact notation

$$i\frac{\partial}{\partial t}\mathbf{F}_\pm(\mathbf{r},t) = \pm c\nabla\times\mathbf{F}_\pm(\mathbf{r},t), \tag{2.110}$$

$$\nabla\cdot\mathbf{F}_\pm(\mathbf{r},t) = 0. \tag{2.111}$$

It appears from Eq. (2.40) that the positive-frequency parts of the Riemann–Silberstein vectors have the plane-wave expansions

$$\mathbf{F}_\pm^{(+)}(\mathbf{r},t) = \int_{-\infty}^{\infty}\mathbf{F}_\pm(\mathbf{q},cq)e^{i(\mathbf{q}\cdot\mathbf{r}-cqt)}\frac{\mathrm{d}^3q}{(2\pi)^3}, \tag{2.112}$$

and by combining Eqs. (2.110) and (2.112) it follows that the Fourier amplitudes must satisfy the algebraic equations

$$\mathbf{F}_\pm(\mathbf{q},cq) = \pm i\boldsymbol{\kappa}\times\mathbf{F}_\pm(\mathbf{q},cq). \tag{2.113}$$

To determine the polarization states of the Riemann–Silberstein vectors one writes these in the form

$$\mathbf{F}_\pm(\mathbf{q},cq) = F_\pm(\mathbf{q},cq)\mathbf{e}_\pm(\boldsymbol{\kappa}), \tag{2.114}$$

where $\mathbf{e}_\pm(\boldsymbol{\kappa}) = \mathbf{p}_\pm + i\mathbf{q}_\pm$ are the relevant unit polarization vectors. Since these vectors obey the relations

$$\mathbf{e}_\pm(\boldsymbol{\kappa}) = \pm i\boldsymbol{\kappa}\times\mathbf{e}_\pm(\boldsymbol{\kappa}), \tag{2.115}$$

their real ($\mathbf{p}_\pm$) and imaginary ($\mathbf{q}_\pm$) parts must be connected by

$$\mathbf{q}_\pm = \pm\boldsymbol{\kappa}\times\mathbf{p}_\pm, \tag{2.116}$$

and $|\mathbf{p}_\pm| = |\mathbf{q}_\pm| = 1/\sqrt{2}$. From the triple products

$$\mathbf{p}_\pm\times\mathbf{q}_\pm\cdot\boldsymbol{\kappa} = \pm\frac{1}{2} \tag{2.117}$$

it then appears that the Riemann–Silberstein $\mathbf{F}_+(\mathbf{q},cq)$ and $\mathbf{F}_-(\mathbf{q},cq)$ relate to states which are right ($\mathbf{e}_+$)- and left ($\mathbf{e}_-$)-hand circular polarized, respectively. Expressed in terms of a set of real basis vectors, and with a phase parameter δ included, one has

$$\mathbf{e}_+(\boldsymbol{\kappa}) = \frac{1}{\sqrt{2}}\left(\boldsymbol{\varepsilon}_1^R(\boldsymbol{\kappa}) + i\boldsymbol{\varepsilon}_2^R(\boldsymbol{\kappa})\right), \tag{2.118}$$

$$\mathbf{e}_-(\boldsymbol{\kappa}) = \frac{e^{i\delta}}{\sqrt{2}}\left(-\boldsymbol{\varepsilon}_1^R(\boldsymbol{\kappa}) + i\boldsymbol{\varepsilon}_2^R(\boldsymbol{\kappa})\right). \tag{2.119}$$

In photon wave mechanics, $\mathbf{F}_+^{(+)}(\mathbf{r},t)$ and $\mathbf{F}_-^{(+)}(\mathbf{r},t)$ may be used to describe single-photon wave packets composed of positive (+) and negative (−) helicity species. The helicity concept will be introduced in Sec. 10.3 and employed throughout this book.

Since the two Riemann–Silberstein vectors $\mathbf{F}_+$ and $\mathbf{F}_-$, given in Eqs. (2.104) and (2.107), and now divided into their positive- and negative-frequency parts, i.e.,

$$\mathbf{F}_\pm(\mathbf{r},t) = \mathbf{F}_\pm^{(+)}(\mathbf{r},t) + \mathbf{F}_\pm^{(-)}(\mathbf{r},t), \tag{2.120}$$

are each other's complex conjugate, the following relations must be satisfied:

$$\mathbf{F}_\pm^{(-)}(\mathbf{r},t) = \left(\mathbf{F}_\mp^{(+)}(\mathbf{r},t)\right)^*. \tag{2.121}$$

In Sec. 2.7, expressions were given for the energy, moment of energy, momentum, and angular momentum of the transverse electromagnetic field. The reader may prove to himself that these conserved quantities can be given in forms of integrals involving only one of the Riemann–Silberstein vectors. Thus, for instance

$$H_T = \int_{-\infty}^{\infty} \mathbf{F}_+^*(\mathbf{r},t)\cdot\mathbf{F}_+(\mathbf{r},t)\mathrm{d}^3r, \tag{2.122}$$

$$\mathbf{K}_T = \int_{-\infty}^{\infty} \mathbf{r}\mathbf{F}_+^*(\mathbf{r},t)\cdot\mathbf{F}_+(\mathbf{r},t)\mathrm{d}^3r, \tag{2.123}$$

$$\mathbf{P}_T = \frac{1}{ic}\int_{-\infty}^{\infty} \mathbf{F}_+^*(\mathbf{r},t)\times\mathbf{F}_+(\mathbf{r},t)\mathrm{d}^3r, \tag{2.124}$$

and

$$\mathbf{J}_T = \frac{1}{ic}\int_{-\infty}^{\infty} \mathbf{r}\times\left(\mathbf{F}_+^*(\mathbf{r},t)\times\mathbf{F}_+(\mathbf{r},t)\right)\mathrm{d}^3r. \tag{2.125}$$

It turns out that these classical field quantities lead directly to proper quantum mechanical mean values of the photon energy wave function [16, 123]; see also Secs. 13.3-13.5.

2.9 Propagation of analytical signal

If a vector field $\mathbf{W}(\mathbf{r},t)$ satisfies the free-space wave equation $\Box\mathbf{W}(\mathbf{r},t)=\mathbf{0}$, we have seen that the analytical signal $\mathbf{W}^{(+)}(\mathbf{r},t)$ may be given in terms of the integral representation in Eq. (2.40). What kind of propagation equation does the analytical signal satisfy? To answer this question, it is useful first to operate with $-\nabla^2$ on Eq. (2.40). This gives one the formula

$$-\nabla^2\mathbf{W}(\mathbf{r},t) = \int_{-\infty}^{\infty} q^2\mathbf{W}(\mathbf{q},cq)e^{i(\mathbf{q}\cdot\mathbf{r}-cqt)}\frac{\mathrm{d}^3q}{(2\pi)^3}. \tag{2.126}$$

This suggests that we use the symbolic notation $\sqrt{-\nabla^2}\mathbf{W}^{(+)}(\mathbf{r},t)$ to denote the integral that appears if q^2 is replaced by q on the right-hand side of Eq. (2.126):

$$\sqrt{-\nabla^2}\mathbf{W}^{(+)}(\mathbf{r},t) \equiv \int_{-\infty}^{\infty} q\mathbf{W}(\mathbf{q},cq)e^{i(\mathbf{q}\cdot\mathbf{r}-cqt)}\frac{\mathrm{d}^3q}{(2\pi)^3}. \tag{2.127}$$