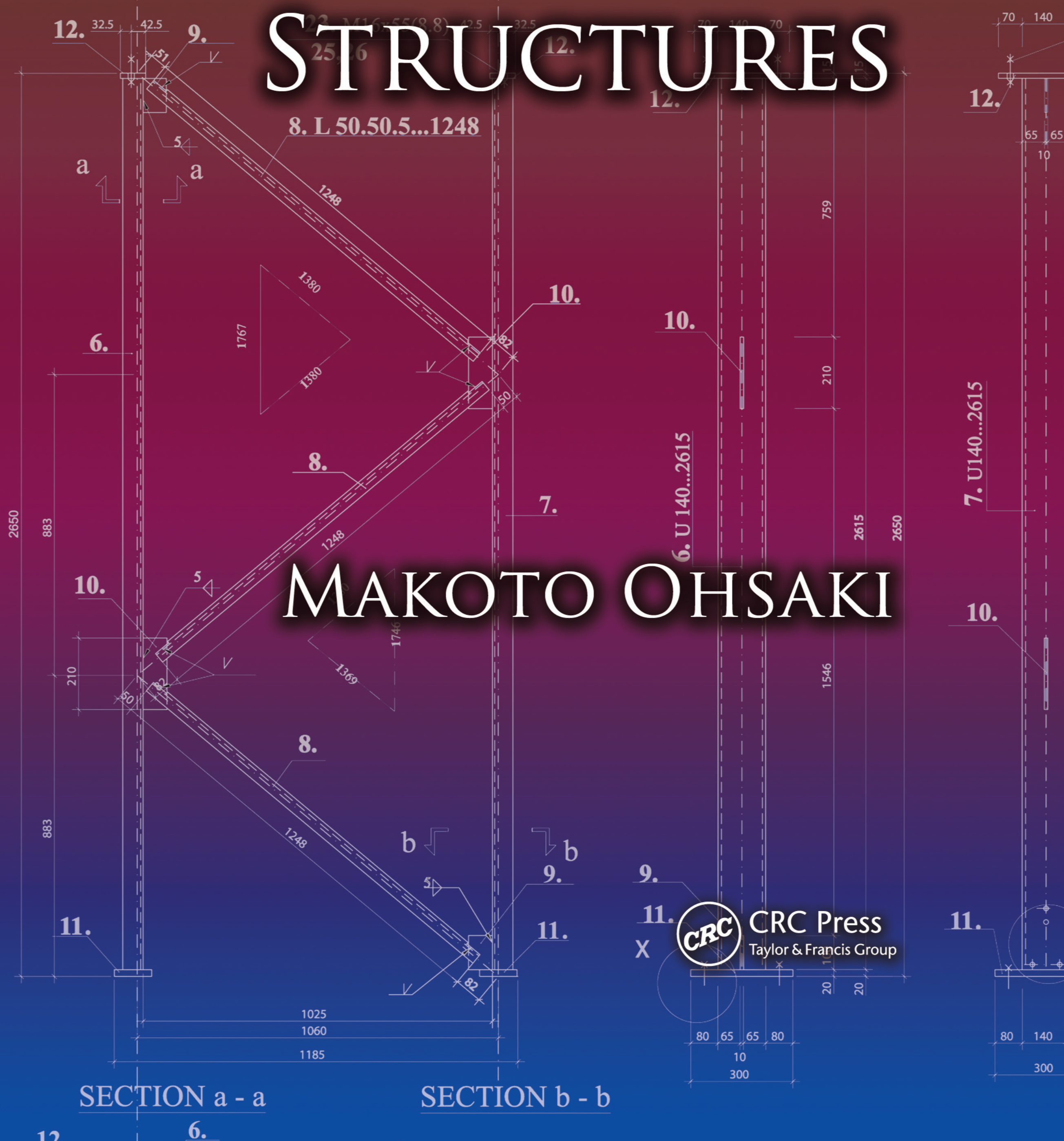
Optimization of Finite Dimensional Structures

Makoto Ohsaki



CRC Press
Taylor & Francis Group

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Makoto Ohsaki
Hiroshima University
Higashi-Hiroshima, Japan

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-13: 978-1-4398-2004-9 (Ebook-PDF)

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Preface

The attempt to find mechanically efficient structural designs and shapes was initiated mainly in the fields of mechanical engineering and aeronautical engineering, which established the field known as *structural optimization*. Many practically acceptable results have been developed for application to automobiles and aircraft. Some examples are structural components, including the wings of aircraft and engine mounts of automobiles, which can be fully optimized using efficient shape optimization techniques.

In contrast, regarding civil engineering and architectural engineering, structural optimization is difficult to apply because structures in these fields are not mass products: structures are designed in accordance with their specific design requirements. Furthermore, the structure's shape and geometry are determined by a designer or an architect in view of nonstructural performance, including the aesthetic perspective. Therefore, the main role of structural engineers is often limited to selection of materials, determination of member sizes through structural analyses, planning details of the construction process, and so on. However, for special structures, such as shells, membrane structures, spatial long-span frames, and highrise buildings, the structural shape should be determined in view of the responses against static and dynamic loads. In truth, the beauty of the structural form is related closely to the mechanical performance of the structure. Therefore, cooperation between designers and structural engineers is very important in designing such structures.

Even for building frames, because of the recent trend of *performance-based design*, optimization has been identified as a powerful tool for designing structures under constraints imposed on practical performance measures, including elastic/plastic stresses and displacements under static/dynamic design loads. Furthermore, recent rapid advancements in the areas of computer hardware and software enabled us to carry out structural analysis many times to obtain optimal or approximately optimal designs. Optimization of real-world structures with realistic objective function and constraints is possible through quantitative evaluation of nonstructural performance criteria, e.g., aesthetic properties, and life-cycle costs, including costs of construction, fabrication, and maintenance.

Many books describing structural optimization have been published since the 1960s; e.g., Hemp (1973), Rozvany (1976), Haug and Cea (1981), Haftka, Gürdal, and Kamat (1990), Papalambos and Wilde (2000), Bendsøe and Sigmund (2003), Arora (2004), etc. These books are mainly classifiable into the following three categories:

1. Basic theories and methodologies for optimization with examples of small structural optimization problems.

2. Continuum-based approaches for application to mechanical and aeronautical structures.

3. Theoretical and analytical results of structural optimization in earlier times without the assistance of computer technology.

Using books of the first category, readers can learn only the concepts and some difficult theories of structural optimization without application to large-scale structures. On the other hand, for the books of the second category, a good background in applied mathematics and continuum mechanics is needed to fully understand the basic concepts and methods. Unfortunately, most researchers, practicing engineers, and graduate students in the field of civil engineering have no such background and are not strongly interested in the basic theories or methods of structural optimization. Also, in mechanical engineering, the finite element approach is used for practical applications, and complex practical design problems are solved in a finite dimensional formulation.

The derivatives of objective and constraint functions, called design sensitivity coefficients, should be computed if a gradient-based approach is used for structural optimization. However, most methods of *design sensitivity analysis* are developed mainly for a continuum utilizing variational principles, for which sensitivity coefficients are to be computed for a functional, such as compliance that can be formulated in an integral of a bilinear form of response. For finite dimensional structures, including trusses and frames, variational formulations are not needed, and sensitivity coefficients can be found simply by differentiating the governing equations in a matrix-vector form.

Another important aspect of optimization in civil engineering is that the design variables often have discrete values: the frame members are usually selected from a pre-assigned list or catalog of available sections. Furthermore, some traditional layouts are often used for plane and spatial trusses and for latticed domes. Therefore, the optimization problem often turns out to be a combinatorial problem, a fact that is not fully introduced into most books addressing the study of structural optimization.

This book introduces methodologies and applications that are closely related to design problems of *finite dimensional structures*, to serve thereby as a bridge between the communities of structural optimization in mechanical engineering and the researchers and engineers in civil engineering. The book provides readers with the basics of optimization of frame structures, such as trusses, building frames, and long-span structures, with descriptions of various applications to real-world problems.

Recently, many efficient techniques of optimization have been developed for convex programming problems, e.g., semidefinite programming and interior point algorithms, which are extensions of the approaches used for linear and quadratic programming problems. The book introduces application of these methods to optimization of finite-dimensional structures. Approximate methods resembling the conventional optimality criteria approaches have also been developed with no reference to the pioneering papers in the 1960s and 1970s. Therefore, it is extremely important to describe their development history to young researchers so that similar methods are not re-developed with no knowledge related to conventional approaches. For that reason, another purpose of this book is to present the historical development of the methodologies and theorems on optimization of frame structures.

The book is organized as follows:

In Chapter 1, the basic concepts and methodologies of optimization of trusses and frames are presented with illustrative examples. Traditional problems with constraints on limit loads, member stresses, compliance, and eigenvalues of vibration are described in detail. A brief introduction is also presented for multiobjective structural optimization, and the shape and topology optimization of trusses.

In Chapter 2, the method of design sensitivity analysis, which is a necessary tool for optimization using nonlinear programming, is presented for various response quantities, including static response, eigenvalue of vibration, transient response for dynamic load, and so on. All formulations are written in matrix-vector form without resort to variational formulation to support ready comprehension by researchers and engineers.

In Chapter 3, details of truss topology optimization are described, including historical developments and difficulties in problems with stress constraints and multiple eigenvalue constraints. Recently developed formulations by semidefinite programming and mixed integer programming are introduced. Applications to plane and spatial trusses are demonstrated.

Chapter 4 presents methods for configuration optimization for simultaneously optimizing the geometry and topology of trusses. Difficulties in optimization of regular trusses are extensively discussed, and an application to generating a link mechanism is presented.

Chapter 5 summarizes various results of optimization of building frames. Uniqueness of the optimal solution of a regular frame is first investigated, and applications of parametric programming are presented. Multiobjective optimization problems are also presented for application to seismic design, and a simple heuristic method based on local search is presented.

In Chapter 6, as a unique aspect of this book, optimization results are presented for spatial trusses and latticed domes. Simple applications of nonlinear programming and heuristic methods are first introduced, and the spatial variation of seismic excitation is addressed in the following sections. The trade-off designs between geometrical properties and stiffness under static loads are shown for arch-type frames and latticed domes described using parametric curve and surface.

Mathematical preliminaries and basic methodologies are summarized in the Appendix, so that readers can understand the details, if necessary, without the exposition of tedious mathematics presented in the main chapters. Various methodologies specifically utilized in some of the sections, e.g., the response spectrum approach for seismic response analysis, are also explained in the Appendix. Also, many small examples that can be solved by hand or using a simple program are presented in the main chapters. Therefore, this book is self-contained, and easily used as a textbook or sub-textbook in a graduate course.

The author would like to deliver his sincere appreciation to Prof. Tsuneyoshi Nakamura, Prof. Emeritus of Kyoto University, Japan, for supervising the author's study for master's degree and Ph.D. dissertation on structural optimization. Supervision by Prof. Jasbir S. Arora of The University of Iowa during the author's sabbatical leave is also acknowledged.

The numerical examples in this book are a compilation of the author's work on structural optimization at Kyoto University, Japan, during the period 1985–2010. The author would like to extend his appreciation to researchers for collaborations on the studies that appear as valuable contents in this book, namely, Prof. Naoki Katoh of the Dept. of Architecture and Architectural Engineering, Kyoto University; Prof. Shinji Nishiwaki of the Dept. of Mechanical Engineering and Science, Kyoto University; Prof. Hiroshi Tagawa of the Dept. of Environmental Engineering and Architecture, Nagoya University; Prof. Yoshihiro Kanno of the Dept. of Mathematical Informatics, University of Tokyo; Prof. Peng Pan of the Dept. of Civil Engineering, Tsinghua University, P. R. China; Dr. Takao Hagishita of Mitsubishi Heavy Industries; Mr. Yuji Kato of JSOL Corporation; Mr. Takuya Kinoshita, Mr. Shinnosuke Fujita, and Mr. Ryo Watada, graduate students in the Dept. of Architecture and Architectural Engineering, Kyoto University. The author would also like to thank again Prof. Yoshihiro Kanno of University of Tokyo for checking the details of the manuscript.

The assistance of Ms. Kari Budyk and Ms. Leong Li-Ming of CRC Press and Taylor & Francis in bringing the manuscript to its final form is heartily acknowledged.

January 2010 Makoto Ohsaki

Contents

Chapter 1

Various Formulations of Structural Optimization

Various formulations of optimization of finite dimensional structures are presented in this chapter. The concepts of structural optimization are first presented in Sec. 1.1 followed by historical review in Sec. 1.2. The basic formulations are presented in Sec. 1.3 with an illustrative example. The simple optimization approach to plastic design that is formulated as a linear programming problem is presented in Sec. 1.4. Optimization results under stress constraints are shown in Sec. 1.5. The approximate method called fully-stressed design (FSD) is presented in Sec. 1.6 with investigation of the relation between optimum design and FSD. The optimality criteria approach to a problem with displacement constraints is presented in Sec. 1.7. Problems concerning the compliance and frequency of free vibration as measures of static and dynamic stiffness are extensively studied in Secs. 1.8 and 1.9, respectively. An example of shape and topology optimization of a truss is presented in Sec. 1.10 as an introduction to Chaps. 3 and 4. The basic formulation of multiobjective structural optimization programming and various methodologies of heuristics are shown in Secs. 1.11 and 1.12, respectively, as an introduction to several sections in the following chapters. Finally, developments in simultaneous analysis and design are summarized in Sec. 1.13.

1.1 Overview of structural optimization

In the process of designing structures in various fields of engineering, the designers and engineers make their best decisions at every step in view of structural and non-structural aspects such as stiffness, strength, serviceability, constructability, and aesthetic property. In other words, they make their *optimal* decisions to realize their best designs; hence, the process of structural design may be regarded as an *optimum design* even though *optimality* is not explicitly pursued.

Structural optimization is regarded as an application of *optimization methods* to structural design. The typical structural optimization problem is formally formulated to minimize an objective function representing the structural

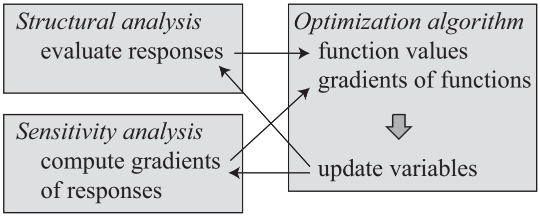


FIGURE 1.1: Relations among structural analysis, optimization algorithm, and design sensitivity analysis for optimization using a nonlinear programming approach.

cost under constraints on mechanical properties of the structure. The total structural weight or volume is usually used for representing the structural cost. Even for the case in which the structural weight is not strongly related to the cost, it is very important that a feasible solution satisfying all the design requirements can be automatically found through the process of optimization. The mechanical properties include nodal displacements, member stresses, eigenvalues of vibration, and linear buckling loads. The structural optimization problem can be alternatively formulated to maximize a mechanical property under constraint on the structural cost.

Although there are many possible formulations for structural optimization, e.g., *minimum weight design* and *maximum stiffness design*, the term *structural optimization* or *optimum design* is usually used for representing all types of optimization problems corresponding to structural design.

In this book, we consider finite dimensional structures, such as frames and trusses, which are mainly used in civil and architectural engineering. In the typical process of structural optimization of finite dimensional structures, the cross-sectional properties, nodal locations, and member locations are chosen as design variables. There are many methods for structural optimization that are classified into

- Nonlinear programming based on the gradients (*sensitivity coefficients* or derivatives) of the objective and constraint functions, which is the most popular and straightforward approach.

- Heuristic approaches, including genetic algorithm and simulated annealing, that do not need gradient information.

In a nonlinear programming approach, the design variables are updated in the direction defined by the sensitivity coefficients of the objective function and constraints. The relations among structural analysis, optimization algorithm, and design sensitivity analysis for optimization using a nonlinear programming approach are illustrated in Fig. 1.1, where the arrows represent the direction of data flow; i.e., sensitivity analysis is carried out at each step of optimization to provide gradients of responses for the optimization algorithm, and structural analysis is needed for sensitivity analysis and function evaluation at an optimization step (see Chap. 2 and Appendix A.2.2 for details of sensitivity analysis and nonlinear programming, respectively).

There are several approaches to the classification of structural optimization problems. In the field of continuum structural optimization, shape optimization means the optimization of boundary shape, whereas the addition and/or removal of holes are allowed in topology optimization (Bendsøe and Sigmund 2003). In this book, we present various methodologies and results for optimization of finite dimensional structures, including rigidly jointed frames and pin-jointed trusses. Since optimization of trusses and frames was developed gradually in 1960s and 1970s by academic groups in different geographical locations, several different terminologies, e.g., *configuration*, *geometry*, and *layout*, were used for representing the similar processes of shape and topology optimization; see, e.g., Dobbs and Felton (1969), Svanberg (1981), Lin, Che, and Yu (1982), Imai and Schmit (1982), Zhou and Rozvany (1991), Twu and Choi (1992), Bendsøe, Ben-Tal, and Zowe (1994), Dems and Gatkowski (1995), Ohsaki (1997b), Bojczuk and Mróz (1999), Stadler (1999), Evgrafov (2006), and Achtziger (2007). On the other hand, optimization of cross-sectional areas of trusses was traditionally called *optimum design*, *design optimization*, or *structural optimization* (Hu and Shield 1961; Prager 1974a; Rozvany 1976). However, the term *sizing optimization* was often used recently to distinguish it from shape optimization (Grierson and Pak 1993; Lin, Che, and Yu 1982; Zou and Chan 2005; Schutte and Groenwold 2003), and structural optimization covers all areas related to optimization of structures.

In this chapter, we present a historical review and various formulations of optimization of finite dimensional structures.

1.2 History of structural optimization

The origin of structural optimization is sometimes credited to Galileo Galilei (1638), who investigated the optimal shape of a beam subjected to a static load. However, his approach was rather intuitive, and he did not establish any theoretical foundation of structural optimization.

The intrinsic properties of minimizing or maximizing functions or functionals in physical phenomena in nature were noticed from ancient times as various minimum/maximum principles. The theoretical basis of minimum principles as a foundation of modern optimization was investigated in the 18th century and established as the *calculus of variation.* The principle of minimum potential energy that leads to the shape of a hanging cable called *catenary* is extensively used nowadays for the design of flexible structures, e.g., cable nets and membrane structures (Krishna 1979). The surface of the minimum area for the specified boundary shape in three-dimensional space is called *minimal surface*, which is equivalent to the surface with vanishing mean curvature, and can be achieved by a membrane with a uniform tension field without external load or pressure. Therefore, the minimal surface is effectively used as the ideal self-equilibrium shape for designing a membrane structure that does not have bending stiffness (Otto 1967, 1969).

Papers by Michell (1904), Maxwell (1890), and Cilly (1900) are often cited as the first paper that mentioned the basic idea of topology optimization; see Sec. 3.1 for the history of topology optimization. However, the so-called *Michell truss* or *Michell structure* has an infinite number of members; hence, it did not lead to any practical development until the 1950s, when the properties of the optimal plastic design of frames were investigated (Foulkes 1954; Drucker and Shield 1961; Heyman 1959). We do not discuss the history of optimization of continuum structures such as plates and shells, because the scope of this book is limited to finite dimensional structures. A comprehensive literature review of early developments of structural optimization is found in Bradt (1986), which was originally published by the Polish Academy of Science, and includes about 300 entries up to the 1950s starting with the book by Galileo Galilei (1638), and more than 1800 entries for the period 1960–1980.

In the 1950s, optimality conditions were studied for the plastic design of frames (Foulkes 1954; Drucker and Shield 1961). In the 1960s, conditions or criteria of optimality were derived utilizing minimum principles for several performance measures of structures (Sewell 1987). Hu and Shield (1961) investigated the uniqueness of optimal plastic design. Taylor (1967) derived the optimality condition for a vibrating rod with specified natural frequency using Hamilton's principle or the *principle of least action*. Prager and Taylor (1968) developed optimality conditions for sandwich beams considering constraints on compliance, natural frequency, buckling load, and plastic limit load, using minimal total potential energy, Rayleigh's principle, and lower- and upper-bound theorems of limit analysis, respectively. Prager (1972, 1974a) summarized the optimality conditions corresponding to various types of constraints, including the case of multiple constraints.

Plastic design of frames was extensively studied in the 1960s and 1970s, because analytical and/or computationally inexpensive methods can be used for this problem. Prager (1971) developed conditions for an optimal frame, subjected to alternative loads, exhibiting the so-called *Foulkes mechanism*. Adeli and Chyou (1987) presented a kinematic approach using automatic generation of independent mechanisms (see Hemp (1973) for various early developments in optimal plastic design).

In the 1970s, when the computer power was still not strong enough to use mathematical programming approaches to optimization of real-world structures, optimality criteria (OC) approaches were widely used for finite dimensional structures. The modern discrete OC approaches to trusses and frames were initiated by Venkayya, Khot, and Berke (1973). Dobbs and Nelson (1975) developed the OC approach to truss design. Reviews of OC approaches are found in Berke and Venkayya (1974) and Venkayya (1978).

Owing to the rapid development of computer hardware and software technologies, many numerical approaches were developed in the 1980s and 1990s to obtain optimization results for practical problems. Developments in this period can be found in many books, e.g., Arora (2007), Adeli (1994), Burns (2002), and Haftka, Gürdal, and Kamat (1990).

It should be noted that the preferred terminologies for structural optimization vary with age. As noted earlier, *structural optimization* of trusses covered only optimization of cross-sectional properties in the 1950s and 1960s. However, *sizing optimization* was recently used to distinguish it from shape and topology optimization. Optimality conditions were first called *Kuhn-Tucker conditions*; however, the name was corrected to *Karush-Kuhn-Tucker conditions* in the 1980s. Multiple load sets for formulation of constraints on static responses were called *alternative loads* until the 1970s; however, they are now usually called *multiple loading conditions* or multiple load sets. Furthermore, *framed structure* was used for representing finite dimensional structures, including pin-jointed trusses and rigidly jointed frames; however, they are classified into trusses and frames, respectively, in recent literature. In this book, we use up-to-date terminology, for consistency, even for describing the results of papers in the early stages of development.

1.3 Structural optimization problem

1.3.1 Continuous problem

If the design variables can vary continuously, i.e., can have real values, and the objective and constraint functions are continuous and differentiable with respect to the variables, the structural optimization problem can be formulated as a nonlinear programming (NLP) problem. Let $\mathbf{A} = (A_1, \ldots, A_m)^\top$ denote the vector of m design variables. For a sizing design optimization problem, $\mathbf{A}$ represents the cross-sectional areas of truss members, heights of the sections of frame members, etc. For a geometry optimization problem, $\mathbf{A}$ may represent the nodal coordinates of trusses and frames. All vectors are assumed to be column vectors throughout this book.

The number of design variables is often reduced using the approach called *design variable linking*, utilizing, e.g., the symmetry properties of the structure. The requirements to be considered in practical applications can also be used for reducing the number of variables; e.g., the beams in the same story of a building frame should have the same section. However, in the following, we assume that each variable can vary independently, and, for trusses and frames, A_i belongs to member i, for simplicity.

Consider an elastic finite dimensional structure subjected to static loads. The vector of state variables representing the nodal displacements is denoted by $\mathbf{U} = (U_1, \dots, U_n)^\top$, where n is the number of degrees of freedom. In most of the design problems in various fields of engineering, the design requirements for responses such as stresses and displacements are given with inequality constraints specified by design codes:

$$H_j(\mathbf{U}(\mathbf{A}), \mathbf{A}) \leq 0, \quad (j = 1, \dots, n^{\mathrm{I}}) \tag{1.1}$$

where n^{I} is the number of inequality constraints. Generally, there exist equality constraints on the response quantities; e.g., an eigenvalue of vibration should be exactly equal to the specified value. However, we consider inequality constraints only, for simple presentation of formulations.

The constraint function $H_j(\mathbf{U}(\mathbf{A}), \mathbf{A})$ depends on the design variables implicitly through the displacement (state variable) vector $\mathbf{U}(\mathbf{A})$ and also directly on the design variables. For example, the axial force N_i of the ith member of a truss is given using a constant n-vector $\mathbf{b}_i$, defining the stress-displacement relation as

$$N_i = A_i \mathbf{b}_i^\top \mathbf{U}(\mathbf{A}) \tag{1.2}$$

which depends explicitly on A_i and implicitly on $\mathbf{A}$ through $\mathbf{U}(\mathbf{A})$.

The upper and lower bounds, which are denoted by A_i^{U} and A_i^{L}, respectively, are usually given for the design variable A_i due to the restriction in fabrication and construction. The objective function, e.g., the total structural volume, is denoted by $F(\mathbf{A})$. Then the structural optimization problem is formulated as

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{A}) && (1.3\text{a}) \\ \text{subject to} \quad & H_j(\mathbf{U}(\mathbf{A}), \mathbf{A}) \leq 0, \quad (j = 1, \dots, n^{\mathrm{I}}) && (1.3\text{b}) \\ & A_i^{\mathrm{L}} \leq A_i \leq A_i^{\mathrm{U}}, \quad (i = 1, \dots, m) && (1.3\text{c}) \end{aligned}$$

Problem (1.3) is classified as an NLP problem, because $\mathbf{U}(\mathbf{A})$ is a nonlinear function of $\mathbf{A}$; see Appendix A.2.2 for details of NLP. The constraints (1.3c) are called *side constraints*, *bound constraints*, or *box constraints*, which are treated separately from the general inequality constraints (1.3b) in most of the optimization algorithms.

As is seen from the definition of constraints in (1.3b), the differential coefficients of $\mathbf{U}(\mathbf{A})$ with respect to $\mathbf{A}$, called design sensitivity coefficients, are needed when solving Problem (1.3) using a gradient-based NLP algorithm. For convenience in deriving the conditions to be satisfied at the optimal solution, the constraint function with respect to $\mathbf{A}$ only is defined as

$$\widetilde{H}_j(\mathbf{A}) = H_j(\mathbf{U}(\mathbf{A}), \mathbf{A}) \tag{1.4}$$

If the side constraints are treated separately from the general inequality constraints, the conditions for optimality are derived using the Lagrangian

$\psi(\mathbf{A}, \boldsymbol{\mu})$ defined as

$$\psi(\mathbf{A}, \boldsymbol{\mu}) = F(\mathbf{A}) + \sum_{j=1}^{n^{\mathrm{I}}} \mu_j \widetilde{H}_j(\mathbf{A}) \tag{1.5}$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{n^{\mathrm{I}}})^\top$ $(\geq \mathbf{0})$ is the vector of Lagrange multipliers.

The necessary conditions for local optimality, which are called *Karush-Kuhn-Tucker conditions* or simply KKT conditions, are given as

$$\begin{cases} \dfrac{\partial \psi}{\partial A_i} \geq 0 \quad \text{for} \quad A_i = A_i^{\mathrm{L}} \\ \dfrac{\partial \psi}{\partial A_i} = 0 \quad \text{for} \quad A_i^{\mathrm{L}} < A_i < A_i^{\mathrm{U}} \\ \dfrac{\partial \psi}{\partial A_i} \leq 0 \quad \text{for} \quad A_i = A_i^{\mathrm{U}} \end{cases} \tag{1.6}$$

where

$$\frac{\partial \psi}{\partial A_i} = \frac{\partial F}{\partial A_i} + \sum_{j=1}^{n^{\mathrm{I}}} \mu_j \frac{\partial \widetilde{H}_j}{\partial A_i}, \quad (i = 1, \ldots, m) \tag{1.7}$$

$$\widetilde{H}_j \leq 0, \quad \mu_j \geq 0, \quad \mu_j \widetilde{H}_j = 0, \quad (i = 1, \ldots, n^{\mathrm{I}}) \tag{1.8}$$

The third equation in (1.8) is called *complementarity conditions* (see Appendix A.2.2.3 for details of the optimality conditions).

Conditions (1.6)–(1.8) are the necessary and sufficient conditions for local optimality, if all the objective and constraint functions are locally convex. Furthermore, (1.6)–(1.8) are sufficient conditions for global optimality, if all the objective and constraint functions are globally convex.

For problems with real variables and continuously differentiable functions, the optimal solutions are found using various approaches of mathematical programming. If the objective function and the constraints are linear functions of the design variables, the problem is formulated as a linear programming (LP) problem, and the optimal solutions are easily found using the standard approach called the simplex method (Luenberger 2003) or the relatively new approach called the interior-point method (Karmarkar 1984; Gondzio 1995).

If the objective and the constraint functions are nonlinear, various approaches of the NLP problem can be used (Fiacco and Cormic 1968; Mangasarian 1969; Pierre and Lowe 1975; Peressini, Sullivan, and Uhl 1988; Ben-Israel, Ben-Tal, and Zolbec 1981; Bersekas 1982). However, there is no approach that is applicable to any type of NLP problem; i.e., the most suitable method should be appropriately chosen for each problem at hand. Furthermore, the method should be selected with regard to the desired accuracy and computational cost for optimization. One of the most popular approaches is sequential quadratic programming (Gill, Murray, and Saunders 2002), which

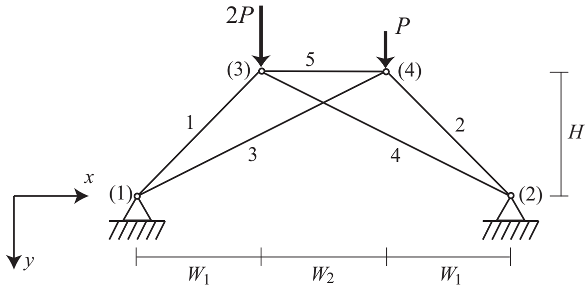


FIGURE 1.2: A five-bar plane truss.

is used for most of the examples of the application of NLP in this book. Readers may refer to Appendix A.2.2 for details of NLP.

Example 1.1
As a simple example of structural optimization, consider a five-bar plane truss, as shown in Fig. 1.2, subjected to vertical static loads. The intersecting members 3 and 4 are not connected with each other. The five bars are classified into Groups 1 and 2, consisting of members $\{1, 2, 5\}$ and $\{3, 4\}$, respectively. The members in Group i $(i = 1, 2)$ have the same cross-sectional area A_i^{g}, and let $\mathbf{A}^{\mathrm{g}} = (A_1^{\mathrm{g}}, A_2^{\mathrm{g}})^\top$. The sum of the lengths of members in the ith group is denoted by L_i^{g}. Then the total structural volume $C(\mathbf{A}^{\mathrm{g}})$, which is taken as the objective function, is defined as

$$C(\mathbf{A}^{\mathrm{g}}) = A_1^{\mathrm{g}} L_1^{\mathrm{g}} + A_2^{\mathrm{g}} L_2^{\mathrm{g}} \tag{1.9}$$

For a simple illustration of the problem, the constraints are given, as follows, for the y-directional displacement U_3 of node 3, which is assumed to be positive, and the stress σ_4 of member 4, which is assumed to be negative:

$$U_3 \leq U_3^{\mathrm{U}}, \quad \sigma_4^{\mathrm{L}} \leq \sigma_4 \tag{1.10}$$

where U_3^{U} and σ_4^{L} are the specified upper bound of U_3 and the lower bound of σ_4, respectively. The constraints are formulated using the function $\widetilde{H}_1(\mathbf{A}^{\mathrm{g}})$ of $\mathbf{A}^{\mathrm{g}}$ only:

$$\widetilde{H}_1(\mathbf{A}^{\mathrm{g}}) = U_3(\mathbf{A}^{\mathrm{g}}) - U_3^{\mathrm{U}} \leq 0, \quad \widetilde{H}_2(\mathbf{A}^{\mathrm{g}}) = -\sigma_4^{\mathrm{L}} - \sigma_4(\mathbf{A}^{\mathrm{g}}) \leq 0 \tag{1.11}$$

Let $W_1 = W_2 = H = 1$ m in Fig. 1.2. The elastic modulus is 200 kN/mm^2, and $P = 10.0$ kN. The bounds for the displacement and stress are $U_3^{\mathrm{U}} = 1.25$ mm and $\sigma_4^{\mathrm{L}} = -0.06$ kN/mm^2.

The set of solutions satisfying $U_3 = U_3^{\mathrm{U}}$ and $\sigma_4 = \sigma_4^{\mathrm{L}}$ is shown in the solid lines in Fig. 1.3 that are drawn in the design variable space. The gray area

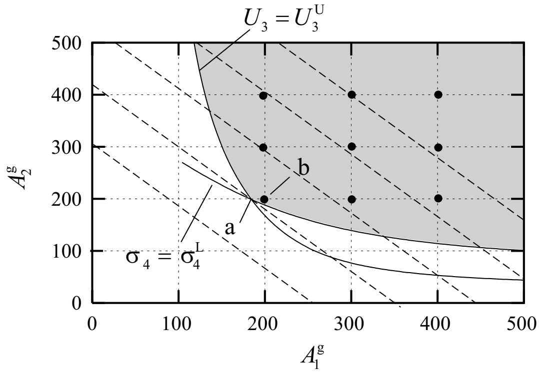


FIGURE 1.3: Feasible region and optimal solutions of the five-bar truss.

is the feasible region satisfying the two constraints with equality. From (1.9), we obtain

$$A_2^g = \frac{1}{L_2^g} C(\mathbf{A}^g) - \frac{L_1^g}{L_2^g} A_1^g \tag{1.12}$$

The solution on each dotted line in Fig. 1.3 has the same values of C. Therefore, if A_1^g and A_2^g can take real values, the point 'a' with the coordinates $(A_1^g, A_2^g) = (184.33, 198.90)$ in the design variable space corresponds to the optimal solution.

In order to verify the optimality of the solution, the sensitivity coefficients are obtained at the optimal solution as

$$\begin{aligned}
&\frac{\partial U_3(\mathbf{A}^g)}{\partial A_1^g} = -0.013112, \quad \frac{\partial U_3(\mathbf{A}^g)}{\partial A_2^g} = -0.018015, \\
&\frac{\partial \sigma_4(\mathbf{A}^g)}{\partial A_1^g} = 0.45650, \quad \frac{\partial \sigma_4(\mathbf{A}^g)}{\partial A_2^g} = 0.20541
\end{aligned} \tag{1.13}$$

The sensitivity coefficients of the objective function are easily computed from the member lengths as

$$\frac{\partial C}{\partial A_1^g} = 3828.4, \quad \frac{\partial C}{\partial A_2^g} = 4472.1 \tag{1.14}$$

Then, from the second equation in (1.6) with $i = 1$ and 2, while A_i is replaced by A_i^g, the positive Lagrange multipliers are found as $\lambda_1 = 1.8680 \times 10^5$ and $\lambda_2 = 2.2694 \times 10^7$. Hence, the optimality conditions are satisfied at the solution $(A_1^g, A_2^g) = (184.33, 198.90)$.

As is seen in the above example, the optimal solution can be found for a simple truss graphically in the design variable space, if we have only two design variables. However, for larger structures with more design variables, the optimal solutions are to be found numerically using a mathematical programming approach or a heuristic approach.

1.3.2 Discrete problem

Suppose a list or catalog of the available standard sections is given for a sizing optimization problem of a frame, and the list $\mathcal{A}_i$ of the cross-sectional properties of the ith member is given as

$$\mathcal{A}_i = \{(A_i^1, I_i^1, Z_i^1), \dots, (A_i^r, I_i^r, Z_i^r)\} \tag{1.15}$$

where A_i^j is the cross-sectional area, I_i^j is the second moment of inertia, Z_i^j is the section modulus of the jth candidate section for member i, and r is the number of available sections, which is the same for all members, for brevity. Note that other properties such as fully-plastic moment should be included if elastoplastic responses are to be considered; see Appendix A.8 for examples of section lists.

Suppose that $J_i = j$ $(i = 1, \dots, m)$ indicates that the jth section in the list is assigned to the ith member, where m is the number of members. This way, the mechanical properties of the frame are defined by the vector $\mathbf{J} = (J_1, \dots, J_m)^\top$ of integer variables. Hence, the nodal displacement vector is a function of $\mathbf{J}$ that is denoted by $\mathbf{U}(\mathbf{J})$. The objective and the constraint functions are also functions of $\mathbf{J}$, which are written as $F(\mathbf{J})$ and $\widetilde{H}_j(\mathbf{J}) = H_j(\mathbf{J}, \mathbf{U}(\mathbf{J}))$, respectively. Then the optimization problem with inequality constraints only is formulated as

$$\begin{aligned} &\text{Minimize} \quad F(\mathbf{J}) && \text{(1.16a)} \\ &\text{subject to} \quad \widetilde{H}_j(\mathbf{J}) \leq 0, \quad (j = 1, \dots, n^{\mathrm{I}}) && \text{(1.16b)} \\ &\qquad\qquad J_i \in \{1, \dots, r\}, \quad (i = 1, \dots, m) && \text{(1.16c)} \end{aligned}$$

Since Problem (1.16) is an integer programming problem, which is equivalently called a combinatorial optimization problem, various methods, e.g., the branch-and-bound method and the branch-and-cut method, can be used (Horst and Tuy 1985; Horst, Pardalos, and Thoai 1995) (see Sec. 3.5 for application of the branch-and-bound method to topology optimization of trusses).

For the example of the five-bar truss in Fig. 1.2, suppose A_1^{g} and A_2^{g} can take only integer values 100, 200, ... Then, the feasible designs satisfying (1.10) are plotted in the filled circle in Fig. 1.3, and the optimal solution exists at point 'b'.

Since the state variables are continuous functions of the design variables, a structural optimization problem turns out to be a mixed integer nonlinear programming (MINLP) problem (Floudas 1995) if the formulation of simultaneous analysis and design, see Sec. 1.13, is used considering the nodal displacements as independent variables. Arora (2002) classified the structural optimization problems into the following six categories:

1. Continuous design variables; functions are twice continuously differentiable (standard NLP problem).

2. Mixed design variables; functions are twice continuously differentiable; discrete variables can have non-discrete values during the solution process (functions can be evaluated at non-discrete points). A configuration optimization problem of a truss with discrete cross-sectional areas and continuous nodal coordinates belongs in this category.

3. Mixed design variables; functions are non-differentiable; discrete variables can have non-discrete values during the solution process. A configuration optimization problem of a truss with discrete cross-sectional area, continuous nodal coordinates, and nodal cost defined as a non-differential function of cross-sectional areas belongs in this category (see Sec. 4.3).

4. Mixed design variables; functions may or may not be differentiable; some of the discrete variables must have only discrete value in the solution process. A configuration optimization problem with a list of candidate topologies and continuous nodal coordinates belongs in this category.

5. Mixed design variables; functions may or may not be differentiable; some of the discrete variables are linked to others; assignment of a value to one variable specifies values for others. A frame optimization problem with discrete cross-sectional properties such as second moment of inertia linked with cross-sectional area belongs to this category.

6. Combinatorial problems; purely discrete non-differentiable problems. Optimization problems for selection of materials, location of supports, etc. belong to this category.

Arora, Huang, and Hsieh (1994) summarized various methods of optimization with discrete variables.

1.4 Plastic design

Optimal plastic design is the simplest and classical problem of optimization of trusses and frames, which was extensively studied in the 1960s. Consider a truss consisting of a perfectly rigid-plastic material; i.e., the strain before yielding is negligibly small, and the stress after yielding is constant at the yield stress, which is assumed to be the same for all members. The truss is subjected to a vector of quasistatic proportional loads $\mathbf{P} = \Lambda \mathbf{P}^0$ defined by the load factor Λ and the constant load pattern vector $\mathbf{P}^0$.

The axial force vector is given as $\mathbf{N} = (N_1, \ldots, N_m)^\top$, where m is the number of members. Let n denote the number of degrees of freedom. The equilibrium equations are formulated in terms of the $n \times m$ equilibrium matrix $\mathbf{D}$ as

$$\mathbf{D}\mathbf{N} = \Lambda \mathbf{P}^0 \tag{1.17}$$

Let $\mathbf{N}^{\mathrm{p}} = (N_1^{\mathrm{p}}, \ldots, N_m^{\mathrm{p}})^\top$ denote the vector of tensile yield axial forces of the members. The yield axial force in compression is given for the ith member, ignoring member buckling, as $-N_i^{\mathrm{p}}$. Then the yield condition is written as

$$-N_i^{\mathrm{p}} \leq N_i \leq N_i^{\mathrm{p}}, \quad (i = 1, \ldots, m) \tag{1.18}$$

Note that N_i^{p} is proportional to the cross-sectional area A_i as $N_i^{\mathrm{p}} = A_i \sigma^{\mathrm{p}}$, where σ^{p} is the tensile yield stress.

First the plastic limit analysis problem is formulated as a linear programming (LP) problem. Utilizing the lower-bound theorem of plastic limit analysis (Shames and Cozzarelli 1997), we can obtain the plastic limit load factor through maximization of the load factor under constraints on the equilibrium equations and the yield conditions:

$$\begin{aligned} &\text{Maximize} && \Lambda && (1.19\text{a}) \\ &\text{subject to} && -\mathbf{N}^{\mathrm{p}} \leq \mathbf{N} \leq \mathbf{N}^{\mathrm{p}} && (1.19\text{b}) \\ & && \mathbf{D}\mathbf{N} = \Lambda \mathbf{P}_0 && (1.19\text{c}) \end{aligned}$$

which is an LP problem with variables Λ and $\mathbf{N}$. Therefore, the plastic limit load can easily be obtained using a standard method of LP such as the simplex method.

The problem of minimizing the total structural volume under constraint on limit load factor can also be formulated as an LP problem. Since N_i^{p} is proportional to A_i, the optimal design that minimizes the total structural volume can be obtained by minimizing $\mathbf{N}^{\mathrm{p}\top}\mathbf{L}$, where $\mathbf{L} = (L_1, \ldots, L_m)^\top$ is the vector of member lengths. The upper and lower bounds for N_i^{p} are denoted by N_i^{pU} and N_i^{pL}, respectively, with the vectors $\mathbf{N}^{\mathrm{pU}} = (N_1^{\mathrm{pU}}, \ldots, N_m^{\mathrm{pU}})^\top$ and $\mathbf{N}^{\mathrm{pL}} = (N_1^{\mathrm{pL}}, \ldots, N_m^{\mathrm{pL}})^\top$. The specified limit load factor is denoted by Λ^{p}. Then, the optimization problem is formulated as

$$\begin{aligned} &\text{Minimize} && \mathbf{N}^{\mathrm{p}\top}\mathbf{L} && (1.20\text{a}) \\ &\text{subject to} && -\mathbf{N}^{\mathrm{p}} \leq \mathbf{N} \leq \mathbf{N}^{\mathrm{p}} && (1.20\text{b}) \\ & && \mathbf{D}\mathbf{N} = \Lambda^{\mathrm{p}} \mathbf{P}^0 && (1.20\text{c}) \\ & && \mathbf{N}^{\mathrm{pL}} \leq \mathbf{N}^{\mathrm{p}} \leq \mathbf{N}^{\mathrm{pU}} && (1.20\text{d}) \end{aligned}$$

where the variables are $\mathbf{N}$ and $\mathbf{N}^{\mathrm{p}}$. Because Problem (1.20) is also an LP problem, this problem was extensively studied in the 1960s and is still important for application to the plastic design of trusses. Note that the plastic collapse mechanisms can be found as the Lagrange multipliers at the optimal

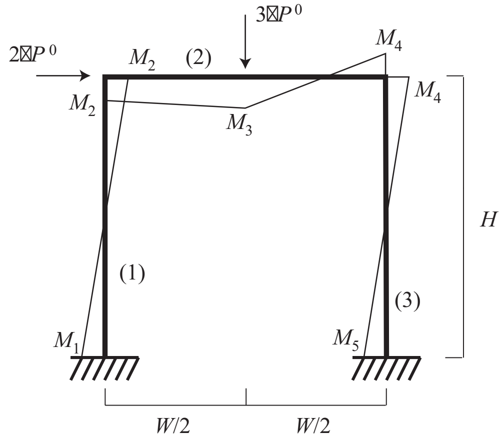


FIGURE 1.4: A simple plane frame.

solution of Problem (1.20), or by solving the dual of Problem (1.20) that is formulated on the basis of the upper-bound theorem of plastic limit analysis, which states that the smallest load factor corresponding to admissible strain and displacement rates defines the collapse load.

The plastic design problem of a frame with concentrated plastic hinges can also be formulated as an LP problem, as follows, if the interaction between the axial force and bending moment on the yield condition is ignored (Adeli and Chyou 1987):

Example 1.2

Consider a plane frame, as shown in Fig. 1.4, subjected to a proportional horizontal load $2\Lambda P^0$ and a vertical load $3\Lambda P^0$ simultaneously. The bending moments at the member ends and the center of the beam are denoted by $M_1, \dots, M_5$, as shown in Fig. 1.4, which illustrates the state where $M_1, \dots, M_5$ are all positive. The numbers in parentheses are member numbers.

The equilibrium equations are given as

$$\begin{aligned} \frac{M_2 + M_1}{H} + \frac{M_4 + M_5}{H} &= 2\Lambda P^0, \\ \frac{M_2 - M_3}{W/2} - \frac{M_3 + M_4}{W/2} &= 3\Lambda P^0 \end{aligned} \tag{1.21}$$

The fully-plastic moment of member i is denoted by M_i^{p}. The yield conditions are then given as

$$\begin{aligned} &-M_1^{\mathrm{p}} \leq M_1 \leq M_1^{\mathrm{p}}, \quad -M_1^{\mathrm{p}} \leq M_2 \leq M_1^{\mathrm{p}}, \\ &-M_2^{\mathrm{p}} \leq M_2 \leq M_2^{\mathrm{p}}, \quad -M_2^{\mathrm{p}} \leq M_3 \leq M_2^{\mathrm{p}}, \quad -M_2^{\mathrm{p}} \leq M_4 \leq M_2^{\mathrm{p}}, \\ &-M_3^{\mathrm{p}} \leq M_4 \leq M_3^{\mathrm{p}}, \quad -M_3^{\mathrm{p}} \leq M_5 \leq M_3^{\mathrm{p}} \end{aligned} \tag{1.22}$$

We assume that the tensile yield stress σ^{p} and the compressive yield stress $-\sigma^{\mathrm{p}}$ are the same, respectively, for all members, and the cross-section of each member is modeled as a *sandwich section*; i.e., the half of the cross-sectional area A_i is concentrated at each flange, and M_i^{p} is proportional to A_i as $M_i^{\mathrm{p}} = A_i r_i \sigma^{\mathrm{p}}$, where r_i is the distance between the flanges.

Hence, the objective function that is proportional to the total structural volume of the frame is formulated as a function of $\mathbf{M}^{\mathrm{p}} = (M_1^{\mathrm{p}}, M_2^{\mathrm{p}}, M_3^{\mathrm{p}})^\top$:

$$F(\mathbf{M}^{\mathrm{p}}) = M_1^{\mathrm{p}} H + M_2^{\mathrm{p}} W + M_3^{\mathrm{p}} H \tag{1.23}$$

Since both the objective function and the constraints are linear functions of the variables $M_1, \dots, M_5$ and $\mathbf{M}^{\mathrm{p}}$, the optimal solution can easily be found by solving an LP problem. For a simple case with $P^0 = 1$, $\Lambda^{\mathrm{p}} = 1$, and $H = W = 1$, we obtain the optimal solution as $M_1^{\mathrm{p}} = 3/8$ and $M_2^{\mathrm{p}} = M_3^{\mathrm{p}} = 5/8$ with $M_1 = M_2 = 3/8$, $M_3 = M_4 = M_5 = 5/8$, and $F(\mathbf{M}^{\mathrm{p}}) = 13/8$.

It is well known that the Foulkes mechanism satisfying the following conditions exists at the optimal solution of a frame for the case in which M_i^{p} is proportional to A_i (Foulkes 1954):

$$\begin{cases} \theta_i = \lambda L_i & \text{for} \quad M_i^{\mathrm{p}} > M_i^{\mathrm{pL}} \\ \theta_i \leq \lambda L_i & \text{for} \quad M_i^{\mathrm{p}} = M_i^{\mathrm{pL}} \end{cases} \tag{1.24}$$

where θ_i is the sum of absolute values of the rotation rate of the plastic hinges in the ith member, and λ is a positive constant. Note that the upper bound for M_i^{p} is not considered, for brevity. Condition (1.24) suggests that the plastic energy dissipation rate per unit volume is the same for members with $M_i^{\mathrm{p}} > M_i^{\mathrm{pL}}$.

There have been many studies on plastic design since the 1960s (Tam and Jennings 1989). Multiple (alternative) loads are considered in some papers, e.g., Prager (1967) and Chan (1969). Munro and Chuang (1986) presented a fuzzy LP approach for the case in which uncertainty exists in the loads. A probabilistic LP approach to limit design under uncertainty was developed by Gavarini and Veneziano (1972).

1.5 Stress constraints

In view of structural design procedure in civil engineering based on allowable stress design criteria, it is very important to obtain an optimal design that satisfies stress and displacement constraints against design loads. In this section, we consider stress constraints only, for simple presentation of the optimization procedure. An approach to optimization of a truss under displacement constraints is demonstrated in Sec. 1.8. Another important aspect in structural design is that several loads, including static loads (self-weight, service load, snow load, etc.) and dynamic loads (wind load, seismic load, etc.), should be considered, and, in the practical design process, the dynamic loads are represented by equivalent static loads. Furthermore, the self-weight and service load are classified as long-term loads, while others are short-term loads. Therefore, different bounds should be given for the stresses against each loading condition.

Consider n^{P} loading conditions (load patterns), and let the superscript k denote the variables and parameters corresponding to the kth loading condition. The upper and lower bounds for σ_i^k are denoted by $\sigma_i^{k\mathrm{U}}$ and $\sigma_i^{k\mathrm{L}}$, respectively. Then the optimization problem for minimizing the total structural volume of a truss under stress constraints is formulated as

$$\text{Minimize} \quad \sum_{i=1}^{m} A_i L_i \tag{1.25a}$$

$$\text{subject to} \quad \sigma_i^{k\mathrm{L}} \leq \sigma_i^k \leq \sigma_i^{k\mathrm{U}}, \quad (i = 1, \dots, m;\ k = 1, \dots, n^{\mathrm{P}}) \tag{1.25b}$$

$$A_i^{\mathrm{L}} \leq A_i \leq A_i^{\mathrm{U}}, \quad (i = 1, \dots, m) \tag{1.25c}$$

where A_i^{L} and A_i^{U} are the lower and upper bounds for A_i, respectively. Note again that the n^{P} load patterns are applied independently, and the stress constraints are assigned for each loading condition.

Example 1.3

Optimum designs are found for a 10-bar truss, as shown in Fig. 1.5, subjected to vertical loads P_1 and P_2, where the numbers with and without parentheses are node numbers and member numbers, respectively (Katoh, Ohsaki, and Tani 2002). Note that the intersecting members are not connected at their centers. A small lower bound $A_i^{\mathrm{L}} = 0.1\ \mathrm{mm}^2$ is given for A_i to prevent instability of the truss, while the upper bound is not given for A_i. The bounds for the stresses are $\sigma_i^{k\mathrm{U}} = 0.2\ \mathrm{N/mm}^2$ and $\sigma_i^{k\mathrm{L}} = -0.2\ \mathrm{N/mm}^2$. Optimal solutions are obtained using the optimization software package SNOPT Ver. 7.2 (Gill, Murray, and Saunders 2002), which utilizes the sequential quadratic programming; see Appendix A.2.2.5.

First, consider a single loading condition $(P_1, P_2) = (0.0, 100.0\ \mathrm{kN})$. The optimal cross-sectional areas and the optimal objective value are shown in the second column in Table 1.1. The optimal solution is also illustrated in Fig. 1.6, where the width of each member is proportional to its cross-sectional area. Note that A_i is equal to its lower bound in members 4, 5, 6, 8, and 10, which may be removed to obtain the statically determinate truss of optimal topology after fixing the unstable node 4. The stress is equal to its upper or lower bound in each member with $A_i > A_i^{\mathrm{L}}$. This process of topology optimization is called the ground structure approach; it is extensively studied in Chap. 3.

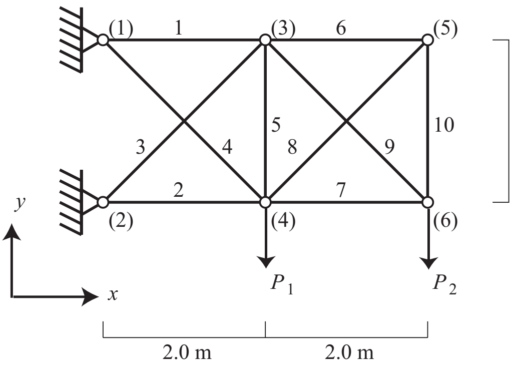


FIGURE 1.5: A 10-bar truss.

TABLE 1.1: Optimal cross-sectional areas and structural volume of the 10-bar truss under stress constraints.

Member number	A_i (mm^2)	
	Single loading	Multiple loading
1	999.931	825.107
2	500.069	674.893
3	707.010	459.771
4	0.100	421.531
5	0.100	211.499
6	0.100	0.100
7	499.937	499.909
8	0.100	0.129
9	707.017	706.978
10	0.100	0.100
Total volume (mm^3)	8.00051×10^6	8.91591×10^6

Next, we obtain the optimal solution under stress constraints against multiple loading conditions $(P_1, P_2) = (0, 100.0\text{ kN})$ and $(100.0\text{ kN}, 0)$. The optimal cross-sectional areas, which are also illustrated in Fig. 1.7, and the objective value are listed in the last column of Table 1.1. The optimal objective value is 8.91591×10^6 mm^3, which is larger than that for the single loading condition. Only the members 6 and 10 connected to node 5 satisfy $A_i = A_i^{\rm L}$, and node 5 cannot be removed, because the cross-sectional area of member 8 is larger than its lower bound. Note that a very strict tolerance of 10^{-10} is assigned for the constraints and optimality conditions in SNOPT. In fact, we can confirm that the stresses for the first load $(P_1, P_2) = (0, 100.0\text{ kN})$ are $\sigma_6^1 = \sigma_{10}^1 = 0.18265$ and $\sigma_8^1 = -0.2$; the member 8 is fully stressed. Since the equilibrium condition $N_8 = -\sqrt{2}(N_6 + N_{10})$ should be satisfied for member forces N_i of members 6, 8, and 10, the cross-sectional area should be larger

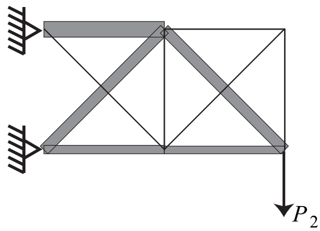


FIGURE 1.6: Optimal design of the 10-bar truss under single loading condition $(P_1, P_2) = (0.0, 100.0 \text{ kN})$.

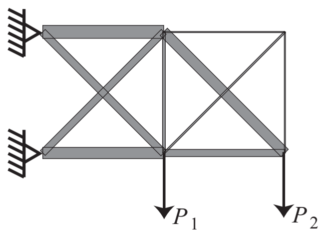


FIGURE 1.7: Optimal solution for multiple loading conditions $(P_1, P_2) = (0, 100.0 \text{ kN})$ and $(100.0 \text{ kN}, 0)$.

than its lower bound if the stresses of members 6 and 10 are close to their upper or lower bounds. Hence, it should be noted that the member with $A_i = A_i^{\mathrm{L}}$ sometimes cannot be removed in the conventional ground structure approach of topology optimization, and the optimal topology may consist of many members with small cross-sectional areas.

The result in Fig. 1.7 demonstrates that the optimal truss under multiple loading conditions is statically indeterminate even when the removal of members 6, 8, and 10 is allowed. It has been confirmed that the stress is equal to its upper or lower bound against at least one loading condition for a member with $A_i > A_i^{\mathrm{L}}$.

1.6 Fully-stressed design

1.6.1 Stress-ratio approach

Consider again a simple optimization problem of a truss with stress constraints, and suppose only the lower bound A_i^{L} is given for the cross-sectional area A_i of the ith member. In a practical design process, obtaining a feasible

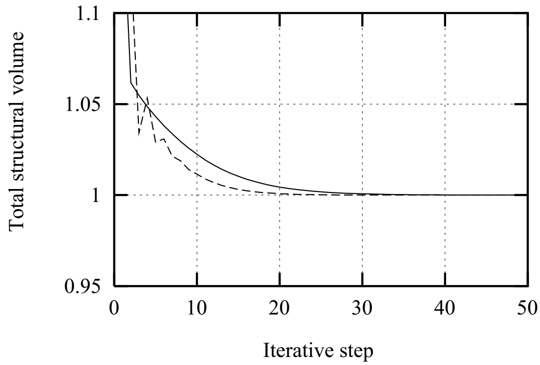


FIGURE 1.8: Convergence history of the total structural volume of FSD of the 10-bar truss; solid line: $r = 1$, dashed line: $r = 1.5$.

solution is sometimes more important than minimizing an objective function. Furthermore, the results of the example in the previous section suggest that an optimal design can be obtained by finding the cross-sectional areas so that the stress of a member with $A_i^{\mathrm{L}} < A_i$ is equal to its upper or lower bound for at least one of the n^{P} loading conditions. The design satisfying this condition is called a *fully-stressed design* (FSD). Note that the inequality constraints $\sigma_i^{k\mathrm{L}} \leq \sigma_i^k \leq \sigma_i^{k\mathrm{U}}$ $(k = 1, \ldots, n^{\mathrm{P}})$ are to be satisfied by the member with $A_i = A_i^{\mathrm{L}}$. In the fully-stressed design approach, a design satisfying these conditions is obtained by iteratively modifying the design variables.

For a simple case of a single loading condition with $\sigma_i^{1\mathrm{L}} = -\sigma_i^{1\mathrm{U}}$ for all members, the FSD can be obtained by the following simple iterative algorithm for updating the cross-sectional areas:

$$A_i^{(k+1)} = A_i^{(k)} \left(\frac{|\sigma_i^1|}{\sigma_i^{1\mathrm{U}}} \right)^r, \quad (i = 1, \ldots, m) \tag{1.26}$$

where $A_i^{(k)}$ is the value of A_i at the kth step of iteration, and r is the parameter for controlling the convergence property, which is usually between 1 and 2. Note that $A_i^{(k+1)}$ is replaced by A_i^{L} if $A_i < A_i^{\mathrm{L}}$ is satisfied as the result of application of (1.26).

The design update rule (1.26) called the *stress-ratio approach*, assumes that the modification of the cross-sectional area of a member does not have any strong effect on the axial forces of the members. For example, the axial forces of a statically determinate truss are determined only from the equilibrium equations and are independent of the cross-sectional areas. In this case, the stress of a member against the specified set of loads is inversely proportional to its cross-sectional area, and the FSD can be found within only one step of application of (1.26) with $r = 1$.

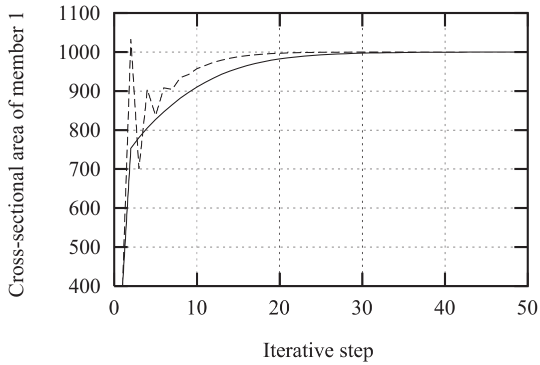
The stress-ratio approach can also be effectively used for building frames, for which constraints are given for the stress at each edge of the section at the member ends due to the bending moment and axial force. It is very convenient for investigating the nearly optimal load paths of a plane regular frame from the loaded nodes to the supports. If the variable A_i defines the size of the section with the dimension of length, e.g., height and width of the wide-flange section, the appropriate value of the parameter r in (1.26) ranges between 1/3 and 1/2 (Mueller, Liu, and Burns 2002).

Example 1.4

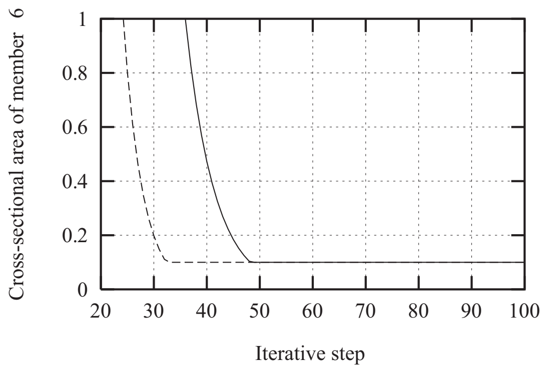
An FSD is found for the 10-bar truss in Fig. 1.5 under single loading condition $(P_1, P_2) = (0.0, 100.0 \text{ kN})$. A small lower bound, 0.1 mm^2, which is the same as the value in Example 1.3 in Sec. 1.5, is given for A_i to prevent instability of the truss and to compare the results. Fig. 1.8 shows the convergence history of the total structural volume divided by the value 8.91591×10^6 in Table 1.1 for the optimal solution under stress constraints subjected to the same single loading condition. The histories of A_1 and A_6 are also plotted in Figs. 1.9 (a) and (b), respectively, where only a small range is plotted for A_6, because A_6 is equal to the lower bound, 0.1 mm^2, at the converged FSD. As is seen from these figures, the total structural volume and the cross-sectional areas converge monotonically to the optimal values within 30 steps, if $r = 1$. The convergence property is improved if a larger value, 1.5, is assigned for r; i.e., an approximate optimal solution can be found within 20 steps; however, some oscillation is observed at the early stage of iteration. Note that the total structural volume converges to the optimal value under stress constraints, and the cross-sectional areas of the FSD are the same as the optimal values in Table 1.1. This way, the optimal truss under stress constraints for a single loading condition can easily be found by the simple stress-ratio approach (1.26) if the absolute values of the upper- and lower-bound stresses are the same.

The relation between the FSD and optimum design under stress constraints has been extensively studied since the 1960s (Razani 1965; Kicher 1966; Patnaik and Dayaratnam 1970; McNeil 1971; Chern and Prager 1972; Nagtegaal 1973; Gunnlaugsson and Martin 1973) and was revisited mainly in the community of applied mathematics in the 1990s (Bendsøe and Sigmund 2003). However, it seems that the FSDs are not clearly defined each case with $A_i^{\mathrm{L}} > 0$ and $A_i^{\mathrm{L}} = 0$. Here we do not assign an upper bound for the cross-sectional area, and define the FSD as follows (Nagtegaal 1973):

- If $A_i^{\mathrm{L}} > 0$, then the stress σ_i^k of a member with $A_i > A_i^{\mathrm{L}}$ should be equal to $\sigma_i^{k\mathrm{U}}$ or $\sigma_i^{k\mathrm{L}}$ for at least one loading condition; whereas $\sigma_i^{k\mathrm{L}} \leq \sigma_i^k \leq \sigma_i^{k\mathrm{U}}$ should be satisfied by a member with $A_i = A_i^{\mathrm{L}}$.



(a) member 1



(b) member 6

FIGURE 1.9: Convergence history of cross-sectional areas of FSD of the 10-bar truss; solid line: $r = 1$, dashed line: $r = 1.5$.

- If $A_i^{\mathrm{L}} = 0$, then the stress σ_i^k of a member with $A_i > 0$ should be equal to $\sigma_i^{k\mathrm{U}}$ or $\sigma_i^{k\mathrm{L}}$ for at least one loading condition; whereas no constraint exists on the stress of a nonexistent member with $A_i = 0$.

Therefore, there is a discontinuity in the FSDs between the cases with $A_i^{\mathrm{L}} = 0$ and $A_i^{\mathrm{L}} = e$, where e is a small positive value. Note that the case with $A_i^{\mathrm{L}} = 0$ corresponds to the topology optimization problem that is extensively investigated in Sec. 3.5.3.

1.6.2 Single loading condition

First we consider a truss subjected to a single loading condition, and let n denote the number of degrees of freedom. Suppose the truss consisting of m members is statically indeterminate; i.e., $n < m$ with n being the number of degrees of freedom. The vectors of nodal displacements $\mathbf{U}^1 = (U_1^1, \dots, U_n^1)^\top$

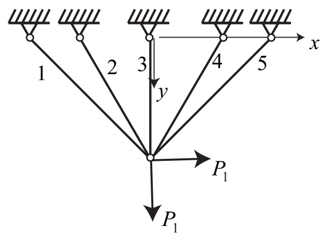


FIGURE 1.10: A statically indeterminate five-bar truss.

and member strains $\boldsymbol{\varepsilon}^1 = (\varepsilon_1^1, \dots, \varepsilon_m^1)^\top$ should satisfy the compatibility conditions

$$\boldsymbol{\varepsilon}^1 = \mathbf{C}\mathbf{U}^1 \tag{1.27}$$

where $\mathbf{C}$ is an $m \times n$ matrix that is defined by the kinematic relations only. Suppose the truss is stable and $\mathbf{C}$ is full-rank; i.e., the rank of $\mathbf{C}$ is equal to n because $n < m$. Hence, we can eliminate $\mathbf{U}^1$ using (1.27) to express $m - n$ components of $\boldsymbol{\varepsilon}^1$ with respect to the remaining n components.

Therefore, using the constitutive relation $\sigma_i^1 = E\varepsilon_i^1$ with the elastic modulus E, the $m - n$ equations are obtained for the stresses $\sigma_1^1, \dots, \sigma_m^1$. Hence, the stress can be independently assigned only for n members, and the stresses of all the members of a statically indeterminate truss cannot generally be equal to the upper or lower bound. Consequently, for a truss to be fully-stressed, the cross-sectional areas of at least $m - n$ members should be equal to their lower bounds, and the stresses of the remaining n members should be appropriately assigned so that $\sigma_i^1 = \sigma_i^{1\mathrm{L}}$ or $\sigma_i^1 = \sigma_i^{1\mathrm{U}}$ for those members, and $\sigma_i^{1\mathrm{L}} \leq \sigma_i^1 \leq \sigma_i^{1\mathrm{U}}$ for the remaining $m - n$ members with $A_i = A_i^{\mathrm{L}}$.

Example 1.5

Consider a symmetric five-bar truss, as shown in Fig. 1.10, where $n = 2$ and $m = 5$, i.e., the truss is statically indeterminate, and we can assign two member stresses independently. In the following examples, the units of load and length are omitted for brevity. The truss is symmetric with respect to the y-axis. The angles of members 1 and 2 from the x-axis are $\pi/4$ and $\pi/3$, respectively. If σ_1^1 and σ_3^1 are chosen as the independent variables, the stresses of the remaining members are given as

$$\begin{aligned} \sigma_2^1 &= \frac{\sqrt{3}}{2}\sigma_1^1 + \frac{3-\sqrt{3}}{4}\sigma_3^1, \\ \sigma_4^1 &= -\frac{\sqrt{3}}{2}\sigma_1^1 + \frac{3+\sqrt{3}}{4}\sigma_3^1, \\ \sigma_5^1 &= \sigma_3^1 - \sigma_1^1 \end{aligned} \tag{1.28}$$

which show that the stresses cannot have the same absolute value for all members. Therefore, generally $A_i = A_i^{\mathrm{L}}$ should be satisfied for three members

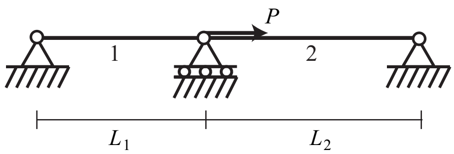


FIGURE 1.11: A statically indeterminate two-bar truss.

so that the truss is fully stressed. Hence, the FSD is statically determinate if $A_i^{\mathrm{L}} = 0$ for all members. Note that the stress of a nonexistent member can be computed from the strain because the two nodes (supports) connected to any member of this truss exist (see Sec. 3.5.3 for more details).

For example, the total structural volume is minimized under conditions $P_1 = 1$, $P_2 = 0$, and $A_i^{\mathrm{L}} = 0$ for all members. If the bounds of stresses are given as $\sigma_i^{\mathrm{U}} = -\sigma_i^{\mathrm{L}} = 1/\sqrt{2}$ for all members, the optimal cross-sectional areas are obtained as $A_1 = A_5 = 1$ and $A_2 = A_3 = A_4 = 0$. Then the stresses are obtained as $\sigma_1 = 1/\sqrt{2}$. Accordingly, $\sigma_2 = \sqrt{6}/4$, $\sigma_3 = 0$, $\sigma_4 = -\sqrt{6}/4$, $\sigma_5 = -1/\sqrt{2}$, and the truss is statically determinate and fully stressed. For this example, the truss is fully stressed even if a very small positive value e is assigned for A_i^{L} of all members because $\sigma_i^{\mathrm{L}} \leq \sigma_i \leq \sigma_i^{\mathrm{U}}$ is satisfied by the nonexistent member 3.

Example 1.6

As another illustrative example, consider a statically indeterminate truss, as shown in Fig. 1.11, that has two colinearly located members, and assume $P > 0$. The bounds for the stress are given as $\sigma_i^{1\mathrm{L}} = -\sigma_i^{1\mathrm{U}}$, where $\sigma_1^{1\mathrm{U}}$ and $\sigma_2^{1\mathrm{U}}$ are not necessarily the same. The lower bounds for A_i are given as $A_1^{\mathrm{L}} = A_2^{\mathrm{L}} = e$, where e has a sufficiently small positive value.

If $L_2 = 2L_1$ and $\sigma_1^{1\mathrm{U}} = \sigma_2^{1\mathrm{U}} = \overline{\sigma}$ for a specified positive value $\overline{\sigma}$, then the optimization for minimizing the total structural volume V leads to $A_1 \simeq P/\overline{\sigma}$ and $A_2 = e$, because member 2 is longer than member 1, and $\sigma_1^1 = \overline{\sigma}$ and $\sigma_2^1 = -\overline{\sigma}/2$ are satisfied from the compatibility condition. Hence, the optimal solution is fully stressed. By contrast, if $\sigma_1^{1\mathrm{U}} = \overline{\sigma}$ and $\sigma_2^{1\mathrm{U}} = 4\overline{\sigma}$, then the optimization leads to $A_1 = e$ and $A_2 \simeq P/(4\overline{\sigma})$, because the larger length of member 2 is compensated by the larger absolute value of the allowable stress; consequently, V is approximately equal to $PL/(2\overline{\sigma})$. In fact, if we assume $A_2 = e$, then $A_1 \simeq P/\overline{\sigma}$ and, accordingly, V is approximately equal to $PL/\overline{\sigma}$, which is larger than $PL/(2\overline{\sigma})$.

At the optimal solution with $A_2 \simeq P/(4\overline{\sigma})$, $\sigma_1^1 = 8\overline{\sigma}$, and $\sigma_2^1 = -4\overline{\sigma}$ are satisfied. Hence, the optimal solution is not fully stressed; i.e., the stress constraint is violated by member 1. Therefore, the optimal solution may not be fully stressed if $A_i^{\mathrm{L}} > 0$ and the stress bounds are not the same for all the members of a statically indeterminate truss. However, if $A_i^{\mathrm{L}} = 0$, then the

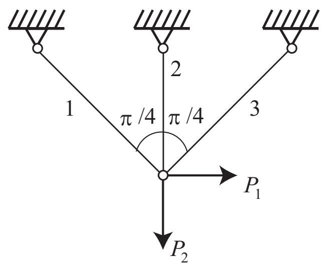


FIGURE 1.12: A three-bar truss (Type 1).

optimal solution is $(A_1, A_2) = (0, P/(4\overline{\sigma}))$, which is fully stressed, because the stress constraint need not be satisfied by the nonexistent member 1.

1.6.3 Multiple loading conditions

Next we consider the truss under multiple loading conditions. The optimization problem under stress constraints is formulated as (1.25) without upper-bound cross-sectional area. Let n^{A} denote the number of members for which the stress is equal to its lower or upper bound for at least one loading condition. If the truss is statically determinate, the axial force is independent of the cross-sectional areas, and $n^{\mathrm{A}} = m$ should be satisfied; i.e., the truss is fully stressed.

For a statically indeterminate truss, we can specify stresses for at most $n \times n^{\mathrm{P}}$ members, because the stresses of n members can be specified for each loading condition, as demonstrated in Example 1.5. Therefore, the stress can be equal to its lower or upper bound for all members if $n^{\mathrm{P}} \geq m/n$ (Patnaik and Dayaratnam 1970). However, it is well known that the optimal solution under multiple loading conditions is not generally fully stressed even for the case where $\sigma_i^{1\mathrm{L}} = -\sigma_i^{1\mathrm{U}}$ and $\sigma_i^{1\mathrm{U}}$ is the same for all members; i.e., $\sigma_i^{1\mathrm{L}} \leq \sigma_i^1 \leq \sigma_i^{1\mathrm{U}}$ may be satisfied by a member with $A_i > 0$ (Kicher 1966; Patnaik and Dayaratnam 1970; McNeil 1971; Gunnlaugsson and Martin 1973; Patnaik and Hopkins 1998).

For a topology optimization problem with $A_i^{\mathrm{L}} = 0$ for all members, the stress constraints may be violated by the nonexistent members, and the optimal truss may be statically determinate even for the multiple loading conditions; see Sec. 3.5 for details.

Example 1.7

Consider a three-bar truss (Type 1), as shown in Fig. 1.12, which is subjected to two independent loads, P_1 and P_2, respectively. From $n^{\mathrm{P}} = 2$, $m = 3$, and $n = 2$, we can see $n^{\mathrm{P}} \geq m/n$ is satisfied. Suppose $P_1 = P_2 = 10$, and the lengths of members 1, 2, and 3 are $\sqrt{2}$, 1, and $\sqrt{2}$, respectively. The

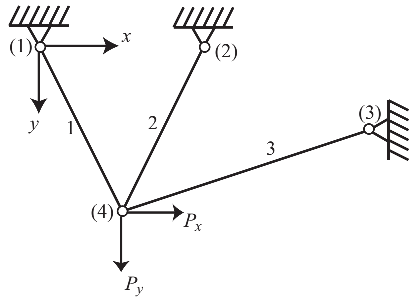


FIGURE 1.13: A three-bar truss (Type 2).

lower-bound cross-sectional areas are given as $A_i^{\mathrm{L}} = 0.1$ for all members; i.e., removal of a member is not allowed.

The optimal solution for $\sigma_i^{\mathrm{U}} = -\sigma_i^{\mathrm{L}} = 10$ for all members is $(A_1, A_2, A_3) = (\sqrt{2}/2, 1/2, \sqrt{2}/2)$, where the total structural volume is 2.5. Note that the value of A_i^{L} does not have any effect on the optimal solution if it is positive and not more than 1/2. The stresses are computed as $(\sigma_1^1, \sigma_2^1, \sigma_3^1) = (10, 0, -10)$ and $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (5, 10, 5)$. Therefore, the optimal truss is fully stressed. However, if $A_i^{\mathrm{L}} = 0$, then a member may be removed to obtain a statically determinate optimal truss, and the stress constraint may be violated by a nonexistent member. In fact, if we remove member 2, then the optimal solution is $(A_1, A_3) = (\sqrt{2}/2, \sqrt{2}/2)$, where the total structural volume is 2.0, which is less than 2.5 for the three-bar truss. If we assume that the elastic modulus E is equal to 1, then the vertical displacement of node 2 is 20. Therefore, the strain of member 2, although it does not exist, is 20, which is double the upper-bound stress.

Example 1.8

Next, consider a three-bar truss (Type 2), as shown in Fig. 1.13, where the coordinates of nodes and supports 1–4 are (0,0), (1000,0), (2000,500), and (500,1000), respectively. Three independent load sets $(P_x, P_y) = (5, 10)$, $(-5, 10)$, and $(-20, 10)$ are applied. The upper-bound stress is 0.2 and $A_i^{\mathrm{L}} = 0$ for all members. The optimal cross-sectional areas and the maximum absolute value of stress of each member under three loading conditions are listed in the second column of Table 1.2. As is seen, the optimal truss is statically indeterminate. Because the absolute value of the stress of member 2 is less than 0.2 for any loading condition, the optimal truss is not fully stressed. In fact, the optimal solutions of two-bar trusses after removal of members 1, 2, and 3, respectively, have larger objective values, as shown in the third, fourth, and fifth columns of Table 1.2, than that of the three-bar truss in the second column.

TABLE 1.2: Optimal cross-sectional areas, total structural volume, and maximum absolute values of stresses of the three-bar truss (Type 2) and corresponding two-bar trusses.

	three-bar	two-bar		
		2, 3	1, 3	1, 2
A_1	42.717	0	63.888	83.853
A_2	36.063	89.443	0	139.75
A_3	84.219	94.868	112.94	0
$\max\|\sigma_1^k\|$	0.20000	0.60000	0.20000	0.20000
$\max\|\sigma_2^k\|$	0.19709	0.20000	0.26429	0.20000
$\max\|\sigma_3^k\|$	0.20000	0.20000	0.20000	0.30000
Total volume	2.3539	2.5000	2.5000	2.5000

1.7 Optimality criteria approach

An optimization method that directly solves the optimality criteria (optimality conditions) is called the *optimality criteria approach* (OC approach) (Venkayya, Khot, and Berke 1973; Berke and Venkayya 1974; Dobbs and Nelson 1975; Khot, Berke, and Venkayya 1978). This approach is very effective for the case where the optimality conditions are written in a simple manner with explicit expressions of sensitivity coefficients with respect to the design variables and the state variables. Furthermore, this approach is more efficient in view of computational time and required memory than the gradient-based nonlinear programming (NLP) approaches; see Sec. 2.2 for sensitivity analysis of static responses, and Appendix A.2.2 for details of optimality conditions for general NLP problems.

Another advantage of the OC approach is that the computer program is very simple. Therefore, in the 1960s and 1970s, when computer power was not sufficient for computing sensitivity coefficients of the responses of moderately large structures many times for optimization, various studies on theoretical and computational aspects of the OC approaches were presented. For problems with general equality and inequality constraints, the OC approach is classified as a dual approach of NLP (Fleury 1979, 1980). Since the purpose here is to find a solution that satisfies the constraints and optimality conditions, it is possible to use a Newton-Raphson iteration for solving these nonlinear equations (Khot, Berke, and Venkayya 1978). However, the recursive formulas, as presented below, are generally used in an OC approach.

For the problem under stress constraints only, an approximate optimal solution can be easily found using the stress-ratio approach of fully-stressed design, as discussed in the previous section. Therefore, in this section, we consider an optimization problem of a truss under displacement constraints.

Suppose, for simplicity, a constraint is given only for the jth displacement component as

$$U_j \leq U_j^{\mathrm{U}} \tag{1.29}$$

where U_j is assumed to be positive. The objective function is the total structural volume, which is a monotonically increasing function of the cross-sectional areas $\mathbf{A} = (A_1, \ldots, A_m)^\top$. On the other hand, the nodal displacements are generally decreasing functions of $\mathbf{A}$, if the loads do not depend on $\mathbf{A}$. Therefore, the displacement constraint (1.29) is considered to be active, i.e., satisfied with equality, at the optimal solution assuming that the lower bounds A_i^{L} for the cross-sectional areas are sufficiently small. The upper bounds A_i^{U} are assumed to be sufficiently large to ensure the existence of a feasible solution. In this case, the following condition is obtained from the optimality condition (1.6) for members with $A_i^{\mathrm{L}} < A_i < A_i^{\mathrm{U}}$:

$$L_i + \mu_j \frac{\partial U_j}{\partial A_i} = 0 \tag{1.30}$$

where μ_j (≥ 0) is the Lagrange multiplier for the constraint (1.29).

The term $\partial U_j / \partial A_i$ in (1.30) is the sensitivity coefficient of U_j with respect to A_i that can be obtained efficiently using the adjoint variable method described in Sec. 2.2.2, because we have only one displacement component to be constrained. Let n denote the number of degrees of freedom. The axial force and the $n \times n$ stiffness matrix with respect to the global coordinates of the ith member are denoted by N_i and $\mathbf{K}_i$, respectively. The displacement vector against the specified loads is denoted by $\mathbf{U}$. The values corresponding to the virtual unit load at the jth displacement component are indicated by the superscript $(\,\cdot\,)^j$. Then the following relation is derived from the adjoint variable method of design sensitivity analysis of static response:

$$\begin{aligned} \frac{\partial U_j}{\partial A_i} &= -\mathbf{U}^{j\top} \frac{\partial \mathbf{K}_i}{\partial A_i} \mathbf{U} \\ &= -\frac{L_i}{A_i^2 E} N_i^j N_i \end{aligned} \tag{1.31}$$

where E is the elastic modulus.

Define Z_i as

$$Z_i = \mu_j \frac{N_i^j N_i}{A_i^2 E} \tag{1.32}$$

Then the optimality condition (1.30) is written as

$$Z_i = 1 \tag{1.33}$$

Let the superscript $(\,\cdot\,)^{(k)}$ denote a value at the kth step of iteration. For a statically determinate truss, N_i^j and N_i are independent of A_i. Therefore, assuming that μ_j is constant, the cross-sectional areas can be updated from (1.32) as follows:

$$A_i^{(k+1)} = (Z_i^{(k)})^{\frac{1}{2}} A_i^{(k)} \tag{1.34}$$

For a statically indeterminate truss, the cross-sectional area is updated by

$$A_i^{(k+1)} = (Z_i^{(k)})^{r} A_i^{(k)} \tag{1.35}$$

where r is a parameter between 0 and 1 for controlling the convergence property. Note that A_i is replaced with A_i^{L} and A_i^{U}, respectively, if $A_i < A_i^{\mathrm{L}}$ or $A_i > A_i^{\mathrm{U}}$ after application of (1.35).

Next, we derive the update rule of the Lagrange multiplier. From (1.32) and (1.33), we obtain

$$A_i = \sqrt{\mu_j \frac{N_i^j N_i}{E}} \tag{1.36}$$

By using the principle of virtual unit load and the active constraint $U_j = U_j^{\mathrm{U}}$, we have

$$U_j^{\mathrm{U}} = \sum_{i=1}^{m} \frac{L_i}{A_i E} N_i^j N_i \tag{1.37}$$

Through incorporation of A_i in (1.36) into (1.37), the Lagrange multiplier μ_j is updated as

$$\mu_j^{(k+1)} = \left(\sum_{i=1}^{m} \frac{L_i \sqrt{N_i^j N_i}}{U_j^{\mathrm{U}} \sqrt{E}} \right)^2 \tag{1.38}$$

Then we move to the next step of iteration. This way, the solution satisfying the constraint and optimality conditions is found by an iterative approach.

Alternatively, a linear approximation can be used for recursively updating A_i and μ_j (Khot, Berke, and Venkayya 1978). Multiplying $(1-\alpha)A_i^{(k)}$ on both sides of (1.33) with a parameter $0 < \alpha < 1$, letting

$$(1-\alpha)A_i^{(k)} = A_i^{(k+1)} - \alpha A_i^{(k)} \tag{1.39}$$

and rearranging the equation, we have the following update rule for A_i:

$$A_i^{(k+1)} = A_i^{(k)}[\alpha + (1-\alpha)Z_i^{(k)}] \tag{1.40}$$

Linear approximation of U_j leads to the following requirement for the displacement constraint at the $(k+1)$st step:

$$U_j^{(k)} + \sum_{i=1}^{m} \frac{\partial U_j}{\partial A_i}(A_i^{(k+1)} - A_i^{(k)}) = U_j^{\mathrm{U}} \tag{1.41}$$

Then, from (1.31), (1.37), (1.40), and (1.41), we obtain the following recursive formula for μ_j:

$$\mu_j \sum_{i=1}^{m} \frac{L_i (N_i^j N_i)^2}{E^3 A_i^3} = \frac{(2-\alpha)(U_j^{(k)} - U_j^{\mathrm{U}})}{1-\alpha} \tag{1.42}$$

The OC approach assumes that the member forces against the applied loads and the virtual unit load are insensitive to variation of the design variables, which is the same as the assumption for the stress-ratio approach (1.26) for fully-stressed design. This is better achieved if the responses are approximated with respect to the reciprocals of the cross-sectional areas (Schmit and Farshi 1974; Zhou and Haftka 1995). Let $a_i = 1/A_i$, and regard A_i as a function of a_i. Then we have

$$\frac{\partial A_i}{\partial a_i} = -\frac{1}{a_i^2} \tag{1.43}$$

and the sensitivity coefficient of U_j with respect to a_i is obtained from (1.31), (1.43), and $A_i = 1/a_i$ as

$$\begin{aligned} \frac{\partial U_j}{\partial a_i} &= \frac{\partial U_j}{\partial A_i}\frac{\partial A_i}{\partial a_i} \\ &= -\frac{L_i}{E} N_i^j N_i \end{aligned} \tag{1.44}$$

which does not explicitly depend on a_i. Although the sensitivity of the total structural volume turns out to be dependent on a_i, convergence of the recursive formulation is improved by using a_i as a design variable.

Because the number of analyses for computing the displacements against virtual unit load is proportional to the number of active or nearly active displacement constraints, this approach is effective for the case with a small number of active displacement constraints. For extension of the OC approach to problems with stress constraints, a pair of unit self-equilibrium forces is applied at the two ends of each member with an active stress constraint. However, the OC approach can be successfully combined with the fully-stressed design approach for problems with stress and displacement constraints.

Pereyra, Lawver, and Isenberg (2003) used an OC approach with a penalty function to optimize a building frame. For continuum structures such as beams, plates, and shells, a continuum-type optimality criteria (COC) approach was developed in the 1960s (Prager and Taylor 1968; Olhoff and Taylor 1979; Rozvany 1989). Furthermore, OC and COC were combined to a discretized form of COC approach termed DCOC (Zhou and Rozvany 1992, 1993; Rozvany and Zhou 1994).

Since the OC approach is simple and easy to implement, it is widely applied in many areas of heuristics, e.g., evolutionary structural optimization (ESO) (Yang, Xie, Steven, and Querin 1999a; Xie and Steven 1993) and cellular automaton (Canyurt and Hajela 2005).