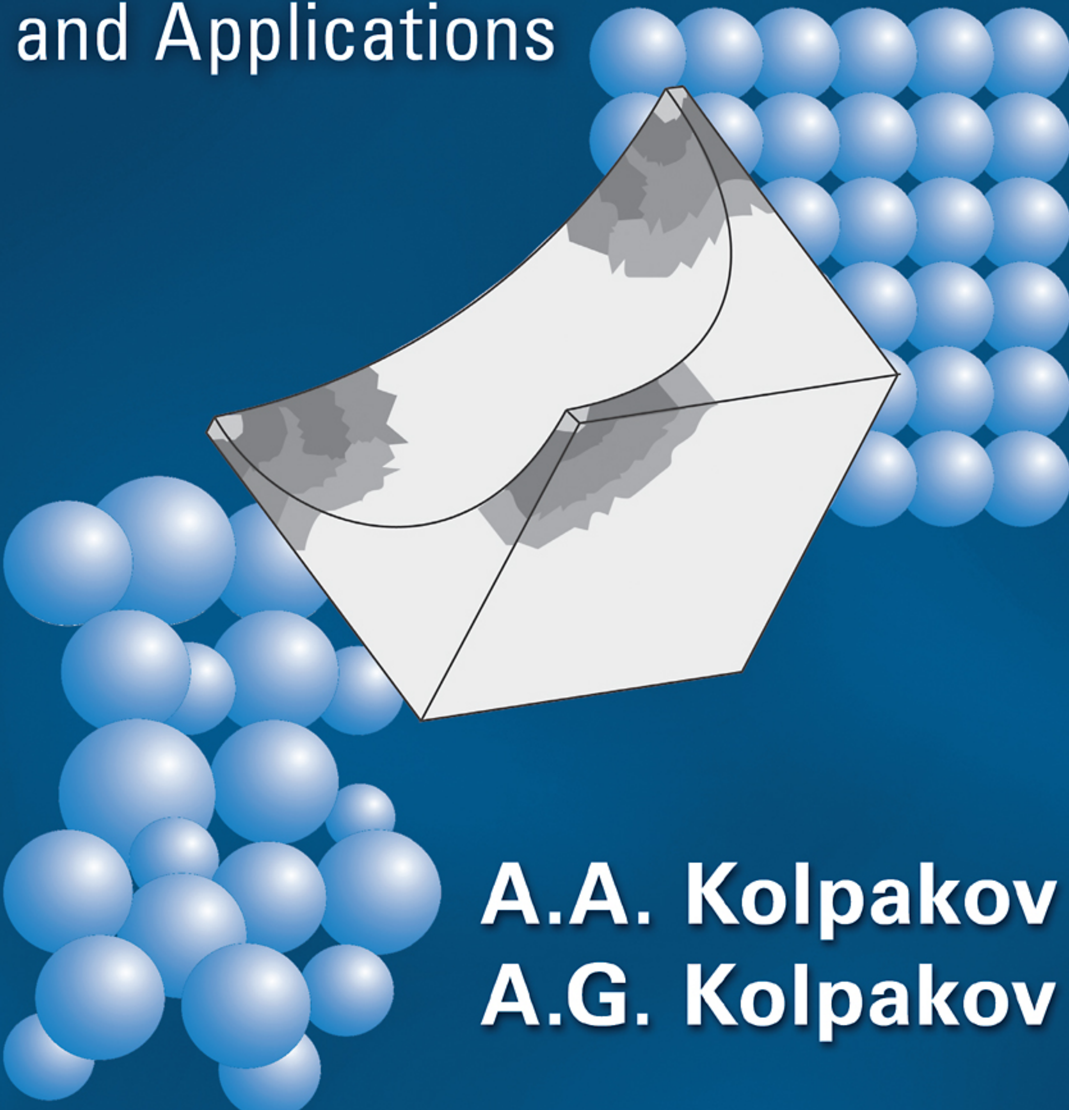
Capacity and Transport in Contrast Composite Structures

Asymptotic Analysis and Applications



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PREFACE

This book is devoted to the analysis of the capacity of systems of closely placed bodies and the transport properties of high-contrast composite structures. This title covers many similar problems well known in natural science, material science and engineering.

The term “transport problem” implies problems of thermoconductivity, diffusion, electrostatics and many other similar problems, which can be described with a scalar linear elliptic equation or a nonlinear equation of elliptic type. For a linear inhomogeneous medium, the transport problem consists of balance equation

$$\mathrm{div}\mathbf{q} = f(\mathbf{x}),$$

constitutive equation

$$q_i = c_{ij}\frac{\partial \varphi}{\partial x_j},$$

which is often written in the form $q_i = -c_{ij}\dfrac{\partial \varphi}{\partial x_j}$, and boundary conditions.

Here φ is the potential, $\nabla\varphi = \left(\dfrac{\partial \varphi}{\partial x_1}, ..., \dfrac{\partial \varphi}{\partial x_n}\right)$ is the driving force, $\mathbf{q} = (q_1, ..., q_n)$ is the flux, c_{ij} is a tensor describing local (microscopic) transport property of the medium (tensor of dielectric constants, tensor of thermoconductivity constants, etc.), n is the dimension of the problem (in the book n takes values 2 or 3).

The equations above can be transformed into one elliptic equation

$$\frac{\partial}{\partial x_i}\left(c_{ij}\frac{\partial \varphi}{\partial x_j}\right) = f(\mathbf{x}),$$

which must be supplied with an appropriate boundary condition.

Table 1 lists several transport problems that are mathematically equivalent. Due to this equivalence we can treat these problems within a common theoretical framework.

In some cases, it is necessary to take into account the nonlinearity of local properties of component(s) of composite. In practice and in nature, we meet various types of nonlinearities. In thermoconductivity, usually, coefficients of thermoconductivity depend on the temperature: $c_{ij} = c_{ij}(\varphi)$ (φ means the temperature). In electrostatics, usually, dielectric constants depend on the electric field: $c_{ij} = c_{ij}(\nabla\varphi)$, ($\varphi$ means the potential of electric field).

Table 1. *List of Phenomena (* asymmetric deformation or torsion).*

Phenomenon	*Potential*	*Driving force*	*Flux*	*Local tensor*
Heat conduction	Temperature	Temperature gradient	Heat flux	Thermal conductivity
Electrical conduction	Electric potential	Electric field	Current density	Electrical conductivity
Diffusion	Density	Density gradient	Diffusion current density	Diffusivity
Electrostatics	Electric potential	Electric field	Electric displacement	Dielectric permittivity
Magnetostatics	Magnetic potential	Magnetic field	Magnetic induction	Magnetic permittivity
Elasticity theory*	Displacement	Strain	Stress	Elastic moduli
Flow in porous media	Pressure	Weighted fluid velocity	Pressure gradient	Fluid permittivity

The term "composite material" means that the local transport properties (described by the tensor c_{ij}) depend on spatial variable $\mathbf{x}$. Thus, for linear composite materials $c_{ij} = c_{ij}(\mathbf{x})$. For nonlinear composite materials $c_{ij} = c_{ij}(\mathbf{x}, \varphi)$ or $c_{ij} = c_{ij}(\mathbf{x}, \nabla\varphi)$. It would not be correct to call an arbitrary inhomogeneous material a composite material. The term composite material assumes an existence of some structure in material. The structures can be very different: from regular to random, from particles-filled to laminated. Often, the term composite material assumes a property to be solid (to represent a unity). At the same time, systems of bodies / particles in air and liquids (powders, aerosol, suspensions, slurries) should not be separated from the composite material (the mentioned systems consist of at least two components, one of which is bodies / particles and the other component the surrounding medium). This is a reason why we use the term "composite structure" in this book, which designates both composite material and system of bodies / particles.

The systems of bodies and particle-filled composite materials can be treated in the framework of a unique approach. The mathematical models for bodies and particle-filled composites are the same; they are differential equations with discontinuous coefficients (see the equation above). The difference between problems for systems of bodies and composite materials is related to the type of the boundary-value problem: inner boundary-value problems correspond to composite materials and outer boundary-value problems correspond to systems of bodies.

A composite structure has some characteristic dimensions. One dimension is the size of the structure as a whole (so-called macroscopic dimension). We assume the macroscopic dimensional has the order of unity. Another dimension is the size of the structural elements of a composite (so-called microscopic dimension). We denote this dimension $\delta \ll 1$. Note that in many publications devoted to the homogenization theory, the macroscopic dimension is denoted by the symbol ε. Since we present our theory in terms of electrostatics (see below), the symbol ε is reserved in this book for dielectric constant. The number of sizes (often referred to as scales) is not restricted by two. Multi-scale structures are well-known (see, e.g., [30, 283]).

The term “high-contrast” means that transport properties of components of composite material are strongly different. The extremal (and widely used in physics and engineering, see, e.g., [340, 354]) case of high-contrast structures is a system of perfectly conducting bodies / particles.

The book is written in the terms of electrostatics, i.e., we call the solution of the transport problem potential, but not temperature or density, although all the results are valid for thermoconductivity and diffusion problems (as well as for all the problems listed in Table 1). A reason for using the electrostatic terminology is that the transport property of densely packed systems is determined by capacity of the pairs of neighbor bodies (it will be demonstrated below). It explains why capacity stands before the transport properties in the title of the book. It also explains why we discuss most problems keeping in mind the electrostatic problem.

The book presents mathematical treatment to phenomena intensively discussed in literature on natural sciences and engineering. For some problems (for example, the problem of effective properties of nonlinear dielectric) the intensive discussion was started in the last decade. Some problems were known and discussed for more than a century (for example, the problem of the capacity of a system of densely placed bodies). The current progress in the analysis of the mentioned problems was stimulated by progress in the mathematical methods (progress in the theory of partial differential equations, development of the homogenization method, etc.), in computer techniques and finite element computer programs. It is why a considerable part of the book is devoted to mathematics calculations and the presentation of results of numerical computations.

Many problems analyzed in the book were initiated by real world problems. For example, the theory of asymptotic behavior of capacity of a system of closely placed bodies was initiated by a project supported by a consortium of industrial companies (the names of the companies in 1999 were Polyclad and Hadco). The theory of nonlinear high-contrast dielectrics–ferroelectrics composites was initiated by a project supported by the U.S. Department of Energy. The initial stages of the mentioned projects are described in [39, 40, 41, 191].

This book is written on the basis of the authors’ results published in Russian and international journals in the 1990s to 2000s. Most Russian scientific journals are translated to English from cover to cover by international publishers. English versions of all the authors’ Russian papers included in the list of references can be found on the Internet at *http://www.springer.de* (Springer-Verlag) and *http://*

www.elsevier.com (Elsevier Science Publishers).

The book is structured as follows:

Chapter 1 presents a brief exposition of some asymptotic methods used for analysis of composite structures (composite materials and systems of bodies / particles) with brief historical comments.

Chapter 2 presents results of numerical analysis, which demonstrate specific properties of distributions of local fields in high-contrast composite structures and systems of closely placed bodies. In particular, the existence of "energy necks" in a system of densely packed bodies and closeness of potentials of the bodies determined from solution of the original continuum problem and the "potentials of nodes" determined from the corresponding network model are demonstrated.

Chapter 3 presents asymptotic analysis of the capacity of a system of closely placed bodies. In this chapter, we establish a relationship between the transport problem and the problem of asymptotic behavior of the capacity of a system of closely placed bodies. We do it on the basis of our generalization and mathematic interpretation of the "Tamm shielding effect" for a system of closely placed bodies (for two bodies, the phenomenon was described by the Soviet physicist, Nobel Prize Laureate I.E. Tamm in his book [353] published in 1927). Analysis of the problem leads us to the conclusion that the unique universal property of a system of closely placed bodies is the impossibility of localization of energy outside the channels between the neighbor bodies. As far as Tamm shielding, we found that it is a conditional effect. We demonstrate that the necessary and sufficient condition for existence of Tamm shielding (and, as a result, arising of "energy channels" between neighbor bodies, energy decomposition, network approximation, etc.) is the infinite increasing capacity of a pair of neighbor bodies when the distance between them tends to zero. This is a pure geometrical condition (it depends on the geometry of the bodies only). We note that this condition is not valid for the arbitrary geometry of bodies. As a result, network approximation (network modeling) is not possible for any system of closely placed bodies. Then the capacity (and transport property) of a system of closely placed bodies is controlled not only by material contrast and interparticle distances. The geometry of bodies is an additional necessary control parameter.

In Chapter 4, we put the question: "Do the total flux, energy and capacity (which are characteristics of integral nature) exhaust characteristics of the original continuum model which can be approximated with the corresponding network model?" We demonstrate that the potentials of the bodies can be added to this list (under the condition that the Tamm shielding effect takes place for the bodies under consideration!).

Chapter 5 presents a description of expansion of the method developed in Chapters 3 and 4 for systems of bodies to highly filled contrast composites. In this chapter, we also present some examples of numerical analysis of transport properties of high-contrast highly filled disordered composite material with the network model. The authors think that it would be difficult, if possible, to obtain similar results with a continuum model even using a large computer.

Chapter 6 deals with the mathematical and numerical analysis of special homogenization problems for a nonlinear composite with high-contrast components. The specificity of the problem considered is related not with any restrictions on the original problem (it is just a problem of general form) but with analysis of a special characteristic named the homogenized tunability of composite material. This characteristic is well-known in the electronics industry. From the mathematical point of view, this is (roughly speaking) the measure of nonlinearity of the problem under consideration. This chapter demonstrates that the behavior of effective characteristics of nonlinear composites can differ from the behavior of effective characteristics of linear composites qualitatively. For example, effective (homogenized) tunability can increase significantly when one dilutes nonlinear material with linear inclusions. No analog of this effect exists in linear homogenization theory. The data on the homogenized permittivity presented in this chapter may be of interest for the general theory of composite materials, because they clearly demonstrate that homogenized characteristics can show no correlation with the volume fraction of components of the composite.

Chapter 7 deals with the problem of loss of high-contrast composites.

Chapter 8 is devoted to transport and elastic properties of thin layers, which cover or join solid bodies. This theme is related to the problems considered in Chapters 3 and 4. In particular, the trial functions developed for analysis of thin joints were predecessors of the trial functions used in Chapters 3 and 4.

The authors thank Dr. S.I. Rakin (STU, Novosibirsk) for assistance in research. The authors thank Prof. I.V. Andrianov (RWTH–Aachen), Prof. L. Berlyand (Pennsylvania State University), Prof. V.V. Mityushev (Uniwersytet Pedagogiczny w Krakowie), Prof. A. Gaudiello (Università degli Studi di Cassino), Prof. V.V. Zikov (Vladimir State Humanitarian University) for providing references, useful comments and discussions. The research was supported through Marie Curie actions FP7, project PIIF2-GA-2008-219690.

The authors hope that the book will be used by both applied mathematicians interested in new mathematical methods and engineers interested in prospective materials and design methods. The authors would be happy if the book stimulates the interest of engineering students in mathematics as well as the interest of mathematical students in the problems arising in modern engineering and natural science.

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Chapter 1

IDEAS AND METHODS OF ASYMPTOTIC ANALYSIS AS APPLIED TO TRANSPORT IN COMPOSITE STRUCTURES

When we consider a medium formed of a large number of small components, a system of closely placed bodies or a medium formed of components with strongly different (contrast) properties, we usually find small or large parameters naturally related to the structures under consideration. Sometimes we found not one but two or even more small or large parameters. For a composite body formed of large number of small components, the natural small parameter is a characteristic dimension of the components (usually, as compared with the dimension of the body). If, in addition, composite material is formed of contrast components, there appears one more parameter — ratio of material characteristics of the components.

If characteristics (either material or geometrical) depend on small or large parameters, the corresponding mathematical models account for these dependences. The mathematical models containing small or large parameters often can be analyzed by using asymptotic methods. The asymptotic methods strongly depend on the specific type of parameter and specific problem. We can divide (very roughly) the asymptotic methods arising in applied sciences into two groups:

1) problems in which geometry depends on a parameter,

2) problems in which material characteristics depend on a parameter.

Examples of the first group problems are asymptotic methods developed for analysis of problems in thin or small diameter domains [70, 75, 182, 282, 336, 360], in singularly perturbed domains [228, 264], in thin layers [229, 294, 317, 337, 338], in junctions of structural elements [44, 90, 130]. Examples of the second group of the methods are classical theory of small perturbation of coefficients of differential equations and integral functionals [153, 164, 334] and the homogenization theory [30, 21, 157]. If material characteristics are periodic with period depending on small parameter, we arrive at the classical theory of homogenization [21, 91, 157]. If material characteristics can be described by random fast oscillation functions, we arrive at the random homogenization [157, 194, 195, 286, 393]. If the variation of material characteristics, in addition, is large, we arrive at so-called "stiff" problems [25, 58, 60, 65, 73, 98, 149, 211, 218, 219, 284, 289] and problems of transmitting through strongly inhomogeneous structures [103, 129].

We present below a brief overview of asymptotic methods, which can be useful for the reader.

1.1. Effective properties of composite materials and the homogenization theory

The problem of computation of overall properties of composite materials has a long history and it has attracted attention of some of outstanding scientists. Historically, analysis of overall properties of composite materials was started with a model of material filled with particles. For example, Poisson [295] constructed a theory of induced magnetism in which the body was assumed to be composed of non-conducting material filled with conducting spheres. Faraday [117] proposed a model for dielectric materials that consists of metallic globules separated by insulating materials. Significant contributions to solution of the problem of computation of overall properties of composite materials were done by Maxwell [227] and Rayleigh [348]. Other well-known 19th century contributors to the field were Clausius [92], Mossotti [261] and Lorenz [215].

In the 20th century many prominent scientists paid attention to the computation of overall properties of mixtures [64, 93, 128, 150, 214], suspensions [111, 112, 202, 310, 375] and systems of bodies and particles [50, 51]. The significant achievement was the theory of bound for effective characteristics of composite materials. The foundations of this theory were laid in the works by Reuss, Voight and Hill [150, 305, 367].

In the 1970s to 1980s, the so-called homogenization method was elaborated and applied to the analysis of composite materials. The foundations of the homogenization theory were laid in the pioneering papers by Spagnolo and Marino [224, 343, 344] published in 1960s, followed by numerous works published in 1970s–1980s. Mention the papers [20, 21, 30, 32, 108, 194, 221, 280, 317, 325, 397] (list is not complete, for additional bibliography information see [30, 21, 157]). The applied directions of the homogenization method are presented in [4, 5, 13, 27, 28, 29, 52, 56, 69, 78, 91, 97, 132, 134, 142, 159, 205, 278, 283, 285, 287, 314, 360, 382]. Applications of the homogenization method provided many important results of both theoretical and engineering significance. Mention theoretical prediction [6, 178] and manufacturing [201] of materials with negative Poisson's ratio and application of the homogenization method to design of composites possessing required overall properties [27, 28, 29].

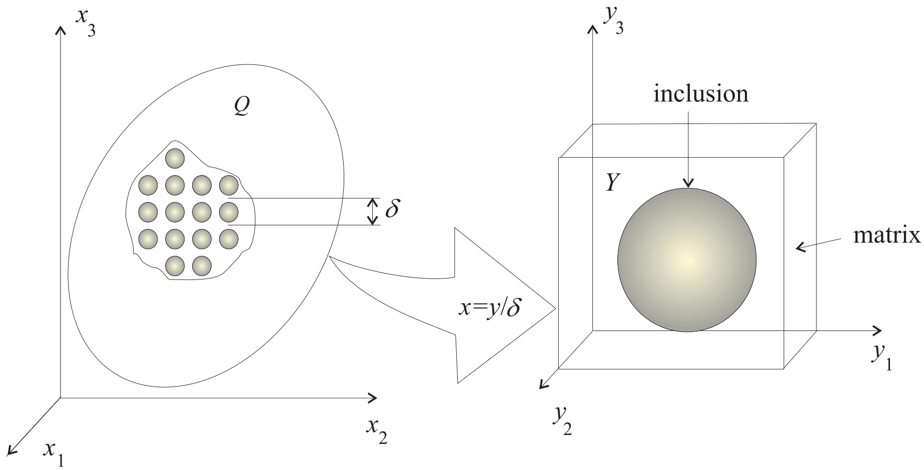

Figure 1.1. *A body of periodic structure and its periodicity cell Y in fast variables.*

The homogenization method for composites of a periodic structure uses various mathematical techniques. The basic techniques are presented in [5, 30, 157, 317]. In the present, various multiscale techniques are developed (see, e.g., [243, 283, 288]) and widely used in applied sciences (see, e.g., [14, 104, 198, 207, 213]).

1.1.1. Homogenization procedure for linear composite materials

In this section, we present basic ideas of the asymptotic expansions method. Consider an inhomogeneous body with a regular distribution of transport properties of those components, see Fig. 1.1.

The transport problem (thermoconductivity, diffusion, etc.) for that body has the form

$$L_\delta T^\delta = \frac{\partial}{\partial x_i}\left(c_{ij}^\delta(\mathbf{x})\frac{\partial T^\delta}{\partial x_j}\right) = f(\mathbf{x}) \text{ in } Q, \tag{1.1}$$

$$T^\delta(\mathbf{x}) = 0 \text{ on } \partial Q. \tag{1.2}$$

Here Q designates the region occupied by the composite material, ∂Q designates its boundary. Here δ is a parameter, which will be associated with the characteristic dimension of inhomogeneity of a composite, see Fig. 1.1. Thus, we consider a problem with parameter (or, in other words, a sequence of problems).

The following standard conditions are applied to the coefficients $c_{ij}^\delta(\mathbf{x})$: for all $\mathbf{x} \in Q$,

$$c_1|\mathbf{z}|^2 \leq c_{ij}^\delta(\mathbf{x}) z_i z_j \leq c_2|\mathbf{z}|^2$$

for any $\mathbf{z} \in R^n$ $(n = 2, 3)$.

Here $0 < c_1, c_2 < \infty$ do not depend on δ. The uniform boundary condition (1.2) does not lead to the loss of generality of our consideration because the homogenized constants do not depend on the type of boundary conditions [30, 157]. We consider here uniform boundary conditions for simplicity.

Let us note that problem (1.1) and (1.2) permits the following formulation: find $T^\delta(\mathbf{x})$ from the solution of the minimization problem,

$$J_\delta(T) + \langle f, T\rangle \to \min,\ T(\mathbf{x}) \in H_0^1(Q), \tag{1.3}$$

where

$$J_\delta(T) = \frac{1}{2}\int_Q c_{ij}^\delta(\mathbf{x})\frac{\partial T}{\partial x_i}(\mathbf{x})\frac{\partial T}{\partial x_j}(\mathbf{x})d\mathbf{x}$$

is a quadratic functional.

In this book, $H_0^1(Q)$ means closure according to the norm

$$||f||_{H^1(Q)} = \sqrt{\sum_{i=1}^{3}\int_Q \left|\frac{\partial f}{\partial x_i}(\mathbf{x})\right|^2 d\mathbf{x} + \int_Q f(\mathbf{x})^2 d\mathbf{x}}$$

of a set $C^\infty(Q)$ of finite functions, which are infinitely smooth in Q and vanish in a neighbor of ∂Q (other equivalent definitions of $H_0^1(Q)$ can be found in Appendix A).

The pointed branches in (1.3) signify dual coupling of the elements from $H_0^1(Q)$ and $H^{-1}(Q)$ in the standard duality of these spaces (for details see, e.g., [212]). The dual coupling coincides for sufficiently smooth functions with an inner product in $H_0^1(Q)$ (see [113, 212] for details). The equivalence of problems (1.1) and (1.2), and (1.3) is a well-known fact, see, e.g., [113, 212]. Problem (1.1) and (1.2) can be written in the following (so-called weak) form [212] :

$$-\int_Q q_i^\delta(\mathbf{x})\frac{\partial \varphi}{\partial x_i}(\mathbf{x})d\mathbf{x} = \int_Q f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \tag{1.4}$$

for any $\varphi(\mathbf{x}) \in H_0^1(Q)$.

In (1.4)

$$q_i^\delta(\mathbf{x}) = c_{ij}^\delta(\mathbf{x})\frac{\partial T^\delta}{\partial x_j}(\mathbf{x}). \tag{1.5}$$

If a body is formed of many small components, the characteristic dimension δ of components is small: $\delta \ll 1$. Mathematically, this fact is formalized in the form $\delta \to 0$.

As is well known from engineering practice, materials and structures formed of many small components (concrete, wool, suspensions, aerosols) can be regarded as homogeneous ones. Note that most engineering handbooks (except special handbooks on composite structures, see, e.g., [74]) usually present technical constants

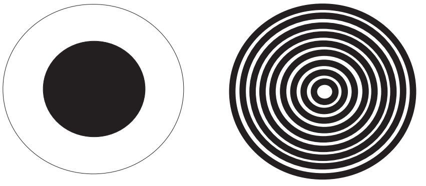

Figure 1.2. *A ring with a circular cartridge (left) and wood-like structure (right).*

of inhomogeneous materials (thermoconductivity coefficients, viscosity, elastic constants) as characteristics of homogeneous material, see, e.g., [197]. It means that one replaces an original inhomogeneous material for a (fictitious) homogeneous material. It is clear that such kind of substitution is possible only in asymptotic sense, when a material is a unity of small components with the characteristics dimension $\delta << 1$ ($\delta \to 0$). Usually, engineer-experimentator does not think about asymptotic, but he cares for taking a specimen for experiment relatively large as compared with the dimension of microstructure [172, 379]. It is clear that choice of specimen is equivalent to acceptance of asymptotic nature of overall characteristics of composite structures. We present two two-component circular structures as example. The volume fractions of the constitutive materials are equal in both structures. The circular structure displayed in Fig. 1.2 (left) cannot be approximated by a homogeneous structure, while the layered circular structure displayed in Fig. 1.2 (right) can be approximated by a homogeneous structure when the characteristic thickness of the layers is small. The layered circular structure displayed in Fig. 1.2 (right) is similar to the structure of wood. The engineering characteristics of woods are given in most hand-books in the form of characteristics of homogeneous materials. These characteristics describe properties of woods adequately and are successfully used in practice.

A homogeneous body (specifically, one which we want to put in correspondence with a composite) is described by the problem

$$LT^{(0)} = \frac{\partial}{\partial x_i}\left(\widehat{c}_{ij}\frac{\partial T^{(0)}}{\partial x_i}\right) = f(\mathbf{x}) \text{ in } Q, \tag{1.6}$$

$$T^{(0)}(\mathbf{x}) = 0 \text{ on } \partial Q, \tag{1.7}$$

or by the minimization problem: find $T^{(0)}(\mathbf{x})$ from the solution of the problem

$$J(T) + \langle f, T\rangle \to \min,\ T(\mathbf{x}) \in H_0^1(Q), \tag{1.8}$$

where

$$J(T) = \frac{1}{2}\int_Q \widehat{c}_{ij}\frac{\partial T}{\partial x_i}(\mathbf{x})\frac{\partial T}{\partial x_j}(\mathbf{x})d\mathbf{x}.$$

Here, $\widehat{c}_{ij}$ are homogenized transport constants describing a homogeneous material (we use the hat symbol "$\widehat{\ }$" to mark homogenized characteristic corresponding to local characteristic under consideration). It is clear that the homogenized constants depend on the local transport constants of the composite.

The coupling of operators L_δ and L and functionals J_δ and J is well known. The functionals J_δ and J are the potentials of the corresponding operators, and operators L_δ and L are derivatives, in the sense of Gâteaux [113], of the corresponding functionals.

Asymptotic expansion based approach to the analysis of media with a periodic structure

Periodic materials are less prevalent in nature, where we meet disordered structures of combination of periodicity with various random derivations from it. On the other hand, we meet numerous periodic structures among artificial (man-made) structures.

Let us consider the case when an inhomogeneous body has a periodic structure in coordinates $\mathbf{x}$, with a period (called periodicity cell, unit cell, or basic cell) δY, see Fig. 1.1. For a periodic structure, the factor δ is the dimension of the periodicity cell, see Fig. 1.1. The material characteristics of the indicated medium are described by periodic functions in spatial variable of the following type [30]

$$c_{ij}^{\delta}(\mathbf{x}) = c_{ij}(\mathbf{x}/\delta), \tag{1.9}$$

where $c_{ij}(\mathbf{y})$ are periodic functions with a periodicity cell Y.

The method of asymptotic expansions is based on the ideas of solving the problem with rapidly oscillating coefficients in the form of the following special series:

$$T^{\delta}(\mathbf{x}) = T^{(0)}(\mathbf{x}) + \sum_{n=1}^{\infty} \delta^n T^{(n)}(\mathbf{x}, \mathbf{y}), \tag{1.10}$$

where $\mathbf{y} = \mathbf{x}/\delta$ is a "fast" variable and $\mathbf{x}$ is a "slow" variable, i.e., a two-scale expansion is considered. Functions $T^{(n)}(\mathbf{x}, \mathbf{y})$ in (1.10) are assumed to be periodic in variable $\mathbf{y}$ with periodicity cell Y. Function $T^{(0)}(\mathbf{x})$ is a function only of "slow" variable $\mathbf{x}$. By substituting $\mathbf{x}/\delta$ for $\mathbf{y}$, the functions become periodic in $\mathbf{x}$ with periodicity cell δY.

We will seek the solution of problem (1.1) and (1.2) in the form of an asymptotic expansion (1.10). While differentiating, we will separate the variables according to the formula

$$\frac{\partial f(\mathbf{x}, \mathbf{x}/\delta)}{\partial x_i} = f_{,ix}(\mathbf{x}, \mathbf{y}) + \delta^{-1} f_{,iy}(\mathbf{x}, \mathbf{y}), \tag{1.11}$$

$$\mathbf{y} = \mathbf{x}/\delta. \tag{1.12}$$

Here and afterward, subscript $,ix$ means $\dfrac{\partial}{\partial x_i}$ and the subscript $,iy$ means $\dfrac{\partial}{\partial y_i}$.

The operator L_δ on the left-hand side of equation (1.1), allowing for the differentiating rule (1.11), can be written as

$$L_\delta = \delta^{-2} L_0 + \delta^{-1} L_1 + L_2, \tag{1.13}$$

where

$$L_0 = \frac{\partial}{\partial y_i}\left(c_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right),$$

$$L_1 = \frac{\partial}{\partial x_i}\left(c_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right) + \frac{\partial}{\partial y_i}\left(c_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right),$$

$$L_2 = c_{ij}(\mathbf{y})\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}.$$

We change in (1.1) operator L_δ for its representation (1.13) and $T^\delta(\mathbf{x})$ for the series (1.10) and then equate the terms of the same order in δ. As a result, we obtain an infinite sequence of problems. The first three of them have the following form:

$$L_0 T^{(0)} = 0, \tag{1.14}$$

$$L_0 T^{(1)} + L_1 T^{(0)} = 0, \tag{1.15}$$

$$L_0 T^{(2)} + L_1 T^{(1)} + L_2 T^{(0)} = f(\mathbf{x}). \tag{1.16}$$

Equation (1.14) is satisfied identically because function

$$T^{(0)} = T^{(0)}(\mathbf{x}) \tag{1.17}$$

does not depend on the variable $\mathbf{y}$, see (1.10).

By virtue of (1.17), (1.15) takes the form

$$\left(c_{ij}(\mathbf{y})T^{(1)}_{,jy}\right)_{,iy} + \left(c_{ik}(\mathbf{y})\right)_{,iy} T^{(0)}_{,kx}(\mathbf{x}) = 0. \tag{1.18}$$

By separating the variables $\mathbf{x}$ and $\mathbf{y}$, the solution of (1.18) can be set up as

$$T^{(1)}(\mathbf{x},\mathbf{y}) = N^k(\mathbf{y})T^{(0)}_{,kx}(\mathbf{x}) + V(\mathbf{x}), \tag{1.19}$$

where $N^k(\mathbf{y})$ represents a solution of the problem

$$\begin{cases} \left(c_{ij}(\mathbf{y})N^k_{,jy} + c_{ik}(\mathbf{y})\right)_{,iy} = 0 \text{ in } Y, \\ N^k(\mathbf{y}) \text{ is periodic in } \mathbf{y} \text{ with periodicity cell } Y, \end{cases} \tag{1.20}$$

and $V(\mathbf{x})$ is an arbitrary function of the argument $\mathbf{x}$.

We call problem (1.20) a cellular problem. It is also called a basic cell or a unit cell problem [30, 157, 318]. Let us consider equations (1.16), in which the function is unknown and $\mathbf{x}$ is a parameter. This problem has a periodic solution if the following equality is fulfilled [21]:

$$\langle L_1 T^{(1)} + L_2 T^{(0)} \rangle = f(\mathbf{x}), \tag{1.21}$$

where

$$\langle \bullet \rangle = \frac{1}{|Y|} \int_Y \bullet d\mathbf{x} \tag{1.22}$$

indicates the average value over periodicity cell Y. The average value symbol $\langle\rangle$ should not be confused with the dual coupling symbol $\langle , \rangle$.

From the homogenized equation (1.21), on account of (1.19), we obtain the homogenized (called also averaged or macroscopic) equation (1.6) for $T^{(0)}(\mathbf{x})$ with the boundary conditions (1.7). Simultaneously, we obtain from (1.19) and (1.21) the following formula for computation of the homogenized coefficients $\widehat{c}_{ij}$:

$$\widehat{c}_{ij} = \langle c_{ij}(\mathbf{y}) + c_{ik}(\mathbf{y}) N^j_{,ky}(\mathbf{y}) \rangle. \tag{1.23}$$

It is known (see, e.g., [30]) that

$$T^\delta(\mathbf{x}) \to T^{(0)}(\mathbf{x}) \text{ weakly in } H^1(Q), \tag{1.24}$$
$$T^\delta(\mathbf{x}) \to T^{(0)}(\mathbf{x}) \text{ in } L_2(Q)$$

as $\delta \to 0$, where $T^\delta(\mathbf{x})$ is the solution of the original problem (1.1) and (1.2), and $T^{(0)}(\mathbf{x})$ is the solution of the homogenized problem (1.6) and (1.7). We note that the second limit in (1.24) is the consequence of the first limit and Sobolev embedding theorem [342].

It is also known [30] that

$$T^\delta(\mathbf{x}) - \left(T^{(0)}(\mathbf{x}) + \delta N^j(\mathbf{x}/\delta) \frac{\partial T^{(0)}}{\partial x_j}(\mathbf{x}) \right) \to 0 \text{ in } H^1(Q), \text{ as } \delta \to 0. \tag{1.25}$$

From relations (1.5), (1.11) and (1.25), for the local flux

$$q_i^\delta(\mathbf{x}) = c_{ij}(\mathbf{x}/\delta) \frac{\partial T^\delta}{\partial x_j}(\mathbf{x})$$

the following approximation can be derived:

$$q_i^\delta(\mathbf{x}) - \left(c_{ij}(\mathbf{x}/\delta) + c_{ik}(\mathbf{x}/\delta) N^j_{,ky}(\mathbf{x}/\delta) \right) \frac{\partial T^{(0)}}{\partial x_j}(\mathbf{x}) \to 0 \text{ in } L_2(Q) \text{ as } \delta \to 0. \tag{1.26}$$

It is known (see, e.g., [30, 278]) that

$$\langle \mathbf{q}^\delta(\mathbf{x}) \rangle \to \mathbf{q}^0(\mathbf{x}) \text{ as } \delta \to 0, \tag{1.27}$$

where $\langle \mathbf{q}^\delta \rangle$ is the average value of the local flux $\mathbf{q}^\delta$ and $\mathbf{q}^0$ is the flux determined from the solution of the homogenized problem,

$$q_i^0(\mathbf{x}) = \widehat{c}_{ij} \frac{\partial T^{(0)}}{\partial x_j}(\mathbf{x}). \tag{1.28}$$

The flux determined by equality (1.28) is called the homogenized flux. One can derive formula (1.27) by averaging (1.26) over periodicity cell with regard to the definition of the homogenized constants (1.23). It follows from (1.24) that in the limit (as $\delta \to 0$), potential (electric potential, temperature, etc.) in a nonhomogeneous material will behave like potential in a homogeneous material with effective transport coefficients given by (1.23). But, it is seen from (1.26) and (1.28), the local flux can differ (and usually it differs) from the homogenized flux. As seen from (1.27), the averaged values of the local flux coincides with the homogenized flux. Note that the actual (substantively existing) flux in a composite is the local flux. The average value of local flux characterizes the overall (macroscopic) transport property of composite material and it can be used to compute the homogenized characteristics of composite.

1.1.2. Homogenization procedure for nonlinear composite materials

The foundations of the homogenization theory for nonlinear operators and functionals were laid in the works [45, 221, 223]. Now, there exist some homogenization procedures for nonlinear composite materials based on $G(\Gamma)$-convergence and multiscale technique [21, 96].

G-limit based approach to analysis of media with a periodic structure

The G-limit approach is a sophisticated mathematical method used in the analysis of homogenization problems and proof of convergence theorems [45, 72, 100, 221]. It was not widely used in applied sciences and engineering previously. Recently, the situation has changed, see, e.g. [300].

We need certain mathematical notations, which were introduced in [221]. Let us denote by V the Banach reflexive space and V^* is a space topologically conjugated with V [113]. Following [221], let us denote by $C_0(V)$ a set of convex functionals in V, which are assumed to take values in $(-\infty, +\infty]$, not identically equal to $+\infty$ and lower semicontinuous (see Appendix A). Let us also denote

$$C(\alpha, v_0, M) = \{f(\mathbf{x}) \in C_0(V) : f(v) \leq \alpha(v) \text{ for all } v(\mathbf{x}) \in V, f(v_0) \leq M < \infty\},$$

where the functional $\alpha(\mathbf{x}) \in C_0(V)$ is such that $\alpha(v) - \langle v^*, v \rangle$ reaches a minimum in V for any $v^* \in V^*$.

The functional f^*, defined in $v^* \in V^*$ through the equality

$$f^*(v^*) = \sup_{v \in V} (\langle v^*, v \rangle - f(v))$$

is called a conjugate to f.

Definition 1. *The sequence of functionals $\{f_\delta\} \subset C(\alpha, v_0, M)$ G-converges to the functional f, if*

$$\lim_{\delta\to 0} f^*_\delta(v^*) = f^*(v^*) \text{ as } \delta \to 0 \text{ for any } v^* \in V^*.$$

Definition 2. *The sequence of operators $L_\delta : V \to V^*$ G-converges to the operator $L : V \to V^*$, if operators L_δ and L are invertible and for any $v^* \in V^*$*

$$\lim_{\delta\to 0} L_\delta^{-1} v^* = L^{-1} v^* \text{ weakly in } V \text{ as } \delta \to 0.$$

The equivalence of these two definitions of G-convergence was proved in [45].

The abstract G-limit (G-convergence) approach briefly described above can be applied to both linear and nonlinear problems. Mention densely related Γ-convergence method [57, 100].

In the terms of physics, Definition 1 implies the convergence of the energy of the original composite body to the energy of the homogenized body when dimension of inhomogeneities δ becomes small (that is formalized in the form $\delta \to 0$). Definition 2 implies the convergence of the solution (electric potential, temperature, etc., see Table 1) corresponding to the original composite body to the solution corresponding to the homogenized body when $\delta \to 0$. These convergences take place for arbitrary "mass source" v^* in the composite. Note that Definition 2 says nothing about strong convergence of derivatives of the solution of the original problem.

Homogenization procedure for nonlinear composite materials of periodic structure

We consider the following problem: find $T^\delta(\mathbf{x})$ from the solution of the minimization problem,

$$J_\delta(T) + \langle f, T\rangle \to \min,\ T(\mathbf{x}) \in H^1_0(Q), \tag{1.29}$$

where

$$J_\delta(T) = \int_Q G^\delta\left(\mathbf{x}, \nabla T(\mathbf{x})\right) d\mathbf{x}.$$

A homogeneous body is described by the minimization problem: find $T^{(0)}(\mathbf{x})$ from the solution of the minimization problem,

$$J(T) + \langle f, T\rangle \to \min,\ T(\mathbf{x}) \in H^1_0(Q), \tag{1.30}$$

where

$$J(T) = \int_Q \widehat{G}\left(\mathbf{x}, \nabla T(\mathbf{x})\right) d\mathbf{x}.$$

The corresponding boundary-value problems are obtained by computation of Gâteaux derivatives (or computation of variations) [113] of the functionals in the left-hand sides of (1.29) and (1.30) and have the form

$$\frac{\partial}{\partial x_i}\left(q_i^\delta\left(\mathbf{x}, \frac{\partial T^\delta}{\partial x_1}(\mathbf{x}), ..., \frac{\partial T^\delta}{\partial x_n}(\mathbf{x})\right)\right) = f(\mathbf{x}) \text{ in } Q,\ T^\delta(\mathbf{x}) = 0 \text{ on } \partial Q,$$

and

$$\frac{\partial}{\partial x_i}\left(q_i^0\left(\mathbf{x}, \frac{\partial T^{(0)}}{\partial x_1}(\mathbf{x}), ..., \frac{\partial T^{(0)}}{\partial x_n}(\mathbf{x})\right)\right) = f(\mathbf{x}) \text{ in } Q,\ T^{(0)}(\mathbf{x}) = 0 \text{ on } \partial Q,$$

where fluxes

$$q_i^\delta(\mathbf{x}, \mathbf{z}) = \frac{\partial G^\delta}{\partial z_i}(\mathbf{x}, \mathbf{z}),$$

$$q_i^0(\mathbf{x}, \mathbf{z}) = \frac{\partial \widehat{G}}{\partial z_i}(\mathbf{x}, \mathbf{z}).$$

Here $\mathbf{z}$ corresponds to driving force ∇T.

The nonlinear homogenization problem is most investigated in the periodic case when the function $G^\delta(\mathbf{x}, \mathbf{z})$ is periodic in spatial variable $\mathbf{x}$ and has the form

$$G^\delta(\mathbf{x}, \mathbf{z}) = G^{(0)}(\mathbf{x}/\delta, \mathbf{z}),$$

$G^{(0)}(\mathbf{y}, \mathbf{z})$ is periodic in $\mathbf{y}$ with periodicity cell Y. The sufficient conditions for nonlinear homogenization can be found in [222, 223]. We note that the theory of nonlinear homogenization is not as detailed as the linear homogenization theory (it is naturally, because the nonlinear problems usually are more difficult than linear ones). An exception is the homogenization theory for nonlinear ordinary differential equations, where a relatively complete homogenization theory was developed (in material science, this case corresponds to laminated materials as shown in [159], see also Appendix B).

For composite of periodic structure the function $G(\mathbf{z})$ is determined as follows (we assume that all functions under consideration exist, for details see, e.g., [99, 157]):

$$\widehat{G}(\mathbf{z}) = \min_{N \in V_Y} \int_Y G^{(0)}\left(\mathbf{y}, \nabla(N(\mathbf{y}) + \mathbf{z}\mathbf{y})\right) d\mathbf{y}, \tag{1.31}$$

where

$$V_Y = \{f(\mathbf{y}) \in H^1(Y) : f(\mathbf{y}) \text{ is periodic in } \mathbf{y} \text{ with periodicity cell } Y\}.$$

In the case under consideration the function $G(\mathbf{z})$ depends on the variable $\mathbf{z}$ only.

The problem (1.31) (under some additional conditions, see, e.g., [100, 222, 223]) is equivalent to a periodic problem for nonlinear partial differential equation corresponding to the functional in the right-hand part of (1.31).

Formula (1.31) expresses the energy form corresponding to the homogenized body through the local energy of the original body. In many cases other relationships of the homogenized and local characteristics can be useful. Mention the relationship of the homogenized and local fluxes, which has the form

$$\int_{\Gamma} \mathbf{q}^{\delta}(\mathbf{y})\mathbf{n}d\mathbf{y} \to \int_{\Gamma} \mathbf{q}^{0}(\mathbf{y})\mathbf{n}d\mathbf{y} \text{ as } \delta \to 0, \tag{1.32}$$

for any side Γ of the periodicity cell Y ($\mathbf{n}$ means the normal vector to Γ). Formula (1.32) implies that the fluxes through the volume Y in the original inhomogeneous body and the homogenized body are equal to one another.

The above mentioned relationships between the homogenized and local characteristics are in agreement and predict the same characteristics of the homogenized structure.

1.2. Transport properties of periodic arrays of densely packed bodies

For a history of the problem of mathematical analysis of transport properties of periodic arrays of bodies, turn to Maxwell's book [227], where electric fields in a periodic array of bodies is considered. Maxwell's analysis opens a stage in analysis of the problem, which can be characterized as analysis of transport properties of periodic array of spheres, cylinders and disks. An outstanding contribution in the mathematical analysis of the problem was made by Rayleigh [348] who first studied the problem in mathematically rigorous way, as well as Mossotti [261], Clausius [92], Garnett [128], and Lorenz [215].

Now, one can distinguish two basic methods used to analyze the problem discussed. One is the homogenization method described in Section 1.1.1. We note that the original version of the homogenization method does not assume high contrast of components of composite (nontrivial modifications are required to adopt the homogenization method for contrast composites, see, e.g., [279, 281]) and (it is the main restriction) the homogenization method assumes proportional scaling of all components of composite (see formula (1.12) and Fig. 1.1 illustrating the scaling in the homogenization method). When the inclusions (bodies, particles, etc.) are almost touching, standard homogenization procedures lose the convergence property. It is why other asymptotic methods were developed for analysis of systems of almost touching bodies and similar structures. We present the main ideas, which lay the foundation of the methods developed for analysis of periodic arrays of almost touching bodies and highly filled contrast composite materials.

1.2.1. Periodic media with piecewise characteristics and periodic arrays of bodies

The inhomogeneous media can demonstrate different overall properties in dependence on the local geometry and topology of components of the media. The most well-known example of dependence of the overall properties on the geometry and topology of components of composite material is the percolation phenomenon [138, 170, 293]. Another example is the topology design theory [29].

Periodic and disordered structures are the main types of structures we deal with in practice. A periodic structure is a deterministic structure, which can be obtained by periodic repetition of a typical element (called periodicity or basic cell [30, 159]). The disordered structures (also called topologically disordered structures [143]) are not deterministic. The level of disorder varies in the range from small disorder (small random perturbation of a deterministic, for example periodic, structure) to complete disorder (random structures) [31, 46, 89, 194, 275, 359]. Most of materials, both natural and artificially produced, are partially or completely disordered. Limited number of solid composite materials (we call solid the essentially three-dimensional bodies) of periodic structures are produced using high technologies, see, for example, [88, 230]. At the same time many artificial structures (frames, structural elements of airplanes, ships, etc.) have deterministic structures, usually periodic or quasiperiodic [61, 89, 159, 182, 217, 265, 272, 273, 303].

In periodic systems we can naturally select a typical element — periodicity cell, see Fig. 1.1, which determines the property of the system in whole. It means that the local properties of the periodic system or periodic material (solution of the problem for typical element, usually called local problem) completely determine the overall properties of the periodic system in whole.

In this book, we consider media with piecewise constant characteristics, which correspond to systems of bodies and particle filled composite materials. Material properties of such composites are described by discontinuous (thus, non-differentiable) functions. A typical form of a function describing material properties of particle filled composite material is

$$a(\mathbf{x}) \;=\; \begin{cases} a_I \text{ in inclusions,} \\ a_m \text{ in matrix.} \end{cases} \tag{1.33}$$

The typical graph of the function (1.33) for one inclusion is presented in Fig. 1.3.

Function of the form (1.33) also describes the body. If we consider electrostatic problem for a system of bodies, the distribution of the dielectric characteristics is described just by the function (1.33), where a_I means the dielectric constant of the material of bodies / particles and a_m means the dielectric constant of a substance surrounding the bodies.

In the last decades so called graded composite materials were reported (see, e.g., [139, 155, 255, 333]). In graded composites, there exists a relatively thick intercomponent layer between the basic components of composite. The function $a(\mathbf{x})$, which describes the local material properties of graded composite, changes continuously and has the form

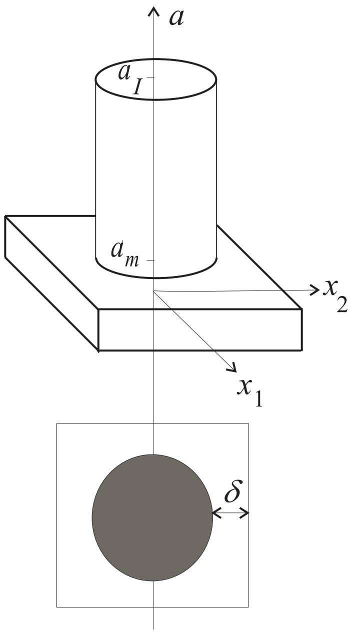

Figure 1.3. *The typical graphs and distribution over the periodicity cell of the functions* $a(\mathbf{x})$ *for two-dimensional composite of periodic structure described by the function (1.33).*

$$a(\mathbf{x}) = \begin{cases} a_I \text{ in inclusions,} \\ a_0(\mathbf{x}) \text{ in the interface ``particle–matrix'' region,} \\ a_m \text{ in matrix.} \end{cases} \tag{1.34}$$

The typical graph of the functions (1.34) for one inclusion is presented in Fig. 1.4.

If $a_I \ll a_m$ (the properties of the particles are vastly different from the properties of the matrix) then the composite is called a high-contrast composite. In many cases, the highly conducting bodies / particles are replaced by perfectly conducting bodies / particles (with infinite conductivity or, equivalently, zero resistivity). In this case, there remains one parameter, which describes the microstructure of composite. This is called the interparticle distance parameter δ, see Figs 1.3 and 1.4.

1.2.2. Problem of computation of effective properties of a periodic system of bodies

After Maxwell and Rayleigh, the problem of transport properties of periodic arrays of bodies attracted attention of many investigators (it is impossible to present a complete list of references here because it would be very long, some references can be found in [315]). The problem of transport properties of arrays of closely placed

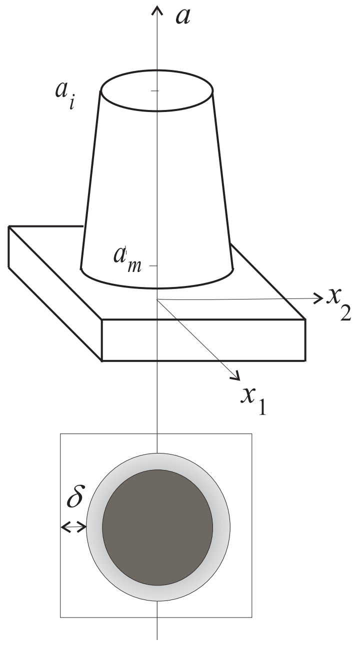

Figure 1.4. *The graphs and distribution over the periodicity cell of the functions* $a(\mathbf{x})$ *for two-dimensional "graded" composite.*

bodies was analyzed in [36, 114, 166, 231, 232, 235, 236, 253, 296, 349, 399] (the list is not complete). A major contribution into the problem was made by McPhedran and his co-authors, see, e.g., [233, 234, 238, 269, 291, 389]. We emphasize two specific features of problems considered in the above-mentioned papers:

(i) the bodies form a periodic array, which is obtained by periodic translation of one body;

(ii) the bodies have simple geometry (spheres, cylinders, etc).

In the frameworks of directions, many works were devoted to transport properties of array of spheres [10, 106, 166, 232, 236, 319, 320] and circular cylinders (disks) [80, 81, 166, 233, 234, 235]. Transport properties of array of elliptic cylinders [237, 269, 389] and cylinders having square cross-section [11] were considered, see also [383, 384, 398]. Comparison of transport properties of array of periodic spheres and an array of periodic cubes was presented in [8]. In [270] transport properties of square array of coated cylinders were analyzed. Recently arrays of rhombic fibers [9] were considered. In [234] a problem for closely placed, highly conducting cylinders was considered using a technique of complex analysis. Later the methods of complex analysis were effectively applied to analysis of systems of closely placed disks by Mityushev and co-authors [220, 248, 249, 250, 251, 253, 254], see also [292, 311, 312]. In [23] conduction through a granular material was investigated using ensemble averaging and approximate solutions for closely packed spheres.

Most problems were analyzed under the condition of perfect contact between matrix and inclusions, which assumes no jump conditions with respect to potential and normal flux on the surface of conjugation of the matrix and inclusion. The arrays of bodies with not perfect contact were considered in [26, 147, 163, 206, 226, 369].

Method of power series as applied to the computation of transport properties of periodic arrays of bodies

We present the basic ideas of the method of power series widely used for computation of overall properties of periodic structures. We consider a simple-cubic array of identical spheres of radii R embedded into a homogeneous matrix. Denote the periodicity cell of the array by Y. Following to [315], we associate the problem considered with the problem of electrostatic of composite material. An overall external electric field $\mathbf{E}$ is assumed to be applied parallel to Ox_1-axis:

$$\mathbf{E} = (E_0, 0, 0).$$

The potential distribution in the composite satisfies Laplace equation

$$\Delta\varphi = 0 \tag{1.35}$$

both inside and outside the spheres.

Consider a sphere with the center at the point 0. The general power series for the potential in spherical coordinates (r, θ, φ) is the following (see, e.g., [315]): inside the sphere

$$\varphi_I(r, \theta, \varphi) = A_0 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (A_{lm} r^l + A_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi), \tag{1.36}$$

outside the sphere

$$\varphi_m(r, \theta, \varphi) = C_0 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} C_{lm} r^l Y_{lm}(\theta, \varphi). \tag{1.37}$$

In (1.36) and (1.37), $Y_{lm}(\theta, \varphi)$ is the spherical harmonics of order (l, m), i.e., solution of Laplace equation in spherical coordinates, which can be represented in the terms of Legendre functions $P_l^m(\cos\theta)$, see [315]

$$Y_{lm}(\theta, \varphi) = \frac{(l - |m|)!}{(l + |m|)!} P_l^{|m|}(\cos\theta) e^{im\varphi}.$$

The conjugation conditions at the interface surface $|\mathbf{x}| = R$ between the sphere and matrix are

$$\varphi_I(\mathbf{x}) = \varphi_m(\mathbf{x}), \tag{1.38}$$

$$a_I \frac{\partial \varphi_I}{\partial \mathbf{n}}(\mathbf{x}) = a_m \frac{\partial \varphi_m}{\partial \mathbf{n}}(\mathbf{x}),$$

where a_I and a_m mean the permittivity of the materials of sphere and matrix, correspondingly. In addition, the function $\varphi_m(\mathbf{x})$ is periodic with periodicity cell Y.

From these conditions (the conjugation conditions and the periodicity condition, as well as the symmetry of solution and the absence of singularities in solution), one can determine the coefficients C_{lm}, A_{lm} and B_{lm}. We present the scheme of solution of the problem following to [315].

From the second condition in (1.38), it follows that

$$A_{lm} = \frac{B_{lm}\left[\dfrac{a_s}{a_m} + \dfrac{l+1}{l}\right]}{R^{2l+1}\left(1 - \dfrac{a_s}{a_m}\right)}. \tag{1.39}$$

Due to symmetry of the array only odd values l in (1.37) and (1.36) and only values of m that are multiples of 4 must be allowed.

The condition of the absence of singularity leads to the equality

$$A_0 + \sum_{l=1}^{\infty}\sum_{m=0}^{2l-1} A_{2l-1,m} r^{2l-1} P_{2l-1}^{m}(\cos\theta)\cos(m\varphi) - E_0 x_1 =$$

$$= \sum_{l=1}^{\infty}\sum_{m=0}^{2l-1}\sum_{i=0}^{\infty} B_{2l-1,m}\rho_i^{-2l} P_{2l-1}^{m}(\cos\theta_i)\cos(m\varphi_i). \tag{1.40}$$

In (1.40) the coordinates $(\rho_i, \theta_i, \varphi_i)$ are measured to the center of the i-th sphere. In his treatment of this problem, Rayleigh truncated terms higher than those involving r^3 from the Legendre polynomials, see [315].

The equations with respect to the unknown coefficients $B_{2l-1,m}$ are obtained by equating the partial derivatives of all orders with respect to variable x_1 in (1.40) (see for details [315]). They have the following form

$$\sum_{l=n+1}^{\infty}\sum_{m=0}^{2l-2n-2}\begin{pmatrix} m+2l-1 \\ 2n+1 \end{pmatrix} P_{2l-2n-1}^{m}(\cos\theta_0)\cos(m\varphi_0)A_{2l-1,m} +$$

$$+\sum_{l=1}^{\infty}\sum_{m=0}^{2l-1}\sum_{i=1}^{\infty}\begin{pmatrix} 2l+2n-m \\ 2n+1 \end{pmatrix}\rho_i^{-2l-2n-1} P_{2l+2n}^{m}(\cos\theta_i)\cos(m\varphi_i)B_{2l-1,m} = E_0\delta_{n0},$$

where

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!},$$

$(\rho_0, \theta_0, \varphi_0)$ corresponds to a point at the boundary of the periodicity cell Y, and $\delta_{00} = 1$, $\delta_{0n} = 0$ $(n \geq 1)$.

Detailed analysis of the problem can be found in [236]. Analysis of similar problems can be found in [23, 232, 233, 239, 291, 399]. We do not discuss the details of the solution and only note that it is evident that the method of power series works well for small concentrations of spherical or circular inclusions. In this case distribution of potential is similar to distribution of potential in the problem about unique inclusion. Solution for unique inclusion with corrector containing few additional harmonics provides us with accurate solution. If diameter $2R$ of the sphere is close to the size of the periodicity cell Y, the solution of the problem discussed is strongly different from solution for unique inclusion, it is necessary to account for all terms in series and solve the corresponding problems. This fact can be explained easily using some results, which will be discussed in detail below.

1.2.3. Keller analysis of conductivity of medium containing a periodic dense array of perfectly conducting spheres or cylinders

As was noted above, when the bodies are placed relatively far from one another, the approach based on the harmonic series works well and it is necessary to save only a few harmonics in the series to obtain accurate formulas. When the bodies are placed densely, it is necessary to save a large number of harmonics in the series. The results of the paper [174] (see also Chapter 2) explain this fact. We consider a periodic system of disks as an example. When the disks are placed relatively far apart, the distribution of energy around each disk is a smooth function of polar angle close to distribution of energy around a single disk, see Fig. 1.5 (left).

When the disks are placed densely, the energy as a function of polar angle looks like a singular function, see Fig. 1.5 (right). It is known that a function like that shown in Fig. 1.5 (left) usually can be approximated well with a small number of harmonics and it is necessary to save a large number of harmonics in series to approximate a function like shown in Fig. 1.5 (right).

It would be natural to manipulate with functions like those shown in Fig. 1.5 (right) without using technique of power series. This was done in [166]. In the previously mentioned paper Keller reported that *"The previous results of Maxwell, of Rayleigh and Meredith and Tobias are not valid near the singularity"* (this is for almost touching bodies) and derived formulas for transport properties of periodic arrays of circular disks and spheres different from the Maxwell's formula. Also it was reported that new asymptotic formulas derived in [166] agree well with the numerical results [165].

Keller's analysis was based on a hypothesis (formally incorrect, see below, nevertheless very fruitful) about the form of flux between two closely placed disks (spheres). We employ Keller method to derive an approximate formula for the flux between two disks (the i-th and the j-th) of radii R placed at the distance δ_{ij}, see Fig. 1.6. Although the original Keller analysis was given for two spheres, we present corresponding computations for two disks. We do it in order to demonstrate the dimensional sensitivity of the problem (i.e., existence of some differences in properties

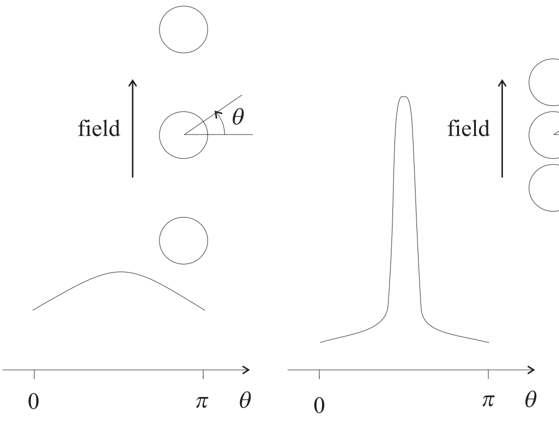

Figure 1.5. *Typical distribution of density of the local energy around disk as a function of polar angle θ: left – dilute composite, right – composite with densely packed disks.*

of solutions of two- and three-dimensional problems).

We approximate the disks by the tangential parabolas

$$y = \frac{\delta_{ij}}{2} + \rho\frac{x^2}{2},$$

and

$$y = -\left(\frac{\delta_{ij}}{2} + \rho\frac{x^2}{2}\right),$$

where

$$\rho = \frac{1}{R} \tag{1.41}$$

is curvature of the disks.

The distance $H(x)$ between the parabolas is

$$H(x) = \delta_{ij} + \rho x^2. \tag{1.42}$$

We assume that the matrix is uniform and dielectric constant of the material of the matrix is equal to ε. Following [166], we assume that the potential in the region between the disks has the form

$$\varphi(\mathbf{x}) = \frac{(t_i - t_j)y}{H(x)}.$$

Then the local flux in the region between the disks has the form (here $\mathbf{x} = (x, y)$)

$$\mathbf{v}(\mathbf{x}) = \varepsilon\nabla\varphi(\mathbf{x}) = \varepsilon\left(0, \frac{t_i - t_j}{H(x)}\right). \tag{1.43}$$