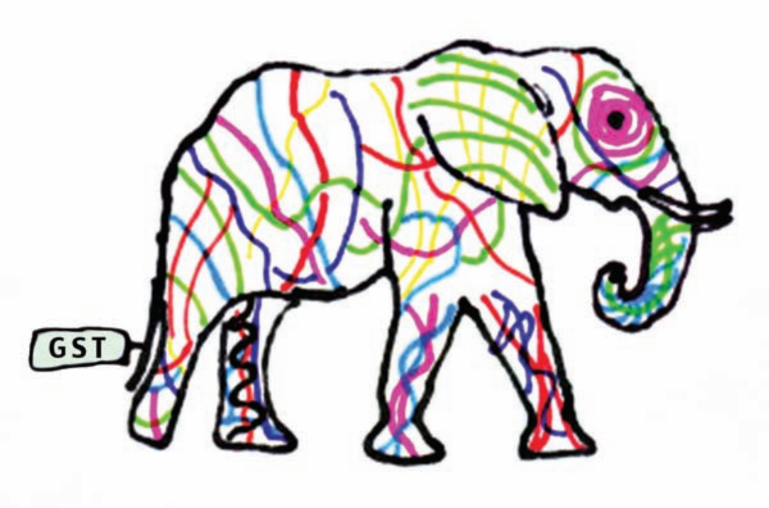
Peter Buchen

An Introduction to Exotic Option Pricing



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An Introduction to Exotic Option Pricing

Peter Buchen

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Dedication

For my students:

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Symbols and Abbreviations

aBm	Arithmetic Brownian motion
ADS	Arrow–Debreu security
AFM	Arbitrage Free Measure
a.s.	Almost surely
atm	At-the-money
BS	Black–Scholes
ccf	Correlation coefficient
cdf	Cumulative density function
EMM	Equivalent martingale measure
$\stackrel{d}{=}$	Equal in distribution
ESO	Executive stock option
FBV	Free boundary value (problem)
FK	Feynman–Kac
FTAP	Fundamental Theorem of Asset Pricing
FX	Foreign exchange
gBm	Geometrical Brownian motion
GM	Geometric mean
GFC	Global Financial Crisis
GST	Gaussian Shift Theorem
hot	Higher order terms
iid	Independent and identically distributed
iff	If and only if
IVP	Initial Value Problem
itm	In-the-money
ode	Ordinary differential eqn.
lhs	Left-hand side
OTC	Over-The-Counter
otm	Out-of-the-money
mgf	Moment-generating function
MoI	Method of Images
MR	Martingale Restriction
pax	Pay-at-expiry
pde	Partial differential equation
pdf	Probability density function
PSR	Principle of Static Replication
pv	Present value
rhs	Right-hand side
rms	Root mean square
RNM	Risk Neutral Measure
rv	Random variable
sde	Stochastic differential eqn.
TBV	Terminal Boundary Value (problem)
TV	Terminal Value (problem)
wlog	Without loss of generality
wrt	With respect to
ZCB	Zero Coupon Bond

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Preface

This book is a collection of a large amount of material developed from my teaching, research, and supervision of student projects and PhD theses. It also contains a significant quantity of original unpublished work.

One of my main interests in financial mathematics was to seek elegant methods for pricing derivative securities. Although the literature on derivatives is vast, virtually none outside the academic journals concentrates solely on pricing methods. Where it is considered, details are often glossed over, with comments like: "$\cdots$ *and after a lengthy integration, we arrive at the result*," or "$\cdots$ *this partial differential equation can be solved to yield the answer.*" In my experience, many students, even the mathematically gifted ones, found the subject of pricing anything but the simplest derivatives somewhat unsatisfactory and often quite daunting. One aim of this book is to correct the impression that exotic option pricing is a subject only for technophiles. My plan is to present it in a mathematically elegant and easily understood fashion. To this end:

> *I show in this book how to price, in a Black–Scholes economy, the standard exotic options, and a host of non-standard ones as well, without generally performing a single integration, or formally solving a partial differential equation.*

How is this to be achieved? In a nutshell, the book devotes a lot of space to developing specialized methods based on no-arbitrage concepts, the Black–Scholes model and the Fundamental Theorem of Asset Pricing. These include the Principal of Static Replication, the Gaussian Shift Theorem and the Method of Images. The last of these, which has been borrowed from theoretical physics, is ideally suited to pricing barrier and lookback options. But don't let this technical stuff deter you from reading further!

While the book is certainly targeted to the mathematically capable reader, it is written in non-technical language, in which theorems and proofs are delivered in heuristic rather than formal mathematical terms. This is the *applied*, as opposed to the *pure* mathematical approach. That is not to say that formal methods are not important — they are. But the technical approach is not the focus of this book. Anyone competent in junior year university calculus should be able to understand this book without difficulty.

The following quote from Freedman [23] recently caught my eye: "*The market crash of 2008 that plunged the world into economic recession from which it is still reeling had many causes. One of them was mathematics.*" Statements such as these should serve to remind the reader that this introductory text is not about risk management, but pricing. The GFC should be fair warning to everyone that no mathematical model yet captures the complexities of real markets.

How is this book different from other books written on exotic options? Most other books concentrate on listing or quoting formulae, computing such formulae, applying them to hedging and risk management or combinations of these. As mentioned above, few devote much space to the actual derivation of prices, and those that do generally follow standard practice by employing integration (or partial differential equation) methods. Given the diversity of exotic options, these integrations can be quite cumbersome and are often very complicated.

This book, by contrast, focuses entirely on pricing exotic options. With very few exceptions, no prices within this book are quoted without derivation. Generally, full details of the calculations are provided. The book contains many ideas and techniques which are perhaps new to the general quantitative finance community.

The book is divided into two parts. The first three chapters contain the necessary financial, mathematical and statistical background for the rest of the book. While much of the topics covered is standard, there is also a significant component of specialized material which might be unfamiliar to the reader.

The ensuing chapters contain the applications to exotic option pricing. They include dual-expiry options such as compound and chooser options, multi-asset rainbow options, barrier and lookback options, Asian options, and much more. Chapter 10 of the book introduces and derives a very powerful formula for pricing a class of multi-asset, multi-period binary options called $\mathbb{M}$-binaries. These include all the standard binary (digital) options which are the basic building blocks for complex exotic options. In a very real sense, these $\mathbb{M}$-binaries represent the limit to which Black–Scholes technology can be pushed.

Some might criticize this book on the grounds that its focus on Black–Scholes pricing is too narrow, or even not relevant in today's rumble-tumble financial world. True, the markets are more complex than can be modeled by a Black–Scholes view of the world. Volatilities are not constant, but highly variable and most likely stochastic. But although some stochastic volatility models have been put forward to explain market behavior, and even price exotic options, none has yet to attain any widespread acceptance. Certainly not to the same extent that the Black–Scholes framework has penetrated the consciousness of practitioners and academics. In any case, stochastic volatility models rarely have closed-form solutions, so option pricing largely becomes an exercise in Monte Carlo simulation. It could be argued that the analytic solutions obtained in the Black–Scholes model, which are so readily produced in this book, can be used as control variates, to reduce the errors of such Monte Carlo simulations.

The book is basically an applied mathematics one, with exotic option pricing as its application area. As such, it is foremost a textbook for advanced university students, typically at the honors and postgraduate levels. Each chapter contains a set of exercise problems to assist in the understanding of the techniques introduced. However, the book should be useful to researchers and practitioners as well, and hopefully not only an addition to their library shelves. The book is not intended to be a complete historical account of exotic option pricing. Some relevant references may unintentionally have been omitted. I take full responsibility for such omissions and for the inevitable typos that always seem to escape detection.

Acknowledgements
My interest and fascination with financial mathematics was sparked by my friend and one-time futures trader Bobby Richman in 1985.

This book is dedicated to my many past students, some of whose work has contributed significantly to its content. I take the opportunity here to acknowledge my colleagues who have provided inspiration and collegiality over the years. These include Alan Brace, Carl Chiarella, David Colewell, Jeff Dewynne, David Edelman, Robert Elliott, Volf Frishling, Ben Goldys, Andrew Grant, Michael Kelly, Hugh Luckock, Marek Musiela, Alex Novikov, Eckhard Platen, Marek Rutkowski, Erik Schloegl, Pavel Shevchenko, David Stump, John van der Hoek, and Song-Ping Zhu. Of course, many of my past students have since become my colleagues and personal friends.

Particular thanks also go to David Johnstone and Michael McKenzie who kindly provided the opportunity and resources by which this book was written within the finance discipline of the School of Business, University of Sydney.

Sophie Ashton, Matthew Hardman, Kevin Ho-Shon, Otto Konstandatos, Tim Kyng, Charles Magri, Hamish Malloch, Max Skipper and Tony Vassallo ably assisted with the proofreading. Their generosity of time spent on the task and valuable suggestions are acknowledged with gratitude.

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Sydney, Australia

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Part I

Technical Background

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Chapter 1

Financial Preliminaries

The first three chapters constitute Part I of this book and provide the necessary financial, mathematical and statistical background upon which later chapters on pricing exotic options depend. As an introductory text, the book aims to present a comprehensive treatment of exotic option pricing in the Black–Scholes (BS) framework. Readers who are familiar with this background may skip directly to the applications in Part II and refer back to relevant sections of Part I when necessary.

Part I is divided into three sections: the Financial Preliminaries, the Mathematical Preliminaries and Gaussian Random Variables. The Financial Preliminaries include important concepts such as the no-arbitrage principle, static replication as a pricing tool, the derivation of the Black–Scholes approach to option pricing and a non-technical presentation on the Fundamental Theorem of Asset Pricing.

The chapter on mathematical preliminaries presents many of the technical details, mostly without proof, that underpin modern quantitative finance. These include elements of the stochastic calculus, arithmetic and geometric Brownian motion, martingales and techniques for solving the BS partial differential equation (pde).

The chapter on Gaussian random variables is included because from a statistical viewpoint, the BS model can be expressed almost exclusively in terms of these fundamental quantities. A good working knowledge of Gaussian statistics permits many short-cuts to price evaluation and hence readers are strongly advised to be familiar with the contents of this chapter. In particular, as later chapters will affirm, many exotics can be priced using the Gaussian Shift Theorem, without recourse to any formal integration or pde solving. No attempt is made to present a complete and coherent account of all these topics. Such a task would be just too daunting and other books are available to fill the gaps. Hence it is best to regard these opening chapters as a collection of useful and interesting results which impact on exotic option pricing.

Let us also be clear what this book is not about: it is not concerned with legal, institutional or trading aspects of derivatives, nor is it concerned with the design and risk management of structured products. The book is almost exclusively concerned with the arbitrage free pricing of derivative securities within a standard Black–Scholes framework and its extensions. These derivatives can be highly non-standard and come under the umbrella term: *exotic options*.

This opening chapter on Financial Preliminaries is essentially a review of derivative security generalities, such as no-arbitrage conditions and the two main techniques for pricing derivative securities. These are the Black–Scholes partial differential equation (BS-pde) method and the Equivalent Martingale Measure (EMM) method entailed by the Fundamental Theorem of Asset Pricing (FTAP). Many of the mathematical details that underlie these concepts will be found in the following chapter on Mathematical Preliminaries.

Readers of this book are assumed however, to be familiar with derivative security basics and have a working knowledge of University level calculus and mathematical statistics. Our approach, although mathematical, is nevertheless heuristic rather than technical. The book is an introductory treatise on the subject of exotic option pricing and is targeted to senior undergraduate, post-graduate finance and mathematics students, practitioners and researchers in the field of Quantitative Finance.

1.1 European Derivative Securities

A derivative security is any contract whose future payoff depends in some way on the price evolution of one or more underlying assets.

Simple European derivatives have a single future payoff date T, called the expiration date (expiry date for short) and the payoff depends on the price $X = X(T)$ of a single underlying asset on that date. The term expiry date is treated in this book as synonymous with maturity date, settlement date, and payoff date, though in some contexts these may have subtle differences. For European options, the payoff is a function of X only and we shall write $V(X,T) = F(X)$, where $F(X)$ is called the *payoff function* of the derivative and the function V stands for the value of the derivative.

Generally the underlying asset price evolves according to some stochastic process, which makes X a random variable (rv), and $F(X)$ a function of that random variable. We shall take $F(X)$ to be a measurable function of X, which means in practice that given X, $F(X)$ is uniquely determined. That is, the payoff depends only on X and not on any other extraneous random variables. Pricing the derivative entails finding the present value (pv), $V(x,t)$

of the derivative before expiry, where $x = X(t)$ denotes the underlying asset price at any time $t < T$. That the price of the derivative $V(x,t)$ depends only on the current asset price x, is a consequence the Markov property discussed in the next chapter.

Vanilla Option Payoffs		
	Contract	*Payoff Function at T*
1.	Long Forward	$(X-k)$
2.	Short Forward	$(k-X)$
3.	Call Option	$(X-k)^+$
4.	Put Option	$(k-X)^+$

Four well-known examples of European options, together with their payoff functions are listed in the above table. The long/short forward contract is the obligation to buy/sell the underlying asset for k dollars (the settlement price) at date T. The call/put option is the right (but not the obligation) to buy/sell the underlying asset for k dollars (the strike price) at date T.

The payoffs for the call and put option are written in terms of the plus function, defined by:

$$(x-k)^+ = (x-k)\mathbb{I}(x>k); \qquad (k-x)^+ = (k-x)\mathbb{I}(x<k) \tag{1.1}$$

and $\mathbb{I}(x>k) = (1 \text{ if } x > k;\ 0 \text{ otherwise})$, denotes the usual indicator or step function. The plus function captures the optionality in the call and put contracts. The call is exercised at T only if $x > k$ at expiry, while the put option is exercised at T only if $x < k$.

American options are similar to European options with the additional feature that they may be exercised at any time prior to expiry, as well as on their expiry date T. American options are therefore *path dependent* options, while European options are strictly *path independent*. It is well known that pricing American options is a much harder task than pricing their European counterparts. We shall mainly be concerned in this book with pricing European options.

1.2 Exotic Options

European and American calls and puts are often called *plain vanilla* options and are traded both in organized exchanges and in over-the-counter (OTC) markets.

Exotic options are any derivative securities which are not plain vanilla. Exotics can be classified in several different ways. As for European and American options, they can be either path-dependent, or path-independent. Examples of path-dependent exotic options include barrier lookback and Asian options, all of which are analyzed in detail in later chapters of this book.

Some exotic options are simple portfolios of vanilla options. These are usually referred to as *packages*. An example is a *range-forward* derivative consisting of long position in a European call option and a short position in a European put option. These are the easiest to price, as we shall demonstrate later in this chapter, using the Principle of Static Replication.

There is also a family of exotic options on a single underlying asset, whose payoff depends on the asset price at two future dates T_1 and T_2 say, with $T_1 < T_2$. We refer to such options as *dual-expiry* options. Examples include compound options such as a call-on-call option, and chooser options where at time T_1 the holder chooses either a call option or a put option, both of which expire at time T_2. Such dual-expiry options are considered in detail in Chapter 5 of this book.

Other exotic options that have been traded and also appear in the academic literature include derivatives whose payoffs at the single future expiry date T depend on the prices of two distinct, but perhaps correlated, underlying assets. Such derivatives are often referred to as *two-asset rainbow* options or *correlation options*. The best known example is the *exchange option* which gives its holder the right to exchange one asset for another asset at time T. Chapter 6 considers several examples of these exotic options.

We shall also study in this book extensions of dual-expiry and two-asset rainbow options, including, in Chapter 10, general multi-period and multi-asset exotics.

1.3 Binary Options

A particularly important class of exotic options is the family of *binary options*. These options pay out at one or more future dates if and only if some exercise condition is met. If not met, they expire worthless. Binary options are also called *digital options* in the derivatives industry. The best known examples are the cash-or-nothing digital, which we shall refer to as a *bond binary*, and the asset-or-nothing digital, which we shall refer to simply as an *asset binary*. The bond binary pays one dollar or nothing according to whether the exercise condition is met or not. The asset binary pays one unit of the underlying asset if the exercise condition is met, and nothing otherwise.

Binary options are particulary important, as they are often the basic building blocks of more complex exotic options. So if we can price the binary options, we can then price the exotic option on which they are constructed.

In this context, a standard European call option is actually a binary option on a long forward contract. From the previous table of vanilla option payoffs, it is clear that the payoff of the call option is $(X - k)$, that of a long forward, but only if the exercise condition $X > k$ is met. In similar fashion, a standard European put option is seen to be a binary option on a short forward contract with exercise condition $X < k$.

We shall meet many different types of binary options in this book, including dual-expiry binaries and two-asset rainbow binaries. The final chapter is concerned with pricing a general multi-period, multi-asset binary which includes all other binaries studied in this book as special cases.

1.4 No-Arbitrage

When pricing derivative securities, whether plain vanilla or exotic, we seek the fair price of the derivative at equilibrium. By fair price we mean the *arbitrage free* price. Indeed, the concept of no-arbitrage in derivative pricing is no less important than Newton's laws of motion are for particle dynamics.

In the Black–Scholes framework, the no-arbitrage assumption leads to unique prices for both the underlying asset and derivatives written on this asset. This uniqueness derives from the fact that a Black–Scholes market is complete. In such a market, there exists a set of assets which span the risks in that market. This means the cash flows of any contract over any market asset or group of assets can be replicated by a (usually dynamic) portfolio of the spanning set.

It is generally accepted that real financial markets are incomplete, which has the consequence that derivative prices, though possibly arbitrage free, are not unique. There may exist a continuum of prices, all of which are arbitrage free. Recent advances in Behavioral Finance attempt to price financial assets in an economy which is neither complete, nor arbitrage free. A recent survey of such models can be found in Shefrin [68]. Thus, relative to these wider contexts, the Black–Scholes world is very much an idealized one. Yet, despite the considerable efforts of both academics and practitioners, the Black–Scholes framework continues to be a benchmark against which other pricing models are currently measured.

The assumption of no-arbitrage has a number of important consequences for derivative pricing, three of which we discuss immediately below.

1.4.1 The Law of One Price

Perhaps the simplest and best known outcome of the no-arbitrage principle is the *Law of One Price*, which can loosely be stated as follows.

> *If two securities have exactly the same pattern of future cash flows, then the securities must have the same price today.*

The proof of this statement is straightforward. Suppose the two securities are labeled A and B and that their current prices satisfy $V_A > V_B$. Then we could short sell security A and go long in (i.e., buy) security B. This strategy gives a positive cash flow now of amount $(V_A - V_B)$ and all future cash flows generated by the two securities exactly cancel. Hence we receive a positive amount now and have zero cash flow for all future states of the world. This is an arbitrage. Similarly, if $V_A < V_B$, we could obtain another arbitrage by reversing the above strategy. That is, we now buy security A and short sell security B. In order to avoid either of these arbitrages, there is only one conclusion, namely we must have $V_A = V_B$. □

The simple two security situation above illustrates other important features of general pricing models.

1. It is assumed there are no impediments in the market to short selling the security. Of course in real markets there are often significant restrictions to short selling. In such cases, theoretical arbitrages may develop and persist because there is no practical way to take advantage of them. For example, if there are no short sales allowed, the theoretical price equilibrium may lie outside the set of real market positions.

2. In setting up the arbitrage strategy, there is an immutable rule:

   > *In every arbitrage strategy, buy the relatively under-priced security and short sell the relatively over-priced security.*

   In other words, it can never be optimal in an arbitrage strategy to purchase the (relatively) over-priced or sell the (relatively) under-priced security.

1.4.2 The Principle of Static Replication

Another consequence of the no-arbitrage condition is the *Principle of Static Replication* (or PSR), which can be stated as follows.

If the future payoff of a derivative security can be expressed as a portfolio of elementary securities, then the price of the derivative must be equal to the price of the replicating portfolio.

This statement can be made more precise by expressing it mathematically. Let $v_i(X,T)$ for $i = 1, 2, \ldots$ denote the payoff at time T of a set of elementary derivatives and let $v_i(x,t)$ denote their arbitrage-free prices at any time $t < T$ where $X = X_T$ and $x = X_t$ are the underlying asset prices at the designated times T and t. Suppose further, the market admits another derivative security whose payoff at time T can be represented as the portfolio

$$V(X,T) = \sum_i \alpha_i v_i(X,T), \tag{1.2}$$

where α_i are the portfolio weights, which may be positive or negative. The Principle of Static Replication states that the price of the new derivative at any time $t < T$ is given by

$$V(x,t) = \sum_i \alpha_i v_i(x,t). \tag{1.3}$$

REMARK 1.1 The PSR essentially states that the pricing functional, i.e., the rule which determines the price $V(x,t)$ from its payoff $V(X,T)$, must be linear. Observe in particular, that if all the portfolio weights $\alpha_i = 0$, then zero payoff must be worth zero value today, but the converse is not necessarily true. A derivative with the possibility of both positive and negative payoffs at time T, may have zero value today, without violating the condition of no-arbitrage.

The Principle of Static Replication gets its name from the following observation. At time t we may replicate the decomposable derivative security with the above portfolio. Once set up, this portfolio may be held to expiry without further adjusting the weights α_i. Such a constant weight portfolio is an example of a static portfolio, in contrast to the dynamic portfolios which underlie the Black–Scholes approach to option pricing.

The above principle should not be confused with the idea of *static hedging*, which has become popular in recent times (e.g., see Carr et al. [14]). While there are similarities, the respective contexts are quite different. Static hedging is concerned with decomposing certain exotic options as portfolios of other *traded* securities. There is no requirement in the PSR that the portfolio securities are actually traded: they may be a set of entirely theoretical securities which have never seen the light of a real market. Nevertheless, the PSR is an indispensable tool for pricing certain classes of exotic option, and this is precisely how it shall be used in this book. □

Example 1.1
Consider two elementary derivatives on a stock X. The first is an up-type asset binary (an asset-or-nothing digital) with expiry T payoff

$$A_k^+(X,T) = X\,\mathbb{I}(X > k). \tag{1.4}$$

The second is an up-type bond binary (a cash-or-nothing digital) with expiry payoff

$$B_k^+(X,T) = \mathbb{I}(X > k). \tag{1.5}$$

The asset binary therefore pays one unit of the underlying asset at time T, but only if the asset price is above a given exercise price k. On the other hand, the bond binary pays one unit of cash (a dollar say) at time T, again only if the asset price is above k at time T.

Observe that the bond binary is similar to a zero coupon bond (ZCB) with a one-dollar face value. The main difference is that the bond binary delivers the one dollar face value only if the underlying asset satisfies the exercise condition $X > k$. The ZCB is not really a derivative security at all, because its guaranteed one-dollar payoff is independent of any underlying asset. The bond binary, however, is a genuine derivative security whose payoff has strong dependence on the underlying asset price.

The superscripted plus-signs on A_k^+ and B_k^+ are not just decoration, but have significance. We shall also consider binaries with the following payoffs

$$A_k^-(X,T) = X\mathbb{I}(X < k) \qquad \text{and} \qquad B_k^-(X,T) = \mathbb{I}(X < k)$$

with an obvious interpretation. These binaries are referred to as down-type binaries, which are in-the-money (itm) when the underlying asset price is below the exercise price.

Suppose for the present, we have priced all these binary options in some asset price model and that for time $t < T$ these prices are given respectively by

$$A_k^\pm(x,t) \qquad \text{and} \qquad B_k^\pm(x,t)$$

where $x = X_t$ is the asset price at time t. Now consider a European call option with expiry date T and strike price k. The payoff at time T can be expressed as

$$\begin{aligned} C_k(X,T) &= (X-k)^+ = (X-k)\mathbb{I}(X > k) \\ &= X\mathbb{I}(X > k) - k\mathbb{I}(X > k) \\ &= A_k^+(X,T) - kB_k^+(X,T). \end{aligned}$$

That is, the European call option payoff is seen to be equivalent to that of the binary option portfolio: long 1 up-type asset binary and short k up-type bond binaries. Hence, by the PSR, the price of the call option today (at time $t < T$) is given by

$$C_k(x,t) = A_k^+(x,t) - kB_k^+(x,t). \qquad (1.6)$$

A similar argument leads to the following representation for the European put option price in terms of down-type binaries

$$P_k(x,t) = -A_k^-(x,t) + kB_k^-(x,t). \qquad (1.7)$$

Observe that Equations (1.6) and (1.7) are model independent and are therefore valid in any arbitrage-free economy, Black–Scholes or otherwise. Chapter 4 of this book gives more details of the analysis and application of binary options. ▯

1.4.3 Parity Relations

We shall meet throughout this book many different parity relations. Probably the best known parity relation is the European *put-call parity* relation

$$C_k(x,t) - P_k(x,t) = x - ke^{-r(T-t)}; \qquad (t \leq T) \qquad (1.8)$$

where C_k, P_k denote the present values of a European call and put option, of strike price k, expiry date T and $x = X(t)$ is the underlying asset price at the current time t.

This parity relation, like all others we shall meet, is just a particular example of the Principle of Static Replication. In the above case, a portfolio of the long call option and the short put option is statically equivalent to a long forward contract. This is readily seen from the identity

$$\begin{aligned} &(X-k)^+ - (k-X)^+ \equiv (X-k) \\ \text{or} \quad &C_k(X,T) - P_k(X,T) = F_k(X,T). \end{aligned}$$

The result (1.8) is established since $(x - ke^{-r\tau})$ is the pv of the forward contract payoff, $F_k(X,T) = (X-k)$.

Example 1.2

As another example of simple parity relations, we mention here the case of *up-down parity* of the asset and bond binaries (on a non-dividend paying asset) considered in example 1.1. For all $t \leq T$,

$$A_k^+(x,t) + A_k^-(x,t) = x \qquad (1.9)$$

$$B_k^+(x,t) + B_k^-(x,t) = e^{-r(T-t)} \qquad (1.10)$$

We leave the proofs of these straightforward results to the reader. ▯