Transforms and Applications Primer for Engineers with Examples and MATLAB®

Alexander D. Poularikas

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Transforms and Applications Primer for Engineers with Examples and MATLAB®

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Preface

This book presents the most common and useful mathematical transforms for students and practicing engineers. It can be considered as a companion for students and a handy reference for practicing engineers who will need to use transforms in their work.

The Laplace transform, which undoubtedly is the most familiar example, is basic to the solution of initial value problems. The Fourier transform, being suited to solving boundary-value problems, is basic to the frequency spectrum analysis of time-varying signals. For discrete signals, we develop the z-transform and its uses. The purpose of this book is to develop the most important integral transforms and present numerous examples elucidating their use. Laplace and Fourier transforms are by far the most widely and most useful of all integral transforms. For this reason, they have been given a more extensive treatment in this book when compared to other books on the same subject.

This book is primarily written for seniors, first-year graduate students, and practicing engineers and scientists. To comprehend some of the topics, the reader should have a basic knowledge of complex variable theory. Advanced topics are indicated by a star (*).

The book contains several appendices to complement the main subjects. The extensive tables of the transforms are the most important contributions in this book. Another important contribution is the inclusion of an ample number of examples drawn from several disciplines. The included examples help the readers understand any of the transforms and give them the confidence to use it. Furthermore, it includes, wherever needed, MATLAB® functions and Book MATLAB functions developed by the author, which are included in the text.

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Alexander D. Poularikas received his PhD from the University of Arkansas, Fayetteville, and became a professor at the University of Rhode Island, Kingston. He became the chairman of the engineering department at the University of Denver, Colorado, and then became the chairman of the electrical and computer engineering department at the University of Alabama in Huntsville.

Dr. Poularikas has authored seven books and has edited two. He has served as the editor in chief of the Signal Processing series (1993–1997) with Artech House, and is now the editor in chief of the Electrical Engineering and Applied Signal Processing series as well as the Engineering and Science Primer series (1998 to present) with Taylor & Francis. He was a Fulbright scholar, is a lifelong senior member of the IEEE, and is a member of Tau Beta Pi, Sigma Nu, and Sigma Pi. In 1990 and in 1996, he received the Outstanding Educators Award of the IEEE, Huntsville Section. He is now a professor emeritus at the University of Alabama in Huntsville.

Dr. Poularikas has authored, coauthored, and edited the following books:

Electromagnetics, Marcel Dekker, New York, 1979.

Electrical Engineering: *Introduction and Concepts*, Matrix Publishers, Beaverton, OR, 1982.

Workbook, Matrix Publishers, Beaverton, OR, 1982.

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The Transforms and Applications Handbook, CRC Press, Boca Raton, FL, 1995.

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Adaptive Filtering Primer with MATLAB, Taylor & Francis, Boca Raton, FL, 2006.

Signals and Systems Primer with MATLAB, Taylor & Francis, Boca Raton, FL, 2007.

Discrete Random Signal Processing and Filtering Primer with MATLAB, Taylor & Francis, Boca Raton, FL, 2009.

1

Signals and Systems

1.1 Introduction

The term **systems**, in general, has many meanings such as electronic systems, biological systems, communication systems, etc. The same is true for the term **signals**, since we talk about optical signals, intelligence signals, radio signals, bio-signals, etc.

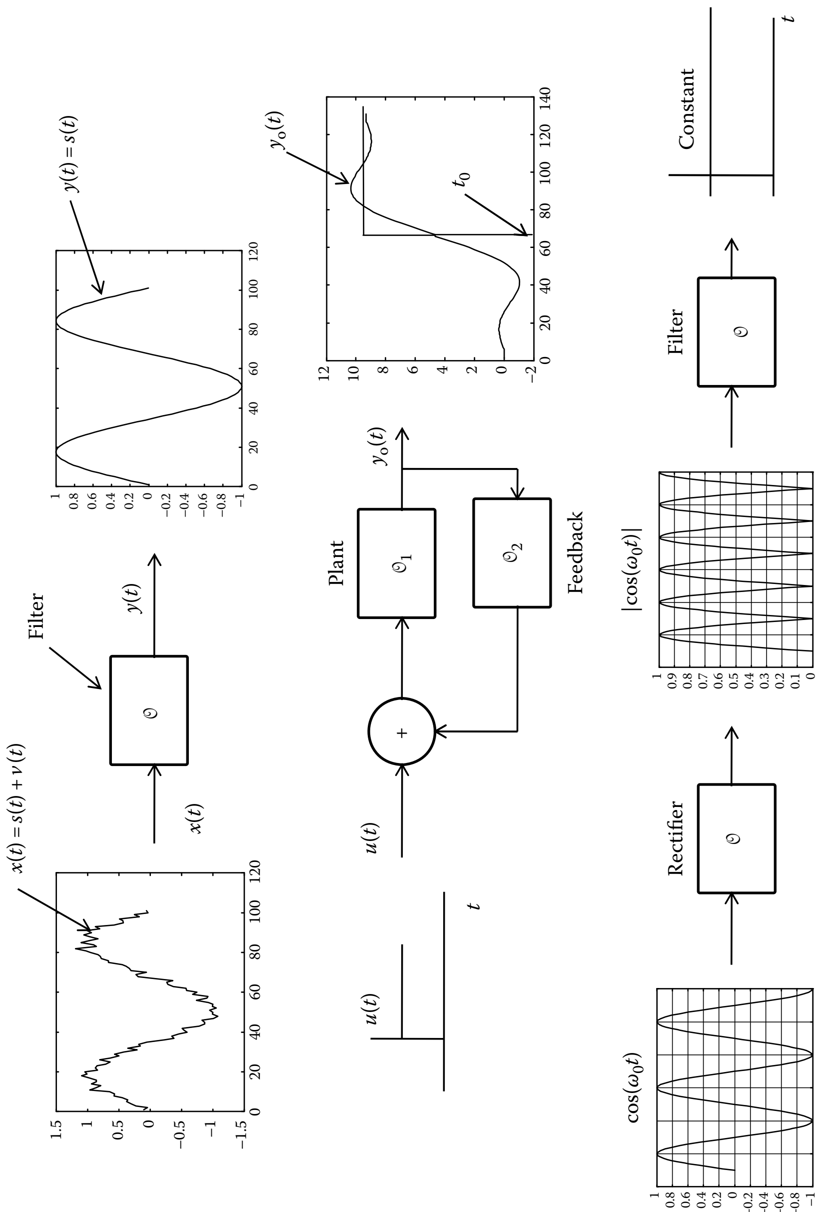
The two terms mentioned above can have the following three interpretations: (1) An electric system is considered to be made of resistors, inductors, capacitors, and energy sources. Signals are the currents and voltages in the electric system. The signals are a function of time and they are related by a set of equations that are the product of physical laws (Kirchhoff's voltage and current laws). (2) We interpret the system based on the mathematical function it performs. For example, a resistor is a multiplier, an inductor is a differentiator, and a capacitor is an integrator. The signals are the result of the rules of the interconnected elements of the system. (3) If the operations can be performed digitally and in real time, then the analog system can be substituted by a computer. The system, under these circumstances, is a digital device (computer) whose input and output are sequences of numbers. Figure 1.1 illustrates three systems and their responses. The top part of the figure represents the ability of a filter to clear a signal from a superimposed noise. The middle part of the figure shows how a feedback configuration affects an input pulse. This is known as the step response of systems. The bottom part of the figure shows how a rectifier and a filter can produce a DC (direct current) source when the input is a sinusoidal signal as the one present in power transmission lines.

In addition to analog systems, we also have digital ones. These systems deal only with discrete signals and are presented later on in the book. A basic, but sophisticated, instrument is the **analog-to-digital** (A/D) converter, which most instruments nowadays contain.

1.2 Signals

A **signal** is a function representing a physical quantity. This can be a current, a voltage, heart signals (EKG), velocities of motion, music signals, economic time series, etc. In this chapter, we will concentrate only on one-dimensional signals, although images, for example, are two-dimensional signals.

A **continuous-time signal** is a function whose domain is every point in a specified interval.



$x(t) = s(t) + v(t)$
$x(t)$
Filter
$\mathcal{O}$
$y(t)$
$y(t) = s(t)$
$u(t)$
t
$u(t)$
+
Plant
$\mathcal{O}_1$
$\mathcal{O}_2$
Feedback
$y_o(t)$
$y_o(t)$
t_0
$\cos(\omega_0 t)$
Rectifier
$\mathcal{O}$
$|\cos(\omega_0 t)|$
Filter
$\mathcal{O}$
Constant
t

FIGURE 1.1

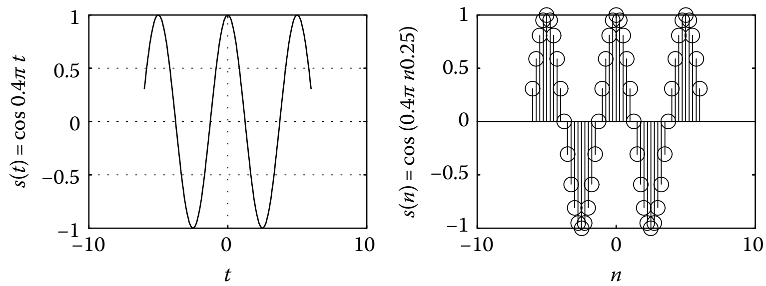


FIGURE 1.2

A **discrete-time signal** is a function whose domain is a set of integers. Therefore, this type of signal is a sequence of numbers denoted by $\{x(n)\}$. It is understood that the discrete-time signal is often formed by **sampling** a continuous-time signal $x(t)$. In this case and for equidistance samples, we write

$$x(n) = x(nT) \quad T = \text{sampling interval} \tag{1.1}$$

Figure 1.2 shows a transformation from a continuous-time signal to a discrete-time signal.

Some important and useful functions are given in Table 1.1.

If the above analog signals are sampled every T seconds, then we will obtain the corresponding discrete ones.

Approximation of a derivative

From Figure 1.3, we observe that we can approximate the samples $y(nT)$ of the derivative $y(t) = x'(t)$ of the signal $x(t)$ for a sufficiently small T as follows:

$$x'(t) \cong \frac{x(t) - x(t-T)}{T} \tag{1.2}$$

$$y(nT) = x'(nT) = \frac{x(nT) - x(nT-T)}{T} = \frac{1}{T}\Delta x(nT) \tag{1.3}$$

We observe that as $T \to 0$, the approximate derivative of $x(t)$, indicated by the inclination of line A, comes closer and closer to the exact one, indicated by the inclination of line B.

Approximation of an integral

The approximation of an integral with its discrete form is shown in Figure 1.4. Therefore, we write

$$y(nT) = \int_0^{nT-T} x(t)dt + \int_{nT-T}^{nT} x(t)dt \tag{1.4}$$

TABLE 1.1 Some Useful Mathematical Functions in Analog and Discrete Format

1. Signum function

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0; \\ -1 & t < 1 \end{cases} \quad \mathrm{sgn}(nT) = \begin{cases} 1 & nT > 0 \\ 0 & nT = 0 \\ -1 & nT < 0 \end{cases}$$

2. Step function

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}; \quad u(nT) = \begin{cases} 1 & nT > 0 \\ 0 & nT < 0 \end{cases}$$

3. Ramp function

$$r(t) = \int_{-\infty}^{t} u(x)dx = tu(t); \qquad r(nT) = nTu(nT)$$

4. Pulse function

$$p_a(t) = u(t+a) - u(t-a) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}; \quad p_a(nT) = u(nT + mT) - u(nT - mT)$$

5. Triangular pulse

$$\Lambda_a(t) = \begin{cases} 1 - \dfrac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}; \quad \Lambda_a(nT) = \begin{cases} 1 - \dfrac{|nT|}{mT} & |nT| < mT \\ 0 & |nT| > mT \end{cases}$$

6. Sinc function

$$\mathrm{sinc}_a(t) = \frac{\sin at}{t} \quad -\infty < t < \infty; \quad \mathrm{sinc}_a(nT) = \frac{\sin anT}{nT}$$

7. Gaussian function

$$g_a(t) = e^{-at^2} \qquad -\infty < t < \infty$$

8. Error function

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n!(2n+1)}$$

properties: $\mathrm{erf}(\infty) = 1, \mathrm{erf}(0) = 0, \mathrm{erf}(-t) = -\mathrm{erf}(t)$

$$\mathrm{erfc}(t) = \text{complementary error function} = 1 - \mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2}\, dx$$

9. Exponential and double exponential

$$f(t) = e^{-t}u(t) \qquad t \geq 0; \quad f(t) = e^{-|t|} \qquad -\infty < t < \infty$$

$$f(nT) = e^{-nT}u(nT) \qquad nT \geq 0; \quad f(nT) = e^{-|nT|} \qquad -\infty < nT < \infty$$

Note: T, sampling time; n, integer.

which becomes

$$y(nT) = y(nT - T) + \int_{nT-T}^{nT} x(t)dt \tag{1.5}$$

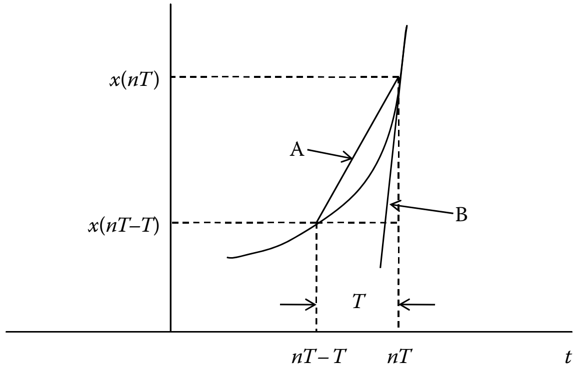


FIGURE 1.3

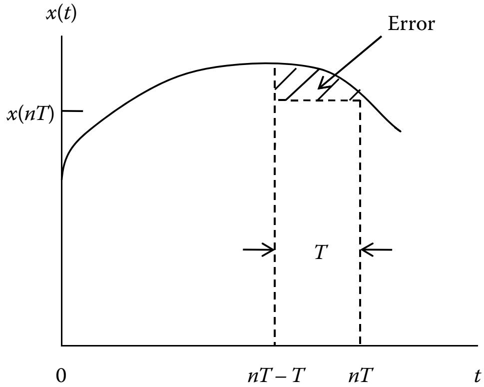


FIGURE 1.4

Approximating the integral in the above equation by the rectangle shown in Figure 1.4, we obtain its approximate discrete form:

$$\boxed{y(nT) \cong y(nT - T) + Tx(nT) \quad n = 0, 1, 2, \ldots} \tag{1.6}$$

Trigonometric functions

Of special interest in the study of linear systems is the class of sine and cosine functions:

$$a \cos \omega t \quad b \sin \omega t \quad r \cos(\omega t + \varphi)$$

These functions are periodic with **a period** $2\pi/\omega$ a**nd a frequency** $f = \omega/2\pi$ cycles/s or Hz.

Complex signals

Signals representing physical quantities are, in general, real. However, in many cases it is convenient to consider complex signals and to use their real or imaginary parts to represent physical quantities. One of these signals is the **complex exponential** $e^{j\omega t}$. This function can be defined by its power series $\left(e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$:

$$e^{j\omega t} = 1 + j\omega t + \frac{(j\omega t)^2}{2!} + \frac{(j\omega t)^3}{3!} + \cdots + \frac{(j\omega t)^n}{n!} + \cdots \tag{1.7}$$

The sum of two sine functions with the same frequency is also a sine function:

$$a\cos\omega t + b\sin\omega t = r\cos(\omega t + \varphi) \tag{1.8}$$

The discrete form of a sine function is

$$x(nT) = \cos\omega nT$$

By separating the real and the imaginary parts of (1.7), we obtain

$$\boxed{e^{j\omega t} = \cos\omega t + j\sin\omega t} \tag{1.9}$$

This fundamental identity can also be used to define the complex exponential $\exp(j\omega t)$ and to derive all its properties in terms of the properties of trigonometric functions.

We observe that $\exp(j\omega t)$ is a complex number with unity amplitude and phase ωt. The sample value of the complex exponential is

$$x(nT) = e^{j\omega nT}$$

This function is a geometric series whose ratio $e^{j\omega T}$ is a complex number of unit amplitude.

From (1.9), it follows that

$$e^{(a+j\omega)t} = e^{at}(\cos\omega t + j\sin\omega t) \tag{1.10}$$

Therefore, if $s = a + j\omega$ is a complex number, then e^{st} is a complex signal whose real part $e^{at}\cos\omega t$ and imaginary part $e^{at}\sin\omega t$ are exponentially decreasing ($a < 0$) and increasing ($a > 0$) sine functions.

From (1.9), we obtain

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

Adding and subtracting the last equation from (1.9), we find Euler's formula:

$$\boxed{\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin \omega t \frac{e^{j\omega t} - e^{-j\omega t}}{2j}} \tag{1.11}$$

A general complex signal $x(t)$ is a function of the form

$$x(t) = x_1(t) + jx_2(t)$$

where $x_1(t)$ and $x_2(t)$ are the real functions of the real variable t. The derivative of $x(t)$ is a complex signal given by

$$\frac{dx(t)}{dt} = \frac{dx_1(t)}{dt} + j\frac{dx_2(t)}{dt}$$

and, in general, for any s, real or complex, we have

$$\frac{de^{st}}{dt} = se^{st} \tag{1.12}$$

Impulse (delta) function

An important function in science and engineering is the **impulse** function also known as Dirac's **delta** function. The signal is represented graphically in Figure 1.5. The delta function is not an ordinary one. Therefore, some fundamental properties of these types of functions, and specifically those of the delta function are presented so that the reader uses it appropriately.

Property 1.1 The impulse function $\delta(t)$ is a signal with a unit area and is zero outside the point at the origin:

$$\begin{cases} \int\limits_{-\infty}^{\infty} \delta(t)dt = 1 \\ \delta(t) = 0 \qquad t \neq 0 \end{cases} \tag{1.13}$$

Property 1.2 The impulse function is the derivative of the step function $u(t)$:

$$\delta(t) = \frac{du(t)}{dt} \tag{1.14}$$

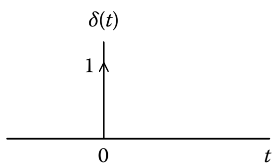


FIGURE 1.5

Property 1.3 The area of the product $\varphi(t)\delta(t)$ equals $\varphi(0)$ for any regular function that is continuous at the origin:

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t)dt = \varphi(0) \tag{1.15}$$

Property 1.4 The delta function can be written as a limit:

$$\delta(t) = \lim v_\varepsilon(t) \quad \varepsilon \to 0 \tag{1.16}$$

where $v_\varepsilon(t)$ is a family of functions with the unit area vanishing outside the interval $\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$:

$$\int_{-\varepsilon/2}^{\varepsilon/2} v_e(t)dt = 1 \quad v_\varepsilon(t) = 0 \quad \text{for } t < -\frac{\varepsilon}{2} \quad \text{and} \quad t > \frac{\varepsilon}{2} \tag{1.17}$$

Figure 1.6 shows the approximation of the delta function by the pulse and the sufficiently small ε.

We can show (see Prob) that the impulse function is even. Hence,

$$\delta(t) = \delta(-t) \tag{1.18}$$

The impulse function $\delta(t - t_0)$ is centered at t_0 of area one. Therefore, from (1.18), we obtain

$$\delta(t - t_0) = \delta(t_0 - t) \tag{1.19}$$

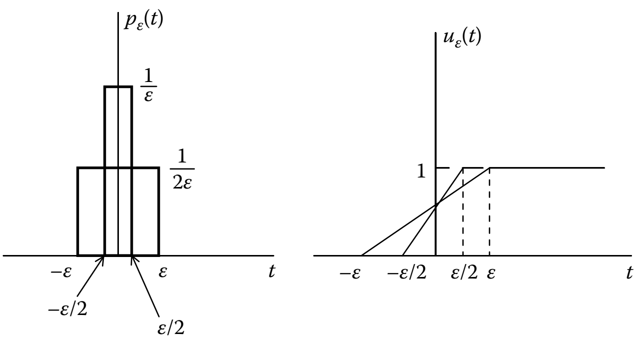


FIGURE 1.6

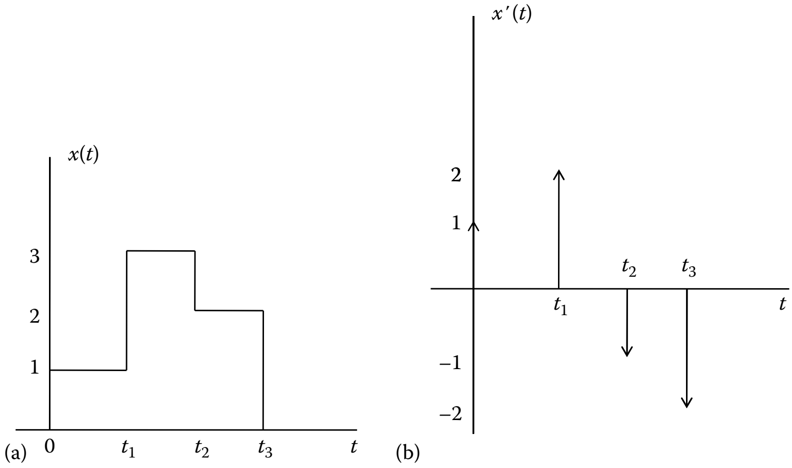


FIGURE 1.7

Using Property 1.2 above, we write

$$\delta(t - t_0) = \frac{du(t - t_0)}{dt} \tag{1.20}$$

Based on the above, the derivative of the function shown in Figure 1.7a is that shown in Figure 1.7b.

Considering Property 1.3, and taking into consideration the evenness of the delta function, we write

$$\boxed{\int_{-\infty}^{\infty} y(x)\delta(t - x)dx = y(t)} \tag{1.21}$$

The above integral is also known as the **convolution** integral. Therefore, we state that the convolution of a function with a delta function reproduces the function. Let us consider the function $y(t - t_0)$ to be convolved with the shifted delta function $\delta(t - a)$. From (1.21), we write

$$\int_{-\infty}^{\infty} y(x - t_0)\delta(t - x - a)dx = y(t - t_0 - a)$$

The identity (1.21) is basic. We can use it, for example, to define the derivative of the delta function. Because the two sides of the equation are functions of t, we can differentiate with respect to t to obtain

$$\int_{-\infty}^{\infty} y(x)\delta'(t-x)dx = y'(t) \tag{1.22}$$

Thus, the derivative of the delta function is such that the area of the product $y(x)\delta'(t-x)$, considered as a function of x, equals $y'(t)$. With $t=0$, (1.22) yields

$$\int_{-\infty}^{\infty} y(x)\delta'(-x)dx = y'(0) \tag{1.23}$$

From calculus we know that when a function is even its derivative is an odd function. Hence,

$$\delta'(-t) = -\delta'(t) \tag{1.24}$$

Inserting (1.24) in (1.23) and changing the dummy variable from x to t, we find

$$\boxed{\int_{-\infty}^{\infty} y(t)\delta'(t)dt = -y'(0)} \tag{1.25}$$

Additional delta functional properties are given in Table 1.2.

TABLE 1.2 Delta Functional Properties

1.	$\delta(at) = \frac{1}{\vert a\vert}\delta(t)$
2.	$\delta\left(\frac{t-t_0}{a}\right) = \vert a\vert\delta(t-t_0)$
3.	$\delta(at-t_0) = \frac{1}{\vert a\vert}\delta\left(t-\frac{t_0}{a}\right)$
4.	$\delta(-t+t_0) = \delta(t-t_0)$
5.	$\delta(-t) = \delta(t)$; $\delta(t) =$ even function
6.	$\int_{-\infty}^{\infty}\delta(t)f(t)dt = f(0)$
7.	$\int_{-\infty}^{\infty}\delta(t-t_0)f(t) = f(t_0)$
8.	$f(t)\delta(t) = f(0)\delta(t)$
9.	$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$
10.	$t\delta(t) = 0$
11.	$\int_{-\infty}^{\infty}A\delta(t)dt = \int_{-\infty}^{\infty}A\delta(t-t_0)dt = A$
12.	$f(t)*\delta(t) = \text{convolution} = \int_{-\infty}^{\infty}f(t-\tau)\delta(\tau)d\tau = f(t)$
13.	$\delta(t-t_1)*\delta(t-t_2) = \int_{-\infty}^{\infty}\delta(\tau-t_1)\delta(t-\tau-t_2)d\tau = \delta[t-(t_1+t_2)]$
14.	$\sum_{n=-N}^{N}\delta(t-nT)*\sum_{n=-N}^{N}\delta(t-nT) = \sum_{n=-2N}^{2N}(2N+1-\vert n\vert)\delta(t-nT)$

TABLE 1.2 (continued) Delta Functional Properties

15.	$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} f(t)dt = -\frac{df(0)}{dt}$
16.	$\int_{-\infty}^{\infty} \frac{d\delta(t-t_0)}{dt} f(t)dt = -\frac{df(t_0)}{dt}$
17.	$\int_{-\infty}^{\infty} \frac{d^n\delta(t)}{dt^n} f(t)dt = (-1)^n \frac{d^n f(0)}{dt^n}$
18.	$f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\delta(t) + f(0)\frac{d\delta(t)}{dt}$
19.	$t\frac{d\delta(t)}{dt} = -\delta(t)$
20.	$t^n \frac{d^m\delta(t)}{dt^m} = \begin{cases} (-1)^n n!\delta(t), & m = n \\ (-1)^n \frac{m!}{m-n!} \frac{d^{m-n}\delta(t)}{dt^{m-n}}, & m > n \\ 0, & m < n \end{cases}$
21.	$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} = 0, \quad \frac{d\delta(t)}{dt} = \text{odd function}$
22.	$f(t) * \frac{d\delta(t)}{dt} = \frac{df(t)}{dt}$
23.	$f(t)\frac{d^n\delta(t)}{dt^n} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{d^k f(0)}{dt^k} \frac{d^{n-k}\delta(t)}{dt^{n-k}}$
24.	$\frac{\partial\delta(yt)}{\partial y} = -\frac{1}{y^2}\delta(t)$
25.	$\delta(t) = \frac{du(t)}{dt}$
26.	$\frac{d^n\delta(-t)}{dt^n} = (-1)^n \frac{d^n\delta(t)}{dt^n}, \left\{ \frac{d^n\delta(t)}{dt^n} \text{ is even if } n \text{ is even, and odd if } n \text{ is odd.} \right\}$
27.	$(\sin\ at)\frac{d\delta(t)}{dt} = -a\delta(t)$
28.	$\frac{d\delta(t)}{dt} = \frac{d^2u(t)}{dt^2}$
29.	$-\delta(t) = \frac{du(-t)}{dt}$
30.	$\delta(t-t_0) = \frac{du(t-t_0)}{dt}$
31.	$\frac{d\text{sgn}(t)}{dt} = 2\delta(t)$
32.	$\delta[r(t)] = \sum_n \frac{\delta(t-t_n)}{\left\lvert\frac{dr(t_n)}{dt}\right\rvert}, \quad t_n = \text{zeros of } r(t), \frac{dr(t_n)}{dt} \neq 0$
33.	$\frac{d\delta[r(t)]}{dt} = \sum_n \frac{\frac{d\delta(t-t_n)}{dt}}{\frac{dr(t)}{dt}\left\lvert\frac{dr(t_n)}{dt}\right\rvert}, \quad t_n = \text{zeros of } r(t), \frac{dr(t_n)}{dt} \neq 0, \frac{dr(t)}{dt} \neq 0$
34.	$\delta(\sin\ t) = \sum_{n=-\infty}^{\infty} \delta(t - n\pi)$
35.	$\delta(t^2 - 1) = \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t+1)$
36.	$\delta(t^2 - a^2) = \frac{1}{2a}[\delta(t+a) + \delta(t-a)]$

(*continued*)

TABLE 1.2 (continued) Delta Functional Properties

37.	$\delta(t) = \lim_{\varepsilon\to 0} \dfrac{e^{-t^2/\varepsilon}}{\sqrt{\varepsilon\pi}}$
38.	$\delta(t) = \lim_{\omega\to\infty} \dfrac{\sin\ \omega t}{\pi t}$
39.	$\delta(t) = \lim_{\varepsilon\to 0} \dfrac{1}{\pi}\dfrac{\varepsilon}{t^2+\varepsilon^2}$
40.	$\delta(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega t\ d\omega$
41.	$\dfrac{df(t)}{dt} = \dfrac{d}{dt}[tu(t) - (t-1)u(t-1) - u(t-1)]$ $= t\delta(t) + u(t) - (t-1)\delta(t-1) - u(t-1) - \delta(t-1)$
42.	$\mathrm{comb}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT), \quad f(t)\mathrm{comb}_T(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)$ $\mathrm{COMB}_{\omega_0}(\omega) = \mathcal{F}\{\mathrm{comb}_T(t)\} = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \quad \omega_0 = \dfrac{2\pi}{T}$

$$
\begin{aligned}
\frac{d}{dt}([2-u(t)]\cos t) &= \frac{d}{dt}(2\cos t - u(t)\cos t)\\
&= -2\sin t - \delta(t)\cos t + u(t)\sin t\\
&= (u(t)-2)\sin t - \delta(t)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\sin t\right) &= \left[\delta\left(t-\frac{\pi}{2}\right) - \delta(t-\pi)\right]\sin t\\
&\quad + \left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\cos t\\
&= \delta\left(t-\frac{\pi}{2}\right) + \left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\cos t
\end{aligned}
$$

Example

The values of the following integrals are

$$
\int_{-\infty}^{\infty} e^{2t}\sin 4t \frac{d^2\delta(t)}{dt^2}dt = (-1)^2\frac{d^2}{dt^2}[e^{2t}\sin 4t]|_{t=0} = 2\times 2\times 4 = 16
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty}(t^3+2t+3)\left(\frac{d\delta(t-1)}{dt} + 2\frac{d^2\delta(t-2)}{dt^2}\right)dt &= \int_{-\infty}^{\infty}(t^3+2t+3)\frac{d\delta(t-1)}{dt^2}dt\\
&\quad + 2\int_{-\infty}^{\infty}(t^3+2t+3)\frac{d^2\delta(t-2)}{dt^2}dt\\
&= (-1)(3t^2+2)|_{t=1} + (-1)^2 2(6t)|_{t=2}\\
&= -5 + 24 = 19
\end{aligned}
$$

Example

The values of the following integrals are

$$
\int_0^4 e^{4t}\delta(2t-3)dt = \int_0^4 e^{4t}\delta\left[2\left(t-\frac{3}{2}\right)\right]dt = \frac{1}{2}\int_0^4 e^{4t}\delta\left(t-\frac{3}{2}\right)dt = \frac{1}{2}e^{4\frac{3}{2}} = \frac{1}{2}e^6
$$

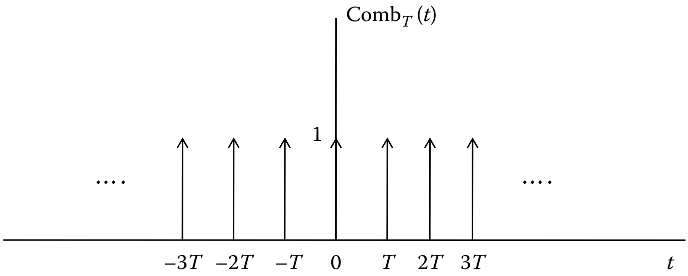


FIGURE 1.8

The comb function

The comb function is represented mathematically as follows:

$$\boxed{\mathrm{comb}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)} \tag{1.26}$$

This function is used extensively for studying the sampling of signals. Figure 1.8 shows the comb function pictorially.

1.3 Circuit Elements and Equation

In this text, we use the idealized model of physical devices, passive or active, which is specified in terms of its terminal properties. In Figure 1.9, we show the passive and active elements of electrical circuits.

A **circuit** or **network** is a combination of connected elements and external sources. The inputs are the sources (voltage or current) and the outputs are voltages or currents across elements and through elements, respectively. A network is a special form of an **analog** system since its inputs and outputs are continuous signals.

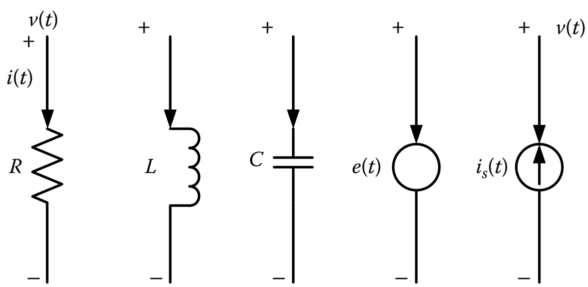


FIGURE 1.9

The **state** of a network at a certain time (taking the time $t=0$ for simplicity) is the set of all the voltages across the capacitors and all the currents through the inductors. If we know the **initial state** of a network at $t=0$ and all its inputs at $t=0$, then we can determine all its responses for an all-time $t \geq 0$. If all the currents through the inductors and all the voltages across the capacitors are zero, the network is at a **zero initial state**. If the network is at a zero initial state, then its response is known as the **zero-state response**. If all the sources are zero, then its response is called the **zero-input response**. The zero-input response is due to the energy stored in the network.

The voltages and the currents of the passive elements are

Resistor

$$v(t) = Ri(t) \quad i(t) = Gv(t) \quad G = \frac{1}{R} \tag{1.27}$$

Inductor

$$v(t) = L\frac{di(t)}{dt} \quad i(t) = \frac{1}{L}\int_0^t v(x)dx + i(0) \tag{1.28}$$

Capacitor

$$i(t) = C\frac{dv(t)}{dt} \quad v(t) = \frac{1}{C}\int_0^t i(x)dx + v(0) \tag{1.29}$$

Voltage source

$$e(t) \equiv \text{known, independent of } i(t)$$

Current source

$$i_s(t) \equiv \text{known, independent of } v(t)$$

Initial conditions

Knowing the initial conditions of a network (voltages across the capacitors and the currents through the inductors) it is sufficient to find its response for $t \geq 0$. In this text we assume that there is a continuation of the initial conditions which means that the currents through the inductors or the voltages across the capacitors are the same at $t(0-)$ and $t(0+)$.

Impulse response

The following simple example will elucidate how a network responds to an impulse input source. Let the input voltage of a simple RL series circuit be a delta function as shown in Figure 1.10. Kirchhoff's voltage law of a network loop is

$$L\frac{di(t)}{dt} + Ri(t) = \delta(t) \tag{1.30}$$

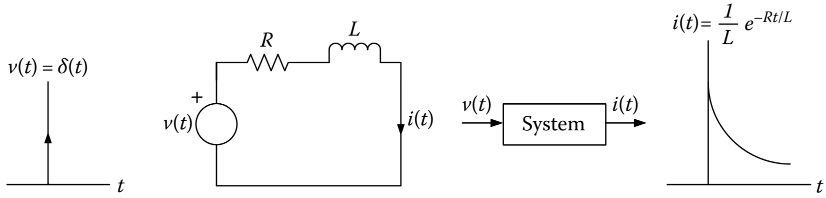


FIGURE 1.10

Integrating the above equation from (0−) to (0+), and taking into consideration that $i(t)$ is a continuous function, we obtain

$$L\int_{0-}^{0+}\frac{di(t)}{dt}dt + R\int_{0-}^{0+} i(t)dt = \int_{0-}^{0+}\delta(t)dt \quad \text{or} \quad L[i(0+) - i(0-)] + R0 = 1$$

$$\text{or} \quad L[i(0+) - i(0-)] = 1 \tag{1.31}$$

Since the input impulse function is a discontinuous one, the current is also a discontinuous function with a discontinuity such that $L[i(0+) - i(0-)]$ is equal to 1. If, in addition, the system (here the network) is causal, $i(0-) = 0$ and hence $i(0+) = 1/L$. Therefore, if the circuit is in the zero state and it is connected to a delta function source, the current $i(t)$ changes instantly from zero to $1/L$.

Derived initial conditions

Derived initial conditions are determined from the circuit equations, and, in general, depend also on the sources. Let us assume that there is an initial current, $i(0) = i_0$, in the RL circuit shown in Figure 1.10. In addition, let the voltage source be a constant, $v(t) = V$. In this case, the solution of (1.30) with a constant voltage is the function

$$i(t) = \underbrace{\frac{V}{R}(1 - e^{-Rt/L})}_{\text{zero-state response}} + \underbrace{i_0 e^{-Rt/L}}_{\text{zero-input response}} \tag{1.32}$$

This result will be derived in Chapter 7.

For an RC series circuit, Kirchhoff's mesh voltage law results in

$$Ri(t) + v_c(t) = v(t) \quad \text{or} \quad Ri(t) + \frac{1}{C}\int_{-\infty}^{0} i(x)dx + \frac{1}{C}\int_{0}^{t} i(x)dx = v(t) \quad \text{or}$$

$$Ri(t) + \frac{1}{C}\int_{0}^{t} i(x)dx + v_c(0) = v(t) \tag{1.33}$$

where $v_c(t)$ is the voltage across the capacitor. This equation can be cast into an ordinary differential equation by differentiating both sides. Hence,

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt} \tag{1.34}$$

To solve (1.34), we must find the initial value of the current, $i(0)$. Setting $t = 0$, we obtain

$$Ri(0) + v_c(0) = v(0) \quad \text{or} \quad i(0) = \frac{v(0) - v_c(0)}{R}$$

This is the **derived** initial condition, and it depends not only on the initial (state) voltage across the capacitor but also on the initial value of the voltage source.

As another example, Kirchhoff's mesh equation for a series RLC circuit with an initial current $i(0)$ through the inductor and the initial voltage $v_c(0)$ across the capacitor is

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(x)dx + v_c(0) = v(t) \quad i(0) \tag{1.35}$$

Taking the derivative with respect to the independent variable t, we find

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt} \tag{1.36}$$

Since the above equation is a second-order differential equation of the dependent variable i, we must find, in addition to its initial value $i(0)$, the initial value of its derivative $i'(0)$. Setting $t = 0$ in (1.35) and assuming that $v(t)$ does not have a discontinuity at $t = 0$, we obtain

$$Li'(0) + Ri(0) + v_c(0) = v(0) \quad \text{or} \quad i'(0) = \frac{1}{L}[v(0) - v_c(0) - Ri(0)]$$

which is a derived initial condition.

State equations of an RLC series circuit

The state variables are the current $i(t)$ through the inductor and the voltage $v_c(t)$ across the capacitor and they satisfy the following two first-order differential equations:

$$\begin{aligned} &C\frac{dv_c(t)}{dt} = i(t) \qquad v_c(0) \\ &L\frac{di(t)}{dt} + Ri(t) + v_c(t) = v(t) \qquad i(0) \end{aligned} \tag{1.37}$$

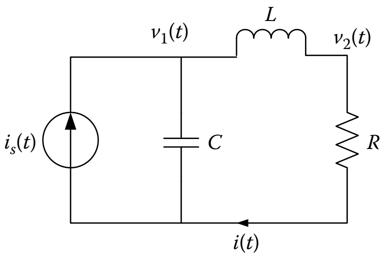


FIGURE 1.11

Node and state equations of the circuit in Figure 1.11

The circuit in Figure 1.11 has two variables: the voltage $v_1(t)$ across the capacitor and the current $i(t)$ through the inductor. The initial voltage across the capacitor is $v_1(0)$ and the initial current through the inductor is $i(0)$.

Node equations (the algebraic sum of currents at a node should be equal to zero)

For the node equation we use as primary unknowns, the node voltages $v_1(t)$ and $v_2(t)$. Hence,

$$\begin{aligned}
&C\frac{dv_1(t)}{dt} + \frac{v_2(t)}{R} = i_s(t) \quad v_1(0) \\
&L\frac{di(t)}{dt} + Ri(t) - v_1(t) = 0 \quad \text{or} \quad \frac{L}{R}\frac{dv_2(t)}{dt} + v_2(t) - v_1(t) = 0 \quad v_2(0) = Ri(0)
\end{aligned} \tag{1.38}$$

State equations

$$\begin{aligned}
C\frac{dv_1(t)}{dt} + i(t) &= i_s(t) \qquad v_1(0) \\
L\frac{di(t)}{dt} + Ri(t) - v_1(t) &= 0 \qquad i(0)
\end{aligned} \tag{1.39}$$

State equations for the circuit in Figure 1.12

State equations

$$\begin{aligned}
L_1\frac{di_1(t)}{dt} + R_1 i_1(t) + v_c(t) &= v(t) \\
L_2\frac{di_2(t)}{dt} + R_2 i_2(t) - v_c(t) &= 0 \\
i_1(t) - i_2(t) - C\frac{dv_c(t)}{dt} &= 0
\end{aligned} \tag{1.40}$$

Block diagrams of systems

Circuit diagrams describe the structure of a network. However, the block diagrams describe the terminal properties of the network (system). Inside the block we present

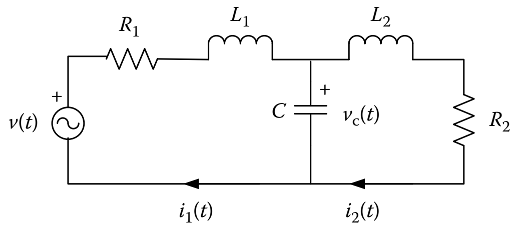


FIGURE 1.12

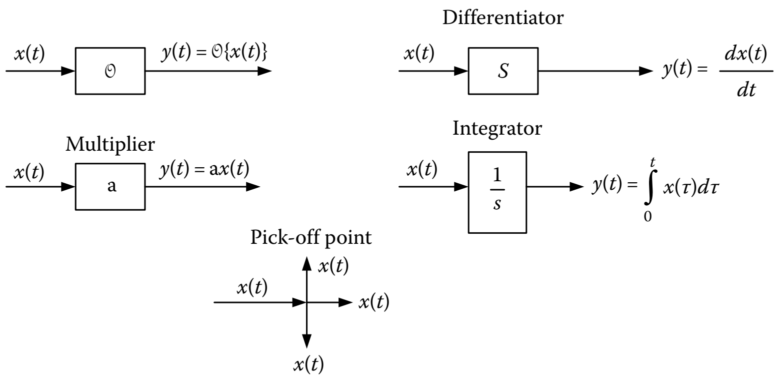


FIGURE 1.13

different identifiers that will characterize the system operation. In general, we introduce in the block a script $\mathcal{O}$ to represent a general operator that operates on the input to the block to produce the output. In Figure 1.13, we show the block-diagram representation of, a **general** system, a **differentiator**, a **multiplier**, an **integrator**, and a **pick-off point**. The significance of s is given later in Chapter 7. Note that at the pick-off point the input quantity appears in all the branches without any variation of its magnitude.

Figure 1.14a depicts three basic ways that systems can be configured. It is assumed that the terminal properties of each system remain unchanged (no loading effect takes place). Figure 1.14b shows the equivalent input–output of the **cascade** and the **parallel** and the feedback configurations. In Figure 1.14a and for the first system, we obtain $y_1(t) = \mathcal{O}_1 x(t)$ or $y(t) = \mathcal{O}_2 y_1(t) = \mathcal{O}_2\mathcal{O}_1 x(t)$. The second expression characterizes the first system of part (b) of the figure. For the second system of part (a) of the figure, we find

$$y_1(t) = \mathcal{O}_1 x(t) \quad \text{and} \quad y_2(t) = \mathcal{O}_2 x(t), \quad \text{and, therefore, } y(t) = y_1(t) + y_2(t) = [\mathcal{O}_1 + \mathcal{O}_2]x(t)$$

which characterizes the second system of part (b) of the figure. For the feedback system of part (a), we obtain: $y_1(t) = x(t) \pm \mathcal{O}_2 y(t)$ or $y(t) = \mathcal{O}_1[x(t) \pm \mathcal{O}_2 y(t)]$. Solving $y(t)$, and keeping in mind that we do not perform divisions with the operators but only use their inverse form, we find $y(t) = [1 \mp \mathcal{O}_1\mathcal{O}_2]^{-1}\mathcal{O}_1 x(t)$. This expression characterizes the third

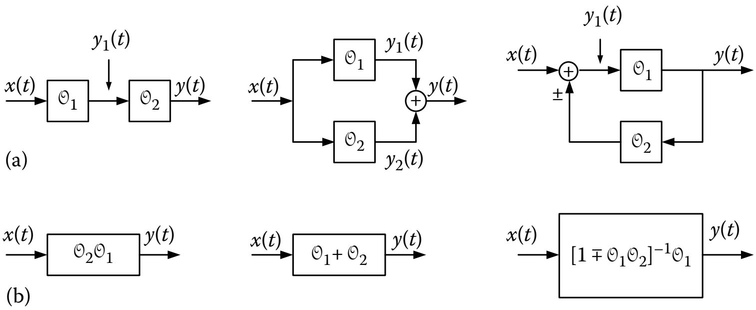


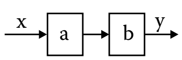
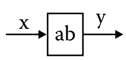
FIGURE 1.14

system of part (b) of the figure. If we substitute the two operators with constants a and b, the transfer functions of the three systems are

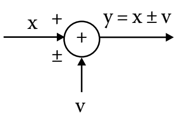
$$H_c = ab, \quad H_p = a + b, \quad H_f = \frac{a}{1 \mp ab} \tag{1.41}$$

Table 1.3 represents block-diagram transformations.

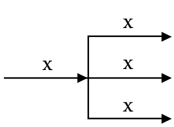
TABLE 1.3 Block-Diagram Transformations of Systems

		Two blocks in cascade

(a)

		Summation point

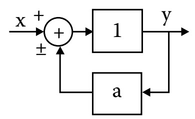
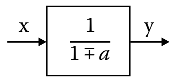
(b)

		Pick-off point

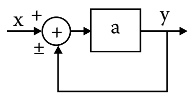
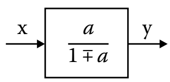
(c)

(*continued*)

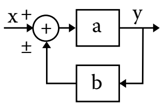
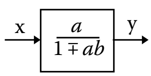
TABLE 1.3 (continued) Block-Diagram Transformations of Systems

		Feedback loop

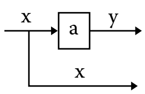
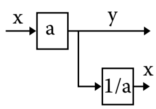
(d)

		Special case of unit feedback loop

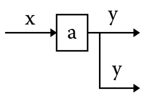
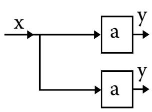
(e)

		Complete feedback loop

(f)

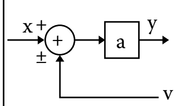
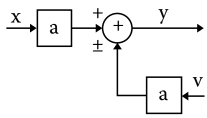
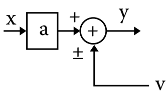
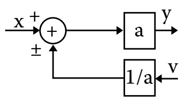
		Moving a pick-off point ahead

(g)

		Moving a pick-off point behind

(h)

TABLE 1.3 (continued) Block-Diagram Transformations of Systems

		Moving a summing point ahead
(i)		
		Moving a summing point behind
(j)		

1.4 Linear Mechanical and Rotational Mechanical Elements

The linear mechanical systems with their equivalent circuit characterizations are shown in Figure 1.15. The rotating fundamental mechanical systems and their equivalent circuit characterizations are shown in Figure 1.16. The terminal properties of these signals are given below.

1.4.1 Linear Mechanical Systems

Damper

$$f(t) = Dv(t), \quad v(t) = \frac{1}{D}f(t), \quad D = \text{damping constant } (\text{N}\cdot\text{s/m}) \\ f(t) = \text{force (N)}, \quad v(t) = \text{velocity (m/s)} \tag{1.42}$$

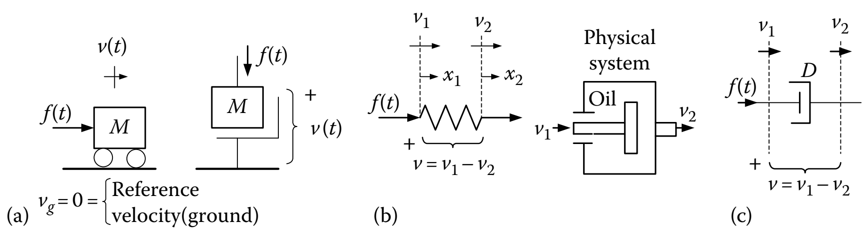


FIGURE 1.15

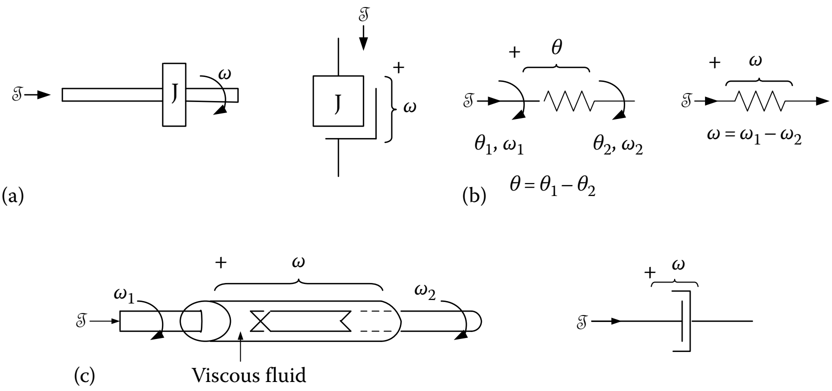


FIGURE 1.16

Spring

$$f(t) = Kx(t), \quad v(t) = \frac{1}{K}\frac{df(t)}{dt}, \quad x(t) = \text{displacement (m)}$$
$$K = \text{spring constant (N/m)} \tag{1.43}$$

Mass

$$f(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2} \quad \text{Newton} = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}\text{(N)}$$
$$v(t) = \frac{1}{M}\int_{-\infty}^{t} f(x)dx \qquad M = \text{mass(kg)} \tag{1.44}$$

From the above equations, we observe the following analogies between the circuit elements and the linear mechanical elements: the mass and the capacitor, the spring and the inductor, and the damper and the resistor.

1.4.2 Rotational Mechanical Systems

Damper

$$\mathfrak{T}(t) = D\omega(t), \quad D = \text{damping costant (N}\cdot\text{s}\cdot\text{m/rad)}$$
$$\omega(t) = \frac{1}{D}\mathfrak{T}(t); \quad \mathfrak{T} = \text{torque (N}\cdot\text{m)} \tag{1.45}$$

Spring

$$\begin{aligned}\mathfrak{T}(t) &= K\theta(t) = K\int_{-\infty}^{t}\omega(x)dx, \quad K = \text{spring constant}\,(\text{N}\cdot\text{m/rad})\\ \omega(t) &= \frac{1}{D}\frac{d\mathfrak{T}(t)}{dt}\end{aligned} \tag{1.46}$$

Moment of inertia

$$\begin{aligned}\mathfrak{T}(t) &= J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}, \quad J = \text{polar moment of inertia}\,(\text{kg}\cdot\text{m}^2)\\ \omega(t) &= \frac{1}{J}\int_{-\infty}^{t}\mathfrak{T}(x)dx\end{aligned} \tag{1.47}$$

For the rotation elements, we observe the following analogies between these elements and the circuit elements: the damper and the resistor, the inductor and the spring, and the mass and the moment of inertia. The current in circuits, the force in linear mechanical systems, and the torque in rotational mechanical elements are the **through variables**. The voltage in circuits, the velocity in linear mechanical systems, and the angular velocity in rotational mechanical systems are the **across variables**.

1.5 Discrete Equations and Systems

A **discrete** system is a process that relates the input discrete-time signal $x(n)$ (or $x(nT)$) with the discrete-time output signal $y(n)$ (or $y(nT)$). The elements with which we create discrete systems are shown in Figure 1.17. The special symbol z^{-1} indicates the delay which we discuss in Chapter 8. The delay element has a **memory.** This means that the output at any particular time depends on the value of the input one unit earlier. In the discrete systems, we also have pick-off points as we have defined them in the circuits case.

A simple first-order system is defined by the following discrete equation:

$$y(n) = -2y(n-1) + x(n) \tag{1.48}$$

Its block-type representation and its solution is shown in Figure 1.18. This is a recursive equation, and its solution is found by iteration, assuming (or defining), of course, its

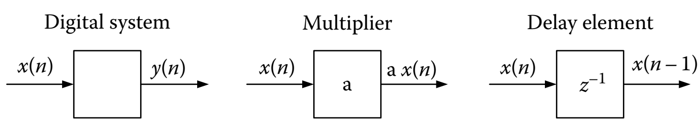


FIGURE 1.17

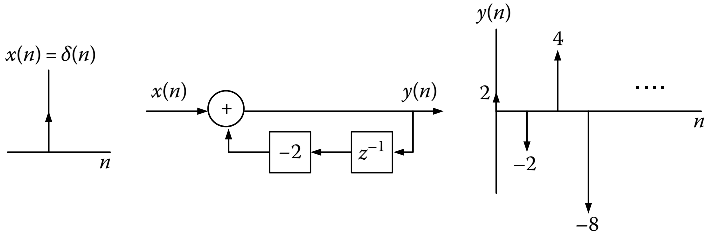


FIGURE 1.18

initial conditions. If we set $y(-1)=0$ and the input function to be a delta function, we obtain

$$
\begin{aligned}
y(0) &= -2y(-1) + \delta(0) = 0 + 1 = 1 \\
y(1) &= -2y(0) + \delta(1) = -2 + 0 = -2 \\
y(2) &= -2y(1) + \delta(2) = -2(-2) + 0 = 4 \\
y(3) &= -2y(2) + \delta(3) = -2(4) + 0 = -8 \\
&\vdots
\end{aligned}
$$

The **state** of a discrete system at a certain time n_0 is the set of values of the outputs $q_i(n-1)$ of all delay elements at $n=n_0$. Therefore, if we know the state of the system at $n=n_0$ and all its sources for $n>n_0$, then we can determine all its responses for any $n>n_0$.

The **initial state** of a system is its state at $n=0$, where this time of origin is taken for convenience. Hence, the initial state of a system is the set of values $q_i(-1)$ of the inputs $q_i(n)$ to all delay elements at $n=-1$. If the system is at the zero state, then its responses for $n>0$ are called **zero-state** responses. We therefore conclude that its responses are due only to its inputs (external sources). On the other hand, if all external sources are zero, its responses are only due to the energy sources of the system and they are called **zero-input** responses.

State equations

State variables are the inputs $q_i(n)$ to all the delay elements (or any linear transformation of these signals). The state variables are determined from the state equations resulting from the rules of the interconnected elementary systems. The **state** equations become a system of a specific number of equations equal to the number of the delay elements present in the system. To find the solution, besides the input sources, we need the **initial conditions** which are the values of the state variables at $n=-1$. It is apparent, for example, that the second-order discrete system

$$y(n) = 3.5y(n-1) + 5y(n-2) + x(n) \tag{1.49}$$

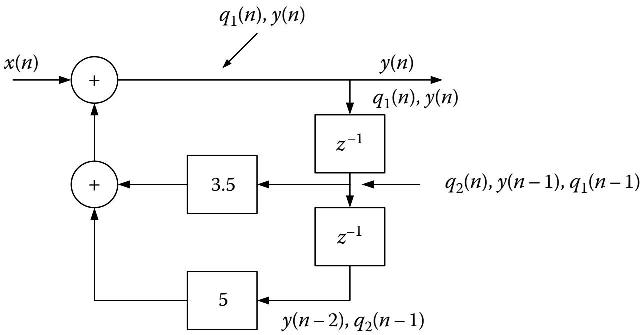


FIGURE 1.19

which is shown in the block-diagrammatic form in Figure 1.19, has the following state variables representation:

$$\begin{aligned}
&q_1(n) - 3.5q_2(n) - 5q_2(n-1) = x(n) \\
&q_2(n) - q_1(n-1) = 0 \\
&\left.\begin{aligned} q_1(-1) = y(-1) \\ q_2(-1) = y(-2) \end{aligned}\right\} \text{initial conditions}
\end{aligned} \tag{1.50}$$

Recursive and non-recursive systems

If a discrete (difference) equation, which represents the system, has one input and an output with additional delayed outputs, it is called a **recursive** one. We also call these systems **infinite impulse systems** (IIR). The difference equation of (1.49) represents a recursive system and it is shown in Figure 1.19.

If, however, we have the discrete system representation by the equation

$$y(n) = b_0x(n) + b_1x(n-1) + b_2x(n-2) \tag{1.51}$$

we say that the system is not recursive. This type of system is also called the **finite impulse response** system (FIR). Figure 1.20 shows such a system. Note the feedback configuration of the IIR systems and the forward configuration of the FIR systems.

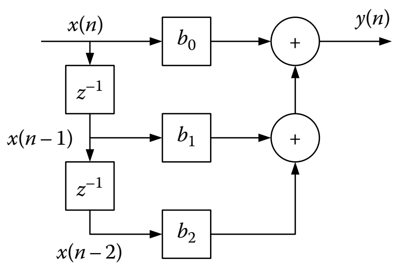


FIGURE 1.20

1.6 Digital Simulation of Analog Systems

Since the physical systems are represented mathematically by differential and integro-differential equations, we must approximate derivatives and integrals (see (1.3) and (1.6)). The approximations are derived by interrogating Figures 1.3 and 1.4. A second-order derivative is approximated in the form

$$\frac{d^2y(t)}{dt^2} \cong \frac{y(nT) - 2y(nT - T) + y(nT - 2T)}{T^2} \tag{1.52}$$

To have the solution of a second-order differential equation, we must have the value of the derivative at $t = 0$. Therefore, we must substitute the analog derivative with an equivalent discrete one. Hence, we write

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = \frac{dy(0)}{dt} \cong \frac{y(0T) - y(0T - T)}{T}$$

or

$$y(-T) = y(0) - T\frac{dy(0)}{dt} \tag{1.53}$$

1.7 Convolution of Analog Signals

The convolution operation on functions is one of the most useful operations encountered in the study of signals and systems. The importance of the convolution integral in systems studies stems from the fact that a knowledge of the output of the system to an impulse (delta) function excitation allows us to find its output to any input function (subject to some mild restrictions).

To help us develop the convolution integral, let us begin with the properties of the delta function. Based on the delta properties, we write

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau \tag{1.54}$$

Observe that, as far as the integral is concerned, the time t is a parameter (constant for the integral although it can take any value) and the integration is with respect to τ. Our next step is to represent the integral with its equivalent approximate form, the summation form, by dividing the τ axis into intervals of ΔT, then the above integral is represented approximately by the sum

$$f_a(t) = \lim_{\Delta T \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta T)\delta(t - n\Delta T)\Delta T \tag{1.55}$$

As ΔT goes to zero and n increases to infinity, the product $n\Delta T$ takes the value of τ, ΔT becomes $d\tau$ and the summation becomes integral, thus recapturing (1.54).

Note: *The function $f(t)$ has been approximated with an infinite sum of shifted delta functions equal to $n\Delta T$* and their area is equal to $f(n\Delta T)\Delta T$.

We define the response of a causal (system that reacts after being excited) and an LTI system to a delta function excitation by $h(t)$, known as the **impulse response** of the system. If the input to the system is $\delta(t)$ the output is $h(t)$, and when the input is $\delta(t-t_0)$ then the output is $h(t - t_0)$. Further, we define the output of a system by $g(t)$ if its input is $f(t)$. Based on the definitions discussed so far, it is obvious that if the input to the system is $f_a(t)$, the output is a sum of impulse functions shifted identically to the shifts of the input delta functions of the summation, and, therefore, the output is equal to

$$g(t) = \lim_{\Delta T \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta T)h(t - n\Delta T)\Delta T$$

In the limit, as ΔT approaches zero, the summation becomes an integral of the form

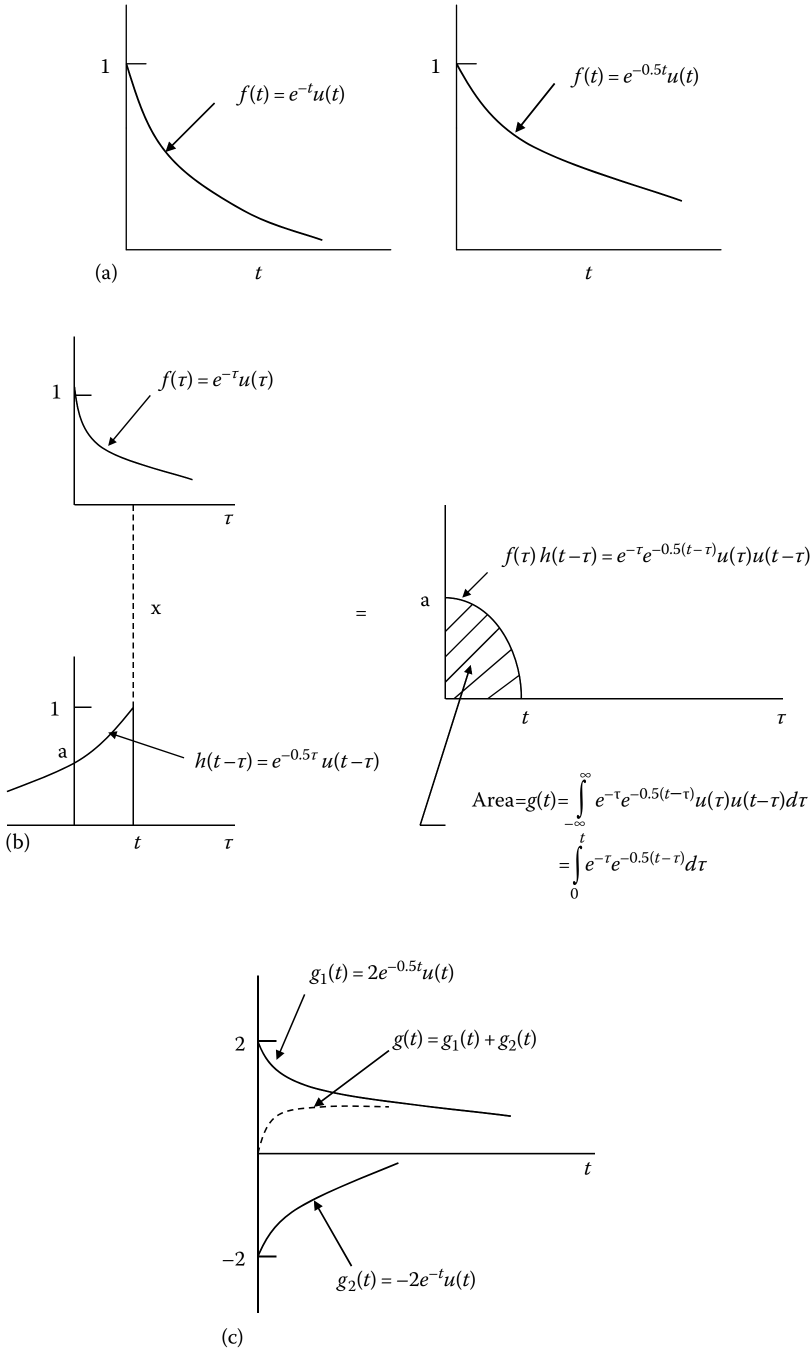
$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{1.56}$$

This is the **convolution integral** for any two functions $f(t)$ and $h(t)$.

Convolution is a general mathematical operation, and for any two real-valued functions, their convolution, indicated mathematically by the asterisk between the functions, is given by

$$\boxed{g(t) \triangleq f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau} \tag{1.57}$$

Note: *Equation* 1.57 *tells us the following: given two functions in the time domain t, we find their convolution g*(t) *by doing the following steps:* (1) *rewrite one of the functions in the τ domain by just setting wherever there is t, the variable τ; the shape of the function is identical to that in the t domain;* (2) *to the second function substitute t-τ wherever there is t; this produces a function in the τ domain which is flipped* (*the minus sign in front of τ*) *and shifted by t* (*positive values of t shift the function to the right and negative values shift the function to the left*); (3) *multiply these two functions and find another function of τ, since t is a parameter and a constant as far as the integration is concerned; and* (4) *next find the area under the product function whose value is equal to the output of the convolution at t* (*in our case it is g*(t)). *By introducing the infinite values of t's, from minus infinity to infinity, we obtain the output function g*(t).



1
$f(t) = e^{-t}u(t)$
(a)
t
1
$f(t) = e^{-0.5t}u(t)$
t
1
$f(\tau) = e^{-\tau}u(\tau)$
τ
x
=
a
$f(\tau)\,h(t-\tau) = e^{-\tau}e^{-0.5(t-\tau)}u(\tau)u(t-\tau)$
t
τ
1
a
$h(t-\tau) = e^{-0.5\tau}\,u(t-\tau)$
(b)
t
τ
Area=$g(t)=\int_{-\infty}^{\infty} e^{-\tau}e^{-0.5(t-\tau)}u(\tau)u(t-\tau)d\tau$
$=\int_{0}^{t} e^{-\tau}e^{-0.5(t-\tau)}d\tau$
$g_1(t) = 2e^{-0.5t}u(t)$
2
$g(t) = g_1(t) + g_2(t)$
t
−2
$g_2(t) = -2e^{-t}u(t)$
(c)

FIGURE 1.21

From the convolution integral, we observe that one of the functions does not change when it is mapped from the t to τ domain. The second function is reversed or folded over (mirrored with respect to the vertical axis) in the τ domain and it is shifted by an amount t, which is just a parameter in the integrand. Figure 1.21a and b shows two functions in the t and τ domains, respectively. We now write

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-0.5(t-\tau)}u(t-\tau)d\tau = \int_{0}^{t} e^{-\tau}e^{-0.5(t-\tau)}d\tau$$

$$= e^{-0.5t}\int_{0}^{t} e^{-0.5\tau}d\tau = 2(e^{-0.5t} - e^{-t})$$

Figure 1.21c shows the results of the convolution.

1.8 Convolution of Discrete Signals

As we have indicated in the above section, the convolution of continuous signals is defined as follows:

$$g(t) = \int_{-\infty}^{\infty} f(x)h(t-x)dx \qquad (1.58)$$

The above equation is approximated as follows:

$$g(t) = \int_{-\infty}^{\infty} f(x)h(t-x)dx = \sum_{m=-\infty}^{\infty} \int_{mT-T}^{mT} f(x)h(t-x)dx \cong \sum_{m=-\infty}^{\infty} Tf(mT)h(t-mT)$$

or

$$\boxed{g(nT) = T\sum_{m=-\infty}^{\infty} f(mT)h(nT-mT) \quad n = 0, \pm 1, \pm 2, \ldots \quad m = 0, \pm 1, \pm 2, \ldots} \qquad (1.59)$$

For $T = 1$, the above convolution equation becomes

$$g(n) = \sum_{m=-\infty}^{\infty} f(m)h(n-m) \quad n = 0, \pm 1, \pm 2, \ldots \quad m = 0, \pm 1, \pm 2, \ldots \qquad (1.60)$$

If the input function to the system is the delta function

$$\delta(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \quad \delta(nT - mT) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \qquad (1.61)$$

then, (1.60) gives $g(n) = h(n)$.

Additional properties of the convolution process are shown in Table 1.4.

TABLE 1.4 Convolution Properties

1. Commutative

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau$$

2. Distributive

$$g(t) = f(t) * [h_1(t) + h_2(t)] = f(t) * h_1(t) + f(t) * h_2(t)$$

3. Associative

$$[f(t) * h_1(t)] * h_2(t) = f(t) * [h_1(t) * h_2(t)]$$

4. Shift invariance

$$g(t) = f(t) * h(t)$$

$$g(t-t_0) = f(t-t_0) * h(t) = \int_{-\infty}^{\infty} f(\tau - t_0)h(t-\tau)d\tau$$

5. Area property

$$\begin{aligned} A_f &= \text{area of } f(t), \\ m_f &= \int_{-\infty}^{\infty} tf(t)dt = \text{first moment} \\ K_f &= \frac{m_f}{A_f} = \text{center of gravity} \\ A_g &= A_f A_h, \quad K_g = K_f + K_h \end{aligned}$$

6. Scaling

$$g(t) = f(t) * h(t)$$

$$f\left(\frac{t}{a}\right) * h\left(\frac{t}{a}\right) = |a| g\left(\frac{t}{a}\right)$$

7. Complex valued functions

$$g(t) = f(t) * h(t) = [f_r(t) * h_r(t) - f_i(t) * h_i(t)] + j[f_r(t) * h_i(t) + f_i(t) * h_r(t)]$$

8. Derivative

$$g(t) = f(t) * \frac{d\delta(t)}{dt} = \frac{df(t)}{dt}$$

9. Moment expansion

$$g(t) = m_{h0}f(t) - m_{h1}f^{(1)}(t) + \frac{m_{h2}}{2!}f^{(1)}(t) + \cdots + \frac{(-1)^{n-1}}{n-1!}m_{h(n-1)}f^{(n-1)}(t) + E_n$$

$$m_{hk} = \int_{-\infty}^{\infty} \tau^k h(\tau)d\tau$$

$$E_n = \frac{(-1)^n m_{hn}}{n!} f^{(n)}(t-\tau_0), \quad \tau_0 = \text{constant in the interval of integration}$$

10. Fourier transform

$$\mathcal{F}\{f(t) * h(t)\} = F(\omega)\ H(\omega)$$

(*continued*)

TABLE 1.4 (continued) Convolution Properties

11. Inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

12. Band-limited function

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \sum_{n=-\infty}^{\infty} Tf(nT)h_\sigma(t-nT)$$

$$h_\sigma(t) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} H(\omega)e^{j\omega t}\,d\omega, \quad f(t) = \sigma - \text{band limited} = 0, |t| > \sigma$$

13. Cyclical convolution

$$x(n)\otimes y(n) = \sum_{m=0}^{N-1} x(\,(n-m)\bmod N)y(m)$$

14. Discrete-time

$$x(n) * y(n) = \sum_{m=-\infty}^{\infty} x(n-m)y(m)$$

15. Sampled

$$x(nT) * y(nT) = T\sum_{m=-\infty}^{\infty} x(nT-mT)y(mT)$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}.$$

Examples

Example 1.1

It is desired to plot the functions $x(t) = -2u(2-t)$, $x(t) = u(t-1) - 2u(t-3)$, and $x(t) = 2\delta(t-1) - \delta(t+1)$. These functions are plotted in Figure E.1.1. ■

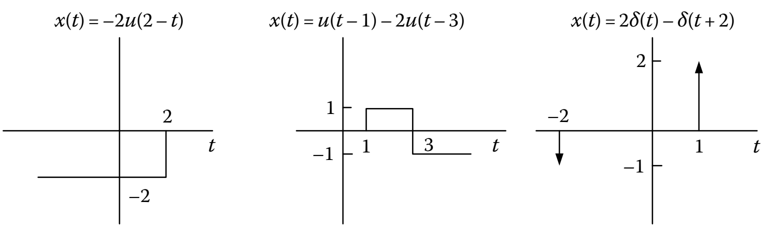


FIGURE E.1.1

Example 1.2

The evaluation of integrals, involving delta functions, is shown in the equations below:

$$\int_{-4}^{5}(t^2+2)[\delta(t)+3\delta(t-2)]dt=\int_{-4}^{5}(t^2+2)\delta(t)dt+\int_{-4}^{5}3(t^2+2)\delta(t-2)dt=2+18=20$$

$$\int_{-4}^{3}t^2[\delta(t+2)+\delta(t)+\delta(t-5)]dt=\int_{-4}^{3}t^2\delta(t+2)dt+\int_{-4}^{3}t^2\delta(t)dt+\int_{-4}^{3}t^2\delta(t-5)dt=4+0+0=4$$

■

Example 1.3

A series RLC circuit is shown in Figure E.1.2 driven by the voltage source $v(t)$. The circuit has two state variables: the capacitor voltage $v_c(t)$ and the inductor current $i(t)$, with the initial conditions $v_c(0)$ and $i(0)$. The circuit has one mesh and one mesh current that satisfies Kirchhoff's voltage law:

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_{0}^{t}i(x)dx+v_c(0)=v(t)\quad i(0) \tag{1.62}$$

We next, reduce the above integrodifferential equation into a differential equation by differentiation:

$$L\frac{d^2i(t)}{dt^2}+R\frac{d(i)}{dt}+\frac{1}{C}i(t)=\frac{dv(t)}{dt} \tag{1.63}$$

To solve (1.63), we need the initial conditions $i(0)$ and its derivative at zero time $i'(0)$ since this is an equation of the second order. The $i(0)$ is the given initial state of the

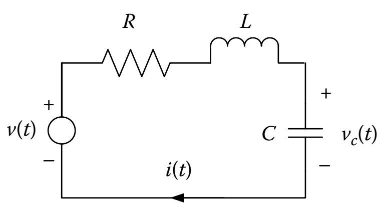


FIGURE E.1.2

system. To find the second initial condition, we must set $t=0$ in (1.62). The substitution gives

$$Li'(0) + Ri(0) + \frac{1}{C}\int_0^0 i(x)dx + v_c(0) = v(0) \quad \text{or} \quad Li'(0) + Ri(0) + v_c(0) = v(0)$$

The above equation gives the desired initial condition:

$$i'(0) = \frac{1}{L}[v(0) - v_c(0) - Ri(0)]$$

State equations

The current through the capacitor based on Kirchhoff's law is equal to the current $i(t)$. Second, the algebraic sum of the voltages in the mesh should be equal to zero. Hence,

$$\begin{aligned} C\frac{dv_c(t)}{dt} &= i(t) \qquad v_c(0) \\ L\frac{di(t)}{dt} + Ri(t) + v_c(t) &= v(t) \qquad i(0) \end{aligned} \tag{1.64}$$

and this is a system of two first-order differential equations. ∎

Example 1.4

Let the circuit (system) shown in Figure E.1.3 have the initial conditions $v_c(0)$, $i_1(0)$, and $i_2(0)$ of its state variables. To find the state equations, we sum algebraically the voltages in the two loops and the currents at the node. Hence,

State equations

$$\begin{aligned} L_1\frac{di_1(t)}{dt} + R_1 i_1(t) + v_c(t) &= v(t) \\ L_2\frac{di_2(t)}{dt} + R_2 i_2(t) - v_c(t) &= 0 \\ i_1(t) - i_2(t) - C\frac{dv_c(t)}{dt} &= 0 \end{aligned} \tag{1.65}$$

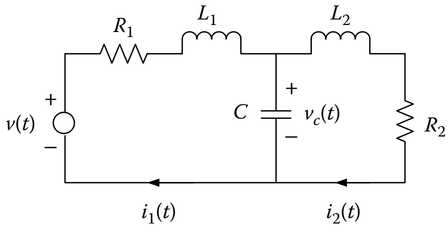


FIGURE E.1.3

Mesh equations

$$
\begin{aligned}
&R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + \frac{1}{C}\int_0^t i_1(x)dx - \frac{1}{C}\int_0^t i_2(x)dx + v_c(0) = v(t) \\
&-\frac{1}{C}\int_0^t i_1(x)dx + L_2 \frac{di_2(t)}{dt} + R_2 i_2(t) + \frac{1}{C}\int_0^t i_2(x)dx - v_c(0) = 0
\end{aligned}
\tag{1.66}
$$

■

Example 1.5

It is desired to create the block-diagram representation of the following differential equation and its equivalent discrete representation. The equation is

$$3\frac{dy(t)}{dt} - y = v(t) \tag{1.67}$$

Its discrete representation is

$$y(nT) = \frac{1}{1-\frac{T}{3}} y(nT - T) + \frac{T}{3\left(1-\frac{T}{3}\right)} v(nT) \tag{1.68}$$

The block-diagram representation of the above two equations are given in Figure E.1.4.

■

Example 1.6

It is required to find the differential equation of the linear mechanical system shown in Figure E.1.5a with respect to the distance traveled by the mass. This system is a rough representation, for example, of the spring-shock absorber system of a car. From the figure, the motion of the mass that is subjected to a spring and a damping force is described by the equation

$$-f(t) + f_M(t) + f_K(t) + f_D(t) = 0 \tag{1.69}$$

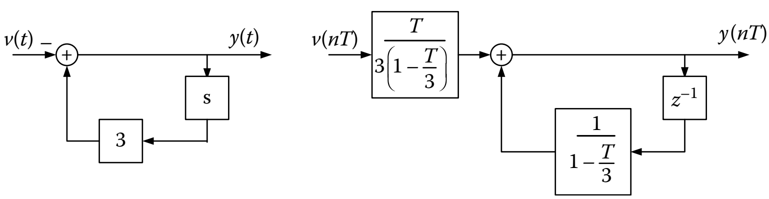


FIGURE E.1.4

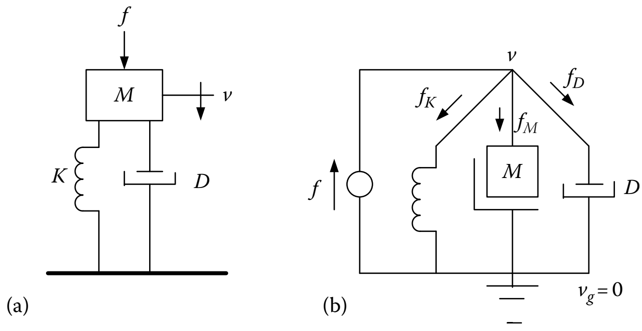


FIGURE E.1.5

or

$$M\frac{dv(t)}{dt} + K\int v(t)dt + Dv(t) = f(t) \tag{1.70}$$

Since the velocity is related to the displacement x by the relation $v(t) = dx(t)/dt$, this equation takes the form

$$\frac{d^2x(t)}{dt^2} + \frac{D}{M}\frac{dx(t)}{dt} + \frac{K}{M}x(t) = \frac{1}{M}f(t) \tag{1.71}$$

Because the velocity is an across variable, the velocity of the mass with respect to the ground, Figure E.1.5b represents the circuit representation of the system. We observe that the force is a through variable and the system is a node equivalent type circuit. ■

Example 1.7

The system shown in Figure E.1.6 represents an idealized model of a stiff human limb as a step in assessing the passive control process of locomotive action. We try to find the movement of the system if the input torque is an exponential function. During the movement, we characterize the friction by the friction constant D. Furthermore, we assume that the initial conditions are zero, $\boldsymbol{\theta}(0) = d\boldsymbol{\theta}(0)/dt = 0$.

Applying D'Alembert's principle, which requires that the algebraic sum of the torques must be equal to zero at a node, we write

$$\boxed{\mathfrak{T}(t) = \mathfrak{T}_g(t) + \mathfrak{T}_D(t) + \mathfrak{T}_J(t)} \tag{1.72}$$

where

$\mathfrak{T}(t)$ = input torque

$\mathfrak{T}_g(t)$ = gravity torque = $Mgl\sin\boldsymbol{\theta}(t)$

$\mathfrak{T}_D(t)$ = frictional torque = $D\boldsymbol{\omega}(t) = D\dfrac{d\boldsymbol{\theta}(t)}{dt}$

$\mathfrak{T}_J(t)$ = inertial torque = $J\dfrac{d\boldsymbol{\omega}(t)}{dt} = J\dfrac{d^2\boldsymbol{\theta}(t)}{dt^2}$

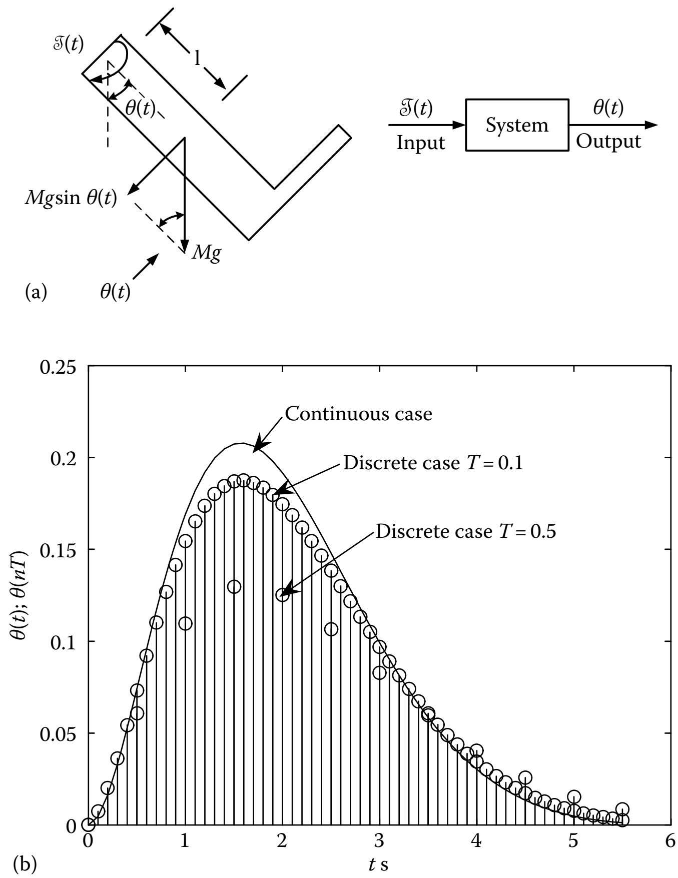


FIGURE E.1.6

Therefore, the equation that describes the system is

$$J\frac{d^2\boldsymbol{\theta}(t)}{dt^2} + D\frac{d\boldsymbol{\theta}(t)}{dt} + Mgl\sin\boldsymbol{\theta}(t) = \mathfrak{J}(t) \tag{1.73}$$

The above equation is nonlinear owing to the presence of the sin $\boldsymbol{\theta}(t)$ term in the expression of the gravity torque. To create a linear equation, we must assume that the system does not deflect much and the deflection angle stays below 30°. Under these conditions, (1.73) becomes

$$J\frac{d^2\boldsymbol{\theta}(t)}{dt^2} + D\frac{d\boldsymbol{\theta}(t)}{dt} + Mgl\boldsymbol{\theta} = \mathfrak{J}(t) \tag{1.74}$$

This is a second-order differential equation and, hence, its solution must contain two arbitrary constants, the values of which are determined from specified initial conditions. For the specific constants $J = 1$, $D = 2$, and $Mgl = 2$, the above equation becomes

$$\frac{d^2\boldsymbol{\theta}(t)}{dt^2} + 2\frac{d\boldsymbol{\theta}(t)}{dt} + 2\boldsymbol{\theta}(t) = e^{-t}u(t) \tag{1.75}$$

We must first find the homogeneous solution from the homogeneous equation (the above equation equal to zero). If we assume a solution of the form $\boldsymbol{\theta}_h(t) = Ce^{st}$, the solution requirements is

$$s^2 + 2s + 2 = 0$$

from which we find the roots $s_1 = -1 + j$ and $s_2 = -1 - j$. Therefore, the homogeneous solution is

$$\boldsymbol{\theta}_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{1.76}$$

where C_i's are arbitrary unknown constants to be found by the initial conditions.

To find the particular solution, we assume a trial solution of the form $\boldsymbol{\theta}_p(t) = Ae^{-t}$ for $t \geq 0$. Introducing the assumed solution in (1.75), we find

$$Ae^{-t} - 2Ae^{-t} + 2Ae^{-t} = e^{-t} \quad \text{or} \quad A = 1$$

The total solution is

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}_h(t) + \boldsymbol{\theta}_p(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + e^{-t} \qquad t \geq 0$$

Applying, next, the initial conditions in the above equation, we find the following system of equations:

$$\begin{aligned} \boldsymbol{\theta}(0) &= C_1 + C_2 + 1 = 0 \\ \frac{d\boldsymbol{\theta}(0)}{dt} &= C_1 s_1 + C_2 s_2 - 1 = 0 \end{aligned}$$

Solving the unknown constants, we obtain $C_1 = (1 + s_2)/(s_1 - s_2)$, $C_2 = (1 + s_1)/(s_1 - s_2)$. Introducing, next, these constants into the total solution and the two roots, we find

$$\boldsymbol{\theta}(t) = -\frac{1}{2}e^{-t}e^{jt} - \frac{1}{2}e^{-t}e^{-jt} + e^{-t} = (1 - \cos t)e^{-1} \qquad t \geq 0 \tag{1.77}$$

The digital simulation of (1.75) is deduced by employing (1.3), (1.52), and (1.53). Hence,

$$\frac{\boldsymbol{\theta}(nT) - 2\boldsymbol{\theta}(nT - T) + \boldsymbol{\theta}(nT - 2T)}{T^2} + 2\frac{\boldsymbol{\theta}(nT) - \boldsymbol{\theta}(nT - T)}{T} + 2\boldsymbol{\theta}(nT) = e^{-nT} \quad n = 0, 1.2, \ldots \tag{1.78}$$

After rearranging the above equation, we obtain

$$\theta(nT) = a(2+2T)\theta(nT-T) - a\theta(nT-2T) + aT^2e^{-nT}$$
$$a = \frac{1}{1+2T+2T^2}, \quad n = 0,1,2,\ldots \tag{1.79}$$

Using (1.53), we obtain that $\theta(-T)=0$. Next, introducing this value and the initial condition $\theta(0T)=0$ in (1.78), we find $\theta(-2T)=T^2$. The following m-file produces the desired output for the continuous case and for the two different sampling values, $T=0.5$ and $T=0.1$.

Book MATLAB® m-file for the Example 1.7: ex_1_5_1

```
%Book m-file for the Example 1.7: ex_1_5_1
t=0:0.1:5.5;
th=(1-cos(t)).*exp(-t);
T1=0.5;N1=5.5/T1;T2=0.1;N2=5.5/T2;
a1=1/(1+2*T1+2*T1^2);a2=1/(1+2*T2+2*T2^2);
thd1(2)=0;thd1(1)=T1^2;thd2(2)=0;thd2(1)=T2^2;
for n=0:N1
     thd1(n+3)=a1*(2+2*T1)*thd1(n+2)-a1*thd1(n+1)+T1^2*a1*exp(-n*T1);
end;
for n=0:N2
     thd2(n+3)=a2*(2+2*T2)*thd2(n+2)-a2*thd2(n+1)+T2^2*a2*exp(-n*T2);
end;
plot([0:55],th,'k');hold on;stem([0:5:N1*5],thd1(1,3:14),'k');
hold on;stem([0:N2],thd2(1,3:58),'k');
```
■

Example 1.8

It is desired to write the state equations for the system shown in Figure E.1.7 and express the output $y(n)$ in terms of the state variables. From the figure, we obtain

$$q_1(n) = 3q_2(n) + 2q_2(n-1) + x(n)$$
$$q_2(n) = q_1(n-1)$$
$$y(n) = 5q_1(n) + 4q_2(n)$$
■

Example 1.9

The convolution of the exponential function $f(t)=\exp(-t)u(t)$ and the pulse symmetric function $p_2(t)$ of width 4 is given by

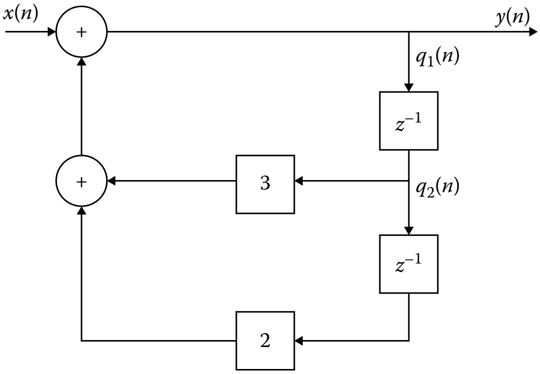


FIGURE E.1.7

$$
\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} [u(x+2) - u(x-2)]e^{-(t-x)}u(t-x)dx \\
&= e^{-t}\int_{-\infty}^{\infty} u(x+2)e^{x}u(t-x)dx - e^{-t}\int_{-\infty}^{\infty} u(x-2)e^{x}u(t-x)dx \\
&= g_1(t) + g_2(t)
\end{aligned}
$$

For $t < -2$, the exponential function and the step function $u(x+2)$ in the x-domain do not overlap and thus the integrand is zero in this range and the integral is also zero. Hence, $g_1(t) = 0$ for $t < -2$. For $t > -2$, there is an overlap from -2 to t for all ts from -2 to infinity. The integration gives

$$g_1(t) = e^{-t}\int_{-2}^{t} e^{x}dx = 1 - e^{-2}e^{-t} \qquad -2 \le t < \infty$$

For the function $g_2(t)$, the exponential function overlaps the step function $-u(x-2)$ from 2 to infinity. Hence,

$$g_2(t) = -e^{-t}\int_{2}^{t} e^{x}dx = -1 + e^{2}e^{-t} \qquad 2 \le t < \infty$$

Therefore, the function $g(t)$ is

$$g(t) = \begin{cases} 0 & t \le -2 \\ 1 - e^{-2}e^{-t} & -2 \le t \le 2 \\ (e^{2} - e^{-2})e^{-t} & 2 \le t < \infty \end{cases}$$

■

Example 1.10

The convolution of the two discrete functions $f(n) = 0.99^n u(n)$ and $h(n) = u(n-2)$ is given by

$$g(n) = \sum_{m=-\infty}^{\infty} 0.99^{n-m} u(n-m)u(m-2) = \sum_{m=2}^{n} 0.99^{n-m} = 0.99^n \sum_{m=2}^{n} 0.99^{-m}$$
$$= 0.99^n (0.99^{-2} + 0.99^{-3} + \cdots + 0.99^{-n}) = 0.99^n 0.99^{-2}(1 + 0.99^{-1} + 0.99^{-2} + \cdots + 0.99^{-n+2})$$
$$= 0.99^{n-2} \frac{1-(0.99^{-1})^{n-1}}{1-0.99^{-1}} = 0.99^{n-2} \frac{1-0.99^{-n+1}}{1-0.99^{-1}} \qquad 2 \le n < \infty$$

■

2

Fourier Series

2.1 Introduction

A **periodic** function is defined by the relation

$$
\begin{aligned}
f(t) &= f(t+T) \qquad T = \text{period} \\
f(t) &= f(t+nT) \quad\; n = \pm 1, \pm 2, \ldots
\end{aligned}
\tag{2.1}
$$

The above relation is true for all t, and the smallest T which satisfies (2.1) is called the **period**.

Knowing the function within a period $f_p(t)$, the above equation can also be written in the form

$$
f(t) = \sum_{n=-\infty}^{\infty} f_p(t-nT) \tag{2.2}
$$

An important feature of a general periodic function is that it can be presented in terms of an infinite sum of sine and cosine functions. The functions that can be expressed by sine or cosine functions must at least obey the Dirichlet conditions, which are (a) only a finite number of maximums and minimums can be present, (b) the number of discontinuities must be finite, and (c) the discontinuities must be bounded, which implies that the function must be absolutely integrable with a value less than infinity.

2.2 Fourier Series in a Complex Exponential Form

Any periodic signal $f(t)$ that satisfies the Dirichlet conditions can be expressed as follows:

$$
\boxed{
\begin{aligned}
f(t) &= \sum_{n=\infty}^{\infty} \alpha_n e^{jn\omega_0 t} = \sum_{n=\infty}^{\infty} |\alpha_n| e^{j(n\omega_0 t + \phi_n)} \qquad -\infty < t < \infty \\
\alpha_n &= \frac{1}{T} \int_a^{a+T} f(t) e^{-jn\omega_0 t}\, dt = \text{complex constant} = |\alpha_n| e^{j\phi_n} \\
&= |\alpha_n| \cos\phi_n + j|\alpha_n| \sin\phi_n \\
\omega_0 &= \frac{2\pi}{T}, \quad \phi_n = \tan^{-1}\left(\mathrm{Im}\{\alpha_n\}/\mathrm{Re}\{\alpha_n\}\right)
\end{aligned}
}
\tag{2.3}
$$

If the function is discontinuous at $t=a$, the function $f(t)$ will converge to $f(a)=[f(a+)+f(a-)]/2$, the mean value at the point of discontinuity (the arithmetic mean of the left-hand and right-hand limits). If $f(t)$ is real, then

$$\alpha_{-n}=\frac{1}{T}\int_{a}^{a+T} f(t)e^{jn\omega_0 t}dt=\left[\frac{1}{T}\int_{a}^{a+T} f(t)e^{-jn\omega_0 t}dt\right]^{*}=\alpha_n^{*} \tag{2.4}$$

This result, when combined with (2.3), yields

$$f(t)=\alpha_0+\sum_{n=1}^{\infty}[(\alpha_n+\alpha_n^{*})\cos n\omega_0 t+j(\alpha_n-\alpha_n^{*})\sin n\omega_0 t] \tag{2.5}$$

2.3 Fourier Series in Trigonometric Form

The trigonometric form of the Fourier series is given by

$$\boxed{\begin{aligned}
&f(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\omega_0 t+B_n\sin n\omega_0 t)\quad -\infty<t<\infty\\
&f(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}C_n\cos(n\omega_0 t+\phi_n)\\
&A_0=2\alpha_0=\frac{2}{T}\int_{a}^{a+T}f(t)dt\\
&A_n=(\alpha_n+\alpha_n^{*})=\frac{2}{T}\int_{a}^{a+T}f(t)\cos n\omega_0 t\,dt=C_n\cos\phi_n\\
&B_n=j(\alpha_n-\alpha_n^{*})=\frac{2}{T}\int_{a}^{a+T}f(t)\sin n\omega_0 t\,dt=-C_n\sin\phi_n\\
&\phi_n=-\tan^{-1}(B_n/A_n)\\
&C_n=\left(A_n^2+B_n^2\right)^{1/2}
\end{aligned}} \tag{2.6}$$

The coefficients C_n are known as the **amplitude spectrum** and the phase ϕ_n is the **phase spectrum**. Therefore, the frequency spectrum of a periodic function is discrete.

2.3.1 Differentiation of the Fourier Series

If $f(t)$ is continuous in $-T/2\le t\le T/2$ with $f(-T/2)=f(T/2)$, and if its derivative $f'(t)$ is piecewise continuous and differentiable, then the trigonometric form of the Fourier series can be differentiated term by term to yield

$$f'(t)=\sum_{n=1}^{\infty}n\omega_0(-A_n\sin n\omega_0 t+B_n\cos n\omega_0 t) \tag{2.7}$$

2.3.2 Integration of the Fourier Series

If $f(t)$ is piecewise continuous in $-T/2 < t < T/2$, then the trigonometric form of the Fourier series can be integrated term by term to yield

$$\int_{t_1}^{t_2} f(t)dt = \frac{1}{2}A_0(t_2 - t_1) + \sum_{n=1}^{\infty} \frac{1}{n\omega_0}[-B_n(\cos n\omega_0 t_2 - \cos n\omega_0 t_1) + A_n(\sin n\omega_0 t_2 - \sin n\omega_0 t_1)] \tag{2.8}$$

2.4 Waveform Symmetries

Even function $[f(t) = f(-t)]$

If $f(t)$ is an even periodic function with a period T, then the trigonometric form of the Fourier series is

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega_0 t, \quad A_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega_0 t\, dt \tag{2.9}$$

Odd function $[f(t) = -f(-t)]$

If $f(t)$ is an odd function, then its trigonometric form is

$$f(t) = \sum_{n=1}^{\infty} B_n \sin n\omega_0 t, \quad B_n = \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega_0 t\, dt \tag{2.10}$$

2.5 Some Additional Features of Periodic Continuous Functions

2.5.1 Power Content: Parseval's Theorem

The power content of a periodic function $f(t)$ in the period T is defined as the mean-square value:

$$\frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2 dt \tag{2.11}$$

If we assume the function as a voltage across an ohm resistor, then (2.11) represents the average power the source delivers to the resistor.

If $f(t)$ and $h(t)$ are two periodic functions with the same period T, then

$$\frac{1}{T}\int_{-T/2}^{T/2} f(t)h(t)dt = \sum_{n=-\infty}^{\infty} (\alpha_f)_n(\alpha_h)_{-n} \tag{2.12}$$

where

$$(\alpha_f)_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt, \quad (\alpha_h)_n = \frac{1}{T}\int_{-T/2}^{T/2} h(t)e^{-jn\omega_0 t}dt \tag{2.13}$$

If $f(t) = h(t)$, then the power content of the periodic function $f(t)$ is

$$\frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2 dt = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \quad \alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt \tag{2.14}$$

For a periodic function expanded in sine and cosine terms, the power content within a period is

$$\boxed{\frac{1}{T}\int_{-T/2}^{T/2} [f(t)]^2 dt = \frac{1}{4}A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)} \tag{2.15}$$

2.5.2 Output of an LTI System When the Input Is a Periodic Function

If the input periodic function is represented by the complex format of the Fourier series, then the output of an LTI system with a transfer function $H(\omega)$ is

$$f_o(t) = \sum_{n=-\infty}^{\infty} \alpha_n H(n\omega_0)e^{jn\omega_0 t} \tag{2.16}$$

If the input to an LTI system is a periodic signal in the form of sine and cosine series, then the output is

$$\boxed{\begin{aligned} f_o(t) &= \frac{A_0}{2}H(0) + \sum_{n=1}^{\infty} |H(n\omega_0)|[A_n \cos[n\omega_0 t + \phi(n\omega_0)] + B_n \sin[n\omega_0 t + \phi(n\omega_0)]] \\ \phi(n\omega_0) &= \tan^{-1}(\mathrm{Im}\{H(n\omega_0)\}/\mathrm{Re}\{H(n\omega_0)\}) \end{aligned}} \tag{2.17}$$

2.5.3 Transmission without Distortion

If an LTI system has a transfer function of the form

$$H(\omega) = h_0 e^{jn\omega_0 t_0} \tag{2.18}$$