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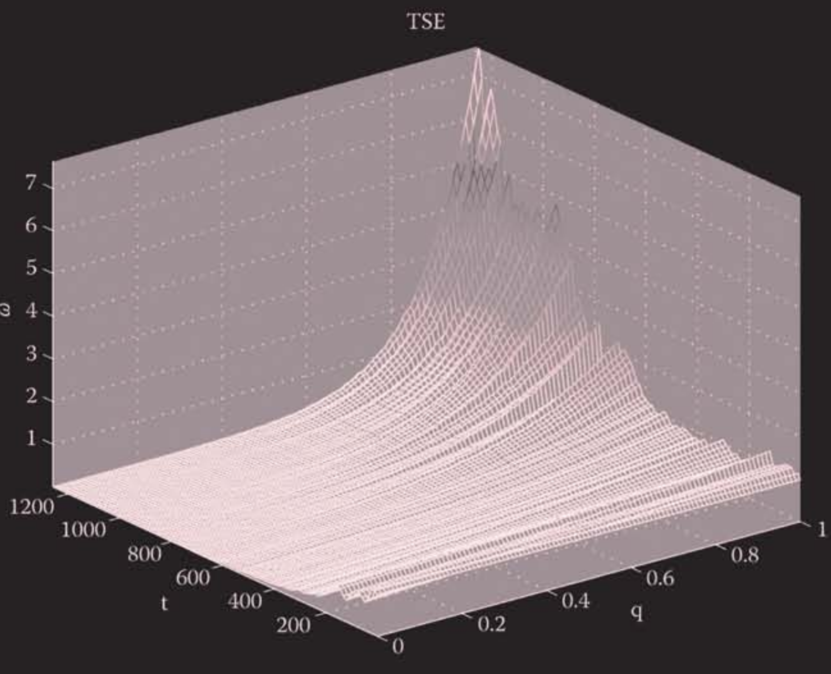
M. A. H. Dempster, Georg Pflug, and Gautam Mitra

Quantitative Fund Management



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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20131121

International Standard Book Number-13: 978-1-4200-8192-3 (eBook - PDF)

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Professor Mitra, prior to his academic career at Brunel University, worked with SCICON, SIA and ICL; in all these organisations he was involved in the (software) development of mathematical programming optimisation and modelling systems. He was the director of advanced NATO Advanced Study Research Institute, which took place in Val-d-Isere (1987) on the topic of Mathematical Models for Decision Support. Professor Mitra has also had substantial involvement with industry-based projects in his role as a director of UNICOM Seminars and OptiRisk Systems. He is the author of two textbooks, four edited books and over ninety-five journal papers. His Web sites are http://www.carisma.brunel.ac.uk and http://www.optirisk-systems.com

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Introduction to Quantitative Fund Management

AT THE TENTH TRIENNIAL INTERNATIONAL CONFERENCE on stochastic programming held at the University of Arizona in October 2004, it was observed that the fund management industry as a whole was far from the leading edge of research in financial planning for asset allocation, asset liability management, debt management and other financial management problems at the strategic (long term) level. This gap is documented in the timely survey of quantitative equity management by Fabozzi, Focardi and Jonas which forms the first chapter of this book. It was therefore agreed to bring out a special issue of *Quantitative Finance* to partially address the imbalance between research and practice by showcasing leading edge applicable theory and methods and their use for practical problems in the industry. A call for papers went out in August and October of 2005. As an outcome of this, we were able to compile a first special issue with the papers forming the ten chapters in Part 1 of this book. In fact, the response to the call was so good that a second special issue focusing on tactical financial planning and risk management is contained in the ten chapters of Part 2.

Taken together, the twenty chapters of this volume constitute the first collection to cover quantitative fund management at both the dynamic strategic and one period tactical levels. They consider optimal portfolio choice for wealth maximization together with integrated risk management using axiomatically defined risk measures. Solution techniques considered include novel applications to quantitative fund management of stochastic control, dynamic stochastic programming and related optimization techniques. A number of chapters discuss actual implemented solutions to fund management problems including equity trading, pension funds, mortgage funding and guaranteed investment products. All the contributors are well known academics or practitioners. The remainder of this introduction gives an overview of their contributions.

In Part I of the book on *dynamic financial planning* the survey by Fabozzi *et al.* (Chapter 1) finds that, at least in the equity world, the interest in quantitative techniques is shifting from basic Markowitz mean-variance portfolio optimization to risk management and trading applications. This trend is represented here with the chapter by Fagiuoli, Stella and Vetura (Chapter 5). The remaining chapters in Part 1 cover novel aspects of lifetime individual consumption investment problems, fixed mix portfolio rebalancing allocation strategies (including Cover-type universal portfolios), debt management for funding mortgages and national debt, and guaranteed return fund construction. Of the ten chapters in Part 1, one is the mentioned survey, three are theoretical, two concern proofs of concept for practical trading or fund management strategies and the remaining four concern real-world implementations for major financial institutions.

Chapter 2 by Pirvu expands on the classical consumption investment problem of Merton to include a value-at-risk constraint. The portfolio selection problem over a finite horizon is a stochastic control problem which is reduced to pathwise nonlinear optimization through the use of the stochastic Pontryagin maximum principal. Numerical results are given and closed form solutions obtained for special cases such as logarithmic utility. The third chapter by Hsuku extends the classical Merton problem in a different direction to study the positive effects of adding derivatives to investors' choices. The model utilizes a recursive utility function for consumption and allows predictable variation of equity return volatility. Both of these theoretical studies concern realistically incomplete markets in which not all uncertainties are priced.

The next three chapters mainly treat variants of the fixed-mix rebalance dynamic asset allocation strategy. The first of these (Chapter 4) by Dempster, Evstigneev and Schenk-Hoppé shows under very general stationary ergodic return assumptions that such a strategy, which periodically rebalances a portfolio to fixed proportions of the current portfolio value, grows exponentially on almost every path even in the presence of suitable transactions costs. Chapter 5 in this group by Fagiuoli, Stella and Ventura develops, and tests on stock data from four major North American indices, an online algorithm for equity trading based on Cover's non-parametric universal portfolios in the situation when some market state information is also available. Chapter 6 by Mulvey, Ural and Zhang discusses return enhancing additions to both fixed mix rebalance strategies and optimal dynamic allocation strategies obtained by dynamic stochastic programming in the context of work for the U.S. Department of Labor. In particular, positive return performance is demonstrated from diversification to non-traditional asset classes, leverage, and overlay strategies which require no investment capital outlay.

The next two chapters concern debt management problems which use dynamic stochastic programming to optimally fund mortgage lending and government spending requirements respectively. These are asset liability management problems in which assets are specified and decisions focus on liabilities, namely, when and how to issue bonds. The first, Chapter 7 by Infanger, is an exemplary study conducted for Freddie Mac which shows that significant extra profits can be made by employing dynamic models relative to static Markovitz mean-variance optimization or traditional duration and convexity matching of assets (mortgage loans) and liabilities (bonds). In addition, efficient out-of-sample simulation evaluation of the robustness of the recommended optimal funding strategies is described, but not historical backtesting. Chapter 8 by Bernaschi, Briani, Papi and Vergni concentrates on yield curve modelling for a dynamic model for funding Italian public debt by government bond issuance. The idea of this contribution, important in an EU context, is to model the basic ECB yield curve evolution together with an orthogonal national idiosyncratic component.

The last two chapters of Part 1 describe the use of dynamic stochastic programming techniques to design guaranteed return pension funds which employ dynamic asset allocations to balance fund return versus guarantee shortfall. Chapter 9 by Hertzog, Dondi, Keel, Schumann and Geering treats this asset liability management problem using a deterministic evolution of the guarantee liability, while Chapter 10 by Dempster, Germano, Medova, Rietbergen, Sandrini and Scrowston treats guarantee shortfall with respect to a stochastic liability which is evaluated from the forward ECB yield curve simulation used to price bonds in the dynamic portfolio. Both chapters employ historical backtesting of their models for respectively a hypothetical Swiss pension fund and (a simplified version of) actual funds backing guaranteed return products of Pioneer Investments.

Taken together, the ten chapters of Part 1 give a current snapshot of state-of-the-art applications of dynamic stochastic optimization techniques to long term financial planning. These techniques range from new pathwise (Pirvu) and standard dynamic programming (Hsuku) methods of stochastic control, through sub-optimal, but easily understood and implemented policies (Dempster *et al.*, Fagiouli *et al.*, Mulvey *et al.*) to dynamic stochastic programming techniques involving the forward simulation of many risk factors (Mulvey *et al.*, Infanger, Bernaschi *et al.*, Hertzog *et al.*, Dempster *et al.*). Although there is currently widespread interest in these approaches in the fund management industry, more than a decade after their commercial introduction they are still in the early stages of adoption by practitioners, as the survey of Fabozzi *et al.* shows. This volume will hopefully contribute to the recognition and wider acceptance of stochastic optimization techniques in financial practice.

Part 2 of this volume on *portfolio construction and risk management* concerns the *tactical* level of financial planning. Most funds, with or without associated liabilities—and explicitly or implicitly—employ a three level hierarchy for financial planning. The top *strategic* level considers asset classes and risk management over longer term horizons and necessarily involves *dynamics* (the topic of Part 1). The middle *tactical* level of the financial planning hierarchy concerns portfolio construction and risk management at the individual security or fund manager level over the period up to the next portfolio rebalance. This is the focus of the ten contributions of the second part of the book. The third and bottom *operational* level of the financial planning hierarchy is actual *trading* which, with the rise of hedge funds, and as the survey of quantitative equity management by Fabozzi *et al.* in Chapter 1 demonstrates, is becoming increasingly informed by tactical models and considerations beyond standard Markowitz *mean-variance optimization* (MVO). This interaction is the evident motivation for many of the chapters in Part 2 with their emphasis on non-Gaussian returns, new risk-return tradeoffs and robustness of benchmarks and portfolio decisions. The first two chapters are based on insights gained from actual commercial applications, while of the remaining eight chapters all but one, which is theoretically addressing an important practical issue, test new theoretical contributions on market data. Another theme of *all* the contributions in this part is that their concern is with techniques which are *scenario*—rather than *analytically—based* (although the purely theoretical chapter uses a limiting analytical approximation). This theme reflects the necessity for nontrivial computational approaches when the classical independent Gaussian return paradigm is set aside in favour of non-equity instruments and shorter term (e.g. daily or weekly) returns.

The first chapter of Part 2, Chapter 11 by Dempster, Germano, Medova, Rietbergen, Sandrini, Scrowston and Zhang treats the problem of benchmarking fund performance using *optimal* fixed mix rebalancing strategies (a theme of Part 1) and tests it relative to earlier work on optimal portfolios for guaranteed return funds described in Chapter 10.

Chapter 12 by Acerbi provides a timely and masterful survey of the recent literature on coherent risk measures, including practical linear programming models for portfolios constructed by their minimization. This theme is elaborated further by Krokhmal in Chapter 13 which treats higher moment coherent risk measures. It examines their theoretical properties and performance when used in portfolio construction relative to standard mean variance and expected shortfall *conditional value at risk* (CVaR) optimization.

The next three chapters treat the robustness properties of the numerical minimization of CVaR using linear programming as employed in practice, for example, for bond portfolios. The first, Chapter 14 by Ciliberti, Kondor and Mezard, uses limiting continuous approximations suggested by statistical physics to define a critical threshold for the ratio of the number of assets to the number of historical observations beyond which the expected shortfall (CVaR) risk measure is not well-defined—a phase-change phenomenon first noted by Kondor and co-authors. Next Kaut, Vladimirou, Wallace and Zenios examine in Chapter 15 the stability of portfolio solutions to this problem with respect to estimation (from historical data) errors. They conclude that sensitivity to estimation errors in the mean, volatility, skew and correlation all have about the same non-negligible impact, while error in kurtosis has about half that of the other statistics. Finally, Chapter 16 by Dupačova and Polívka discusses stress-testing the CVaR optimization problem using the contamination scenario technique of perturbation analysis. They also show that similar techniques may be applied to the minimal analytical *value at risk* (VaR) problem for the Gaussian case, but are not applicable to the corresponding historical scenario based problem.

The next group of three chapters extend the treatment of portfolio construction and risk management beyond the usual simple tradeoff of volatility risk and return embodied in MVO. Chapter 17 by Giacometti, Bertocchi, Rachev and Fabozzi shows that the Black-Litterman Bayesian approach to portfolio construction, incorporating both market and practitioner views, can be extended to Student-t and stable return distributions and VaR and CVaR risk measures. Pflug and Wozabal consider in Chapter 18 the *robust optimization* problem of finding optimal portfolios in the Knightian situation when the distributions underlying returns are not perfectly known. They develop and test an algorithm for this situation based on two level convex optimization. In the last chapter in this group, Chapter 19, Roman, Darby-Dowman and Mitra consider the multi-objective problem of simultaneously trading off expected return with *two* risk measures based on variance and expected shortfall (CVaR). In tests with FTSE 100 index securities they find that an optimal balance with the two risk measures dominates those using either alone.

The final chapter in Part 2, Chapter 20 by Charalambous, Christofides, Constantinide and Martzoukos, treats the basic requirement for pricing exotic and over-the-counter options—fitting vanilla option market price data—using *non-recombining* (binary) trees, a special case of the multi-period scenario trees used in Part 1 for strategic portfolio management. The authors' approach dominates the usual recombining tree (lattice) in that it can easily handle transactions costs, liquidity constraints, taxation, non-Markovian dynamics, etc. The authors demonstrate its practicality using a penalty method and quasi-Newton unconstrained optimization and its excellent fit to the volatility surface—crucial for hedging and risk control.

The ten chapters of Part 2 provide an up-to-date overview of current research in tactical portfolio construction and risk management. Their emphasis on general return distributions and tail risk measures is appropriate to the increasing penetration of hedge fund trading techniques into traditional fund and asset liability management. We hope that this treatment of tactical problems (and its companion strategic predecessor) will make a valuable contribution to the future practical use of systematic techniques in fund management.

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Cambridge, London & Vienna

PART 1

Dynamic Financial Planning

CHAPTER 1

Trends in Quantitative Equity Management: Survey Results

FRANK J. FABOZZI, SERGIO FOCARDI and CAROLINE JONAS

CONTENTS

1.1 INTRODUCTION

IN THE SECOND HALF OF THE 1990S, there was so much skepticism about quantitative fund management that Leinsweber (1999), a pioneer in applying advanced techniques borrowed from the world of physics to fund management, wrote an article entitled: 'Is quantitative investment dead?' In the article, Leinweber defended quantitative fund management and maintained that in an era of ever faster computers and ever larger databases, quantitative investment was here to stay. The skepticism towards quantitative fund management, provoked by the failure of some high-profile quantitative funds, was related to the fact that investment professionals felt that capturing market inefficiencies could best be done by exercising human judgement.

Despite mainstream academic theory that had held that markets are efficient and unpredictable, the asset managers' job has always been to capture market inefficiencies for their clients. At the academic level, the notion of efficient markets has been progressively relaxed. Empirical evidence that began to be accumulated in the 1970s led to the gradual acceptance of the notion that financial markets are somewhat predictable and that systematic market inefficiencies can be detected (see Granger 1992 for a review to various models that accept departures from efficiency). Using the variance ratio test, Lo and MacKinlay (1988) disproved the random walk hypothesis. Additional insights return predictability was provided by Jegadeesh and Titman (1993), who established the existence of momentum phenomena. Since then, a growing number of studies have accumulated evidence that there are market anomalies that can be systematically exploited to earn excess profits after considering risk and transaction costs (see Pesaran 2005 for an up-to-date presentation of the status of market efficiency). Lo (2004) proposed replacing the Efficient Market Hypothesis with the Adaptive Market Hypothesis arguing that market inefficiencies appear as the market adapts to changes in a competitive environment.

The survey study described in this paper had as its objective to reveal to what extent the growing academic evidence that asset returns are predictable and that predictability can be exploited to earn a profit have impacted the way equity assets are being managed. Based on an Intertek 2003 survey on a somewhat different sample of firms, Fabozzi *et al.* (2004) revealed that models were used primarily for risk management, with many firms eschewing forecasting models. The 2006 survey reported in this chapter sought to reveal to what extent modelling has left the risk management domain to become full-fledged asset management methodology. Anticipating the results discussed below, the survey confirms that quantitative fund management is now an industrial reality, successfully competing with traditional asset managers for funds. Milevsky (2004) observes that the methods of quantitative finance have now been applied in the field of personal wealth management.

We begin with a brief description of the field research methodology and the profile of responding firms. Section 1.3 discusses the central finding, that is, that models are being used to manage an increasing amount of equity asset value. Section 1.4 discusses the changing role of modelling in equity portfolio management, from decision-support systems to a fully automated portfolio construction and trading system, and from passive management to active management. Section 1.5 looks at the forecasting models most commonly used in the industry and discusses the industry's evaluation of the techniques. Section 1.6 looks at the use (or lack of use) of high-frequency data and the motivating factors. Section 1.7 discusses risk measures being used and Section 1.8 optimization methodologies. The survey reveals a widespread use of optimization, which is behind the growing level of automation in fund management. The wide use of models has created a number of challenges: survey respondents say that differentiating quantitative products and improving on performance are a challenge. Lastly, in looking ahead, we discuss the issue of the role of models in market efficiency.

1.2 METHODOLOGY

The study is based on survey responses and conversations with industry representatives in 2006. In all, managers at 38 asset management firms managing a total of €3.3 trillion ($4.3 trillion) in equities participated in the survey. Participants include persons responsible for quantitative equity management and quantitative equity research at large and medium-sized firms in North America and Europe.

The home market of participating firms is 15 from North America (14 from U.S. 1 from Canada) and 23 from Europe (U.K. 7, Germany 5, Switzerland 4, Benelux 3, France 2 and Italy 2). Equities under management by participating firms range from €5 bn to €800 bn.

While most firms whose use of quantitative methods is limited to performance analysis or risk measurement declined to participate in this study (only 5 of the 38 participating firms reported no equity funds under quantitative management), the study does reflect the use of quantitative methods in equity portfolio management at firms managing a total of €3.3 trillion ($4.3 trillion) in equities; 63% of the participating firms are among the largest asset managers in their respective countries. It is fair to say that these firms represent the way a large part of the industry is going with respect to the use of quantitative methods in equity portfolio management. (Note that of the 38 participants in this survey, 2 responded only partially to the questionnaire. For some questions, there are therefore 36 (not 38) responses.)

1.3 GROWTH IN EQUITY ASSETS UNDER QUANTITATIVE MANAGEMENT

The skepticism relative to the future of quantitative management at the end of the 1990s has given way and quantitative methods are now playing a large role in equity portfolio management. Twenty-nine percent (11/38) of the survey participants report that more than 75% of their equity assets are being managed quantitatively. This includes a wide spectrum of firms, with from €5 billion to over €500 billion in equity assets under management. Another 58% (22/38) report that they have some equities under quantitative management, though for most of these (15/22) the percentage of equities under quantitative management is less than 25%—often under 5%—of total equities under management. Thirteen percent (5/38) report no equities under quantitative management. Figure 1.1 represents the distribution of percentage of equities under quantitative management at different intervals for responding firms.

Relative to the period 2004–2005, the amount of equities under quantitative management has grown at most firms participating in the survey. Eighty-four percent of the respondents (32/38) report that the percentage of equity assets under quantitative management has either increased with respect to 2004–2005 (25/38) or has remained stable at about 100% of equity assets (7/38). The percentage of equities under quantitative management was down at only one firm and stable at five.

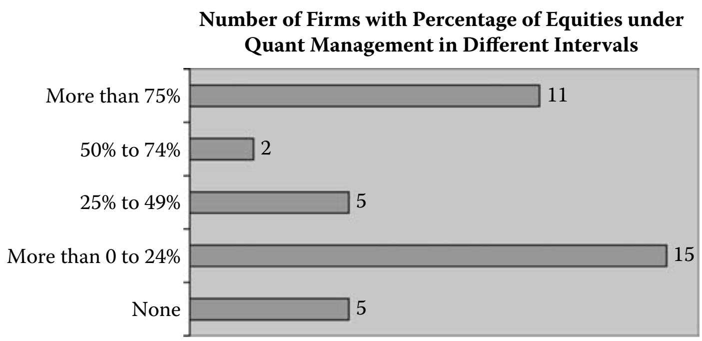

FIGURE 1.1 Distribution of the percentage of equities under quant management.

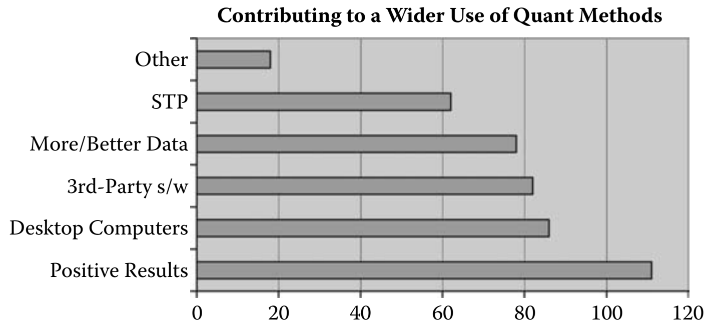

FIGURE 1.2 Score attributed to each factor contributing to a wider use of quant methods.

One reason given by respondents to explain the growth in equity assets under quantitative management is the flows into existing quantitative funds. A source at a large U.S. asset management firm with more than 50% of its equities now under quantitative management said, 'The firm has three distinct equity products: value, growth and quant. Quant is the biggest and is growing the fastest.' The trend towards quantitative management is expected to continue.

According to survey respondents, the most important factor contributing to a wider use of quantitative methods in equity portfolio management is the positive result obtained with these methods. Half of the participants rated positive results as the single most important factor contributing to the widespread use of quantitative methods. Other factors contributing to a wider use of quantitative methods in equity portfolio management are, in order of importance attributed to them by participants, the computational power now available on the desktop, more and better data, and the availability of third-party analytical software and visualization tools. Figure 1.2 represents the distribution of the score attributed to each factor. Participants were asked to rate from 1 to 5 in order of importance, 5 being the most important. Given the sample of 36 firms that responded, the maximum possible score is 180.

Sources identified the prevailing in-house culture as the most important factor holding back a wider use of quantitative methods (this evaluation obviously does not hold for firms that can be described as quantitative): more than one third (10/27) of the respondents at other than quant-oriented firms considered this the major blocking factor. Figure 1.3 represents the distribution of the total score attributed to each factor.

The positive evaluation of models in equity portfolio management is in contrast with the skepticism of some 10 years ago. A number of changes have occurred. First, expectations are now more realistic. In the 1980s and 1990s, traders were experimenting with methodologies from advanced science in hopes of making huge excess returns. Experience of the last 10 years has shown that models can indeed deliver but that their performance must be compatible with a well-functioning market.[1]

[1] There was a performance decay in quantitatively managed equity funds in 2006–2007. Many attribute this decaying performance to the fact that there are now more portfolio managers using the same factors and the same data.

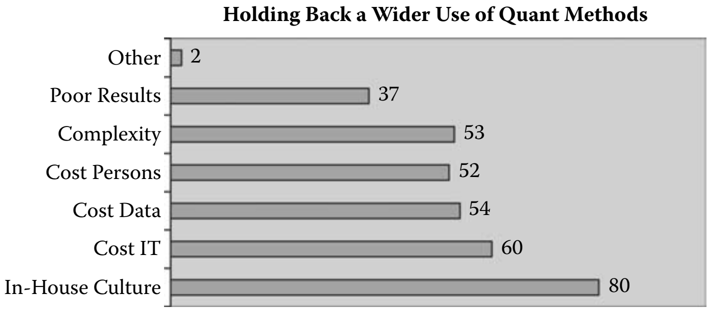

FIGURE 1.3 Score attributed to each factor holding back a wider use of quant methods.

Other technical reasons include a manifold increase in computing power and more and better data. Modellers have now available on their desktop computing power that, at the end of the 1980s, could be got only from multimillion dollar supercomputers. Data, including intraday data, can now be had (though the cost remains high) and are in general 'cleaner' and more complete. Current data include corporate actions, dividends, and fewer errors—at least in developed-country markets.

In addition, investment firms (and institutional clients) have learned how to use models throughout the investment management process. Models are now part of an articulated process that, especially in the case of institutional investors, involves satisfying a number of different objectives, such as superior information ratios.

1.4 CHANGING ROLE FOR MODELS IN EQUITY PORTFOLIO MANAGEMENT

The survey reveals that quantitative models are now used in active management to find alphas (i.e. sources of excess returns), either relative to a benchmark or absolute. This is a considerable change with respect to the past when quantitative models were used primarily to manage risk and to select parsimonious portfolios for passive management.

Another finding of this study is the growing amount of funds managed automatically by computer programs. The once futuristic vision of machines running funds automatically without the intervention of a portfolio manager is becoming a reality on a large scale: 55% of the respondents (21/38) report that at least part of their equity assets are now being managed automatically with quantitative methods; another three plan to automate at least a portion of their equity portfolios within the next 12 months. The growing automation of the equity investment process indicates that that there is no missing link in the technology chain that leads to automatic quantitative management. From return forecasting to portfolio formation and optimization, all the needed elements are in place.

Until recently, optimization represented the missing technology link in the automation of portfolio engineering. Considered too brittle to be safely deployed, many firms eschewed optimization, limiting the use of modelling to stock ranking or risk control functions. Advances in robust estimation methodologies and in optimization now allow a manager to construct portfolios of hundreds of stocks chosen in universes of thousands of stocks with little or no human intervention outside of supervising the models.

1.5 MODELLING METHODOLOGIES AND THE INDUSTRY'S EVALUATION

At the end of the 1980s, academics and researchers at specialized quant boutiques experimented with many sophisticated modelling methodologies including chaos theory, fractals and multi-fractals, adaptive programming, learning theory, complexity theory, complex nonlinear stochastic models, data mining and artificial intelligence. Most of these efforts failed to live up to expectations. Perhaps expectations were too high. Or perhaps the resources or commitment required were lacking. Derman (2001) provides a lucid analysis of the difficulties that a quantitative analyst has to overcome. As observed by Derman, though modern quantitative finance uses some of the techniques of physics, a wide gap remains between the two disciplines.

The modelling landscape revealed by the survey is simpler and more uniform. Regression analysis and momentum modelling are the most widely used techniques: respectively, 100% and 78% of the survey respondents say that these techniques are being used at their firms. Other modelling methods being widely used include cash flow analysis and behavioural modelling. Forty-seven percent (17/36) of the participating firms model cash flows; 44% (16/36) use behavioural modelling. Figure 1.4 represents the distribution of modelling methodologies among participants.

Let us observe that regression models used today have undergone a substantial change since the first multifactor models such as Arbitrage Pricing Theory (APT) were introduced. Classical multifactor models such as APT are static models embodied in linear regression between returns and factors at the same time:

$$r_i = \alpha_i + \sum_{j=1}^{p} \beta_{ij} f_j + \varepsilon_i.$$

Models of this type allow managers to measure risk but not to forecast returns, unless the factors are forecastable. Sources at traditional asset management firms typically use factor models to control risk or build stock screening systems. A source doing regression on factors to capture the risk-return trade-off of assets said, 'Factor models are the most intuitive and most comprehensive models for explaining the sources of risk.'

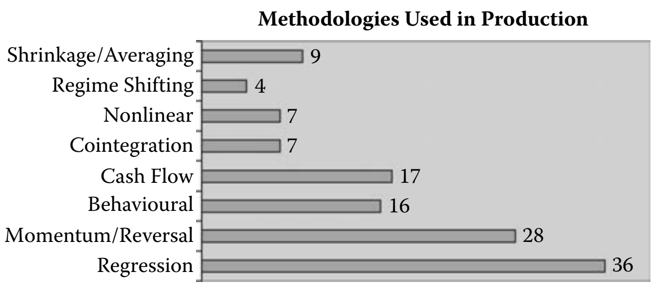

FIGURE 1.4 Distribution of modelling methodologies among participants.

However, modern regression models are dynamic models where returns at time $t+1$ are regressed on factors at time t:

$$r_{i,t+1} = \alpha_i + \sum_{j=1}^{p} \beta_{ij} f_{j,t} + \varepsilon_{i,t}.$$

Models of this type are forecasting models insofar as the factors at time t are predictors of returns at time behaviour $t+1$. In these models, individual return processes might exhibit zero autocorrelation but still be forecastable from other variables.

Predictors might include financial and macroeconomic factors as well as company specific parameters such as financial ratios. Predictors might also include human judgment, for example analyst estimates, or technical factors that capture phenomena such as momentum. A source at a quant shop using regression to forecast returns said, 'Regression on factors is the foundation of our model building. Ratios derived from financial statements serve as one of the most important components for predicting future stock returns. We use these ratios extensively in our bottom-up equity model and categorize them into five general categories: operating efficiency, financial strength, earnings quality (accruals), capital expenditures and external financing activities.'

Momentum and reversals are the second most widely used modelling technique among survey participants. In general, momentum and reversals are used as a strategy not as a model of asset returns. Momentum strategies are based on forming portfolios choosing the highest/lowest returns, where returns are estimated on specific time windows. Survey participants gave these strategies overall good marks but noted that (1) they do not always perform so well, (2) they can result in high turnover (though some use constraints/penalties to deal with this problem) and (3) identifying the timing of reversals is tricky.

Momentum was first reported in Jegadeesh and Titman (1993) in the U.S. market. Jegadeesh and Titman (2002) confirm that momentum continued to exist in the 1990s in the US market throughout the 1990s. Karolyi and Kho (2004) examined different models for explaining momentum and introduced a new bootstrap test. Karolyi and Kho conclude that no random walk or autoregressive model is able to explain the magnitude of momentum empirically found; they suggest that models with time varying expected returns come closer to explaining the empirical magnitude of momentum.

Momentum and reversals are presently explained in the context of local models updated in real time. For example, momentum as described in Jegadeesh and Titman (1993) is based on the fact that stock prices can be represented as independent random walks when considering periods of the length of one year. However, it is fair to say that there is no complete agreement on the econometrics of asset returns that would justify momentum and reversals and stylized facts on a global scale, and not as local models. It would be beneficial to know more about the econometrics of asset returns that sustain momentum and reversals.

Behavioural phenomena are considered to play an important role in asset predictability; as mentioned, 44% of the survey respondents say they use behavioural modelling. Behavioural modellers attempt to capture phenomena such as departures from rationality on the part of investors (e.g. belief persistence), patterns in analyst estimates, and corporate executive investment/disinvestment behaviour. Behavioural finance is related to momentum in that the latter is often attributed to various phenomena of persistence in analyst estimates and investor perceptions.

A source at a large investment firm that has incorporated behavioural modelling into its active equity strategies commented, 'The attraction of behavioural finance is now much stronger than it was just five years ago. Everyone now acknowledges that markets are not efficient, that there are behavioural anomalies. In the past, there was the theory that was saying that markets are efficient while market participants such as the proprietary trading desks ignored the theory and tried to profit from the anomalies. We are now seeing a fusion of theory and practice.'

We remark that the term behavioural modelling is often used rather loosely. Full-fledged behavioural modelling exploits a knowledge of human psychology to identify situations where investors are prone to show behaviour that leads to market inefficiencies. The tendency now is to call 'behavioural' any model that exploits market inefficiency. However, implementing true behavioural modelling is a serious challenge. Even firms with very large, powerful quant teams say that 'considerable work is required to translate [departures from rationality] into a set of rules for identifying stocks as well as entry and exit points for a quantitative stock selection process.'

As for other methodologies used in return forecasting, sources cited nonlinear methods and co-integration. Nonlinear methods are being used to model return processes at 19% (7/36) of the responding firms. The nonlinear method most widely used among survey participants is classification and regression trees (CART). The advantage of CART is its simplicity and the ability of CART methods to be cast in an intuitive framework.

A source using CART as a central part of the portfolio construction process in enhanced index and longer-term value-based portfolios said, 'CART compresses a large volume of data into a form which identifies its essential characteristics, so the output is easy to understand. CART is non-parametric—which means that it can handle an infinitely wide range of statistical distributions—and nonlinear so as a variable selection technique it is particularly good at handling higher-order interactions between variables.'

Only 11% (4/36) of the respondents use nonlinear regime-shifting models; at most firms, judgment is used to assess regime change. Obstacles to modelling regime shifts include the difficulty in detecting the precise timing of a regime switch and the very long time series required for true estimation.

A source at a firm where regime-shifting models have been experimented with commented, 'Everyone knows that returns are conditioned by market regimes, but the potential for overfitting when implementing regime-switching models is great. If you could go back with fifty years of data—but we have only some ten years of data and this is not enough to build a decent model.'

Co-integration is being used by 19% (7/36) of the respondents. Co-integration models the short-term dynamics (direction) and long-run equilibrium (fair value). A perceived plus of co-integration is the transparency that it provides: the models are based on economic and finance theory and calculated from economic data.

1.6 USING HIGH-FREQUENCY DATA

High frequency data (HFD) are being used at only 14% of the responding firms (5/36), to identify profit opportunities and improve forecasts. Another three plan to use HFD within the next 12 months. A source at a large investment firm that is using HFD said, 'We use high-frequency data in event studies. The objective is to gain an understanding of the mechanisms of the market.' A source which is planning to use high-frequency data in the coming 12 months remarked, 'We believe that high-frequency data will allow us to evaluate exactly when it is optimal to trade, for example at close, VWAP, or midday, and to monitor potential market impact of our trades and potential front-running of our brokers.' (VWAP stands for volume-weighted average price.)

Though it is believed that HFD could be useful, cost of data is the blocking factor. Survey participants voiced concerns that the cost of data will hamper the development of models in the future. One source observes, 'The quasi monopolistic positioning of data vendors allows them to charge prices that are incompatible with the revenues structure of all but the biggest firms.' Other reasons cited by the sources not using HFD are a (perceived) unattractive noise-to-signal ratio and resistance to HFD-based strategies on the part of institutional investors.

1.7 MEASURING RISK

Risk is being measured at all the responding firms. Risk measures most widely used among participants include variance (97% or 35/36), Value at Risk (VaR) (67% or 24/36) and downside risk measures (39% or 14/36), Conditional VaR (CVaR), and extreme value theory (EVT) are used at 4 (11%) and 2 (6%) firms, respectively. The considerable use of asymmetric risk measures such as downside risk can be ascribed to the growing popularity of financial products with guaranteed returns. Many firms compute several risk measures: the challenge here is to merge the different risk views into a coherent risk assessment. Figure 1.5 represents the distribution of risk measures used by participants.

It is also interesting to note that among survey participants, there is a heightened attention to model risk. Model averaging and shrinkage techniques are being used by one-fourth (9/36) of the survey participants. The recent take-up of these techniques is related to the fact that most firms are now using multiple models to forecast returns, a trend that is up compared to two or three years ago. Other techniques to mitigate model risk, such as

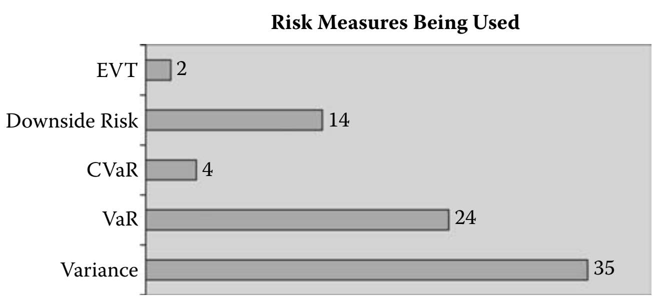

FIGURE 1.5 Distribution of risk measures adopted by participants.

random coefficient models, are not used much in the industry. In dealing with model risk we must distinguish between averaging model results and averaging models themselves. The latter technique, embodied in random coefficient models, is more difficult and requires more data.

1.8 OPTIMIZATION

Another area where much has changed recently is optimization. According to sources, optimization is now performed at 92% (33/36) of the participating firms, albeit in some cases only rarely. Mean-variance is the most widely used technique among survey participants: it is being used by 83% (30/36) of the respondents. It is followed by utility optimization (42% or 15/36) and, more recently, robust optimization (25% or 9/36). Only one firm mentioned that it is using stochastic optimization. Figure 1.6 represents the distribution of optimization methods.

The wider use of optimization is a significant development compared to just a few years ago when many sources reported that they eschewed optimization: the difficulty of identifying the forecasting error was behind the then widely held opinion that optimization techniques were too brittle and prone to 'error maximization.' The greater use of optimization is due to advances in large-scale optimization coupled with the ability to include constraints and robust methods for estimation and optimization itself. It is significant: portfolio formation strategies rely on optimization. With optimization now feasible, the door is open to a fully automated investment process. In this context, it is noteworthy that 55% of the survey respondents report that at least a portion of their equity assets is being managed by a fully automated process.

Optimization is the engineering part of portfolio construction. Most portfolio construction problems can be cast in an optimization framework, where optimization is applied to obtain the desired optimal risk-return profile. Optimization is the technology behind the current offering of products with specially engineered returns, such as guaranteed returns. However, the offering of products with particular risk-return profiles requires optimization methodologies that go well beyond the classical mean-variance optimization. In particular one must be able to (1) work with real-world utility functions and (2) apply constraints to the optimization process.

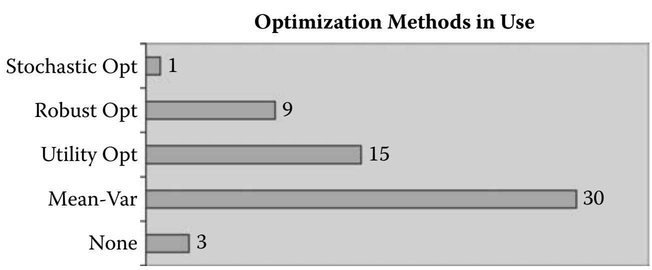

FIGURE 1.6 Distribution of optimization methods adopted by participants.

1.9 CHALLENGES

The growing diffusion of models is not without challenges. Survey participants noted three:

- increasing difficulty in differentiating products;
- difficulty in marketing quant funds, especially to non-institutional investors; and
- performance decay.

Quantitative equity management has now become so widespread that a source at a long-established quantitative investment firm remarked, 'There is now a lot of competition from new firms entering the space [of quantitative investment management]. The challenge is to continue to distinguish ourselves from competition in the minds of clients.'

With many quantitative funds based on the same methodologies and using the same data, the risk is to construct products with the same risk-return profile. The head of active equities at a large quantitative firm with more than a decade of experience in quantitative management remarked, 'Everyone is using the same data and reading the same articles: it's tough to differentiate.'

While sources report that client demand is behind the growth of (new) pure quantitative funds, some mentioned that quantitative funds might be something of a hard sell. A source at a medium-sized asset management firm servicing both institutional clients and high net worth individuals said, 'Though clearly the trend towards quantitative funds is up, quant approaches remain difficult to sell to private clients: they remain too complex to explain, there are too few stories to tell, and they often have low alpha. Private clients do not care about high information ratios.'

Markets are also affecting the performance of quantitative strategies. A recently released report from the Bank for International Settlements (2006) noted that this is a period of historically low volatility. What is exceptional about this period, observes the report, is the simultaneous drop in volatility in all variables: stock returns, bond spread, rates and so on. While the role of models in reducing volatility is unclear, what is clear is that models produce a rather uniform behaviour. Quantitative funds try to differentiate themselves either finding new unexploited sources of return forecastability, for example novel ways of looking at financial statements, or using optimization creatively to engineer special risk-return profiles.

A potentially more serious problem is performance decay. Survey participants remarked that model performance was not so stable. Firms are tackling these problems in two ways. First, they are protecting themselves from model breakdown with model risk mitigation techniques, namely by averaging results obtained with different models. It is unlikely that all models breakdown in the same way in the same moment, so that averaging with different models allows managers to diversify model risk. Second, there is an on-going quest for new factors, new predictors, and new aggregations of factors and predictors. In the long run, however, something more substantial might be required: this is the subject of the next and final section.

1.10 LOOKING AHEAD

Looking ahead, we can see a number of additional challenges. Robust optimization, robust estimation and the integration of the two are probably on the research agenda of many firms. As asset management firms strive to propose innovative products, robust and flexible optimization methods will be high on the R & D agenda. In addition, as asset management firms try to offer investment strategies to meet a stream of liabilities (i.e., measured against liability benchmarking), multistage stochastic optimization methods will become a priority for firms wanting to compete in this arena. Pan *et al.* (2006) call 'Intelligent Finance' the new field of theoretical finance at the confluence of different scientific disciplines. According to the authors, the theoretical framework of intelligent finance consists of four major components: financial information fusion, multilevel stochastic dynamic process models, active portfolio and total risk management, and *financial strategic analysis*.

The future role of high-frequency data is not yet clear. HFD are being used (1) to improve on model quality thanks to the 2000-fold increase in sample size they offer with respect to daily data and (2) to find intraday profit opportunities. The ability to improve on model quality thanks to HFD is the subject of research. It is already known that quantities such as volatility can be measured with higher precision using HFD. Using HFD, volatility ceases to be a hidden variable and becomes the measurable realized volatility, introduced by Torbin *et al.* (2003). If, and how, this increased accuracy impacts models whose time horizon is in the order of weeks or months is a subject not entirely explored. It might be that in modelling HFD one captures short-term effects that disappear at longer time horizons.

Regardless of the frequency of data sampling, modellers have to face the problem of performance decay that is the consequence of a wider use of models. Classical financial theory assumes that agents are perfect forecasters in the sense that they know the stochastic processes of prices and returns. Agents do not make systematic predictable mistakes: their action keeps the market efficient. This is the basic idea underlying rational expectations and the intertemporal models of Merton.

Practitioners (and now also academics) have relaxed the hypothesis of the universal validity of market efficiency; indeed, practitioners have always being looking for asset mispricings that could produce alpha. As we have seen, it is widely believed that mispricings are due to behavioural phenomena, such as belief persistence. This behaviour creates biases in agent evaluations—biases that models attempt to exploit in applications such as momentum strategies.

However, the action of models tends to destroy the same sources of profit that they are trying to exploit. This fact receives attention in applications such as measuring the impact of trades. In almost all current implementations, measuring the impact of trades means measuring the speed at which models constrain markets to return to an unprofitable efficiency. To our knowledge, no market impact model attempts to measure the opposite effect, that is, the eventual momentum induced by a trade.

It is reasonable to assume that the diffusion of models will reduce the mispricings due to behavioural phenomena. However, one might reasonably ask whether the action of models will ultimately make markets more efficient, destroying any residual profitability in excess of market returns, or if the action of models will create new opportunities that can be exploited by other models, eventually by a new generation of models based on an accurate analysis of model biases. It is far from being obvious that markets populated by agents embodied in mathematical models will move toward efficiency. In fact, models might create biases of their own. For example, momentum strategies (buy winners, sell losers) are a catalyst for increased momentum, farther increasing the price of winners and depressing the price of losers.

This subject has received much attention in the past as researchers studied the behaviour of markets populated by boundedly rational agents. While it is basically impossible, or at least impractical, to code the behaviour of human agents, models belong to a number of well-defined categories that process past data to form forecasts. Studies, based either on theory or on simulation, have attempted to analyse the behaviour of markets populated by agents that have bounded rationality, that is, filter past data to form forecasts.[2] One challenge going forward will be to understand what type of inefficiencies are produced by markets populated by automatic decision-makers whose decisions are based on past data. It is foreseeable that simulation and artificial markets will play a greater role as discovery devices.

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CHAPTER 2

Portfolio Optimization under the Value-at-Risk Constraint

TRAIAN A. PIRVU

CONTENTS

2.1 INTRODUCTION

Managers limit the riskiness of their traders by imposing limits on the risk of their portfolios. Lately, the Value-at-Risk (VaR) risk measure has become a tool used to accomplish this purpose. The increased popularity of this risk measure is due to the fact that VaR is easily understood. It is the maximum loss of a portfolio over a given horizon, at a given confidence level. The Basle Committee on Banking Supervision requires U.S. banks to use VaR in determining the minimum capital required for their trading portfolios.

In the following we give a brief description of the existing literature. Basak and Shapiro (2001) analyse the optimal dynamic portfolio and wealth-consumption policies of utility maximizing investors who must manage risk exposure using VaR. They find that VaR risk managers pick a larger exposure to risky assets than non-risk managers, and consequently incur larger losses when losses occur. In order to fix this deficiency they choose another risk measure based on the risk-neutral expectation of a loss. They call this risk measure *Limited Expected Loss* (LEL). One drawback of their model is that the portfolios VaR is never re-evaluated after the initial date, making the problem a static one. In a similar setup, Berkelaar *et al.* (2002) show that, in equilibrium, VaR reduces market volatility, but in some cases raises the probability of extreme losses. Emmer *et al.* (2001) consider a dynamic model with *Capital-at-Risk* (a version of VaR) limits. However, they assume that portfolio proportions are held fixed during the whole investment horizon and thus the problem becomes a static one as well.

Cuoco *et al.* (2001) develop a more realistic dynamically consistent model of the optimal behaviour of a trader subject to risk constraints. They assume that the risk of the trading portfolio is re-evaluated dynamically by using the conditioning information, and hence the trader must satisfy the risk limit continuously. Another assumption they make is that when assessing the risk of a portfolio, the proportions of different assets held in the portfolio are kept unchanged. Besides the VaR risk measure, they consider a coherent risk measure *Tail Value at Risk* (TVaR), and establish that it is possible to identify a dynamic VaR risk limit equivalent to a dynamic TVaR limit. Another of their findings is that the risk exposure of a trader subject to VaR and TVaR limits is always lower than that of an unconstrained trader.

Pirvu (2005) started with the model of Cuoco *et al.* (2001). We find the optimal growth portfolio subject to these risk measures. The main finding is that the optimal policy is a projection of the optimal portfolio of an unconstrained log agent (the Merton proportion) onto the constraint set with respect to the inner product induced by the volatility matrix of the risky assets. Closed-form solutions are derived even when the constraint set depends on the current wealth level.

Cuoco and Liu (2003) study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates. They show that optimal portfolios display a local three-fund property. Leippold *et al.* (2002) analyse VaR-based regulation rules and their possible distortion effects on financial markets. In partial equilibrium the effectiveness of VaR regulation is closely linked to the *leverage effect*, the tendency of volatility to increase when the prices decline.

Vidovic *et al.* (2003) considered a model with time-dependent parameters, but the risk constraints were imposed in a static fashion.

Yiu (2004) looks at the optimal portfolio problem, when an economic agent is maximizing the utility of her intertemporal consumption over a period of time under a dynamic VaR constraint. A numerical method is proposed to solve the corresponding HJB equation. They find that investment in risky assets is optimally reduced by the VaR constraint. Atkinson and Papakokinou (2005) derive the solution to the optimal portfolio and consumption subject to CaR and VaR constraints using stochastic dynamic programming.

This paper extends Pirvu (2005) by allowing for intertemporal consumption. We address an issue raised by Yiu (2004) and Atkinson and Papakokinou (2005) by considering a market with random coefficients. It was also suggested as a new research direction by Cuoco *et al.* (2001). To the best of our knowledge this is the first work in portfolio choice theory with CRRA-type preferences, time-dependent market coefficients, incomplete financial markets, and dynamically consistent risk constraints in a Brownian motion framework.

2.1.1 Our Contribution

We propose a new approach with the potential for numerical applications. The main idea is to consider, on every probabilistic path, an auxiliary deterministic control problem, which we analyse. The existence of a solution of the deterministic control problem does not follow from classical results. We establish it and we also show that first-order necessary conditions are also sufficient for optimality. We prove that a solution of this deterministic control problem is the optimal portfolio policy for a given path. The advantage of our method over classical methods is that it allows for a better numerical treatment.

The remainder of this chapter is organized as follows. Section 2.2 describes the model, including the definition of the Value-at-Risk constraint. Section 2.3 formulates the objective and shows the limitations of the stochastic dynamic programming approach in this context. Section 2.4 treats the special case of logarithmic utility. The problem of maximizing expected logarithmic utility of intertemporal consumption is solved in closed form. This is done by reducing it to a nonlinear program, which is solved pathwise. One finding is that, at the final time, the agent invests the least proportion of her wealth in stocks. The optimal policy is a projection of the optimal portfolio and consumption of an unconstrained agent onto the constraint set. Section 2.5 treats the case of power utility, in the totally unhedgable market coefficients paradigm (see Example 2.7.4, p. 305 of Karatzas and Shreve 1998). The stochastic control portfolio problem is transformed into a deterministic control problem. The solution is characterized by the Pontryagin maximum principle (first-order necessary conditions). Section 2.6 contains an appropriate discretization of the deterministic control problem. It leads to a nonlinear program that can be solved by standard methods. It turns out that the necessary conditions of the discretized problem converge to the necessary conditions of the continuous problem. For numerical experiments, one can use NPSOL, a software package for solving constrained optimization problems that employs a sequential quadratic programming (SQP) algorithm. We end this section with some numerical experiments. The conclusions are summarized in Section 2.7. We conclude the paper with an appendix containing the proofs of the lemmas.

2.2 MODEL DESCRIPTION

2.2.1 The Financial Market

Our model of a financial market, based on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{0\le t\le\infty}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions, consists of $m+1$ assets. The first, $\{S_0(t)\}_{t\in[0,\infty]}$, is a *riskless bond* with a positive interest rate r, i.e. $\mathrm{d}S_0(t) = S_0(t)r\,\mathrm{d}t$. The remaining m are *stocks* and evolve according to the following stochastic differential equation:

$$\mathrm{d}S_i(t) = S_i(t)\left[\alpha_i(t)\mathrm{d}t + \sum_{j=1}^{n}\sigma_{ij}(t)\mathrm{d}W_j(t)\right], \qquad (2.1)$$
$$0 \le t \le \infty,\ i = 1, \ldots, m,$$

where the process $\{W(t)\}_{t\in[0,\infty)} = \{(W_j(t))_{j=1,\ldots,n}\}_{t\in[0,\infty)}$ is an n-dimensional standard Brownian motion. Here, $\{\alpha(t)\}_{t\in[0,\infty)} = \{(\alpha_i(t))_{i=1,\ldots,m}\}_{t\in[0,\infty)}$ is an $\mathbb{R}^m$-valued *mean rate of return* process, and $\{\sigma(t)\}_{t\in[0,\infty)} = \{(\sigma_{ij}(t))_{i=1,\ldots,m}^{j=1,\ldots,n}\}_{t\in[0,\infty)}$ is an $m \times n$ matrix-valued *volatility* process. We impose the following regularity assumptions on the coefficient processes $\alpha(t)$ and $\sigma(t)$.

- All the components of the process $\{\alpha(t)\}_{t\in[0,\infty)}$ are assumed positive, continuous and $\{\mathcal{F}_t\}$-adapted.
- The matrix-valued volatility process $\{\sigma(t)\}_{t\in[0,\infty)}$ is assumed continuous, $\{\mathcal{F}_t\}$-adapted and with linearly independent rows for all $t \in [0,\infty)$, a.s.

The last assumption precludes the existence of a redundant asset and arbitrage opportunities. The *rate of excess* return is the $\mathbb{R}^m$-valued process $\{\mu(t)\}_{t\in[0,\infty)} = \{(\mu_i(t))_{i=1,\ldots,m}\}_{t\in[0,\infty)}$, with $\mu_i(t) = \alpha_i(t) - r$, which is assumed positive. This also covers the case of an incomplete market if $n > m$ (more sources of randomness than stocks).

2.2.2 Consumption, Trading Strategies and Wealth

In this model the agent is allowed to consume. The intermediate consumption process, denoted $\{C(t)\}_{t\in[0,\infty)}$, is assumed positive, and $\{\mathcal{F}_t\}$-progressively measurable. Let $\{(\zeta(t), c(t))\}_{t\in[0,\infty)} = \{(\zeta_i(t)_{i=1,\ldots,m}, c(t)\}_{t\in[0,\infty)}$ be an $\mathbb{R}^{m+1}$-valued *portfolio-proportion* process. At time t its components are the proportions of the agent's wealth invested in stocks, $\zeta(t)$, and her consumption rate, $c(t)$. An $\mathbb{R}^{m+1}$-valued *portfolio-proportion* process is called *admissible* if it is $\{\mathcal{F}_t\}$-progressively measurable and satisfies

$$\int_0^t \left|\zeta^T(u)\mu(u)\right|\mathrm{d}u + \int_0^t \left\|\zeta^T(u)\sigma(u)\right\|^2 \mathrm{d}u$$
$$+\int_0^t c(u)\mathrm{d}u < \infty,\ \text{a.s.},\quad \forall t \in [0,\infty), \qquad (2.2)$$

where, as usual, $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^m$. Given $\{(\zeta(t), c(t))\}_{t\in[0,\infty)}$ is a portfolio-proportion process, the leftover wealth $X^{\zeta,c}(t)(1 - \sum_{i=1}^{m}\zeta_i(t))$ is invested in the riskless bond $S_0(t)$. It may be that this quantity is negative, in which case we are borrowing at rate $r > 0$. The dynamics of the wealth process $\{X^{\zeta,c}(t)\}_{t\in[0,\infty)}$ of an agent using the portfolio-proportion process $\{(\zeta(t), c(t))\}_{t\in[0,\infty)}$ is given by the following stochastic differential equation:

$$\begin{aligned}
\mathrm{d}X^{\zeta,c}(t) &= X^{\zeta,c}(t)\big(\big(\zeta^T(t)\alpha(t) - c(t)\big)\mathrm{d}t + \zeta^T(t)\sigma(t)\mathrm{d}W(t)\big) \\
&\quad + \left(1 - \sum_{i=1}^{m} \zeta_i(t)\right) X^{\zeta,c}(t) r\,\mathrm{d}t \\
&= X^{\zeta,c}(t)\big(r - c(t) + \zeta^T(t)\mu(t)\big)\mathrm{d}t \\
&\quad + \zeta^T(t)\sigma(t)\mathrm{d}W(t)).
\end{aligned}$$

Let us define the p-quadratic correction to the saving rate r:

$$Q_p(t,\zeta,c) \stackrel{\triangle}{=} r - c + \zeta^T\mu(t) + \frac{p-1}{2}\left\|\zeta^T\sigma(t)\right\|^2. \tag{2.3}$$

The above stochastic differential equation has a unique strong solution if (2.2) is satisfied and is given by the explicit expression

$$X^{\zeta,c}(t) = X(0)\exp\left\{\int_0^t Q_0(u,\zeta(u),c(u))\mathrm{d}u + \int_0^t \zeta^T(u)\sigma(u)\mathrm{d}W(u)\right\}. \tag{2.4}$$

The initial wealth $X^{\zeta,c}(0) = X(0)$ takes values in $(0,\infty)$ and is exogenously given.

2.2.3 Value-at-Risk Limits

For the purposes of risk measurement, one can use an approximation of the distribution of the investor's wealth at a future date. A detailed explanation of why this practice should be employed can be found on p. 8 of Cuoco *et al.* (2001) (see also p. 18 of Leippold *et al.* (2002)). Given a fixed time instance $t \geq 0$, and a length $\tau > 0$ of the measurement horizon $[t,\ t+\tau]$, the projected distribution of wealth from trading and consumption is usually calculated under the assumptions that

1. the portfolio proportion process $\{(\zeta(u), c(u))\}_{u\in[t,t+\tau]}$, as well as
2. the market coefficients $\{(\alpha(u)\}_{u\in[t,t+\tau])}$ and $\{(\sigma(u)\}_{u\in[t,t+\tau]}$,

will stay constant and equal their present value throughout $[t,\ t+\tau]$. If τ is small, for example $\tau = 1$ trading day, the market coefficients will not change much and this supports assumption 2. The wealth's dynamics equation yields *the projected wealth* at $t+\tau$:

$$\begin{aligned}
X^{\zeta,c}(t+\tau) = X^{\zeta,c}(t)\exp\big\{&Q_0(t,\zeta(t),c(t))\tau \\
&+ \zeta^T(t)\sigma(t)(W(t+\tau) - W(t))\big\},
\end{aligned}$$

whence *the projected wealth* loss on the time interval $[t,\ t+\tau]$ is

$$\begin{aligned}
X^{\zeta,c}(t) - X^{\zeta,c}(t+\tau) = X^{\zeta,c}(t)[1 - \exp\big\{&Q_0(t,\zeta(t),c(t))\tau \\
&+ \zeta^T(t)\sigma(t)(W(t+\tau) - W(t))\big\}\big].
\end{aligned}$$

The random variable $\zeta^T(t)\sigma(t)(W(t+\tau)-W(t))$ is, conditionally on $\mathcal{F}_t$, normally distributed with mean zero and standard deviation $\|\zeta^T(t)\sigma(t)\|\sqrt{\tau}$. Let the confidence parameter $\alpha \in (0, 1/2]$ be exogenously specified. The α-percentile of the projected loss $X^{\zeta,c}(t) - X^{\zeta,c}(t+\tau)$ conditionally on $\mathcal{F}_t$ is

$$X^{\zeta,c}(t)\left[1-\exp\left\{Q_0(t,\zeta(t),c(t))\tau + N^{-1}(\alpha)\|\zeta^T(t)\sigma(t)\|\sqrt{\tau}\right\}\right],$$

where $N(\cdot)$ denotes the standard cumulative normal distribution function. This prompts the *Value-at-Risk* (VaR) of projected loss

$$\mathrm{VaR}(t,\zeta,c,x) \triangleq x\left[1-\exp\left\{Q_0(t,\zeta,c)\tau + N^{-1}(\alpha)\left\|\zeta^T\sigma(t)\right\|\sqrt{\tau}\right\}\right]^+.$$

Let $a_V \in (0,1)$ be an exogenous risk limit. The *Value-at-Risk constraint* is that the agent at every time instant $t \geq 0$ must choose a portfolio proportion $(\zeta(t), c(t))$ that would result in a *relative* VaR of the projected loss on $[t, t+\tau]$ less than a_V. This, strictly speaking, is the set of all admissible portfolios which, for all $t \geq 0$, belong to $F_V(t)$, defined by

$$F_V(t) \triangleq \left\{(\zeta, c) \in \mathbb{R}^m \times [0,\infty);\ \frac{\mathrm{VaR}(t,\zeta,c,x)}{x} \leq a_V\right\}. \tag{2.5}$$

The fraction VaR/x rather than VaR is employed, whence the name *relative* VaR. If one imposes VaR in absolute terms, the constraint set depends on the current wealth level and this makes the analysis more involved (see Cuoco *et al.* 2001; Pirvu 2005). For a given path ω let us denote $\omega^{(t)} = (\omega_s)_{s\leq t}$ as the projection up to time t of its trajectory. One can see that, for a fixed ω^t, the set $F_V(t)$ is compact and convex, being the level set of a convex, unbounded function $f_V(t, \zeta, c)$,

$$F_V(t) = \left\{(\zeta, c) \in \mathbb{R}^m \times [0,\infty);\ f_V(t,\zeta,c) \leq \log\frac{1}{1-a_V}\right\},$$

where

$$f_V(t,\zeta,c) \triangleq -Q_0(t,\zeta,c)\tau - N^{-1}(\alpha)\left\|\zeta^T\sigma(t)\right\|\sqrt{\tau}. \tag{2.6}$$

The function f_V, although quadratic in ζ and linear in c, may still fail to be convex in (ζ, c) if $\alpha \geq 1/2$, thus $F_V(t)$ may not be a convex set (see Figure 2.1, p. 10 of Cuoco *et al.* 2001). However, the choice of $\alpha \in (0, (1/2)]$ makes $N^{-1}(\alpha) \leq 0$ and this yields convexity.

2.3 OBJECTIVE

Let finite time horizon T and the discount factor δ (the agent's impatient factor) be primitives of the model. Given $X(0)$, the agent seeks to choose an admissible portfolio-proportion process such that $(\zeta(t), c(t)) \in F_V(t)$ for all $0 \leq t \leq T$, and the expected value of her CRRA utility of intertemporal consumption and final wealth,

$$\int_0^T e^{-\delta t} U_p(C(t))\mathrm{d}t + e^{-\delta T} U_p(X^{\zeta,c}(T)), \tag{2.7}$$

is maximized over all admissible portfolios processes satisfying the same constraint. Here, $U_p(x) \triangleq x^p/p$, with $p < 1$ the coefficient of relative risk aversion (CRRA). Let us assume for the moment that the market coefficients are constants. In this case the constraint set $F_V(t)$ does not change over time and we denote it by F_V. Then one can use dynamic programming techniques to characterize the optimal portfolio and consumption policy. The problem is to find a solution to the HJB equation. Define the optimal value function as

$$V(x,t) = \max_{(\zeta,c)\in F_V} \mathbb{E}_t\left[\int_t^T e^{-\delta t} U_p(C(u))\mathrm{d}u + e^{-\delta T} U_p\big(X^{\zeta,c}(T)\big)\right],$$

where $\mathbb{E}_t$ is the conditional operator given the information known up to time t and $X^{\zeta,c}(t) = x$. The HJB equation is

$$\max_{(\zeta,c)\in F_V} J(t,x,\zeta,c) = 0,$$

where

$$J(t,x,c,\zeta) \triangleq e^{-\delta t} U_p(cx) + \frac{\partial V}{\partial t} + x\big(r - c + \zeta^T\mu\big)\frac{\partial V}{\partial x} + \frac{\|\zeta^T\sigma\|^2 x^2}{2}\frac{\partial^2 V}{\partial x^2},$$

with the boundary condition $V(x,T) = e^{-\delta T} U_p(x)$. The value function V inherits the concavity of the utility functions U_p. Being jointly concave in (ζ, c), the function J is maximized over the set F_V at a unique point $(\bar{\zeta},\bar{c})$. Moreover, this point should lie on the boundary of F_V and one can derive first-order optimality conditions by means of Lagrange multipliers. Together with the HJB equation this yields a highly nonlinear PDE that is hard to solve numerically (a numerical scheme was proposed by Yiu (2004), but no convergence result was reported). In the following we approach the portfolio optimization problem by reducing it to a deterministic control problem. We are then able to obtain explicit solutions for logarithmic utility.

2.4 LOGARITHMIC UTILITY

Let us consider the case where the agent is deriving utility from intermediate consumption only. It is straightforward to extend it to also encompass the utility of the final wealth. In light of (2.4),

$$
\begin{aligned}
\log X^{\zeta,c}(t) = & \log X(0) + \int_0^t Q_0(s,\zeta(s),c(s))\mathrm{d}s \\
& + \int_0^t \zeta^T(s)\sigma(s)\mathrm{d}W(s).
\end{aligned} \tag{2.8}
$$

What facilitates the analysis is the decomposition of the utility from intertemporal consumption into a signal, a Lebesque integral and noise, which comes at an Itô integral rate. The decomposition is additive and the expectation operator cancels the noise. Indeed,

$$
\begin{aligned}
\int_0^T e^{-\delta t}\log C(t)\mathrm{d}t = & \int_0^T e^{-\delta t}\log\left(c(t)X^{\zeta,c}(t)\right)\mathrm{d}t \\
= & \frac{1-e^{-\delta T}}{\delta}\log X(0) + \int_0^T e^{-\delta t}\log c(t)\mathrm{d}t \\
& + \int_0^T \int_0^t e^{-\delta t}Q_0(s,\zeta(s),c(s))\mathrm{d}s\,\mathrm{d}t \\
& + \int_0^T e^{-\delta t}\int_0^t \zeta^T(s)\sigma(s)\mathrm{d}W(s)\mathrm{d}t.
\end{aligned}
$$

By Fubini's theorem

$$
\begin{aligned}
\int_0^T \int_0^t e^{-\delta t}Q_0(s,\zeta(s),c(s))\mathrm{d}s\,\mathrm{d}t = & \int_0^T \left(\int_s^T e^{-\delta t}Q_0(s,\zeta(s),c(s))\mathrm{d}t\right)\mathrm{d}s \\
= & \int_0^T \frac{e^{-\delta t}-e^{-\delta T}}{\delta}Q_0(t,\zeta(t),c(t))\mathrm{d}t,
\end{aligned}
$$

hence

$$
\begin{aligned}
\int_0^T e^{-\delta t}\log C(t)\mathrm{d}t = & \frac{1-e^{-\delta T}}{\delta}\log X(0) \\
& + \int_0^T e^{-\delta t}\left(\log c(t) + \frac{1}{\delta}(1-e^{-\delta(T-t)})Q_0(t,\zeta(t),c(t))\right)\mathrm{d}t \\
& + \int_0^T e^{-\delta t}\int_0^t \zeta^T(s)\sigma(s)\mathrm{d}W(s)\mathrm{d}t.
\end{aligned} \tag{2.9}
$$

The linearly independent rows assumption on the matrix-valued volatility process yields the existence of the inverse $(\sigma(t)\sigma^T(t))^{-1}$ and so equation

$$
\sigma(t)\sigma^T(t)\zeta_M(t) = \mu(t) \tag{2.10}
$$

uniquely defines the stochastic process $\{\zeta_M(t)\}_{t\in[0,\infty)} = \{(\sigma(t)(\sigma^T(t))^{-1}\mu(t)\}_{t\in[0,\infty)}$, called the *Merton-ratio process*. It has the property that it maximizes (in the absence of portfolio constraints), the rate of growth, and the log-optimizing investor would invest exactly using the components of $\zeta_M(t)$ as portfolios proportions (see Section 3.10 of Karatzas and Shreve 1991). By (2.10)

$$\left\|\zeta_M^T(t)\sigma(t)\right\|^2 = \zeta_M^T(t)\mu(t) = \mu^T(t)(\sigma(t)\sigma^T(t))^{-1}\mu(t). \tag{2.11}$$

The following integrability assumption is rather technical, but it guarantees that a local martingale (Itô integral) is a (true) martingale (see p. 130 of Karatzas and Shreve 1991). Let us assume that

$$\mathbb{E}\int_0^T \left\|\zeta_M^T(u)\sigma(u)\right\|^2 \mathrm{d}u < \infty. \tag{2.12}$$

This requirement, although imposed on the market coefficients (see Equation (2.11)), is also inherited for all portfolios satisfying the Value-at-Risk constraint.

Lemma 2.1: *For every* $(\zeta(t), c(t)) \in F_V(t)$ *the process* $\int_0^t \zeta^{Ts}(s)\sigma(s)\, dW(s)$, $t \in [0, T]$, *is a martingale, hence* $\mathbb{E}\int_0^t \zeta^T(s)\sigma(s)\, dW(s) = 0$.

Proof: See Appendix 2.A. □

In light of this lemma, the expectation of the noise vanishes, i.e.

$$\mathbb{E}\int_0^T e^{-\delta t}\int_0^t \zeta^T(s)\sigma(s)\mathrm{d}W(s)\mathrm{d}t = 0,$$

after interchanging the order of integration. Thus, taking expectation in the additive utility decomposition (2.9),

$$\begin{aligned}\mathbb{E}\int_0^T e^{-\delta t}\log C(t)\mathrm{dt} &= \frac{1-e^{-\delta T}}{\delta}\log X(0) \\ &\quad + \mathbb{E}\int_0^T e^{-\delta t}\Bigg(\log c(t) + \frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right) \\ &\quad \times Q_0(t, \zeta(t), c(t))\Bigg)\mathrm{d}t.\end{aligned} \tag{2.13}$$

Therefore, to maximize

$$\mathbb{E}\int_0^T e^{-\delta t}\log C(t)\mathrm{d}t$$

over the constraint set, it suffices to maximize

$$g(t, \zeta, c) \triangleq \log c(t) + \frac{1}{\delta}\left(1-e^{-\delta(T-t)}\right)Q_0(t, \zeta(t), c(t)) \tag{2.14}$$

pathwise over the constraint set. For a fixed path ω and a time instance t, we need to solve

$$
\begin{aligned}
\text{(P1)} \quad & \text{maximize } g(t,\zeta,c), \\
& \text{subject to } f_V(t,\zeta,c) \triangleq -Q_0(t,\zeta,c)\tau - N^{-1}(\alpha)\|\zeta^T\sigma(t)\|\sqrt{\tau} \le \log\frac{1}{1-a_V}.
\end{aligned}
$$

The optimal policy for an agent maximizing her logarithmic utility of intertemporal consumption without the risk constraint is to hold the proportion $\{(\zeta_M(t), c_M(t))\}_{t\in[0,T]}$, where $c_M(t) \triangleq \delta/[1-e^{-\delta(T-t)}]$ (the optimum of (P1) without the constraint is $(\zeta_M(t), c_M(t))$).

Lemma 2.2: *The solution of (P1) is given by*

$$\bar{\zeta}(t) = (1 \wedge (\beta(t) \vee 0))\zeta_M(t), \tag{2.15}$$

$$
\begin{aligned}
\bar{c}(t) = {} & u(t,(1\wedge\beta(t)))c_M(t)1_{\{\beta(t)>0\}} \\
& + \left(r + \frac{1}{\tau}\log\frac{1}{1-a_V}\right)1_{\{\beta(t)\le 0\}},
\end{aligned} \tag{2.16}
$$

where $\beta(t)$ is the root of the equation

$$f_V(t, z\zeta_M(t), u(t,z)c_M(t)) = \log\frac{1}{1-a_V} \tag{2.17}$$

in the variable z, with

$$u(t,z) \triangleq 1 + \frac{\sqrt{\tau}\|\zeta_M^T(t)\sigma(t)\|}{N^{-1}(\alpha)}(1-z). \tag{2.18}$$

Proof: See Appendix 2.A. □

Theorem 2.3: *To maximize the logarithmic utility of intertemporal consumption,*

$$\mathbb{E}\int_0^T e^{-\delta t}\log C(t)\,\mathrm{d}t,$$

over processes $(\zeta(t), c(t)) \in F_V(t)$, $0 \le t \le T$, *the optimal portfolio is* $\{(\bar{\zeta}(t), \bar{c}(t))\}_{t\in[0,T]}$.

Proof: This is a direct consequence of (2.13) and Lemma 2.2. □

Remark 1: Since at the final time $c_M(T) = \infty$ and $\bar{c}(t)$ is bounded we must have $\beta(\mathrm{T}) \leq 0$, so $\bar{\zeta}(T) = 0$, and this means that, at the final time, the agent invests the least proportion (in absolute terms) of her wealth in stocks. By (2.15) and (2.16) it follows that $\bar{\zeta}(t) \leq \zeta_M(t)$, and $\bar{c}(t) \leq c_M(t)$, for any $0 \leq t \leq T$, which means that the constrained agent is consuming and investing less in the risky assets than the unconstrained agent. Let T_1 and T_2 be two final time horizons, $T_1 > T_2$. Because $c_M(t, T_1) < c_M(t, T_2)$, from Equations (2.15) and (2.16) we conclude that $\beta(t, T_1) > \beta(t, T_2)$, hence $\bar{\zeta}(t, T_1) > \bar{\zeta}(t, T_2)$, and $\bar{c}(t, T_1) > \bar{c}(t, T_2)$. Therefore, long-term agents can afford to invest more in the stock market and consume more than short-term agents (in terms of proportions).

2.5 NONLOGARITHMIC UTILITY

Let us recall that we want to maximize the expected CRRA utility $(U_p(x) = x^p/p, p \neq 0)$ from intertemporal consumption and terminal wealth,

$$\mathbb{E}\int_0^T e^{-\delta t} U_p(C(t))\mathrm{d}t + \mathbb{E}e^{-\delta T} U_p(X^{\zeta,c}(T)), \tag{2.19}$$

over portfolio-proportion processes satisfying the Value-at-Risk constraint, i.e. $(\zeta(t), c(t)) \in F_V(t)$, $0 \leq t \leq T$. One cannot obtain an additive decomposition into signal and noise as in the case of logarithmic utility. However, a multiplicative decomposition can be performed. By (2.7),

$$\begin{aligned}
U_p(X^{\zeta,c}(t)) &= \frac{X^p(0)}{p} \exp\left(\int_0^t pQ_0(s, \zeta(s), c(s))\mathrm{d}s + \int_0^t p\zeta^T(s)\sigma(s)\mathrm{d}W(s)\right) \\
&= \frac{X^p(0)}{p} \exp\left(\int_0^t \left(pQ_p(s, \zeta(s), c(s)) - \frac{1}{2}p^2\|\zeta^T(s)\sigma(s)\|^2\right)\mathrm{d}s\right. \\
&\quad \left. + \int_0^t p\zeta^T(s)\sigma(s)\mathrm{d}W(s)\right) = \frac{X^p(0)}{p} N^{\zeta,c}(t) Z^{\zeta}(t),
\end{aligned}$$

where

$$N^{\zeta,c}(t) \triangleq \exp\left(\int_0^t pQ_p(s, \zeta(s), c(s))\mathrm{d}s\right), \tag{2.20}$$

$$\begin{aligned}
Z^{\zeta}(t) \triangleq \exp\Bigg(&-\frac{1}{2}\int_0^t p^2\|\zeta^T(s)\sigma(s)\|^2\mathrm{d}s \\
&+ \int_0^t p\zeta^T(s)\sigma(s)\mathrm{dW}(s)\Bigg),
\end{aligned} \tag{2.21}$$

with Q_p defined in (2.3). By taking expectation,

$$\mathbb{E}U_p(X^{\zeta,c}(t)) = \frac{X^p(0)}{p}\mathbb{E}(N^{\zeta,c}(t)Z^{\zeta}(t)). \tag{2.22}$$

The process $N^{\zeta,c}(t)$ is the signal and $Z^{\zeta}(t)$, a stochastic exponential, is the noise. Stochastic exponentials are local martingales, but if we impose the assumption

$$\mathbb{E}\left[\exp\left(\frac{p^2}{2}\int_0^T \left\|\zeta_M^T(u)\sigma(u)\right\|^2 \mathrm{d}u\right)\right] < \infty \tag{2.23}$$

on market coefficients (see Equation (2.11)), the process $Z^{\zeta}(t)$ is a (true) martingale for all portfolio processes satisfying the constraint, as the following lemma shows.

Lemma 2.4: *For every* $(\zeta(t), c(t)) \in F_V(t)$ *the process* $Z^{\zeta}(t)$, $t \in [0, T]$, *is a martingale, hence* $\mathbb{E}Z^{\zeta}(t) = 1$.

Proof: See Appendix 2.A. □

As for utility from intertemporal consumption,

$$\begin{aligned}\int_0^T e^{-\delta t}U_p(C(t))\mathrm{d}t &= \int_0^T e^{-\delta t}U_p(X^{\zeta,c}(t)c(t)\mathrm{d}t \\ &= \frac{X^p(0)}{p}\int_0^T e^{-\delta t}c^p(t)N^{\zeta,c}(t)Z^{\zeta}(t)\mathrm{d}t.\end{aligned} \tag{2.24}$$

We claim that

$$\mathbb{E}\left(\int_0^T e^{-\delta t}c^p(t)N^{\zeta,c}(t)(Z^{\zeta}(t) - Z^{\zeta}(T))\mathrm{d}t\right) = 0.$$

Indeed, by conditioning and Lemma 2.4 we obtain

$$\begin{aligned}\mathbb{E}\big(c^p(t)N^{\zeta,c}(t)\big(Z^{\zeta}(t) - Z^{\zeta}(T)\big)\big) &= \mathbb{E}\big(\mathbb{E}\big[c^p(t)N^{\zeta,c}(t)\big(Z^{\zeta}(t) - Z^{\zeta}(T)\big)|\mathcal{F}_t\big]\big) \\ &= \mathbb{E}\big(c^p(t)N^{\zeta,c}(t)\mathbb{E}\big[\big(Z^{\zeta}(t) - Z^{\zeta}(T)\big)|\mathcal{F}_t\big]\big) \\ &= 0,\end{aligned}$$

and Fubini's theorem proves the claim. Hence, combined with (2.24), we obtain

$$\mathbb{E}\int_0^T e^{-\delta t}U_p(C(t))\mathrm{d}t = \frac{X^p(0)}{p}\mathbb{E}\left(Z^{\zeta}(T)\int_0^T e^{-\delta t}c^p(t)N^{\zeta,c}(t)\mathrm{d}t\right). \tag{2.25}$$

The decomposition for the total expected utility (Equations (2.22) and (2.25)) is

$$\mathbb{E}\int_0^T e^{-\delta t}U_p(C(t))\mathrm{d}t + \mathbb{E}e^{-\delta T}U_p\big(X^{\zeta,c}(T)\big) = \frac{X^p(0)}{p}\mathbb{E}\big(Z^{\zeta}(T)Y^{\zeta,c}(T)\big), \tag{2.26}$$

where the signal $Y^{\zeta,c}(T)$ is given by

$$Y^{\zeta,c}(T) = \int_0^T e^{-\delta t}c^p(t)N^{\zeta,c}(t)\mathrm{d}t + e^{-\delta T}N^{\zeta,c}(T), \tag{2.27}$$

with $N^{\zeta,c}(t)$ defined in (2.20). It appears natural at this point to maximize $Y^{\zeta,c}(T)$ pathwise over the constraint set. For a given path ω, the existence of an optimizer $\{(\bar{\zeta}(t,\omega),\bar{c}(t,\omega))\}_{t\in[0,T]}$ is given by Lemma 2.5. Note that $N^{\zeta,c}(t,\omega)$ depends on the trajectory of $(\zeta(\cdot,\omega),c(\cdot,\omega))$ on $[0,t]$ so one is faced with a deterministic control problem. From now on, to keep the notation simple we drop ω. In the language of deterministic control we can write (2.27) as a cost functional $I\,[x,u]$ given in the form

$$I\,[x,u] = g(x(T)) + \int_0^T f_0(t,x(t),u(t))\mathrm{d}t, \quad g(x) \stackrel{\triangle}{=} e^{-\delta T}x, \tag{2.28}$$

where $u=(\zeta,c)$ is the control, x is the state variable, and the function

$$f_0(t,x,u) \stackrel{\triangle}{=} e^{-\delta t}c^p x \tag{2.29}$$

is defined on the set

$$A = \{(t,x,u)|(t,x)\in[0,T]\times(0,K],u(t)\in F_V(t)\}\subset\mathbb{R}^{m+3}. \tag{2.30}$$

The dynamics of the state variable is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x(t),u(t)), \quad 0\le t\le T, \tag{2.31}$$

with the boundary condition $x(0)=1$, where

$$f(t,x,u) \stackrel{\triangle}{=} x\left(pr - pc + p\zeta^T\mu(t) + \frac{p(p-1)}{2}\left\|\zeta^T\sigma(t)\right\|^2\right). \tag{2.32}$$

The constraints are $(t,x(t))\in[0,T]\times(0,K]$ and $u(t)\in F_V(t)$. Due to the compactness of the set $F_V(t)$, $0\le t\le T$, it follows that $K<\infty$. A pair (x,u) satisfying the above conditions is called admissible. The problem of finding the maxima of $I\,[x,u]$ within all admissible pairs (x,u) is called the Bolza control problem. Classical existence theory for deterministic control does not apply to the present situation and we proceed with a direct proof of existence.

Lemma 2.5: *There exists a solution* $\{\bar{u}(t)\}_{0\le t\le T} \stackrel{\triangle}{=} \{(\bar{\zeta}(t), \bar{c}(t))\}_{0\le t\le T}$ *for the Bolza control problem defined above.*

Proof: See Appendix 2.A. □

An optimal solution $\{\bar{u}(t)\}_{0\le t\le T} \stackrel{\triangle}{=} \{(\bar{\zeta}(t), \bar{c}(t))\}_{0\le t\le T}$ is characterized by a system of forward backward equations (also known as the Pontryagin maximum principle). Let $\tilde{\lambda} = (\lambda_0, \lambda_1)$ be the adjoint variable and

$$H(t, x, u, \tilde{\lambda}) = \lambda_0 f_0(t, x, u) + \lambda_1 f(t, x, u)$$

the Hamiltonian function. The necessary conditions for the Bolza control problem (the Pontryagin maximum principle) can be found in Cesari (1983) (Theorem 2.5.1.i). In general, they are not sufficient for optimality. We prove that, in our context, the necessary conditions are also sufficient, as the following lemma shows.

Lemma 2.6: *A pair* $\bar{x}(t)$, $\bar{u}(t) = (\bar{\zeta}(t), \bar{c}(t)) \in F_V(t)$, $0 \le t \le T$, *is optimal, i.e. it gives the maximum for the functional* $I\,[x, u]$, *if and only if there is an absolutely continuous non-zero vector function of Lagrange multipliers* $\bar{\lambda} = (\lambda_0, \lambda_1)$, $0 \le t \le T$, *with* λ_0 *a constant,* $\lambda_0 \ge 0$, *such that the function* $M(t) \stackrel{\triangle}{=} H(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t))$ *is absolutely continuous and one has*

1. *adjoint equations*:

$$\frac{dM}{dt} = H_t(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t)) \text{ a.e.}; \tag{2.33}$$

$$\frac{d\lambda_1}{dt} = -H_x(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t)) \text{ a.e.}; \tag{2.34}$$

2. *maximum condition*:

$$\bar{u}(t) \in \arg\max_{v\in F_V(t)} H(t, \bar{x}(t), v, \bar{\lambda}(t)) \text{ a.e.}; \tag{2.35}$$

3. *transversality*:

$$\lambda_1(T) = \lambda_0 g'(\bar{x}(T)). \tag{2.36}$$

Proof: See Appendix 2.A. □

The following technical requirement on the market coefficients is sufficient to make $\{(\bar{\zeta}(t), \bar{c}(t))\}_{t\in[0,T]}$ an optimal portfolio process for maximizing the CRRA utility under the Value-at-Risk constraint, as Theorem 2.7 shows. We assume that market coefficients are