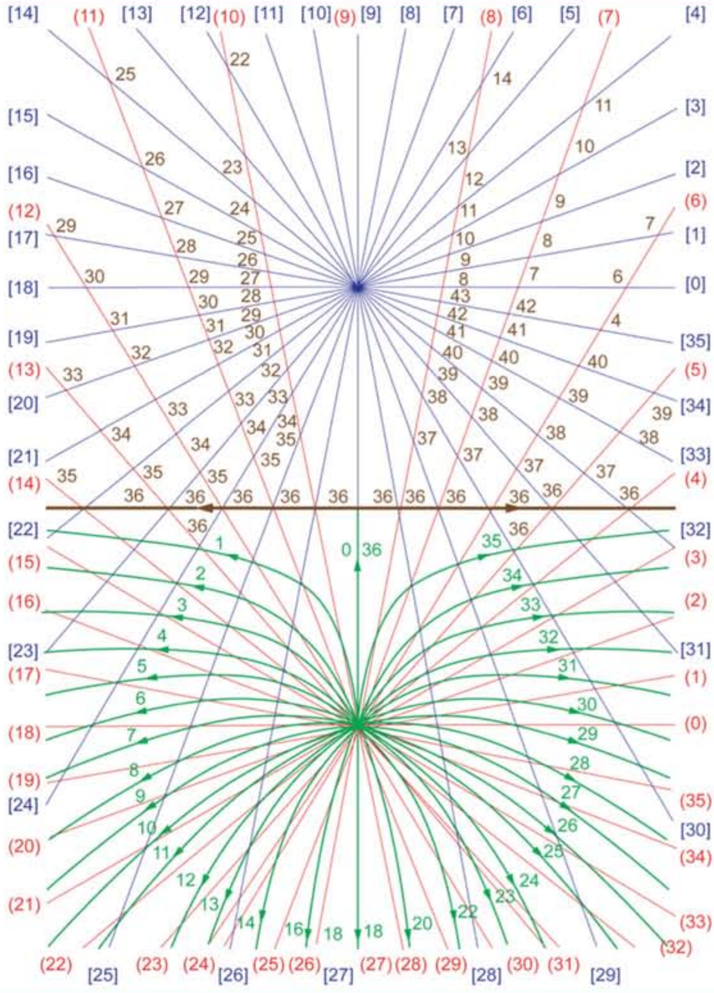
MATHEMATICS AND PHYSICS FOR SCIENCE AND TECHNOLOGY

Complex Analysis with Applications to Flows and Fields



L.M.B.C. Campos

CRC Press
Taylor & Francis Group

Complex Analysis with Applications to Flows and Fields

Mathematical and Physics for Science and Technology

Series Editor

L.M.B.C. Campos

Director of the Center for Aeronautical
and Space Science and Technology

Lisbon Technical University

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20140514

International Standard Book Number-13: 978-1-4200-7120-7 (eBook - PDF)

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Series Preface

The aim of the Mathematics and Physics for Science and Technology series is to describe the mathematical methods as they are applied to model natural physical phenomena and solve scientific and technological problems. The primary emphasis is on the application, including formulation of the problem, detailed solution, and interpretation of results. The mathematical methods are presented in sufficient detail to justify every step of solution, and avoid superfluous assumptions.

The main areas of physics are covered, namely:

- Mechanics of particles, rigid bodies, deformable solids and fluids
- Electromagnetism, thermodynamics, and statistical physics as well as their classical, relativistic, and quantum formulations
- Interactions and combined effects (e.g., thermal stresses, magnetohydrodynamics, plasmas, piezoelectricity, and chemically reacting and radiating flows)

The examples and problems chosen include natural phenomena in our environment, geophysics, and astrophysics; the technological implications in various branches of engineering; and other mathematical models, in biological, economic, and social sciences.

The coverage of areas of mathematics and branches of physics is sufficient to lay the foundations of all branches of engineering, namely:

- *Mechanical*—including machines, engines, structures, and vehicles
- *Civil*—including structures and hydraulics
- *Electrical*—including circuits, waves, and quantum effects
- *Chemical*—including transport phenomena and multiphase media
- *Computer*—including analytical and numerical methods and associated algorithms

Particular emphasis is given to interdisciplinary areas, such as electromechanics and aerospace engineering. These require combined knowledge of several areas and have an increasing importance in modern technology.

Analogies are applied in an efficient and concise way, across distinct disciplines, but also stressing the differences and aspects specific to each area, for example:

- Potential flow, electrostatics, magnetostatics, gravity field, steady heat conduction, and plane elasticity and viscous flow
- Acoustic, elastic, electromagnetic, internal, and surface waves
- Diffusion of mass, electricity and momentum.

In each case the analogies are exploited by common mathematical methods with distinct interpretations in each context.

The series is organized as a sequence of mathematical methods, each with a variety of applications. As the mathematical methods progress, the range of applications widens. For example, complex functions are used to study potential flows and electrostatics in the plane. The three-dimensional extension uses generalized functions. The latter are used with differential equations to describe vibrations and waves. The series and integral transforms are applied to initial and boundary-value problems. Tensor calculus is used for elasticity, viscous fluids, and relativity. Thus each method is consolidated with diverse applications before proceeding to the next.

The presentation of the material is intended to remain accessible to the university student. The subjects are introduced at a basic undergraduate level. The deductions and intermediate steps are detailed. Extensive illustrations and detailed legends promote visual and intuitive memory and understanding. The material is presented like a sequence of lectures and can be used to construct the subjects or disciplines of a university curriculum. It is possible to adjust the level of the curriculum by retaining the basic theory and simpler examples, and using the rest as background material for further reading.

The adaptation of the material for specific lecture courses can be made by selecting the basic theory and examples used as applications. Some topics chosen for greater insight may be included according to the motivation. The bibliography gives a choice of approaches to the subject and the possibility to focus more in specific subareas. The presentation follows a logical rather than historical sequence; some references to the original sources are used to give a historical perspective. The notes at the ends of chapters hint at the broader scope of the subject. The contents of each chapter are previewed in an introduction.

The present series embodies a concept of “interdisciplinary education in science and technology”. The traditional approach is to study each area of mathematics separately (analysis, geometry, differential equations, etc.) as well as each branch of physics (classical mechanics, heat and thermodynamics, electromagnetism, etc.). The student is then expected to “merge” all these sources of information, for example, know all that is needed about partial differential equations for the Maxwell equations of electromagnetism, calculus of variations for the minimum energy methods of elasticity, thermodynamics for the dynamics of compressible fluids, and so on. The time gaps and disjoint nature of this teaching implies a careful sequence of subjects to ensure each subject starts with the required background. Also there is considerable duplication in that similar methods and analogous problem recur in different contexts. Furthermore, the student discovers the utility of most of the mathematics much later, when it is applied to physical and engineering problems. The combined interdisciplinary study aims to resolve these issues.

Although the course starts at undergraduate level it gradually proceeds to research level and to the frontiers of current knowledge. The presentation of each subject takes into account from the very beginning not only the fundamentals but also the major topics of subsequent use. For example, the treatment of complex functions lays the basis for differential equations, integral transforms, asymptotics, and special functions. Linear algebra and analytic geometry lead to tensor analysis, differential geometry, variational calculus, and relativity. An introduction to fluid mechanics via the potential flow is followed by vortical, compressible, viscous, and multiphase flows. Electrostatics and magnetostatics are followed by unsteady electromagnetic fields and waves, magnetohydrodynamics, and plasmas. These apparently disparate subjects are treated at an early stage, analogies are presented, and at subsequent stages they are combined into multidisciplinary applications.

For example, a fluid may be subject to four restoring forces associated with pressure, gravity, rotation, and magnetic fields. The corresponding wave motions are respectively acoustic, internal, inertial, and magnetic waves. They appear combined as magneto–acoustic–gravity–inertial (MAGI) waves in a compressible, ionized, stratified rotating fluid. The simplest exact solutions of the MAGI wave equation require special functions. Thus the topic of MAGI waves combines six subjects: gravity field, fluid mechanics, and electromagnetism and uses a complex analysis, differential equations, and special functions. This is not such a remote subject, since many astrophysical phenomena do involve this combination of several of these effects, as does the technology of controlled nuclear fusion. The latter is the main source of energy in stars and in the universe; if harnessed, it would provide a clean and inexhaustible source of energy on earth. Closer to our everyday experience there is a variety of electromechanical and control systems that use modern interdisciplinary technology. The ultimate aim of the present series is to build up knowledge seamlessly from undergraduate to research level, across a range of subjects, to cover contemporary or likely interdisciplinary needs. This requires a consistent treatment of all subjects so that their combination fits together as a whole.

The approach followed in the present series is a combined study of mathematics, physics, and engineering, so that the practical motivation develops side by side with the theoretical concepts: the mathematical methods are applied without delay to "real" problems, not just to exercises. The electromechanical and other analogies simulate the ability to combine different disciplines, which is the basis of much of modern interdisciplinary science and technology. Starting with the simpler mathematical methods, and consolidating them with the detailed solutions of physical and engineering problems, gradually widens the range of topics that can be covered. The traditional method of separate monodisciplinary study remains possible, selecting mathematical disciplines (e.g., complex functions) or sets of applications (e.g., fluid mechanics). The combined multidisciplinary study has the advantage of connecting mathematics, physics, and technology at an earlier stage. Moreover, preserving that link provides a broader view of the subject and the ability to innovate. Innovation requires an understanding of technological aims, the physical phenomena that can implement them, and the mathematical methods that quantify the expected results. The combined interdisciplinary approach to the study of mathematics, physics, and engineering is thus a direct introduction to a professional experience in scientific discovery and technological innovation.

Preface

The present volume, *Complex Analysis with Applications to Flows and Fields*, consists of four parts; they present the theory of functions of a complex variable, starting with the complex plane (Part 1) and proceeding through the calculus of residues (Part 2) and power series (Part 3) to the conformal mapping (Part 4). The detailed applications cover twice as much space as the mathematical theory and concern potential flows, gravity field, electro- and magnetostatics, steady heat conduction, and other problems. The physical and engineering problems that cover about two-thirds of the text are main motivation; the mathematical results that occupy the remaining about one-third of the material are sufficient to fully justify the solution of problems without additional external references. The self-contained nature of the book concerns both the mathematical background and physical principles needed to formulate problems, justify the solutions, and interpret the results. It is backed by bibliography and other indices.

Organization of the Book

In each part the mathematical theory (physical and engineering applications) appear in alternating odd (even) numbered chapters, for example, Chapters 11, 13, 15, and 17 (12, 14, 16, and 18) in Part 2. The penultimate chapter, for example, Chapter 29 in Part 3, deals with some fundamental mathematical concepts. The last chapter of each part is a collection of 20 detailed examples, for example, Chapter 10 at the end of Part 1 consists of 20 worked out Examples 10.1 to 10.20. The chapters are numbered sequentially (Chapters 1 to 40). The formulae are numbered sequentially in a chapter between curved brackets, for example, (15.20) means formula 20 of Chapter 15. A chapter (e.g., 24) is divided into nine sections (e.g., 24.1 to 24.9); the section may be divided into subsections (e.g., 24.5.1 to 24.5.3). The figures are numbered by chapter (e.g., Figures 12.1 to 12.7 in Chapter 12). The conclusion of each chapter includes references to: (i) the figures as a kind of visual summary; (ii) the note(s), list(s), table(s), diagram(s) and classification(s) as additional support. The latter (ii) apply at the end of each chapter, and are numbered within the chapter (e.g., Note 24.1, List 29.1, Table 24.1, Classification 24.1, Diagram 21.1); if there is more than one they are numbered sequentially (e.g., Notes 24.1 to 24.3). The chapter starts with an introductory preview, and related topics may be mentioned in the notes at the end. The lists of mathematical symbols and physical quantities appear before the main text, and the index of subjects and bibliography at the end of the book.

About the Author

Luis Manuel Braga da Costa Campos was born in Lisbon, Portugal in 1950. He graduated in 1972 as a mechanical engineer from the Instituto Superior Tecnico (IST) of Lisbon Technical University. The tutorials as a student (1970) were followed by a career at the same institution (IST) through all levels: assistant on probation (1972), assistant (1974), auxiliary professor (1978), assistant professor (1982), chair of applied mathematics and mechanics (1985). He has been the coordinator of undergraduate and postgraduate degrees in aerospace engineering since their inception in 1991. He is coordinator of the Applied and Aerospace Mechanics Group in the Department of Mechanical Engineering. He is director and founder of the Center for Aeronautical and Space Science and Technology.

He completed his doctorate on "waves in fluids" at the engineering department of Cambridge University, England (1977). It was followed by a Senior Rouse Ball Scholarship at Trinity College, while on leave from IST. His first sabbatical was as a senior visitor at the Department of Applied Mathematics and Theoretical Physics of Cambridge University, England (1984). His second sabbatical (1991) was as a Alexander von Humboldt Scholar at the Max Planck Institute for Aeronomy in Katlenburg-Lindau, Germany. He could not pursue further sabbaticals abroad owing to major commitments at his home institution, which included extensive travels related to participation in scientific meetings, representation at individual or national levels in international institutions, and involvement in collaborative research projects.

He received the von Karman medal from the Advisory Group for Aerospace Research and Development (AGARD) and Research and Technology Organization (RTO). His participation in AGARD/RTO has been as vice-chairman of the System Concepts and Integration Panel, and as chairman of the Flight Mechanics Panel and of the Flight Vehicle Integration Panel. He has been a member of the Flight Test Techniques Working Group, which is related to the creation of an independent flight test capability active in Portugal for the past 20 years, and which has been used in national and international projects, including those from Eurocontrol and the European Space Agency. He has participated in various committees in the European Space Agency (ESA) as a national representative at the Council and Council of Minister levels.

The author has participated in several activities sponsored by the European Union. He has been involved in 27 research projects with industry, research, and academic institutions. He has been a member of various committees and has been vice-chairman of the Aeronautical Science and Technology Advisory Committee. He has been in the Space Advisory Panel on the future role of the EU in space. He has also been a member of the Space Science Committee of the European Science Foundation, and has been in close coordination with the Space Science Board of the National Science Foundation of the United States. He has been a member of the Committee for Peaceful Uses of Outer Space (COPUOS) of the United Nations. He has been working with these and other institutions as a consultant and advisor.

Regarding his contribution to professional societies, he has been a member and vice-chairman of the Portuguese Academic of Engineering; a fellow of the Royal Aeronautical Society, Astronomical Society, and Cambridge Philosophical Society; an associate fellow of the American Institute of Aeronautics and Astronautics; a founding member of the European Astronomical Society. He has been a member of various other professional associations in aeronautics, engineering, mechanics, acoustics, physics, astronomy, and mathematics. He is or has been a member of Editorial or Honorary Board of *Progress of Aerospace Sciences, Air & Space Europe, International Journal of Aeroacoustics, Revue d'Acoustique, Integral Transforms and Special Functions*. He is a reviewer in *Mathematical Reviews*, and has reviewed two dozen journals.

The author's publications include 4 books, 110 papers in 82 journals, and 160 communications in symposia. His areas of research center on four topics: acoustics, magnetohydrodynamics, special functions, and flight dynamics. His work on acoustics is related to the generation, propagation, and refraction of sound in flows with mostly aeronautical applications. His work on magnetohydrodynamics is related to magneto-acoustic-gravity-inertial waves in solar-terrestrial and stellar physics. The developments on special functions have been mostly based on differintegration operators, generalizing the ordinary derivative, and primitive to complex order. His work on flight dynamics is related to aircraft and rockets, including trajectory optimization, performance, stability, control, and atmospheric disturbances.

The author's interest in topics ranging from mathematics to physics and engineering fits in with the aims and content of the present series; his university teaching and scientific and industrial research relates to the build-up of the series from undergraduate to research level. His professional activities on the technical side are balanced by other cultural and humanistic interests. These are not reflected in publications, except for one book, which is a literary work. His complementary nontechnical interests include classical music (mostly orchestral and choral), plastic arts (painting, sculpture, architecture), social sciences (psychology and biography), history (classical, renaissance, and overseas expansion), and technology (automotive, photo, audio). He speaks four languages (Portuguese, English, French, and Spanish) and reads six (Italian and German). He is listed in various biographical publications, including *Who's Who in the World* since 1986.

Acknowledgments

The present book would require a long list of acknowledgments if all contributions were to be duly recorded. Those which follow are a mere selection based on subjective memory. I would like to first acknowledge the successive generations of university students to whom various parts of the course were taught over the years at Instituto Superior Tecnico; it is hoped that the experience they provided to me is reflected in this book for the benefit of future generations of students. I owe it Mr. Henrique Nuno for the drawings and to the following for typing my manuscript over a period of time: Fernanda Proença, Irene Patriarca, Fernanda Venâncio, Ana Monteiro, Bruno de Souza, Martinha de Sousa, Sónia Marques, Lurdes de Sousa and Sofia Pernadas. The members of my group who have taught similar subjects have made contributions in various ways; in alphabetical order of surname: A.J.M.N. Aguiar, F.S.R.P. Cunha, A.R.A. Fonseca, P.J.S. Gil, M.H. Kobayashi, F.J.P. Lau, J.M.G.S. Oliveira, and P.G.T.A. Serrâo. The final form of the present volume owes most to four persons: Mr. Jorge Coelho for the drawings; Professor J.M. André Junior for very helpful criticisms; Emeritus Professor A.G. Portela for several pages of written general and specific comments and suggestions. At last but not least, to my wife who more than deserves the dedication as the companion of the author in preparing this work.

Mathematical Symbols

The mathematical symbols are those of more common use in the context of (i) sets, quantifiers, and logic; (ii) numbers, ordering, and vectors; (iii) functions, limits, and convergence; (iv) derivatives, integrals and operators. It concludes with a list of functional spaces, most but not all of which appear in the present volume. The section where the symbol first appears may be indicated after a semicolon, for example, "10.2" means Section 10.2.

Sets, Quantifiers, and Logic

Sets

$A \equiv \{x : \ldots\}$ – set A whose elements x have the property.
$A \cup B$ – union of sets A and B.
$A \cap B$ – intersection of sets A and B.
$A \supset B$ – set A contains set B.
$A \subset B$ – set A is contained in set B.

Quantifiers

$\forall_{x \epsilon A}$ – for all x belonging to A holds …
$\exists_{x \epsilon A}$ – there exists at least one x belonging to A such that …
$\exists^1_{x \epsilon A}$ – there exists one and only one x belonging to A such that …
$\exists^\infty_{x \varepsilon A}$ – there exist infinitely many x belonging to A such that …

Logic

$a \wedge b$ – a and b.
$a \vee b$ – or (inclusive): a or b or both.
$a \dot{\vee} b$ – or (exclusive): a or b but not both.
$a \Rightarrow b$ – implication: a implies b.
$a \Leftrightarrow b$ – equivalence: a implies b and b implies a.

Constants

$e = 2.7182\ 81828\ 45904\ 52353\ 60287.$
$\pi = 3.1415\ 92653\ 58979\ 32384\ 62643.$
$\gamma = 0.5772\ 15664\ 90153\ 28606\ 06512.$
$\log 10 = 2.3025\ 85092\ 99404\ 56840\ 179915.$

Numbers, Ordering, and Vectors

Types of numbers

$|C$ – complex numbers: 1.2.
$|C^n$ – ordered sets of n complex numbers.
$|F$ – transfinite numbers.
$|H$ – hypercomplex numbers.
$|I$ – irrational numbers: real nonrational numbers: 1.2.
$|L$ – rational numbers: ratios of integers: 1.1.
$|N$ – natural numbers: positive integers: 1.1.
$|N_0$ – nonnegative integers: zero plus natural numbers: 1.1.
$|P$ – prime numbers: natural numbers without divisors.
$|Q$ – quaternions: 1.9.
$|R$ – real numbers: 1.2.
$|R^n$ – ordered sets of n real numbers.
$|Z$ – integers: 1.1.

Complex numbers

$|\ldots|$ – modulus of complex number...: 1.4.
$\arg(\ldots)$ – argument of complex number...: 1.4.
$\operatorname{Re}(\ldots)$ – real part of complex number...: 1.3.
$\operatorname{Im}(\ldots)$ – imaginary part of complex number...: 1.3.
$\ldots^*$ – conjugate of complex number...: 1.6.

Ordering of numbers

$\sup(\ldots)$ – supremum: smallest number larger or equal than all numbers in the set.
$\max(\ldots)$ – maximum: largest number in set.
$\min(\ldots)$ – minimum: smallest number in set.
$\inf(\ldots)$ – infimum: largest number smaller or equal than all numbers in set.

Vectors

$\vec{A}\cdot\vec{B}$ – inner product.
$\vec{A}\wedge\vec{B}$ – outer product.
$\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)$ – mixed product.
$\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right)$ – double outer product.
$\left|\vec{A}\right|$ – modulus.
$\operatorname{ang}\left(\vec{A},\vec{B}\right)$ – angle of vector $\vec{B}$ with vector $\vec{A}$.

Functions, Limits, and Convergence

Values of functions

$f(a)$ – value of function f at point a.
$f(a+0)$ – right-hand limit at a.
$f(a-0)$ – left-hand limit at a.
$f_{(n)}(a)$ – residue at pole of order n at a: 15.8.
$\overline{B}$ – upper bound: $|f(z)| \leq \bar{B}$ for z in ....
$\underline{B}$ – lower bound: $|f(z)| \geq \underline{B}$ for z in ....

Iterated sums and products

$\sum\limits_{a}$ – sum over a set.
$\sum\limits_{n=a}^{b}$ – sum from $n = a$ to $n =$ b.
$\sum\limits_{n,m=a}^{b}$ – double sum over $n{,}m = a, \ldots, b$.
$\prod\limits_{a}$ – product over a set.
$\prod\limits_{n=a}^{b}$ – product from $n = a$ to $n = b$.

Limits

lim – limit when x tends to a: $x \to a$: 11.2.
l.i.m. – limit in the mean.
$a \sim O(b)$ – a is of order b: $\lim b/a \neq 0{,}\infty$: 19.7.
$a \sim o(b)$ – b is of lower order than a: $\lim b/a = 0$: 19.7.

Convergence

A.C. – absolutely convergent: 21.2.
A.D. – absolutely divergent: 21.2.
C. – convergent: 21.2
C.C. – conditionally convergent: 21.2.
Cn – converges to class n: $C0 \equiv C$.
D. – divergent: 21.1.
N.C. – nonconvergent: divergent or oscillatory: 21.1.
O. – oscillatory: 21.1.
T.C. – totally convergent: 21.7.
U.C. – uniformly convergent: 21.5.
applies to:
- power series: 21.1.
- series of fractions: 27.9.
- infinite products: 27.9.
- continued fractions.

Derivatives, Integrals, and Operators

Differentials and Derivatives

$\mathrm{d}\Phi$ – differential of Φ.

$\mathrm{d}\Phi/\mathrm{d}t$ – derivative of Φ with regard to t.

$\partial\Phi/\partial t \equiv \partial_t\dot{\Phi} \equiv \dot{\Phi}$ – partial derivative of Φ with regard to t.

$\partial\Phi/\partial x_i \equiv \partial_i\Phi \equiv \Phi_{,i}$ – partial derivative of Φ with regard to x_i.

$\partial^n\Phi/\partial x_{i_1}\ldots\partial x_{i_n} \equiv \partial_{i_1}\ldots\partial_{i_n}\Phi \equiv \Phi_{,i_1,\ldots i_n}$ – nth-order partial derivative of Φ with regard to $x_{i_1}, \ldots, x_{i_n}$.

Directed Derivatives

$\partial\Phi/\partial\ell \equiv \vec{\ell}.\nabla\Phi \equiv \ell_i\partial_i\Phi \equiv \ell_i\Phi_{,i}$ – derivative of Φ in the direction of the unit vector $\vec{\ell}$: 18.1

$\partial\Phi/\partial n \equiv \vec{n}\cdot\nabla\Phi$ – derivative in the direction normal to a curve: 18.1.

$\partial\Phi/\partial N \equiv \vec{N}\cdot\nabla\Phi$ – derivative in the direction normal to a surface: 28.1.

$\partial\Phi/\partial s \equiv \vec{s}\cdot\nabla\Phi$ – derivative in the direction of the unit tangent to a curve: 18.1

Invariant Operator

$\nabla\Phi$ – gradient of a scalar Φ: 11.7.

$\nabla.\vec{A}$ – divergence of a vector $\vec{A}$: 11.7.

$\nabla\wedge\vec{A}$ – curl of a vector $\vec{A}$: 11.7.

$\nabla^2\Phi \equiv \Delta\Phi$ – Laplacian of a of scalar Φ: 11.7.

Integrals

$\int\ldots\mathrm{d}x$ – primitive of ... with regard to x: 13.1.

$\int^y\ldots\mathrm{d}x$ – indefinite integral of ... at y: 13.2.

$\int_a^b\mathrm{d}x$ – definite integral of ... between a and b: 13.2.

$⨍_a^b\cdots\mathrm{d}x$ – principal value of integral: 17.8.

$\int^{(z+)}$ – integral along a loop around z in the positive (counterclockwise) direction: 13.5.

$\int^{(z-)}$ – idem in the negative (clockwise) direction: 13.5.

$\int_L$ – integral along a path L: 13.2.

$\int_C^{(+)}$ – integral along a closed path or loop C in the positive direction: 13.5.

$\int_C^{(-)}$ – integral along a closed path or loop C in the negative direction: 13.5.

Functional Spaces

The sets of numbers and spaces of functions are denoted by calligraphic letters, in alphabetical order:
(a, b) – set of functions over interval from a to b.
omission of interval: set of function over real line $)-\infty,+\infty($.
$\mathcal{A}(\ldots)$ – analytic functions in $\ldots$: 27.1.
$\bar{\mathcal{A}}(\ldots)$ – monogenic functions in $\ldots$: 31.1.
$\mathcal{B}(\ldots)$ – bounded functions in $\ldots$: $\mathcal{B} \equiv \mathcal{B}^0$: 13.3.
$\mathcal{B}^n(\ldots)$ – functions with bounded n-th derivative in $\ldots$.
$\mathcal{C}(\ldots)$ – continuous functions in $\ldots$: $\mathcal{C} \equiv \mathcal{C}^0$: 11.2.
$\mathcal{C}^n\ (\ldots)$ – functions with continuous n-th derivative in $\ldots$.
$\bar{\mathcal{C}}(\ldots)$ – piecewise continuous functions in $\ldots$: $\bar{C} \equiv \bar{C}^0$.
$\bar{\mathcal{C}}^n(\ldots)$ – functions with piecewise continuous n-th derivative in $\ldots$.
$\widetilde{\mathcal{C}}(\ldots)$ – uniformly continuous function in…: 13.4.
$\widetilde{\mathcal{C}}^n(\ldots)$ – function with uniformly continuous n-th derivative in….
$\mathcal{D}\ (\ldots)$ – differentiable functions in $\ldots$:$\mathcal{D} \equiv \mathcal{D}^0$: 11.2.
$\mathcal{D}^n\ (\ldots)$ – n-times differentiable functions in $\ldots$.
$\mathcal{D}^\infty\ (\ldots)$ – infinitely differentiable functions or smooth in $\ldots$ 27.1.
$\bar{\mathcal{D}}(\ldots)$ – piecewise differentiable functions in $\ldots$:$\bar{\mathcal{D}} \equiv \bar{\mathcal{D}}^0$.
$\bar{\mathcal{D}}^n(\ldots)$ – functions with piecewise n-th derivative in $\ldots$.
$\mathcal{E}(\ldots)$ – Riemann integrable functions in $\ldots$: 13.2.
$\bar{\mathcal{E}}(\ldots)$ – Lebesgue integrable functions in $\ldots$
$\mathcal{F}(\ldots)$ – functions of bounded fluctuation (or bounded variation) in $\ldots$; $\mathcal{F} \equiv \mathcal{E} \equiv \mathcal{F}^0$.
$\mathcal{F}^n(\ldots)$ – functions with n-th derivative of bounded fluctuation (variation) in $\ldots$.
$\mathcal{G}(\ldots)$ – generalized functions (or distributions) in $\ldots$.
$\mathcal{H}(\ldots)$ – harmonic functions in $\ldots$: 11.6.
$\mathcal{I}(\ldots)$ – integral functions in $\ldots$: 27.9.
$\mathcal{I}_m(\ldots)$ – rational-integral functions of degree m in $\ldots$ $I \equiv I_0$: 27.9.
$\mathcal{J}(\ldots)$ – square integrable functions with a complete orthogonal set of functions – *Hilbert space.*
$\bar{\mathcal{K}}(\ldots)$ – Lipshitz functions in $\ldots$.
$\mathcal{K}^n(\ldots)$ – homogeneous functions of degree n in $\ldots$.
$\mathcal{L}^1(\ldots)$ – absolutely integrable functions in $\ldots$.
$\mathcal{L}^2(\ldots)$ – square integrable functions in $\ldots$.
$\mathcal{L}^p(\ldots)$ – functions with power p of modulus integrable in $\ldots$ - *normed* space: $\mathcal{L}^p \equiv \mathcal{W}^p_0$.
$\mathcal{M}^+(\ldots)$ – monotonic increasing functions in $\ldots$.
$\mathcal{M}^+_0(\ldots)$ – monotonic nondecreasing functions in $\ldots$.
$\mathcal{M}^-_0(\ldots)$ – monotonic nonincreasing functions in….
$\mathcal{M}^-(\ldots)$ – monotonic decreasing functions in $\ldots$.
$\mathcal{N}(\ldots)$ – null functions in $\ldots$.
$\mathcal{O}(\ldots)$ – orthogonal systems of functions in $\ldots$.
$\widetilde{\mathcal{O}}(\ldots)$ – complete orthogonal systems of functions in $\ldots$.
$\mathcal{P}(\ldots)$ – polynomials in $\ldots$: 27.7.
$\mathcal{P}_n(\ldots)$ – polynomials of degree n in $\ldots$: 27.7.
$\mathcal{Q}(\ldots)$ – rational functions in…: 27.7.
$\mathcal{Q}^m_n(\ldots)$ – rational functions of degrees n, m in…: 27.7.
$\mathcal{R}(\ldots)$ – real functions, that is, with the real line as range.
$\mathcal{S}(\ldots)$ – complex functions, that is, with the complex plane as range.

$\mathcal{T}(\dots)$ – functions with compact support, that is, which vanish outside a finite interval.
$\mathcal{T}^n(\dots)$ – temperate functions of order n: n-times differentiable functions with first $(n-1)$ derivatives with compact support.
$\mathcal{T}^\infty(\dots)$ – temperate functions: smooth or infinitely differentiable functions with compact support.
$\mathcal{U}(\dots)$ – single-valued functions in…: 9.1.
$\widetilde{\mathcal{U}}(\dots)$ – injective functions in…: 9.1.
$\bar{\mathcal{U}}(\dots)$ – surjective functions: 9.1.
$\widetilde{\bar{\mathcal{U}}}(\dots)$ – bijective functions: 9.1.
$\mathcal{U}_n(\dots)$ – multivalued functions with n branches in…: 6.1.
$\mathcal{U}_\infty(\dots)$ – many-valued functions in…:6.2.
$\mathcal{U}^1(\dots)$ – univalent functions, in …: 37.4.
$\mathcal{U}^m(\dots)$ – multivalent functions taking m values in …: 37.4.
$\mathcal{U}^\infty(\dots)$ – manyvalent functions in …: note 37.4.
$\mathcal{U}_n^m(\dots)$ – multivalued multivalent functions with n branches and m values in …: note 37.4.
$\mathcal{V}(\dots)$ – good functions, that is, with decay at infinity faster than some power.
$\mathcal{V}^N(\dots)$ – good functions of degree N, that is, with decay at infinity faster than the inverse of a polynomial of degree N.
$\widetilde{\mathcal{V}}(\dots)$ – fairly good functions, that is, with growth at infinity slower than some power.
$\widetilde{\mathcal{V}}^N(\dots)$ – fairly good functions of degree N, that is, with growth at infinity slower than a polynomial of degree N.
$\bar{\mathcal{V}}(\dots)$ – very good or fast decay functions, that is, with faster decay at infinity than any power.
$\mathcal{W}_q^p(\dots)$ – functions with generalized derivatives of orders up to q such that for all the powers p of the modulus is integrable …—*Sobolev* space.
$\mathcal{X}_0(\dots)$ – self-inverse linear functions in …: 37.5.
$\mathcal{X}_1(\dots)$ – linear functions in …: 35.2.
$\mathcal{X}_2(\dots)$ – bilinear, homographic, or Mobius functions in …: 35.4.
$\mathcal{X}_3(\dots)$ – self-inverse bilinear functions in …: 37.5.
$\mathcal{X}_a(\dots)$ – automorphic functions in …: 37.6.
$\mathcal{X}_m(\dots)$ – isometric mappings in …: 35.1.
$\mathcal{X}_r(\dots)$ – rotation mappings in …: 35.1.
$\mathcal{X}_t(\dots)$ – translation mappings in …: 35.1.
$\mathcal{Y}(\dots)$ – meromorphic functions in …: 37.9.
$\mathcal{Z}(\dots)$ – polymorphic functions in …: 37.9.

Physical Quantities

The physical quantities are denoted by lower- or uppercase arabic or greek letters. Some are also represented by mathematical symbols. Calligraphic (Gothic or Old English) uppercase letters are reserved for functional spaces (geometries and associated coordinate transformation groups). The subscripts may be omitted to simplify and lighten the text and equations when there is no ambiguity or risk of confusion; they are introduced again when a distinction is necessary. For example the complex potential for the electrostatic field f_e is denoted by f, except when in comparison with others, e.g., the complex potential for the velocity of a potential flow f_v. The latter is also denoted by f when there is no risk of ambiguity.

Lowercase Arabic Letters

a – radius of cylinder: 24.6.
$\vec{\mathrm{a}}$ – acceleration: 2.1.
b – distance of point multipole from cylinder axis: 24.6.
c – phase speed of waves: 22.1.
c_* – speed of light: 26.1.
c_0 – speed of sound: 22.4.
e – electric charge: 6.1.
f – complex potential: 12.3, for example, for the velocity of a potential flow f_v.
f_e – complex potential of an electrostatic field: 24.3.
f_g – complex potential of a gravity field: 18.4.
f_m – complex potential of a magnetostatic field: 26.3.
f_v – complex potential for the velocity of a potential flow: 12.3.
$\vec{\mathrm{f}}$ – force density per unit volume, area or length: 14.2.
$\vec{\mathrm{f}}_b$ – dilatation force density: 28.3.
$\vec{\mathrm{f}}_e$ – electrical force density: 28.3.
$\vec{\mathrm{f}}_{em}$ – electromagnetic force density: 28.3.
$\vec{\mathrm{f}}_g$ – gravity force density: 28.3.
$\vec{\mathrm{f}}_l$ – vortical force density or Lamb vector: 28.3.
$\vec{\mathrm{f}}_m$ – magnetic force density: 28.3.
$\vec{\mathrm{f}}_n$ – stagnation force density: 28.3.
$\vec{\mathrm{f}}_p$ – hydrodynamic force density: 28.3.
$\vec{\mathrm{g}}$ – acceleration of gravity: 18.3.
g^* – complex conjugate gravity field: 18.4.
h_i – scale factors: 11.9.
i – imaginary unit $i \equiv \sqrt{-1}$: 1.2.
$\vec{\mathrm{j}}$ – electric current per unit area: 26.5.

k – wavenumber: 22.1.
  – factor in the induced drag coefficient: 34.9.
$\vec{k}$ – wavevector: 12.1.
m – mass: 2.9.
m_0 – added mass: mass of fluid extrained by a body in motion: 28.6.
$\overline{m}$ – total mass: mass of a body plus added mass: 28.6.
n – coordinate normal to a curve: 18.1.
$\vec{n}$ – unit normal vector to a curve: 12.1.
p – pressure: 14.2.
p_0 – stagnation pressure: 14.5.
p_n – multipole strength per unit area: 26.4.
q – flow rate of source/sink per unit area or length: 18.1.
  – electric charge density per unit area: 24.1.
q_n – multipole strength per unit length: 26.4.
r – polar coordinate: 1.4.
s – arc length: 11.9.
  – coordinate tangent to a curve: 18.1.
$\vec{s}$ – unit tangent vector to a curve: 12.1.
t – time: 2.1.
v – complex velocity: 6.3.
v^* – complex conjugate velocity: 12.3.
$\vec{v}$ – velocity vector: 6.1.
w – density per unit area of heat source/sink: 32.1.
  – downwash velocity behind wing: 34.7.
x – Cartesian coordinate: 1.2.
$\vec{x}$ – position vector of observer: 6.5.
y – Cartesian coordinate: 1.2.
z – complex number: 1.2
  – Cartesian coordinate: 6.5.

Capital Arabic Letters

A – admittance: 4.4.
  – amplitude of a wave: 22.1.
$\vec{B}$ – magnetic induction vector: 26.1.
C_D – drag coefficient: 28.4, 34.6.
C_L – lift coefficient: 28.5, 34.6.
C_M – pitching moment coefficient: 28.7, 34.6.
D – drag force: 28.2.
$\vec{D}$ – electric displacement vector: 24.1.
E – energy: 8.5.
E_e – electrical energy: 24.2.
E_k – kinetic energy: 8.5.
E_m – magnetic energy: 26.2.
$\vec{E}$ – electric field vector: 24.1.
E^* – complex conjugate electric field: 24.3
F – force vector: inertia force: 2.9.

F_b – dilatation force: 28.3.
F_e – electric force: 24.3, 28.3.
F_{em} – electromagnetic or Laplace–Lorentz force: 6.1, 28.3.
F_g – gravity force: 18.5, 28.3.
F_l – vortical force: 28.3.
F_m – magnetic force: 26.3, 28.3.
F_n – stagnation force: 28.3.
F_p – hydrodynamic force: 28.2.
F_s – suction force: 34.1.
G – gravitational constant: 18.3.
$\vec{\mathrm{G}}$ – heat flux: 32.1.
H – enthalpy: 14.3.
$\vec{\mathrm{H}}$ – magnetic field vector: 26.1.
$\vec{\mathrm{J}}_e$ – electric current: 26.1.
$\vec{\mathrm{J}}_v$ – mass flux: 24.1.
L – lift: 28.2.
M – pitching moment: 28.2.
$\vec{\mathrm{M}}$ – moment of forces: 28.2.
N – coordinate normal to a surface: 28.1.
$\vec{\mathrm{N}}$ – unit vector normal to a surface: 18.1.
P_n – moment of 2^n-multipole: 12.9 (e.g., monopole P_0, dipole P_1, quadrupole P_2).
Q_v – volume flow rate: 12.2.
$\vec{\mathrm{Q}}_e$ – electric polarization vector: 24.1.
$\vec{\mathrm{Q}}_m$ – magnetic polarization vector: 26.1.
R_N – remainder of a series after N terms: 21.1.
S – area element of a surface: 28.1.
– entropy density: 14.3.
$\overline{S}$ – total entropy in a domain: 32.1.
S_N – sum of the first N terms of a series: 21.1.
T – temperature: 32.1.
X – resistance: 4.4.
Y – reactance: 4.4.
Z – impedance: $Z = X + iY$: 4.4.

Lowercase Greek Letters

α – angle-of-attack of a flow: 14.8.
β – internal angle in a corner: 24.9.
χ_e – electric susceptibility: 24.1.
χ_m – magnetic susceptibility: 26.1.
ε – dielectric permittivity: 24.1.
ϕ – phase of a wave: 22.9.
γ – circulation density per unit length: 18.2.
– external angle in a corner: 33.5.
– adiabatic exponent: 14.6.
φ – polar angle: 1.4.
κ – thermal conductivity scalar in isotropic medium: 22.1.

λ – damping: 2.3.
  – wavelength: $\lambda = 2\pi/k$: 12.1.
μ – mass density per length: 18.4.
  – magnetic permeability: 26.1.
ω – angular frequency: 2.2, 22.1.
ρ – mass density per unit area: 14.1.
σ – electric charge density per unit length: 24.2.
τ – period $\tau = 2\pi/\omega$: 2.2.
$\vec{\varpi}$ – vorticity: 14.3.

Capital Greek Letters and Others

Φ – scalar potential: 12.1.
Φ_e – electric potential: 24.2.
Φ_g – gravity potential: 18.3.
Φ_m – magnetic potential: 26.7.
Φ_v – velocity potential: 12.1.
Γ – circulation: 12.1.
ϑ – electric current per unit length: 26.2.
Λ – dilatation: 12.2.
$\vec{\Omega}$ – angular velocity: 6.1.
Ψ_e – field function of electric field: 24.6.
Ψ_{g} – field function of gravity field: 18.3.
Ψ_m – field function of magnetic field: 26.2.
Ψ_v – stream function: 22.2.

Part 1
Complex Domain: Circuits and Stability

The complex numbers are the simplest for which all direct (sum, product, power) and inverse (subtraction, division, root) operations are closed (Chapter 1), that is, when applied to complex numbers these operations always lead to complex numbers (Chapters 3 and 5). Since a complex number is an ordered pair of real numbers, it can be represented on the plane (Chapter 1), and the corresponding geometry has a number of features: (i) the complex plane maps one-to-one to a sphere (Chapter 9), and has only one point-at-infinity; (ii) a multivalued function (Chapter 7) can be represented with each branch on a sheet of a Riemann surface, with the sheets connected at branch-points, and separated by branch-cuts. Since a complex number involves two real numbers, it allows some two-dimensional motions to be represented by one variable, for example, an electron moving transversely to a magnetic field (Chapter 6); it also allows the combination of several properties of mechanical or electrical circuits into a single complex impedance (Chapter 4). Likewise a complex function involves two real functions, and thus can represent in a single expression several kinds of motion (Chapter 2), for example, oscillatory, damped, or unstable regimes (Chapter 8).

1

Complex Numbers and Quaternions

The complex numbers appear at the end of a hierarchy (Sections 1.1 and 1.2) formed by the positive integers, integers, rationals, irrationals, and real numbers; the complex numbers are the simplest for which all three direct (sum, product, power) and inverse (subtraction, division, root) operations are closed, that is, when applied to a complex number the result is also a complex number. A complex number is an ordered pair of real numbers, which can be represented as a point on the plane (Sections 1.3–1.6); this provides a graphical illustration of some properties of elementary real and complex functions (Sections 1.7 and 1.8). A quaternion (Section 1.9) is a generalization of complex number in four dimensions for which the product is noncommutative. Further generalizations of the concept of number (transfinite, hypercomplex) may have less properties. The operations like sum, subtraction, product, division, power, and root may be applied not only to numbers but also to other entities (functions, multiplicities, sets, rings) as long as they retain similar properties.

1.1 Peano (1889, 1891) Postulates for Natural Numbers

An operation $\otimes$ is ***closed*** with regard to a set of numbers iff (if and only if) when applied to any two elements x, y of the set, the result is an element of the set:

$$\otimes \text{ closed in } A\text{: } \forall_{x,y\in A} \Rightarrow x \otimes y \in A. \tag{1.1}$$

Next, consider the three **direct operations** (sum, product, and power) and three **inverse operations** (subtraction, division, and root) to ascertain whether they are closed with regard to sets of numbers (natural, integer, rational, real, complex, and quaternions) in Table 1.1.

The **natural numbers** correspond to the usual counting of objects:

$$|N \equiv \{1, 2, 3, \ldots, \infty\}\,. \tag{1.2}$$

The natural numbers $|N$ can be introduced by the five **Peano postulates** (1889, 1891), via a **successor function**, S, as follows: (i) the number "1" is an integer $1 \in |N$; (ii) the successor of an integer is an integer $n \in |N \Rightarrow Sn \in |N$; (iii) the successor is not unity: $n \in |N \Rightarrow Sn \neq 1$; (iv) identical successors correspond to the same number: $n, m \in |N \wedge Sn = Sm \Rightarrow n = m$; and (v) if a property is true for "one" and every successor it is true for all integers: $1 \in M\ \forall n (n \in M \Rightarrow Sn \in M) \Rightarrow |N \subseteq M$. The last property (v) is the **axiom of induction**, *viz. if a property holds (i) for* $n = 1$*, and if (ii) holding for* n *implies holding for* $n + 1$*, then it holds for all* n *natural numbers.* The five axioms are **independent**, because it is possible to define a system which satisfies all but one, viz. (a) the set {2,3,...} satisfies all properties (ii–v) except (i); (b) the set $|N \cup (1/2)$ with $S1 = 1/2$ and $S{}^1\!/_2 = 2$ only fails (ii); (c) the set {1} fails only (iii); (d) the set {1,2} with $S1 = S2 = 1$ fails only (iv); and (e) the set $|N \cup (-1)$ fails only (v).

TABLE 1.1
Sets of Numbers and Algebraic Operations

Set	Natural	Integers	Rational	Real	Complex	Quaternions
Symbol	$\lvert N$	$\lvert Z$	$\lvert L$	$\lvert R \equiv \lvert L \cup \lvert I$	$\lvert$C	$\lvert Q$
Sum	×	×	×	×	×	×
Subtraction	–	×	×	×	×	×
Product	×	×	×	×	×	× (1)
Division	–	–	×	×	×	×
Power	×	×	–	×	×	×
Root	–	–	–	–	×	–

Note: Successively larger sets of numbers: positive integers or natural $|N$, integers $|Z$, rational $|L$, irrational $|I$, real $|R$, complex $|C$, and quaternions $|Q$. The operations closed (not closed) with regard to a set are marked with a cross x (slash –). The direct operations (sum, product, power) are always closed. The inverse operations (subtraction, division, root) become closed enlarging the set of numbers. The complex numbers are the simplest for which all operations indicated are closed. Beyond the complex numbers, for example, for quaternions, some properties are lost. x: operation is closed with regard to the set; –: operation is not closed with regard to the set; (1): product is noncommutative.

The idea of the Peano postulates is to start with a number and define the remaining through the successor function. The natural numbers are closed with regard to the direct operations: sum, product, power. The number **zero (one)** is the **neutral element** of the sum (product), viz. $n + 0 = n = 0 + n\,(1 \times n = n = n \times 1)$ for all $n \in |N$. Starting the Peano postulate (i) with one (zero) leads to the **positive (nonnegative) integers** $|N$ in (1.2) [$|N_0$ in (1.3)]:

$$|N_0 \equiv \{0, 1, 2, 3, \ldots, \infty\} \equiv \{0\}\, U\, |N. \tag{1.3}$$

Although the natural numbers are closed with regard to the sum, they are not closed with regard to the subtraction. The difference of two positive integers $n - m$ is a positive integer if n is **larger** ($n > m$) than m, but has no solution in the set of positive integers if n is **smaller** ($n < m$) than m. This difficulty is overcome by extending the set of natural numbers to the set of ***integers***:

$$|Z \equiv \{0, \pm 1, \pm 2, \ldots, \pm\infty\}, \tag{1.4}$$

consisting of **positive** $n \in |N$ and **negative** $-n \in |N$ integers that have same *modulus* $|n| = -|n|$ and opposite signs, plus zero.

The integers are closed with respect to the sum, subtraction, and product, but not with regard to the division: the ratio n/m of two integers, apart from sign $|n|\,/\,|m|$, is an integer iff $|n|$ is a **multiple** of $|m|$ or $|m|$ is **a submultiple** of $|n|$, viz.: $|n| = |m|\,p$ with $p \in |N$; otherwise, the ratio does not exist as an integer number. This difficulty is resolved by extending further the set of integers to the set of **rational** numbers:

$$|L \equiv \{x, y \in |N : x/y \in |L\}, \tag{1.5}$$

which are ordered pairs of integers, specified by their ratio. The rational numbers are closed with regard to the three direct operations (sum, product, and power) and also with regard to two of the three *inverse* operations (subtraction and division). However, the rational numbers are not closed with regard to the third inverse operation, viz. the root, for which three cases arise: (i) the Nth root of a rational number is a rational number $q = \sqrt[N]{p}$ iff the Nth power of q is $p = q^N$; (ii) there are positive rational numbers $p > 0$ whose roots are not rational numbers, for example, $\sqrt{2}$ is not rational as will be proved in the sequel

(Section 1.2); and (iii) a negative rational number never has a square root as a rational number. The lifting of the limitation (ii) leads to irrational and real numbers (Section 1.2) and (iii) is resolved by the introduction of complex numbers (Sections 1.3–1.8, Chapter 3).

1.2 Irrational Numbers (Pythagoras, VI b.c.) and Dedekind (1858) Section

In the rational numbers can be introduced a **strict order relation**: of two distinct numbers one must be larger than the other:

$$\forall p, q \in |L : p \neq q \Rightarrow p > q \dot{\vee} q > p; \tag{1.6}$$

the condition (1.6) is (i) **exclusive**, if $p > q$ excludes the possibility $q > p$; and (ii) **exhaustive**, if $p \neq q$ there are no possibilities other than $p > q$ or $q < p$. The proof that $\sqrt{2}$ is irrational (Pythagoras VI b.c.) can be made by "reduction ad absurdum": (i) suppose that $\sqrt{2}$ is rational and specified by the **irreducible fraction** (1.7a), that is, in its lowest terms:

$$\sqrt{2} = \frac{p}{q}; \quad \sqrt{2} = \frac{2q - p}{p - q}; \tag{1.7a,b}$$

(ii) then it is also given by (1.7b) for any p, q, since

$$0 = (2q - p)^2 - 2(p - q)^2 = 2q^2 - p^2; \tag{1.7c}$$

(iii) choosing q such that $0 < p - q < q$ it follows from (1.7b) that the denominator in (1.7a) is not in its lowest terms. This contradiction proves that $\sqrt{2}$ cannot be a rational number. This leads to the introduction of the irrational numbers, in order to make the square root a closed operation for positive arguments.

One way to introduce irrational numbers is through the **Dedekind section** (1858): given a number p, the **upper (lower) class** are sets such that (i) each class has at least one number; and (ii) all numbers of the upper class $y \in p^+$ exceed all numbers of the lower class $x \in p^-$, viz. $y > x$. *A rational number* p *specifies a Dedekind section: it is the largest (or smallest) member, that is,* **supremum (infimum)**, *of the lower (upper) class; if a section is not specified by a rational number, then it defines an* **irrational** *number, for example,* $\sqrt{2}$. The operations on rational numbers may be extended to irrational numbers, as in the example of the sum: (i) let p, q be two numbers, and (p^+, p^-) and (q^+, q^-) their respective (upper, lower) classes; (ii) their sums form a lower class $p^- + q^-$ whose elements are all less than those of the upper class $p^+ + q^+$; and (iii) the resulting Dedekind section is the sum $p + q$ of the two numbers.

Thus, the set of **real** numbers $|R$ can be defined as the union of sets of rational $|L$ and irrational $|I$ numbers:

$$|R \equiv |L \cup |I = \{x : -\infty < x < +\infty\}. \tag{1.8}$$

The set of real numbers establishes a one-to-one correspondence with the points on a real line: some of these points correspond to rational numbers, others to irrational numbers. There are in a given finite interval $a \leq x \leq b$: (i) a **finite** number $M - N + 1$ integers $a \leq N \leq x \leq M \leq b$, where $N - 1 = f(a)$ and $M = f(b)$ is the **integer part** of a real number, obtained by omitting the decimals; (ii) there is **denumerable infinity** of rational numbers x/p, actually less than the number of ordered pairs of integers (x, p), because some may not be irreducible fractions, for example, $2x/2p = x/p$; and (iii) there is a **nondenumerable infinity** of irrational numbers.

1.3 Cartesian Parts: Real and Imaginary (Argand, 1806; Descartes, 1637a; Gauss, 1797)

The extension from the rational to the real numbers makes the square roots $\sqrt{p}$ of a positive number $p > 0$ closed, but there remains another problem: a negative number $p = -|p| < 0$ has no real square root $\sqrt{-|p|}$. For operation square root to be closed, the real numbers have to be extended to the complex numbers, defined (1.9a) as an ordered pair of real numbers:

$$|C \equiv \{x, y \in |R : z = x + iy\}, \quad i^2 = -1, \tag{1.9a,b}$$

where the symbol "i" satisfies (1.9b) as justified subsequently in Section 3.4. Note that (i) the direct operations (sum, product, power) are closed for all sets of numbers, viz. natural $|N$, integer $|Z$, rational $|L$, and real $|R$; (ii) the inverse operations (subtraction, division, root) may not be closed, and their closure may require the extension of the concept of number. Three examples of the process (ii) were given: (i) the subtraction becomes closed (1.1) by extending the natural numbers (1.2, 1.3) to the integers (1.4); (ii) the division becomes closed extending the integers (1.4) to the rational numbers (1.5); and (iii) the square root becomes closed extending the rational (1.5) to the real (1.8) and complex numbers (1.9a,b). Thus, the *complex numbers are the simplest for which all direct (sum, product, power) and inverse (subtraction, division, roots) operations are closed*, as shown in Table 1.1. This justifies the emphasis on the properties of (Sections 1.3–1.8) and operations on (Chapter 3) complex numbers.

A complex number is defined as an ordered pair (x,y) of real numbers, for which the notation usually adopted is (1.10a):

$$z = x + iy : \quad x \equiv \mathrm{Re}\,(z), \quad y \equiv \mathrm{Im}\,(z), \tag{1.10a–c}$$

where Re (Im) denote the **real (imaginary) parts** (1.10b) [(1.10c)]. The real and imaginary parts can be taken as coordinates, respectively, x,y in a Cartesian frame of reference (Descartes, 1637a), so that complex number z is represented by a point in Argand's plane (1806, used before by Gauss in 1797). There exists a one-to-one correspondence between the set $|C$ of complex numbers z, the set $|R \times |R$ of ordered pairs (x,y) of real numbers, and the points P on the complex plane.

1.4 Polar Coordinates: Modulus and Argument

The distance from the point P in Figure 1.1, which represents the complex number z, from the origin defines the **modulus**:

$$z = x + iy : \; r = |z| \equiv \left|x^2 + y^2\right|^{1/2}, \tag{1.11}$$

where $|\ldots|^{1/2}$ means the square root taken with plus sign before the radical, that is, the positive square root. The angle between the straight line OP and the real axis OX, measured

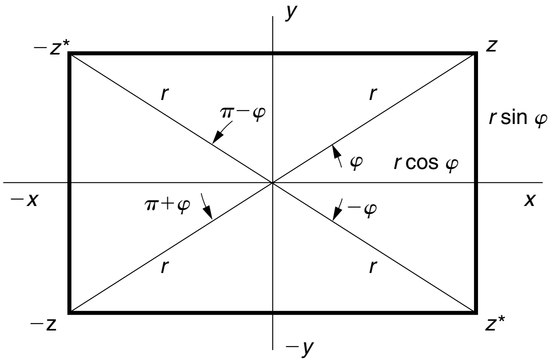

FIGURE 1.1
A complex z is transformed by symmetry relative to the (i) real axis to its conjugate z^*; (ii) the origin to the symmetric $-z$; and (iii) to the imaginary axis to the conjugate symmetric $-z^*$.

from the latter, in the **positive direction**, that is, c*ounterclockwise*, defines the **argument** of the complex number z:

$$\varphi \equiv \arg(z) = \begin{cases} \arctan(y/x) & \text{for } x \geq 0, \quad (1.12\text{a}) \\ \pi + \arctan(y/x) & \text{for } x \leq 0; \quad (1.12\text{b}) \end{cases}$$

in (1.12a,b) it has been taken into account that the inverse circular tangent arc tan is one-to-one in the right-hand half circle $-\pi/2 < \varphi \leq +\pi/2$, corresponding to the upper line (1.12a); the lower line (1.12b) adds left-hand half circle $\pi/2 < \varphi \leq 3\pi/2$ to build-up the full circle $-\pi/2 < \varphi \leq 3\pi/2$, or, equivalently, $-\pi < \varphi \leq +\pi$ or $0 \leq \varphi < 2\pi$. The latter range is denoted by $(0, 2\pi($, and it will henceforth be assumed that the function arc tan will be extended to this range in the form (1.12a,b) whenever necessary. *Thus a complex number can be uniquely represented by (1.11) its modulus $r \geq 0$ and (1.12a,b) argument $0 \leq \varphi < 2\pi$, except at the origin and infinity* r $\neq 0, \infty$, because (i) at origin $r = 0$ the argument φ is undetermined; (ii) at infinity $r = \infty$ there is only one point (as shown in Section 9.2), and the argument φ is again undetermined, because the "point at infinity" can be reached in any direction. The modulus, argument, and real and imaginary parts of complex numbers can be used to represent regions of the plane (Example 10.1).

1.5 Moivre's Formula, Origin and Infinity

The failure of the polar representation (r, φ) at the origin and infinity can be justified analytically by starting from the Cartesian representation (x,y), which is valid everywhere. They are related (Figure 1.1) by

$$x = r\cos\varphi, \quad y = r\sin\varphi; \tag{1.13a,b}$$

this leads to Moivre's formula,

$$z = x + iy = r(\cos\varphi + i\sin\varphi) = r\mathrm{e}^{i\varphi}, \tag{1.14}$$

for the complex number z, where the notation used was

$$e^{i\varphi} \equiv \cos\varphi + i\sin\varphi; \tag{1.15}$$

this can be justified in the context of the theory of elementary transcendental functions. *The polar* (r,φ) *and Cartesian* (x,y) *representations of a complex number (1.14) are related by (1.13a,b) and its inverse (1.11;1.12a,b) for* $r \neq 0,\infty$. The inversion fails if the Jacobian is zero or infinity:

$$\begin{vmatrix} \partial x/\partial r & \partial x/\partial\varphi \\ \partial y/\partial r & \partial x/\partial\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r \neq 0,\infty, \tag{1.16}$$

that is, at the origin $r = 0$ and point at infinity $r = \infty$, where the argument φ is undetermined. The complex numbers gain relative to the real numbers by the property that the root, or inverse of power, is a closed operation; they loose the strict ordering property (1.6), which (i) holds for rational numbers; (ii) extends to real numbers, by the Dedekind section; and (iii) does not extend to complex numbers. For example, (a) complex numbers can be partially ordered by their modulus, argument, real, or imaginary part; (b) they are not strictly ordered, for example, two complex numbers may have the same modulus or argument or real or imaginary part and still not coincide.

1.6 Conjugate and Reflection on the Origin and Axis

The point obtained (Figure 1.1) by reflection of P upon the real axis defines the complex **conjugate**:

$$z = x + iy\colon\ z^* = x - iy = re^{-i\varphi}. \tag{1.17}$$

The reflection on the origin defines the **symmetric** complex

$$-z = -x - iy = re^{i(\pi+\varphi)}, \tag{1.18}$$

and the symmetric conjugate,

$$-z^* = -x + iy = re^{i(\pi-\varphi)}, \tag{1.19}$$

specifies the reflection in the imaginary axis. Making the convention that *the quadrants follow each other cyclically in the positive direction 1,2,3,4,1,2,..., then the complex numbers* z *in (1.14),* $-z^*$ *in (1.19),* $-z$ *in (1.18), and* z^* *in (1.17) lie on successive quadrants of the complex plane (Table 1.2 and Figure 1.1).*

TABLE 1.2
Reflection on the Axis and at the Origin

Transformation	**Conjugate**	**Symmetric**	**Symmetric conjugate**	**Name**
Reflection upon	Real axis	Origin	Imaginary axis	Representation
$(x,y) \to$	$(x,-y)$	$(-x,-y)$	$(-x,y)$	Cartesian
$z \to$	z^*	$-z$	$-z^*$	Complex
$(r,\varphi) \to$	$(r,-\varphi)$	$(r,\pi+\varphi)$	$(r,\pi-\varphi)$	Polar

Note: Reflections of a complex number on the origin and coordinate axis, using Cartesian and polar representations, as shown in Figure 1.1.

1.7 Power with Integral Exponent and Logarithm

The modulus and argument of the polar representation of a complex number, z, can also be applied to any complex function, for example, the **integral power** (1.20b) whose exponent (1.20a) is a positive or negative integer:

$$n \in |Z: \quad z^n = \left(r e^{i\varphi}\right)^n = r^n e^{in\varphi}; \tag{1.20a,b}$$

from (1.20b) follow the formulas:

$$n \in |Z; \quad z \in |C : |z^n| = |z|^n, \quad \arg(z^n) = n \arg(z) \tag{1.21a,b}$$

for *the modulus (1.21a) and argument (1.21b) of an integral power*. The power can be extended to nonintegral exponent (Sections 5.7 and 5.8).

Similarly, the real and imaginary parts of a complex number z can be applied to a function $f(z)$, for example, the Neperian logarithm or logarithm of base e:

$$\log z = \log\left(r e^{i\varphi}\right) = \log r + \log e^{i\varphi} = \log r + i\varphi, \tag{1.22}$$

where the polar representation was used; it leads to the formulas:

$$\operatorname{Re}(\log z) = \log|z|, \quad \operatorname{Im}(\log z) = \arg(z), \tag{1.23a,b}$$

for *the real (1.23a) and imaginary (1.23b) parts of the natural logarithm.*

1.8 Real, Imaginary, and Complex Exponential

The real and imaginary parts, and modulus and argument are four functions that can be applied to any complex expression. For example, in the case of the real $x \neq 0 = y$, imaginary $x = 0 \neq y$ or complex $x \neq 0 \neq y$ exponentia1:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y, \tag{1.24}$$

the modulus (1.25a), argument (1.25b), and the real (1.25c) and imaginary (1.25d) parts are, respectively,

$$|e^z| = e^x, \quad \arg(e^z) = y, \quad \operatorname{Re}, \operatorname{Im}(e^z) = e^x (\cos y, \sin y); \tag{1.25a–d}$$

alternatively

$$|e^z| = \exp\{\operatorname{Re}(z)\}, \quad \arg(e^z) = \operatorname{Im}(z), \tag{1.26a,b}$$

$$\operatorname{Re},\ \operatorname{Im}(e^z) = \exp\{\operatorname{Re}(z)\}\left(\cos\{\operatorname{Im}(z)\}, \sin\{\operatorname{Im}(z)\}\right), \tag{1.26c,d}$$

hold for the *modulus (1.25a) ≡ (1.26a), argument (1.25b) ≡ (1.26b), real and imaginary (1.25c) ≡ (1.26c) and imaginary (1.25d) ≡ (1.26d) parts of the complex exponential.* Two more cases are considered in Example 10.2.

1.9 Noncommutative Product of Quaternions (Hamilton, 1843)

A generalization of the complex number from 2 to 4 dimensions is the **quaternion** *(Hamilton, 1843) defined by*

$$\tilde{x} \equiv x_0 + x_1 i + x_2 j + x_3 k, \tag{1.27}$$

where i, j, k satisfy the following multiplication rules:

$$ii = jj = kk = -1, \quad ij = k = -ji, \quad ki = j = -ik, \quad jk = i = -kj, \tag{1.28a–d}$$

which fall into three groups: (i) squares of i, j, or k equal -1 as in (1.9b); (ii) products are skew-symmetric, for example, $ij = -ji$; and (iii) products follow in cyclic permutations (i,j,k), for example, $jk = i$. The latter two symbolic rules (ii, iii) are similar to the outer product (1.28c,d) of orthogonal base vectors (i, j, k) in three dimensions. The sum of quaternions is commutative:

$$\begin{aligned}\tilde{x} + \tilde{y} &= (x_0 + x_1 i + x_2 j + x_3 k) + (y_0 + y_1 i + y_2 j + y_3 k) \\ &= (x_0 + y_0) + i\,(x_1 + y_1) + j\,(x_2 + y_2) + k\,(x_3 + y_3) = \tilde{y} + \tilde{x};\end{aligned} \tag{1.29}$$

the product of quaternions is not commutative:

$$\begin{aligned}\tilde{x} \times \tilde{y} &\equiv (x_0 + x_1 i + x_2 j + x_3 k) \times (y_0 + y_1 i + y_2 j + y_3 k) \\ &= (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) + i\,(x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2) \\ &\quad + j\,(x_0 y_2 - x_1 y_3 + x_2 y_0 + x_3 y_1) + k\,(x_0 y_3 + x_1 y_2 - x_2 y_1 + x_3 y_0) \\ &\neq \tilde{y} \times \tilde{x},\end{aligned} \tag{1.30}$$

as follows interchanging $(x^i,\ y^i)$ with $i = 0, 1, 2, 3$. A quaternion represents a point (x_1, x_2, x_3, x_4) in a four-dimensional space.

Subtracting the **scalar part** x_0 from the quaternion, $\vec{x}$ specifies the **vector part** in (1.31a), which represents a point in three-dimensional space:

$$\vec{x} \equiv \tilde{x} - x_0 = x_1\, i + x_2\, j + x_3\, k : \quad \tilde{x} = x_0 + \vec{x}, \quad \tilde{x}^* \equiv x_0 - \vec{x}, \tag{1.31a–c}$$

and the **conjugate** (1.31c) of a quaternion (1.31b) is defined by the same (opposite) scalar (vector) part. This is similar to the conjugate (1.17) of a complex number taking the real x (imaginary y) part as the scalar x_0 (vector $\vec{x}$). *The product of a quaternion by its conjugate is commutative, real, and positive and specifies the* **norm** $\|\ldots\|$ *and the* **modulus** $|\ldots|$:

$$\begin{aligned}\|\tilde{x}\| \equiv |\tilde{x}|^2 \equiv \tilde{x} \times \tilde{x}^* &= \left(x_0 - \vec{x}\right) \times \left(x_0 + \vec{x}\right) = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 \\ &= \tilde{x}^* \times \tilde{x},\end{aligned} \tag{1.32}$$

and the **inverse** $\tilde{x}^{-1}$:

$$\tilde{x}^{-1} \equiv \tilde{x}^* \, |\tilde{x}|^{-2}, \quad \tilde{x}^{-1} \times \tilde{x} = |\tilde{x}|^{-2}\, \tilde{x}^* \times \tilde{x} = 1 = \tilde{x} \times \tilde{x}^{-1}. \tag{1.33a,b}$$

This is similar to the modulus (1.11) and inverse (3.19) of a complex number.

Note 1.1. *Types of Numbers*: The extension from complex numbers (1.9a,b) in two dimensions $N = 2$ to quaternions (1.27, 1.28a–d) in four dimensions $N = 4$ "loses" the commutativity of the product, and there is no square root (Table 1.1). Additional properties would be "lost" extending complex numbers to **hypercomplex** numbers, other than quaternions $N = 4$, viz. in higher dimensions $N = 3, 5, 6, \ldots$. Quaternions were used historically to develop vector algebra in three dimensions, until it became clear that it is simpler to do without them. The N-dimensional geometry with $N \geq 3$ is based on N-tuples, or ordered sets of N real numbers, viz. x_n with $n = 1, \ldots, N$, rather than any extension of complex numbers. Complex coordinates $z_n = x_n + iy_n$ are used in spinors in quantum mechanics, both nonrelativistic and relativistic. Although quaternions can be used as coordinates in a four-dimensional space, they are not suitable for the space-time of general relativity because (i) the norm (1.33) of a quaternion corresponds to a Cartesian metric which is positive-definite, that is, has signature $+ + + +$ viz. all signs $+$ in (1.32); (ii) the interval in space-time corresponds to an indefinite metric with signature $- - -+$ or $+ + + -$, that is, one reversed sign in (1.32). Quaternions were invented (Hamilton, 1843) before relativity (Einstein, 1904) and were conceived in connexion with the representation of rotations and rigid displacements in three-dimensional space, which remains their main application. The complex number $\mathrm{e}^{i\varphi}$ [quaternion (1.27) $\equiv$ (1.31a)] can be used to represent a rotation in the plane (in space). The rotations in the plane (space) are (are not) commutative, that is, the final position does not (does) depend on the order, that is, it is not (is) changed if the two rotations are interchanged. Two successive rotations in the plane (space) correspond to the product of complex numbers (quaternions), implying that the product is (is not) commutative. Properties of spinors relate to quaternions. The theory of numbers concerns many other topics, for example, **transfinite** (**prime) numbers** that lie beyond natural numbers (have no natural numbers as divisors). The preceding account suggests that *complex numbers are not only the simplest (Section 1.2) but also the most general, which satisfy all the usual properties of sum and product (Chapter 3) as a continuous set.*

Conclusion 1: The polar and Cartesian representations (Figure 1.1) on the plane, of a complex number z, its conjugate z^*, symmetric $-z$, and symmetric conjugate $-z^*$, also represented in Table 1.2. Table 1.1 lists types of numbers and operations between them.

2

Stability of an Equilibrium Position

When a body (or system) is in a position (or state) of equilibrium, and is subjected to a perturbation, three cases can arise: (i) if it remains in the new, disturbed position, the equilibrium is said to be *indifferent* (Section 2.1); (ii) if it always returns to the old, initial position, the equilibrium is *stable*; and (iii) if it can get farther from the equilibrium position, that is, the perturbation increases, the equilibrium is *unstable.* The return to equilibrium in the case of stability, or deviation from equilibrium in the unstable case, can be monotonic or oscillatory. The words stable, indifferent, and unstable are sometimes reserved for the monotonic case (Section 2.3); the oscillatory case (Sections 2.2 and 2.4) is called *damped*, *oscillatory*, or *overstable*, depending on whether the amplitude of the oscillation decays, stays constant, or grows. All these six cases of motion are included in a single complex expression (Sections 2.5–2.7), which is discussed (Sections 2.8 and 2.9) for a system of arbitrary order. The existence of an equilibrium is not sufficient to ensure that it actually occurs; usually only stable equilibria are found in nature or assure the correct functioning of an engineering device, bearing in mind that perturbations are almost inevitably present. The question of existence of equilibria, and their stability, applies to various kinds of systems, for example, mechanical, electrical, chemical, and so on, both in static and in dynamic conditions.

2.1 Trajectory Following a Perturbation of Equilibrium

A body is in an equilibrium position $x = 0$. At a time $t = 0$ it is displaced to a position x_0, and its subsequent motion for $t > 0$ is specified by the position as a function of time by (2.1b):

$$\zeta \equiv \omega + i\lambda: \quad x(t) = \operatorname{Re}\left(x_0\, \mathrm{e}^{i\zeta t}\right), \tag{2.1a,b}$$

where (2.1a) is generally complex. Since x must be real, the real part of the complex expression in parentheses in (2.1b) is taken. The first and second derivatives of the **displacement** (2.1b) with regard to time specify, respectively, the **velocity** v and **acceleration** a:

$$v(t) \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \operatorname{Re}\left(i\zeta x_0 \mathrm{e}^{i\zeta t}\right), \quad a(t) \equiv \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \operatorname{Re}\left(-\zeta^2 x_0 \mathrm{e}^{i\zeta t}\right); \tag{2.2a,b}$$

here was used $i^2 = -1$ in (1.9b), because $i = \mathrm{e}^{i\pi/2}$ and $i^2 = \mathrm{e}^{i\pi} = -1$. The expressions (2.1, 2.2a,b) are examples of the **complex representation** of a real quantity, and sometimes the function real part of Re is omitted.

The motion (2.1, 2.2a,b) is analyzed next for all possible values of ζ to assess the nature of the equilibrium position $x = 0$. The trivial case of (2.1a,b) is (2.3a):

$$\zeta = 0: \quad x(t) = x_0, \quad v(t) = 0 = a(t), \tag{2.3a,b}$$

that is, *the equilibrium is* **indifferent** *at* x = *0, since the body, after being displaced to* x_0*, remains at rest in the new position (Figure 2.1a).*

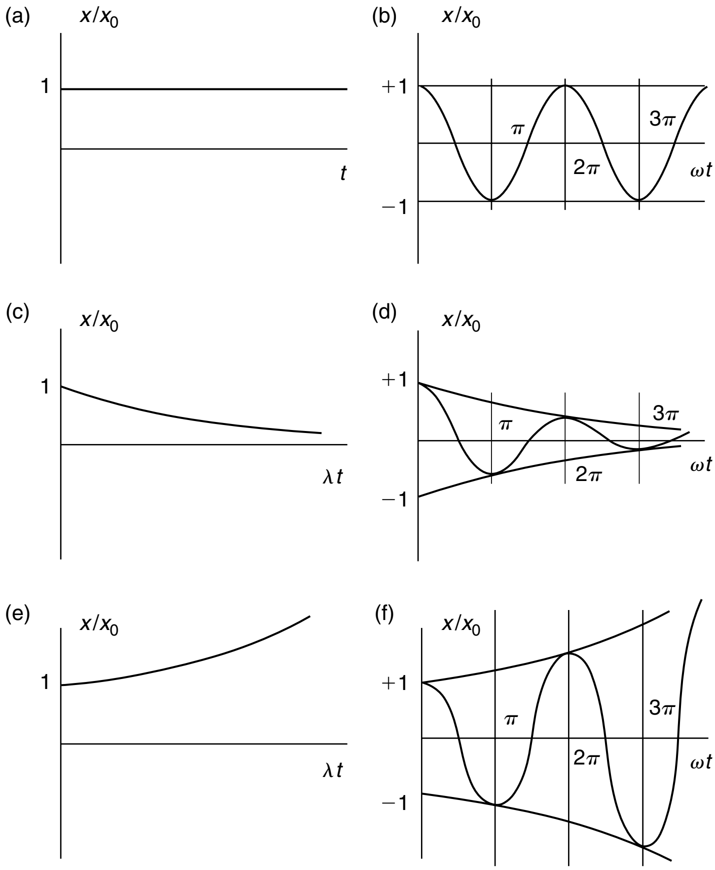

FIGURE 2.1
The six types of motion specified by (2.1a,b) fall in three cases. The first set is (i) indifferent equilibrium, with rest at displaced position (a). The second set is (ii) stability with (ii-1) oscillation around equilibrium position (b) with frequency ω; (ii-2) monotonic decay (c) toward equilibrium position with damping λ; (ii-3) the combination (d) of damped amplitude (ii-2) and harmonic oscillation (ii-1). The third set is (iii) instability with (iii-a) monotonic (e) [(iii-b) oscillatory (f)] growth. The latter (f) is called overstability.

2.2 Oscillatory Motion with Constant Amplitude

The case (II) of real (2.4a), the motion

$$\zeta = \omega: \quad x(t) = \mathrm{Re}\left(x_0 e^{i\omega t}\right) = x_0 \cos(\omega t), \tag{2.4a,b}$$

$$v(t) = \mathrm{Re}\left(i x_0 \omega e^{i\omega t}\right) = -x_0 \omega \sin(\omega t), \tag{2.4c}$$

$$a(t) = \mathrm{Re}\left(-x_0 \omega^2 e^{i\omega t}\right) = -x_0 \omega^2 \cos(\omega t), \tag{2.4d}$$

is **oscillatory**, *with* **frequency** ω, *and* **period** $\tau = 2\pi/\omega$, *with constant amplitudes* $x_0, x_0\omega, x_0\omega^2$, *respectively, for the displacement (2.4b), velocity (2.4c), and acceleration (2.4d) that are out-of-phase by* $\pi/2, \pi$, *that is, a quarter- and a half-period* $\tau/4, \tau/2$. The latter statement corresponds to

$$v(t) = \omega x\left(t + \frac{\pi}{2\omega}\right) = \omega x\left(t + \frac{\tau}{4}\right), \quad a(t) = \omega^2 x\left(t + \frac{\pi}{\omega}\right) = \omega^2 x\left(t + \frac{\tau}{2}\right), \tag{2.5a,b}$$

which follows from (2.4b–d) using

$$\cos\left(\omega t + \frac{\pi}{2}\right) = -\sin(\omega t), \quad \cos(\omega t + \pi) = -\cos(\omega t). \tag{2.6a,b}$$

The displacement from the equilibrium position $x = 0$ never exceeds the initial displacement $|x(t)| \le x_0$ and the body passes through the equilibrium position every half-period $x((n + 1/2)\tau) = 0$ as shown in Figure 2.1b.

2.3 Attenuation or Amplification and Stability or Instability

In the case (III) of pure imaginary (2.7a)

$$\zeta = i\lambda: \quad x(t) = \mathrm{Re}\left(x_0 \mathrm{e}^{i^2\lambda t}\right) = x_0 \mathrm{e}^{-\lambda t}, \tag{2.7a,b}$$

there are two possibilities:

$$v(t) = \mathrm{Re}\left(x_0 i^2 \lambda^2 \mathrm{e}^{i^2\lambda t}\right) = -x_0\lambda \mathrm{e}^{-\lambda t}, \quad a(t) = \mathrm{Re}\left(x_0 i^4 \lambda^2 \mathrm{e}^{i^2\lambda t}\right) = x_0\lambda^2 \mathrm{e}^{-\lambda t}, \tag{2.7c,d}$$

(IIIA) if $\lambda > 0$ *there is* **attenuation**, *and the displacement (2.7b), velocity (2.7c), and acceleration (2.7d) decay monotonically from their initial values, respectively,* $x_0, -\lambda x_0$, *and* $\lambda^2 x_0$, *to zero, as the body returns to the position of equilibrium* $x(\infty) = 0$, *which is* stable (Figure 2.1c); (IIIB) if $\lambda < 0$ *there is* **amplification**, *and the displacement, velocity, and acceleration increase monotonically as the body gets progressively farther from the position of equilibrium that is* **unstable** (Figure 2.1e). The evolution is monotonic, and,

$$x(t) = -\lambda^{-1} v(t) = \lambda^{-2} a(t), \tag{2.8}$$

in *both cases the acceleration has the same sign as the displacement, whereas the velocity has the same (opposite) sign in the case* $\lambda < 0\ (\lambda > 0)$ *of amplification (attenuation)*, that is, the deviation (approach) to equilibrium involves continuing (reversing) the initial motion from $x(0) = x_0$ to $x(\infty) = \infty\ [x(\infty) = 0]$.

2.4 Damped Oscillation or Overstable Growth

The general case (IV) of complex (2.9a) corresponds to

$$\zeta = \omega + i\lambda: \quad x(t) = \mathrm{Re}\left(x_0 \mathrm{e}^{i\omega t} \mathrm{e}^{i^2\lambda t}\right) = x_0 \mathrm{e}^{-\lambda t}\cos(\omega t), \tag{2.9a,b}$$

a *motion which is oscillatory with frequency* ω *and period* $\tau = 2\pi/\omega$, *and has*

$$v(t) = \mathrm{Re}\left(x_0 i(\omega + i\lambda)\right) \mathrm{e}^{i\omega t} \mathrm{e}^{i^2\lambda t} = -x_0 \mathrm{e}^{-\lambda t}\{\omega \sin(\omega t) + \lambda \cos(\omega t)\}, \tag{2.9c}$$

$$a(t) = \mathrm{Re}\left(x_0 i^2 (\omega + i\lambda)^2\right) \mathrm{e}^{i\omega t} \mathrm{e}^{i^2\lambda t} = x_0 \mathrm{e}^{-\lambda t}\left\{\left(\lambda^2 - \omega^2\right)\cos(\omega t) + 2\omega\lambda \sin(\omega t)\right\}, \tag{2.9d}$$

nonconstant amplitude: (IVA) if $\lambda > 0$ *there is an exponential decay* (Figure 2.1d) *corresponding to a* **damped** oscillation that *takes the body back to the equilibrium position* $x(\infty) = 0$, *with progressively smaller overshoots on either side; (IVB) if* $\lambda < 0$ *there is exponential growth* (Figure 2.1f)*, corresponding to an amplified oscillation designated* **overstability**, *because although the body goes through the equilibrium position every half-period, it overshoots progressively more on either side.* The sequence of maxima and minima can be used to define a logarithmic decrement (Example 10.3).

2.5 General Relations for Amplitudes and Phases

The frequency ω and attenuation/amplification factor λ have the same dimensions, namely, the inverse of time, and when they are comparable $\lambda \sim \omega$ there is no simplification of the general formulas (2.9a–d), which specify the following *relations between the amplitudes (2.10a,b) and phases (2.11a,b) of the displacement (2.9b), velocity (2.9c), and acceleration (2.9d):*

$$|v| = |x|\,|\zeta| = |x|\,\sqrt{\omega^2 + \lambda^2}, \quad |a| = |x|\,\left|\zeta^2\right| = |x|\,\left(\omega^2 + \lambda^2\right), \tag{2.10a,b}$$

$$\arg(v) = \arg(x) + \arg(i\zeta) = \arg(x) + \frac{\pi}{2} + \operatorname{arc\,tan}\left(\frac{\lambda}{\omega}\right), \tag{2.11a}$$

$$\arg(a) = \arg(x) + \arg\left(-\zeta^2\right) = \arg(x) - \pi + 2\,\operatorname{arc\,tan}\left(\frac{\lambda}{\omega}\right). \tag{2.11b}$$

The formulas (2.9c,d) have been deduced from (2.9a,b), and (2.7c,d) from (2.7a,b), and (2.4c,d) from (2.4a,b), using the complex representations; they can be checked by direct derivation of the real expressions, for example, to derive (2.9d) from (2.9b):

$$\begin{aligned}\frac{\mathrm{d}^2\left\{x_0 \mathrm{e}^{-\lambda t}\cos(\omega t)\right\}}{\mathrm{d}t^2} &= \frac{\mathrm{d}\left\{-x_0\mathrm{e}^{-\lambda t}\left\{\lambda\cos(\omega t) + \omega\sin(\omega t)\right\}\right.}{\mathrm{d}t} \\ &= x_0 \mathrm{e}^{-\lambda t}\left\{\left(\lambda^2 - \omega^2\right)\cos(\omega t) + 2\lambda\omega\sin(\omega t)\right\}.\end{aligned} \tag{2.12}$$

The deduction of the algebraic relations (2.10a,b and 2.11a,b) from the real representation would require determination of the amplitudes from the maxima of x, v, $a(t)$, and of the phases as the time lags at these maxima; the use of the complex representation is much simpler.

2.6 Predominantly or Weakly Oscillatory Motion

If the frequency ω and attenuation/amplification factor λ are of dissimilar magnitudes, the general results in Sections 2.5 and 2.6 simplify, in the two extremes of (i) motion that is **predominantly oscillatory** (2.13a):

$$\omega \gg |\lambda|: \quad x(t), v(t), a(t) = x_0 \mathrm{e}^{-\lambda t}\left\{\cos(\omega t), -\omega\sin(\omega t), -\omega^2\cos(\omega t)\right\}, \tag{2.13a–d}$$

the amplitude and phase relations are the same as for the oscillatory motion (2.4b–d) but the amplitude grows (decays) for $\lambda < 0\,(\lambda > 0)$ *slowly on the time scale of a period*

$1/|\lambda| \gg 1/\omega = \tau/2\pi$; *(ii) motion that is* **predominantly attenuated/amplified** (2.14a):

$$|\lambda| \gg \omega: \quad x(t), v(t), a(t) = x_0 \cos(\omega t)\, \mathrm{e}^{-\lambda t} \left\{1, -\lambda, \lambda^2\right\}, \tag{2.14a–d}$$

the relations between amplitudes are the same as for the purely attenuated/amplified motion, (2.7a–d), but there is a long-period $\tau = 2\pi/\omega \gg 2\pi/|\lambda|$ of oscillation in phase, which implies small overshoots in the damping, and a large-scale inversion of motion at long intervals of time in the case of overstability.

2.7 Frequency and Attenuation/Amplification Factor

It is clear from the preceding account that the type of motion specified by (2.1b) and the nature of the equilibrium position $x = 0$ are determined solely by the position of (2.1a) in the complex ζ-plane, which acts as a diagnostic diagram in Figure 2.2 and Table 2.1. All six cases are determined by the combination of (i) zero $\omega = 0$ or nonzero $\omega \neq 0$ frequency, for monotonic or oscillatory motion [the sign of the frequency does not matter since $\mathrm{Re}\left(\mathrm{e}^{i\omega t}\right) = \cos(\omega t) = \mathrm{Re}\left(\mathrm{e}^{-i\omega t}\right)$, that is, it can be taken always positive]; (ii) constant $\lambda = 0$, increasing $\lambda > 0$ or decreasing $\lambda < 0$ amplitude, since the sign of the imaginary part of ζ is important to distinguish attenuation from amplification $\mathrm{Re}(\mathrm{e}^{i^2\lambda t}) = \mathrm{e}^{-\lambda t} \neq \mathrm{e}^{\lambda t} = \mathrm{Re}(\mathrm{e}^{-i^2\lambda t})$. The position (Figure 2.2) of the ζ-plane specifies (i) rest at the origin (Figure 2.1a); (ii) oscillation with constant amplitude (Figure 2.1b) on the real axis; (iii/iv) monotonic decay (growth) on [Figure 2.1c (2.1e)] the positive (negative)

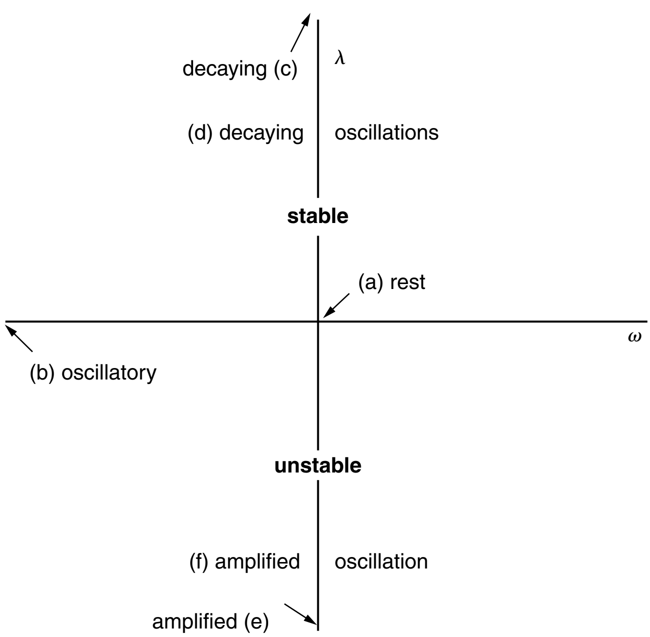

FIGURE 2.2
The position in the (ω, λ)-plane of frequency ω and damping/amplification ($\lambda > / < 0$) indicates the six types of motion and three cases of stability in Table 2.1 and Figure 2.1a–f.

TABLE 2.1
Amplification/Attenuation Factor

Parameters Meaning	$\boldsymbol{\lambda > 0}$ **Attenuation**	$\boldsymbol{\lambda = 0}$ **Constant amplitude**	$\boldsymbol{\lambda < 0}$ **Amplification**
$\omega = 0$ Monotonic	*stable* Figure 2.1c	*indifferent* Figure 2.1a	*unstable* Figure 2.1e
$\omega \neq 0$ Nonzero Frequency	*damped* Figure 2.1d	*oscillatory* Figure 2.1b	*overstable* Figure 2.1f

Note: The equation of motion (2.1b) as the real representation of a complex quantity (2.1a) leads to the six cases illustrated in Figure 2.1a–f. The parameters are initial amplitude x_0, oscillation frequency ω, and damping λ. The six cases are indicated in the diagnostic plane $\omega + i\lambda$ in Figure 2.2.

imaginary axis; and (v/vi) oscillatory decay or stability (oscillatory growth or overstability) in [Figure 2.1d (2.1f)], the upper (lower) half-plane.

2.8 Differential Equation and Stability Criteria

The nature of the equilibrium, and the motion in its neighborhood, depends only on the value of (2.1a) in (2.1b), and it is indicated next how it can be determined in the case of a system described by a linear ordinary differential equation of order N with constant coefficients:

$$\sum_{n=0}^{N} B_n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} = 0, \tag{2.15}$$

without forcing, that is, homogeneous in the sense the r.h.s. is zero. Bearing in mind that (2.1b) satisfies

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = \frac{\mathrm{d}^n \left(x_0 \mathrm{e}^{i\zeta t}\right)}{\mathrm{d}t^n} = (i\zeta)^n x_0 \mathrm{e}^{i\zeta t} = (i\zeta)^n x(t), \tag{2.16}$$

it follows that (2.15) is met by

$$x(t)\, Q_N(\zeta) = 0, \qquad Q_N(\zeta) \equiv \sum_{n=0}^{N} B_n (i\zeta)^n = (i)^N B_N \prod_{n=0}^{N} (\zeta - \zeta_n), \tag{2.17a,b}$$

for all t in (2.17a), provided that ζ be a root ζ_n of the complex polynomial (2.17b) of degree N.

To each distinct ζ_n root of the **characteristic polynomial** (2.17b) corresponds to a particular solution (2.18a):

$$x_n(t) = \exp(i\zeta_n t); \quad x(t) = \sum_{n=1}^{N} A_n x_n(t), \tag{2.18a,b}$$

their linear combination (2.18b) involving N arbitrary constants $A_1, \ldots, A_N$ specifies the general solution. If a root ζ_0 is of multiplicity q, then the q "identical solutions" (2.18a) are replaced by (2.18d):

$$k = 0, \ldots, q-1: \quad x_k(t) = t^k \exp(i\zeta_0 t), \tag{2.18c,d}$$

which are q distinct solutions (2.18c).

From the receding results follow **the stability criteria**: *the equilibrium $x = 0$ of the system described by the differential equation (2.15) with characteristic polynomial (2.17b) is specified by the nature of the roots (2.18a–d) as follows (Figure 2.2): (i) if all roots have positive imaginary parts the equilibrium is stable, and the motion monotonic attenuated if all real parts are zero, or damped oscillatory if at least one real part is nonzero; (ii) if at least one root has negative imaginary part the equilibrium is unstable, and the motion monotonic amplified if all such roots have zero real part, and oscillatory overstable if at least one such root has nonzero real part; and (iii) in the remaining case of some roots with zero imaginary part and none with negative imaginary part, there are three subcases: (a) indifferent equilibrium if the only root is zero (both real and imaginary parts); (b) oscillation with constant amplitude if at least one of the roots with zero imaginary part has nonzero real part; and (c) if there are multiple roots there is instability if their real parts are all zero, and overstability if at least one real part is nonzero.* The arbitrary constants in the general solution (2.18b) are determined from initial conditions, as shown next in the particular case of the harmonic oscillator.

2.9 Initial Conditions for Harmonic Oscillator

The application of initial conditions to render unique the solution of a differential equation describing the stability of a system is illustrated next in the simplest case of a harmonic oscillator (Subsection 2.9.1) leading to three interchangeable pairs of arbitrary constants (Subsection 2.9.2).

2.9.1 Equation of Motion of a Harmonic Oscillator

As a simple example suppose that the displacement satisfies the differential equation

$$m > 0,\ k > 0: \quad m\ddot{x} = -kx, \tag{2.19}$$

which states a balance between (i) the **inertia** force, equal to **mass** m times acceleration; and (ii) the force of a linear **spring** that is proportional to the displacement, through the **resilience** k, and pulls the particle back to the origin $x = 0$; a large (small) resilience means (i) for same displacement x, a strong (weak) pulling force; and (ii) for the same mass m, a higher (lower) acceleration. The mechanical system (2.19) consisting of a mass and a spring omits the damper (included in Section 4.1, Figure 4.1a, and Example 10.4). For the present purpose of application of initial conditions, this case is sufficient. The subsequent solutions of (2.19) assume that there is a spring $k \neq 0$. In the absence of the spring (2.20a), the acceleration is zero (2.20b):

$$k = 0: \quad \ddot{x} = 0, \quad \dot{x}(t) = v_0, \quad x(t) = x_0 + v_0 t, \tag{2.20a–d}$$

and hence the motion is uniform (2.20c,d). The two arbitrary constants in (2.20d) are the initial velocity (2.20c) and initial position $x_0 = x(0)$. The equation of motion (2.19) applies to the harmonic oscillator that involves six assumptions: (i) one-dimensional since there is a single displacement x; (ii) vibration since it depends only on time t; (iii) linear since it involves the displacement x and time derivatives in (2.15), but no powers or cross-products; (iv) second-order since the highest order of derivation is two, corresponding to the acceleration; (v) without damping or amplification, which is considered in Examples 10.4 and 10.5; and (vi) it is a free oscillation due to initial conditions, that is, without applied external forces. The relaxation of these assumptions is considered in Note 2.1.

The equation of motion (2.19) can be written in the form (2.21a):

$$\ddot{x} + \omega^2 x = 0, \quad \omega \equiv \sqrt{\frac{k}{m}}, \tag{2.21a,b}$$

where (2.21b) is the **natural frequency**, because it will be shown next (2.23b) that the motion is an oscillation with this frequency. To prove this, the equation of motion (2.21a) is put into the form

$$0 = \left\{ \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2 \right\} x(t) = \left\{ Q_2 \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \right\} x(t), \tag{2.22a}$$

showing (2.1b) that characteristic polynomial (2.17b) in the case is

$$Q_2(\zeta) = (i\zeta)^2 + \omega^2 = -\zeta^2 + \omega^2 = -(\zeta - \omega)(\zeta + \omega); \tag{2.22b}$$

it has real symmetric roots (2.23a):

$$\zeta_\pm = \pm\omega : \quad x_\pm(t) = \exp(i\zeta_\pm t) = \exp(\pm i\omega t), \tag{2.23a,b}$$

corresponding to solutions (2.23b) that are indeed oscillations with frequency (2.21b).

2.9.2 Three Interchangeable Pairs of Arbitrary Constants

A linear combination (2.18b) of (2.23b), viz.:

$$x(t) = A_+ x_+(t) + A_- x_-(t) = A_+ \mathrm{e}^{i\omega t} + A_- \mathrm{e}^{-i\omega t}, \tag{2.24}$$

where $A_\pm$ are arbitrary constants, is also a solution of the equation of motion (2.19) $\equiv$ (2.22a), viz. it is the general solution. Using the identity (1.15) in the form

$$\exp(\pm i\omega t) = \cos(\omega t) \pm i \sin(\omega t), \tag{2.25}$$

the law of motion (2.24) becomes

$$x(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t), \tag{2.26}$$

where A_1, A_2 are given by

$$A_1 \equiv A_+ + A_-, \quad A_2 \equiv i(A_+ - A_-). \tag{2.27a,b}$$

That is, a second pair of arbitrary constants related to the first pair.

The velocity is specified from (2.26) by

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = \omega\{-A_1 \sin(\omega t) + A_2 \cos(\omega t)\}. \tag{2.28}$$

The arbitrary constants may be determined from **initial conditions**, for example, of **single-point** type, specifying initial displacement (2.29b) and velocity (2.29c) at time (2.29a):

$$t = 0: \quad x_0 = x(0) = A_1, \quad v_0 = v(0) = \omega A_2. \tag{2.29a–c}$$

Thus, the displacement (2.26) [velocity (2.28)] are given by (2.30a) [(2.30b)]:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t), \quad v(t) = -x_0 \omega \sin(\omega t) + v_0 \cos(\omega t), \tag{2.30a,b}$$

confirming the initial values (2.29b) [(2.29c)] for (2.29a).

A third pair (A, α) of arbitrary constants is

$$A\mathrm{e}^{i\alpha} \equiv A_1 + iA_2 = 2A_-, \tag{2.31}$$

where (2.27a,b) was used. Real A and α imply

$$A_1 = A\cos\alpha, \quad A_2 = A\sin\alpha; \tag{2.32a,b}$$

thus, the displacement (2.26) is given by

$$x(t) = A[\cos(\omega t)\cos\alpha + \sin(\omega t)\sin\alpha] = A\cos(\omega t - \alpha) = \mathrm{Re}\left\{A\mathrm{e}^{i(\omega t - \alpha)}\right\}. \tag{2.33}$$

This shows that the **amplitude** (2.34a) and **phase** (2.34b) of the oscillation are given by

$$A = |A_1 + iA_2| = \left|(A_1)^2 + (A_2)\right|^{1/2} = \left|x_0{}^2 + \left(\frac{v_0}{\omega}\right)^2\right|^{1/2}, \tag{2.34a}$$

$$\tan\alpha = \frac{A_2}{A_1} = \frac{v_0}{x_0\,\omega}, \tag{2.34b}$$

in terms of the initial conditions (2.29a–c). If at time $t = 0$ the velocity (displacement) is zero $v_0 = 0\,(x_0 = 0)$, the phase is $\alpha = 0 \ \ (\alpha = \pi/2)$, and the amplitude is $x_0\,(v_0/\omega)$. If at time $t = 0$ neither the displacement nor the velocity is zero, then (2.34a,b) do not simplify. It is possible to eliminate the phase in (2.33) by a suitable choice of origin of time:

$$\bar{t} \equiv t - \frac{\alpha}{\omega} = t - \frac{\alpha\tau}{2\pi}: \quad x(t) = A\cos(\omega\bar{t}) \equiv \bar{x}(\bar{t}), \tag{2.35a,b}$$

where the period τ was used.

Note 2.1. *Analysis of the Stability of Systems*: An example of harmonic oscillator appears in the rolling oscillations of a ship (Chapter 8). The six assumptions (i) to (vi) stated in Subsection 2.9.1 can be relaxed as indicated next. Forced oscillations (vi) are considered (Chapter 4) for a second-order system with damping (v), and the corresponding initial conditions appear in Example 10.5; this problem is an application of the solution ordinary differential equations; the multidimensional (i) case involves the solution of coupled systems of ordinary differential equations. Both linear and nonlinear (ii) oscillations can be considered for mechanical systems with one or several degrees of freedom; the nonlinear oscillations (iii) lead to elliptic functions. If the displacement were to depend on position and time, this would (iv) lead to waves described by partial differential equations.

Note 2.2. *Alternate Forms of the Stability Theorem*: The solution of the linear homogeneous ordinary differential equation with constant coefficients (2.15) can be written in several alternate forms:

$$\frac{x(t)}{x_0} = \mathrm{e}^{i\zeta t} = \mathrm{e}^{-st} = \mathrm{e}^{\vartheta t}, \tag{2.36a–c}$$

for example, (i) the kernel of the Fourier transform uses (2.36a) $\equiv$ (2.1b); (ii) the kernel of the Laplace transform uses (2.36b); and (iii) the characteristic polynomial of the differential equation is usually (2.36c) put in the form

$$P_n(\vartheta) \equiv \sum_{n=0}^{N} B_n \vartheta^n = \sum_{n=0}^{N} B_n (-s)^n = \sum_{n=0}^{N} B_n (i\zeta)^n \equiv Q_N(\zeta). \tag{2.37a–c}$$

The stability theorem (Section 2.8) was stated in terms of ζ, viz.:

$$\vartheta = -s = i\zeta = i(\omega + i\lambda) = -\lambda + i\omega, \tag{2.38}$$

and can be restated in terms of ϑ or s, with $\omega(\lambda)$ having the same meaning of frequency (decay rate).

Conclusion 2: The stability conditions for the motion (2.1a,b) about an equilibrium position $x = 0$, are indicated in Table 2.1, viz. the position in the complex ζ-plane indicates (Figure 2.2) the cases (Figure 2.1) of: (a) indifferent equilibrium and body at rest in the perturbed position; (b) oscillation with constant amplitude about the equilibrium position; (c) monotonic attenuation towards the position of stable equilibrium; (e) monotonic amplification away from the position of unstable equilibrium; (d) oscillation with decreasing amplitude, i.e., damping towards the equilibrium position; (f) oscillation with increasing amplitude, i.e., overstable oscillation, which is an oscillatory form of instability.

3

Addition, Product, and Inverses

An operation upon complex numbers is a relation between two ordered pairs of real numbers and a third ordered pair, and it can be represented graphically on the complex plane. The sum (Sections 3.1 and 3.2) is most conveniently performed in the Cartesian representation, and the product (Sections 3.3 and 3.4) in the polar representation. The operations sum and product, and their inverses subtraction and division, can be used (Sections 3.5 and 3.6) together with the conjugate operator to determine the real and imaginary parts, and the modulus and argument, of complex expressions. The algebra of complex numbers is related to the geometry of the Cartesian plane (Sections 3.8 and 3.9), and implies a number of trigonometrical relations (Section 3.7). The algebra of complex numbers extends to functions of one or more variables, and thus to complex spaces whose real dimension is twice the number of complex variables.

3.1 Complex Addition and Rule of the Parallelogram

Starting with two complex numbers in the Cartesian representation:

$$u = a + ib, \quad v = c + id, \tag{3.1a,b}$$

the operator addition is defined by the sum of each element of the pair,

$$u + v \equiv (a + c) + i\,(b + d), \tag{3.2}$$

and corresponds (Figure 3.1) to the addition of u and v as vectors, according to the **rule of the parallelogram**. Hence, the real (imaginary) part is the sum of the real (imaginary) parts:

$$\operatorname{Re}(u + v) = \operatorname{Re}(u) + \operatorname{Re}(v)\,, \quad \operatorname{Im}(u + v) = \operatorname{Im}(u) + \operatorname{Im}(v)\,. \tag{3.3a,b}$$

The addition of complex numbers has the following properties: (i) it is closed because the sum of complex numbers is a complex number, that is, (1.1) holds for the operation + in $|C$; (ii) it is commutative (3.4a) and associative (3.4b):

$$u + v = v + u, \quad (u + v) + w = u + (v + w)\,; \tag{3.4a,b}$$

(iii) it has a neutral element, namely, zero $0 + 0i$; (iv) each complex number $u = a + ib$ has a symmetric $-u = -a - ib$, given by (1.18). Hence, the ordered pair ($|C$, +), where $|C$ is the set of complex numbers and + the operation addition defined by (3.1a,b; 3.2), is a **commutative group**.

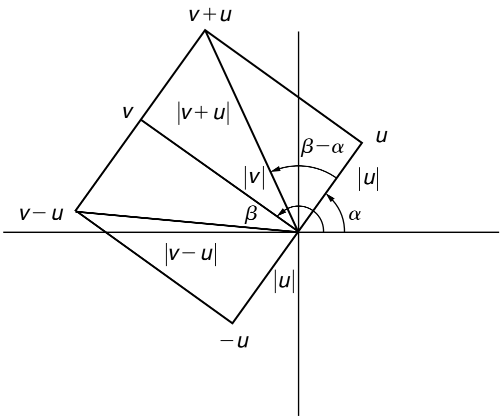

FIGURE 3.1
Taking two complex numbers u, v as the sides of a rectangle their sum is the diagonal. Their difference $u - v$ is the diagonal of the rectangle obtained by reflection on the v-direction.

3.2 Modulus, Argument, and Triangular Equalities (Pythagoras, VI b.c.)

The modulus (1.11) of the sum (3.2) is given by

$$\begin{aligned} |u+v|^2 &= (a+c)^2 + (b+d)^2 = \left(a^2+d^2\right) + 2\,(ac+bd) \\ &= |u|^2 + |v|^2 + 2\,|u|\,|v|\,(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\,, \end{aligned} \tag{3.5}$$

where α, β are the arguments of u,v. The expression (3.5) simplifies

$$\alpha \equiv \arg(u)\,, \beta \equiv \arg(v): \quad |u+v| = \left| |u|^2 + |v|^2 + 2\,|u|\,|v|\cos(\alpha-\beta) \right|^{1/2}, \tag{3.6}$$

which *specifies the length* $|u+v|$ *of one side of a triangle, from the lengths* $|u|, |v|$ *of the other two sides, and the angle* $\alpha - \beta$ *between them* (Figure 3.2a). In the case $\alpha - \beta = \pm\pi/2$, the two perpendicular sides lead to **Pythagoras theorem (VI b.c.)** *for the rectangular triangle (Figure 3.2b):*

$$\arg(u) - \arg(v) = \pm\frac{\pi}{2}: \quad |u+v|^2 = |u|^2 + |v|^2\,. \tag{3.7}$$

From (1.12a) and (3.2) follows the expression:

$$\begin{aligned} \arg(u+v) &= \text{arc}\ \tan\left(\frac{b+d}{a+c}\right) \\ &= \text{arc}\tan\left(\frac{|u|\sin\alpha + |v|\sin\beta}{|u|\cos\alpha + |v|\cos\beta}\right), \end{aligned} \tag{3.8}$$

for *the argument of the sum of complex numbers (3.8);* other alternative expressions are given in Example 10.6.

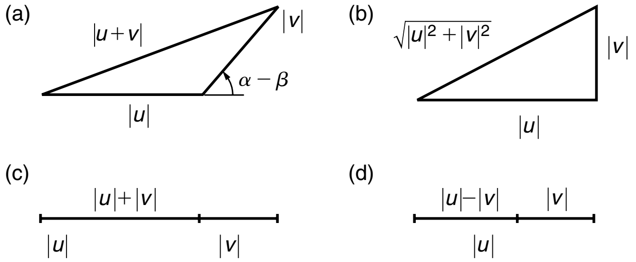

FIGURE 3.2
The sum of complex numbers by the rule of the diagonal of the rectangle (Figure 3.1) is equivalent to summing complex numbers as the sides of a triangle (a). A particular case is a rectangular triangle (b) with orthogonal sides for which holds the Pythagoras theorem. The degenerate cases are collapsed triangles with parallel (c) or antiparallel (d) sides.

3.3 Complex Product, Homothety, and Rotation

Starting with complex numbers in the polar representation:

$$u = |u|\, \mathrm{e}^{i\alpha}, \quad v = |v|\, \mathrm{e}^{i\beta}, \tag{3.9a,b}$$

the operation product is defined by

$$uv \equiv |u|\, |v|\, \mathrm{e}^{i(\alpha+\beta)}, \tag{3.10}$$

and corresponds (Figure 3.3) to (i) multiplying the moduli, that is, multiplying $|u|$ by $|v|$,which yields an extension if $|v| > 1$, a contraction if $|v| < 1$, and leaves the length unchanged if $|v| = 1$; (ii) adding the arguments, that is, adding α to β, which yields a counterclockwise (clockwise) rotation if $\beta > 0\,(\beta < 0)$. Thus, *the product of complex numbers corresponds algebraically to the product (sum) of moduli (arguments):*

$$|uv| = |u|\, |v|\,, \quad \arg{(u+v)} = \arg{(u)} + \arg{|v|}\,, \tag{3.11a,b}$$

and geometrically to a **homothety** (i) [**rotation** (ii)]; these are discussed in more detail subsequently (Section 35.1). Bearing in mind that $|uv|\,/\,|v| = |u|\,/1$ and $\arg{(u)} = \arg{(uv)} - \arg{(v)}$, the *triangles with vertices at* $(0,\,1,\,u)$ *and at* $(0, v, uv)$ *are* **similar**, *that is, have equal angles between the sides, and the lengths of the sides are in the same ratio;* the triangles $(0,\,1,\,v)$ and $(0,\,u,\,uv)$ are also similar, and both pairs are illustrated in Figure 3.4. The product of complex numbers is closed, has neutral element (viz. unity 1), and each element u has an inverse $1/u$ (as shown in Section 3.5). It is commutative (3.12a) and associative (3.12b), and distributive (3.12c) with regard to the sum:

$$uv = vu, \quad (uv)\, w = u\, (vw)\,, \quad (u+v)\, w = uw + vw. \tag{3.12a–c}$$

It follows that *the ordered pair* $(|C, \times)$ [*triad* $(|C, +, \times)$], *where* $|C$ *is the set of complex numbers, and* $+, \times$ *are the operations addition and product, is a commutative group* **(ring)**.

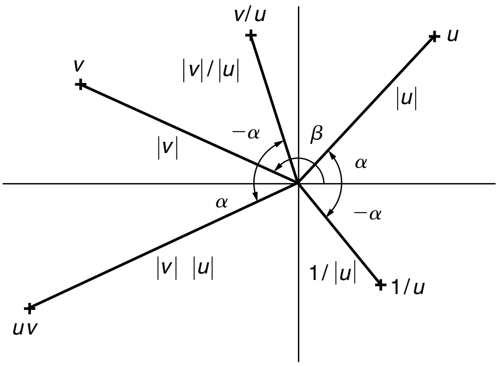

FIGURE 3.3
The inverse $1/u$ of a complex number has the inverse modulus and argument with opposite sign. The product u, v of complex numbers multiplies the moduli and adds the arguments. It follows that the ratio of complex numbers divides the moduli and subtracts the arguments.

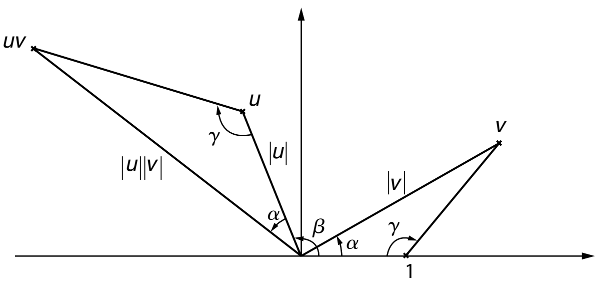

FIGURE 3.4
The triangles $(0, 1, v)$ and $(0, u, uv)$ are similar because the latter is obtained from the former by (i) rotating through $\beta = \arg(u)$ and (ii) multiplying all sides by $|u|$.

3.4 Meaning of the Imaginary Symbol "i"

The real (1.13a) [imaginary (1.13b)] parts of the product of complex numbers in polar form (3.10) are given, respectively, by

$$\operatorname{Re}(uv) = |u|\,|v| \cos(\alpha + \beta) = |u| \cos\alpha\, |v| \cos\beta - |u| \sin\alpha\, |v| \sin\beta = ac - bd; \qquad (3.13a)$$
$$\operatorname{Im}(uv) = |u|\,|v| \sin(\alpha + \beta) = |u| \sin\alpha\, |v| \cos\beta + |u| \cos\alpha\, |v| \sin\beta = bc + ad; \qquad (3.13b)$$

hence, the product of complex numbers in the Cartesian representation is specified by

$$(a + ib)(c + id) = \left(ac + i^2 bd\right) + i(ad + bc) = (ac - bd) + i(ad + bc); \qquad (3.14)$$

this corresponds to the ordinary rules of multiplication, and agrees with (3.13a,b) $\equiv$ (3.14), provided that

$$i^2 = -1, \quad \sqrt{-1} = \pm i. \qquad (3.15a,b)$$

The *condition (3.15a,b) specifying the imaginary symbol* "i" coincides with (1.9b); it can also be justified by noting that lies on the positive imaginary axis at distance unity from the origin in the complex plane $i = e^{i\pi/2}$, so that $i^2 = e^{i\pi} = -1$.

3.5 Conjugate of the Sum, Product, and Inversion

From (3.2) [(3.9a,b; 3.10)] and (1.17), it follows that *the conjugate of the sum (product) is the sum (product) of the conjugates:*

$$(u+v)^* = u^* + v^*, \quad (uv)^* = u^* v^*; \tag{3.16a,b}$$

from the expressions of a complex number and its conjugate (1.17), follow the *formulas for the real and imaginary parts:*

$$x \equiv \mathrm{Re}(z) = \frac{z+z^*}{2}, \quad y = \mathrm{Im}\,(z) = \frac{z-z^*}{2i}, \tag{3.17a,b}$$

and for the modulus and argument:

$$r \equiv |z| = (z^* z)^{1/2}, \quad \varphi \equiv \arg\,(z) = \frac{1}{2}\arg\left(\frac{z}{z^*}\right). \tag{3.18a,b}$$

The **inverse** $1/z$ *of a complex number* z *has the inverse modulus and symmetric argument:*

$$\frac{1}{z} = r^{-1} \mathrm{e}^{-i\varphi} = \frac{z^*}{z^* z} = z^* \, |z|^{-2}; \tag{3.19}$$

using the conjugate leads to

$$\mathrm{Re}, \mathrm{Im}\left(\frac{1}{z}\right) = r^{-1}\left\{\cos\varphi, -\sin\varphi\right\} = \{x, -y\}\left|x^2+y^2\right|^{-1/2}, \tag{3.20a,b}$$

for the real and imaginary parts of the inverse.

3.6 Complex Representation of Real Quantities

The formulas (3.17a,b) are examples of the complex representation, using z^*, z of real quantities x,y. The sum of complex representations is the complex representation of the sum (3.3a,b), but the product does not commute with the complex representation because $\mathrm{Re}\,(uv) \neq \mathrm{Re}\,(u)\,\mathrm{Re}\,(v)$ in (3.13a) and $\mathrm{Im}\,(uv) \neq \mathrm{Im}\,(u)\,\mathrm{Im}\,(v)$ in (3.13b). From (3.13a)

$$\mathrm{Re}\,(uv + uv^*) = (ac - bd) + (ac + bd) = 2ac = 2\mathrm{Re}\,(u)\,\mathrm{Re}\,(v), \tag{3.21a}$$

which can also be proved in the reverse direction from (3.17a):

$$\begin{aligned}\mathrm{Re}\,(u)\,\mathrm{Re}\,(v) &= \frac{1}{2}(u+u^*)\,\frac{1}{2}(v+v^*) = \frac{1}{4}\left[(uv+uv^*)+(uv+uv^*)^*\right] \\ &= \frac{1}{2}\,\mathrm{Re}\,(uv+uv^*).\end{aligned} \tag{3.21b}$$

Thus, the *complex representation of real quantities (3.17a,b) satisfies the rules (3.3a,b) for the sum, and for the product:*

$$\mathrm{Re}\,(uv + uv^*) = 2\,\mathrm{Re}\,(u)\,\mathrm{Re}\,(v) = \mathrm{Re}\,(uv + u^* v), \tag{3.22a}$$

$$\mathrm{Im}\,(uv - uv^*) = 2\,\mathrm{Re}\,(u)\,\mathrm{Im}\,(v) = \mathrm{Im}\,(uv + u^* v), \tag{3.22b}$$

$$\mathrm{Re}\,(uv^* - uv) = 2\,\mathrm{Im}\,(u)\,\mathrm{Im}\,(v) = \mathrm{Re}\,(u^* v - uv); \tag{3.22c}$$

all formulas can be proved as (3.21a,b). The complex representation is used in Section 4.5.

As a check on the formulas of complex representation of real quantities, for example, u,v in (3.22a) are put in polar form:

$$u \equiv \mathrm{e}^{i\alpha}, v \equiv \mathrm{e}^{i\beta}: \quad \operatorname{Re}\left(\mathrm{e}^{i(\alpha+\beta)} + \mathrm{e}^{i(\alpha-\beta)}\right) = 2\operatorname{Re}\left(\mathrm{e}^{i\alpha}\right)\operatorname{Re}\left(\mathrm{e}^{i\beta}\right), \tag{3.23}$$

this leads to the **trigonometric multiplication formulas**:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta), \tag{3.24a}$$

$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta), \tag{3.24b}$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta), \tag{3.24c}$$

where (3.24b,c) can be deduced similarly from (3.22b,c) as (3.24a) was deduced (3.23) from (3.22a). Adding and subtracting among (3.24a–c) leads the **trigonometric addition formulas:**

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \tag{3.25a}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \tag{3.25b}$$

these were used in (3.13a,b). These formulas (3.24a–c; 3.25a,b) are generalized next from two to any number of arguments.

3.7 Trigonometric Addition and Multiplication Formulas

The starting point is the identity:

$$\exp\{i(z_1 + \cdots + z_N)\} = \exp(iz_1)\cdots\exp(iz_N), \tag{3.26a}$$

in the form

$$\cos(z_1 + \cdots + z_N) + i\sin(z_1 + \cdots + z_N) = (\cos z_1 + i\sin z_1)\cdots(\cos z_N + i\sin z_N). \tag{3.26b}$$

The cosine (sine) of the sum on the l.h.s. of (3.26a) [(3.26b)] is the real (imaginary) part of the r.h.s., which is obtained as

$$\cos(z_1 + \cdots + z_N) = \sum_{p=0}^{\leq N/2}{}' (-)^p \sin z_1 \ldots \sin z_{2p} \cos z_{2p+1} \ldots \cos z_N, \tag{3.27a}$$

$$\sin(z_1 + \cdots + z_N) = \sum_{p=1}^{\leq N/2}{}' (-)^p \sin z_1 \ldots \sin z_{2p+1} \cos z_{2p+2} \ldots \cos z_N, \tag{3.27b}$$

by taking (i) all terms where sin appears an even $2p$ (odd $2p+1$) number of times, with factor $i^{2p} = (-)^p \left[i^{-1} i^{2p+1} = (-)^p\right]$; (ii) the remaining $N - 2p$ $(N - 2p + 1)$ factors are the cosines; and (iii) all permutations of $z_1, \ldots, z_N$ are included in the sum as indicated by $\sum'$. This proves (3.27a,b), and setting (3.28a) leads to (3.28b,c):

$$z_1 = \cdots = z_N \equiv z: \quad \cos(Nz) = \sum_{p=0}^{\leq N/2} \binom{N}{2p} (-)^p \sin^{2p} z \cos^{N-2p} z, \tag{3.28a,b}$$

$$\sin(Nz) = \sum_{p=0}^{\leq N/2} \binom{N}{2p+1} (-)^p \sin^{2p+1} z \cos^{N-2p-1} z, \tag{3.28c}$$

where the number of permutations is

$$\binom{N}{2p} \equiv \frac{N!}{(2p)!\,(N-2p)!}, \quad \binom{N}{2p+1} = \frac{N!}{[(2p+1)!\,(N-2p-1)!]}, \tag{3.29a,b}$$

respectively, in (3.28b, 3.29a) and (3.28c, 3.29b), and thus a normal sum $\sum$ is used in (3.28b,c).

The **addition (multiplication) formulas** *for the circular cosine (3.27a) [(3.28b, 3.29a)] and sine (3.27b) [(3.28c, 3.29b)] with any number of arguments* are simplest for $N = 2$ and $N = 3$. The former are the **double addition** formulas:

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \tag{3.30a}$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \tag{3.30b}$$

confirming (3.25a,b) ≡ (3.30a,b); these include the **duplication formulas** (3.31a–c):

$$z_1 = z_2 \equiv z: \quad \cos(2z) = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z, \tag{3.31a,b}$$

$$\sin(2z) = 2\cos z \sin z, \quad \cos^2, \sin^2(z) = \frac{1 \pm \cos 2z}{2}, \tag{3.31c,d}$$

from which follow (3.31d) ≡ (3.31b).

For $N = 3$, the following **triple addition** formulas are obtained:

$$\begin{aligned} \cos(z_1 + z_2 + z_3) = {} & \cos z_1 \cos z_2 \cos z_3 - \cos z_1 \sin z_2 \sin z_3 \\ & - \cos \mathrm{z}_2 \sin z_1 \sin z_3 - \cos z_3 \sin z_1 \sin z_2, \end{aligned} \tag{3.32a}$$

$$\begin{aligned} \sin(z_1 + z_2 + z_3) = {} & -\sin z_1 \sin z_2 \sin z_3 + \sin z_1 \cos z_2 \cos z_3 \\ & + \sin \mathrm{z}_2 \cos z_1 \cos z_3 + \sin z_3 \cos z_1 \cos z_2, \end{aligned} \tag{3.32b}$$

which include the **triplication** formulas:

$$z_1 = z_2 = z_3 \equiv z: \quad \cos(3z) = \cos^3 z - 3\cos z \sin^2 z = \cos \mathrm{z}\left(4\cos^2 z - 3\right), \tag{3.33a,b}$$

$$\sin(3z) = 3\sin z \cos^2 z - \sin^3 z = \sin \mathrm{z}\left(3 - 4\sin^2 z\right). \tag{3.33c}$$

These formulas can also be deduced by recurrence, for example, applying twice (3.30a,b) to

$$\begin{aligned} \cos(z_1 + z_2 + z_3) &= \cos(z_1)\cos(z_2 + z_3) - \sin(z_1)\sin(z_2 + z_3) \\ &= \cos(z_1)\left[\cos(z_2)\cos(z_3) - \sin(z_2)\sin(z_3)\right] \\ &\quad - \sin(z_1)\left[\sin(z_2)\cos(z_3) + \sin(z_3)\cos(z_2)\right], \end{aligned} \tag{3.34}$$

which leads to (3.34) ≡ (3.32a). The quadruple addition and quadruplicating formulas are indicated in Example 10.7.

3.8 Conjugate Complex and Triangular Inequalities

Squaring (3.17a,b) leads to

$$(z + z^*)^2 \geq 0 \geq (z - z^*)^2, \tag{3.35a,b}$$

the *complex conjugate inequalities, where the equality sign in (3.35a)* [(3.35b)] *holds only for imaginary (real)* z. The conjugate can be used to calculate the modulus of the sum:

$$\begin{aligned} |u + v|^2 = (u + v)(u^* + v^*) &= u^*u + v^*v + u^*v + v^*u \\ &= |u|^2 + |v|^2 + u^*v + (u^*v)^*, \end{aligned} \tag{3.36a}$$

where the real representation (3.22a) may be used:

$$u^*v + (u^*v)^* = \mathrm{Re}\,(u^*v) = \mathrm{Re}\left\{|u|\,|v|\,\mathrm{e}^{i(\beta-\alpha)}\right\} = |u|\,|v|\cos(\alpha - \beta)\,; \tag{3.36b}$$

substitution of (3.36b) in (3.36a) yields (3.6).

Bearing in mind that $-1 \leq \cos(\alpha - \beta) \leq +1$, from (3.6) $\equiv$ (3.36a,b)

$$|u|^2 + |v|^2 - 2\,|u|\,|v| \leq |u + v|^2 \leq |u|^2 + |v|^2 + 2\,|u|\,|v|\,, \tag{3.37a}$$

follows the relation:

$$|u| > |v|: \quad |u| - |v| \leq |u + v| \leq |u| + |v|\,, \tag{3.37b}$$

which is designated the **triangular inequality**, *since it shows* (Figure 3.2a) *that the length of one side of a triangle cannot exceed the sum of the lengths of the other two sides, and cannot be less than their difference; the right- (left-) hand side equality holds in the case in* Figure 3.2c (3.2d) *of parallel (antiparallel) collinear sides.*

3.9 Generalized Schwartz (1890) or Polygonal Inequality

The result (3.37b) can be generalized to finite sum:

$$|z_k| - \sum_{\substack{n=1 \\ n \neq k}}^{N} |z_n| \leq \left|\sum_{n=1}^{N} z_n\right| \leq \sum_{n=1}^{N} |z_n|, \tag{3.38}$$

which is the generalized Schwartz or (1890) **polygonal inequality**, *showing that* (Figure 3.5a) *the length of a straight line between two points cannot exceed the length of any polygonal line joining the two points, and equality on the r.h.s. can only occur for collinear parallel segments* (Figure 3.5b); *the equality on the l.h.s. applies to collinear segments with z_k antiparallel to all others* (Figure 3.5c). *Also the length of one side of the polygon cannot be less than the longest side with the lengths of all remaining sides subtracted from it.* The proof of (3.38) is made by induction: (i) since (3.38) applies for $N = 2$, in which case it

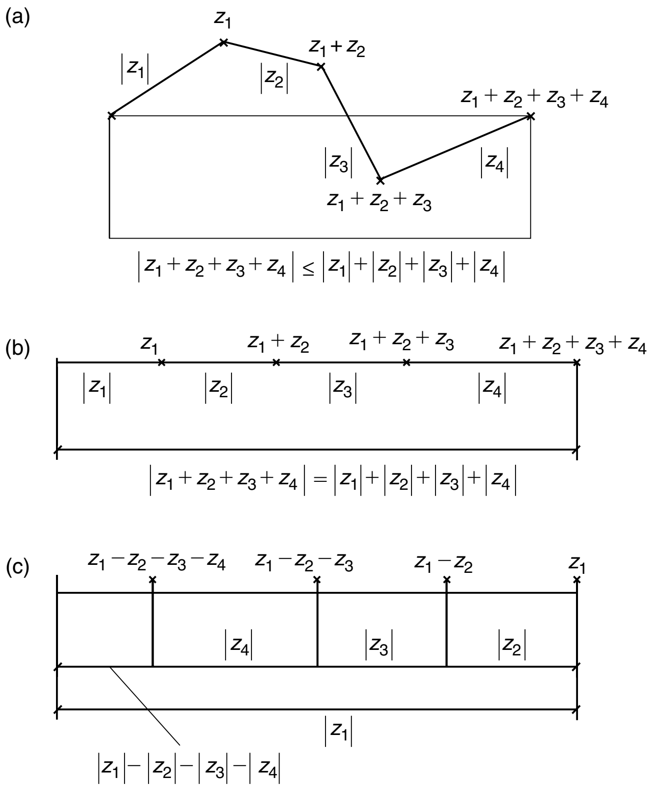

FIGURE 3.5
The polygonal inequality (3.38) states that the length of a polygonal line cannot be less than the length of a straight line joining the ends (a); it is equal only if all sides are parallel (b); if some segments are antiparallel, they are subtracted from the total length (c).

coincides with (3.37b) for $u \equiv z_1$, $v \equiv z_2$; (ii) besides if (3.38) holds for N, then it also holds for $N + 1$. To prove the latter statement: (ii-1) choose (u,v) in (3.39a):

$$u \equiv z_{N+1},\ v \equiv \sum_{n=1}^{N} z_n : \quad \left| \sum_{n=1}^{N+1} z_n \right| = |u+v| \le |u| + |v| = |z_{N+1}| + \left| \sum_{n=1}^{N+1} z_n \right| \le \sum_{n=1}^{N+1} |z_n| \,, \tag{3.39a,b}$$

where (3.37b) and (3.38) were used to prove the right-hand inequality (3.39b); (ii-2) for the left-hand inequality choose (u,v) in (3.40a):

$$u \equiv z_k,\ v \equiv \sum_{\substack{n=1\\ n\neq k}}^{N+1} z_n : \quad |z_k| - \sum_{\substack{n=1\\ n\neq k}}^{N+1} |z_n| = |u| - |v| \le |u+v| = \left| \sum_{n=1}^{N+1} z_n \right| \le \sum_{n=1}^{N+1} |z_n| \,, \tag{3.40a,b}$$

where (3.37b) and (3.38) were used to prove (3.40b). Note that the left- (3.40b) and right-hand (3.40a) inequalities operate independently.

Note 3.1. *Formal Inequalities and Equalities*: The Schwartz inequality applies not only to numbers (3.38) but also to the integrals of functions. It can also be extended to exponents other than $1/2 = p$ in the modulus, by the Holder and Minkowski inequalities; the latter arise in the context of functional analysis with application to generalized functions, and Fourier series and integrals, and other orthogonal expansions and integral representations. The addition and multiplication formulas in Section 3.7 can be extended from the circular cosine and sine to the circular tangent and to the corresponding hyperbolic functions. Some higher transcendental functions also have addition and/or multiplication formulas, for example, (i) the gamma and polygamma functions; (ii) the elliptic functions of Jacobi and Weierstrass; and (iii) some special functions, for example, Bessel.

Conclusion 3: Graphical representation of operations and relations between complex numbers: (Figure 3.1) sum, difference, and symmetry; (Figure 3.3) product, division, and inverse. The product of complex numbers specifies (Figure 3.4) two pairs of similar triangles, with corresponding angles γ and δ. The (Figure 3.2) triangular inequality (a), has the orthogonal (b), parallel (c), and antiparallel (d) subcases; it is generalized by (Figure 3.5) the polygonal inequality (a), which also has a parallel (b) and antiparallel (c) subcases.

4

Impedance of Associations of Circuits

A mechanical circuit consists (Section 4.1) of masses, dampers, and springs, and is analogous to an electrical circuit consisting (Section 4.3) of inductors, resistors, and capacitors. A mechanical (electrical) circuit is characterized by the impedance (Section 4.2), which is the ratio of the mechanical (electromotive) force to the velocity (electric current). The impedance is generally complex (Section 4.4), since it appears as a coefficient of proportionality between two quantities that may have different amplitudes and phases (Section 4.5). Several circuits, electrical or mechanical, may be associated in series, in parallel (Section 4.6) or in hybrid arrangements (Section 4.9), and the total impedance is specified by the partial impedances of each separate element. The calculation of impedances of complex circuits (Section 4.7) is an application of the algebra of complex numbers, and the laws of association are different (Section 4.8) for mechanical and electrical circuits. The mechanical circuits are used in the suspensions of vehicles (e.g., cars, trains, etc.) and in the "soft" or "shock absorbing" mountings of machinery; the electrical circuits are used in a variety of electrical devices, for example, radios, computers, appliances, and so on. The electromechanical circuits also have applications like control systems, actuators, and so on.

4.1 Inertia, Friction, and Elastic Forces

Consider a basic **mechanical circuit**, represented in Figure 4.1a and consisting of a mass, damper, and spring. When the mass is displaced from its equilibrium position, three **forces** arise: (i) the **inertia** force, equal to the product of mass m and acceleration; (ii) the **friction** force, which, for weak damping, is proportional to the velocity through the **damping coefficient** b; (iii) the **elastic** force, which, for small displacement, is proportional to the latter through the **resilience** k of the spring. Adding all forces leads to the **mechanical force**, which is specified by

$$m > 0, b > 0, k > 0 : \quad F = m\ddot{x} + b\dot{x} + kx, \tag{4.1}$$

where dot denotes derivative with regard to time, viz., $x, \dot{x}, \ddot{x}$ are, respectively, the displacement, velocity, and acceleration. The friction force was not considered in the harmonic oscillator (Section 2.9), that is, (2.19) corresponds to (4.1) with $b = 0$. There may be two causes for the motion: free motion if there is no mechanical force $F = 0$, and the initial conditions are a nonequilibrium position (as in Section 2.9); (ii) forced motion, which is independent of initial conditions, and specified by the mechanical force (Section 4.2).

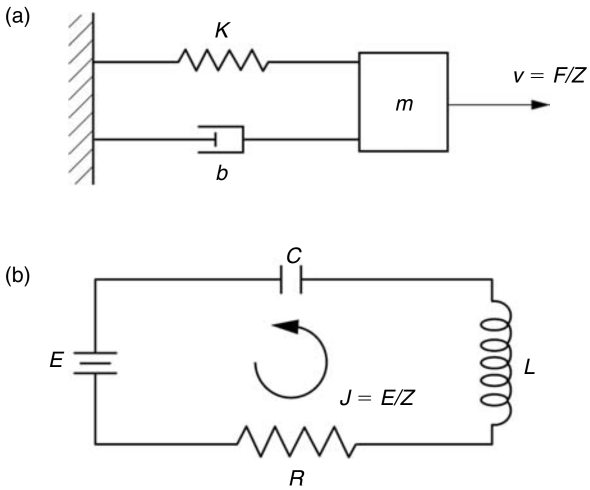

FIGURE 4.1
The analogy between mechanical (a) [electrical (b)] circuits replaces the (i) mass m (induction L) of body (self); (ii) damping b (resistance R) of a damper (resistor); and (iii) resilience k (inverse capacity $1/C$) of a spring (capacitor). These specify the mechanical Z_m (electrical Z_e) impedance of the circuit relating the mechanical force F (electromotive force E) to the velocity v (electric current J).

4.2 Free and Forced Motion of Circuit

In the **free motion** of a circuit, for which the mass is released from a disturbed position, in the absence of mechanical force $F = 0$, the relation (4.1) becomes a homogeneous or unforced differential equation:

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0; \tag{4.2}$$

it is linear with constant coefficients (2.15) and specifies the nature of the equilibrium (Section 2.8) and the type of motion in its vicinity (2.1a,b). Given this type of displacement x, it is related to the velocity v and acceleration a by (2.2a,b), so that

$$v \equiv \dot{x}, \quad x = -i\frac{v}{\zeta}, \quad a = i\zeta v. \tag{4.3a–c}$$

Given a velocity $v\,(t)$, imposed upon the circuit by an external influence, the associated mechanical force (4.1) is specified as a function of time by (4.4a):

$$F\,(t) = Z_m v\,(t)\,, \quad Z_m \equiv b + i\left(\zeta m - \frac{k}{\zeta}\right), \tag{4.4a,b}$$

as a linear function of velocity $v(t)$, through a complex constant (4.4b), designated **mechanical impedance**. For example, the wheel of a car travels over an undulating ground, by measuring its velocity v, and knowing the impedance Z_m of the suspension, the mechanical force F being exerted can be determined. In the forced motion, the frequency ω and decay or amplification λ in (4.5a) $\equiv$ (2.1a) are imposed by the mechanical force (4.1) and lead to the mechanical impedance (4.4b) $\equiv$ (4.5b):

$$\zeta \equiv \omega + i\lambda: \quad Z_m = b + m\,(i\omega - \lambda) - k\frac{\lambda + i\omega}{\omega^2 + \lambda^2}. \tag{4.5a,b}$$

Using (4.3a–c) the mechanical force (4.1) can also be expressed in terms of the acceleration (4.6a) or displacement (4.6b):

$$F\,(t) = \left(m - i\frac{b}{\zeta} - \frac{k}{\zeta^2}\right) a\,(t) = \left(-m\zeta^2 + ib\zeta + k\right) x\,(t). \tag{4.6a,b}$$

In all cases the coefficient is independent of time, and is specified by (i) the mass m of the body, the coefficient b of friction, and the resilience k of the spring; and (ii) the frequency ω and decay/growth λ of the forcing.

4.3 Electrical Induction, Resistance, and Capacity

The **electromechanical analogy** (Table 4.1) replaces a mechanical (Figure 4.1a) by an electrical circuit, represented in Figure 4.1b, with the displacement x replaced by the electric charge q, and the velocity $v \equiv \dot{x}$ by the electric current $j = \dot{q}$. *The devices forming the basic electrical (mechanical) circuit are a capacitor (spring), resistance (damper), and induction (mass), and the circuit is driven by* **the electromotive** *(mechanical)* **force** $E\,(F)$, *applied externally, for example, through a battery (engine). The total electromotive force is the sum of the contribution of the capacitor* $1/C$, *resistance* R, *and self* L, *multiplying, respectively, the charge, current, and its time derivative:*

$$E = L\ddot{e} + R\dot{e} + \frac{e}{C}. \tag{4.7}$$

The expression (4.7) is analogous to the relation (4.1), exchanging mechanical F for electromotive E force, and velocity $v \equiv \dot{q}$ for current, so that

$$J \equiv \dot{e}: \quad E\,(t) = Z_e\; J\,(t)\,, \quad Z_e \equiv R + i\left(\zeta L - \frac{1}{\zeta C}\right), \tag{4.8a,b}$$

TABLE 4.1
Analogy of Mechanical and Electrical Circuits*

Circuit	**Mechanical**	**Electrical**
Force	Mechanical: F	Electromotive: E
Variable	Velocity: v	Current: j
Zero phase	Damper: b	Resistor: R
Phase $+\pi/2$	Mass: m	Induction: L
Phase $-\pi/2$	spring: k	Capacitor: $1/C$
Impedance	Z_m: (4.4b)	Z_e: (4.8b)

* Corresponds to Figure 4.1.

the **electrical impedance**, relating electromotive force to current (4.8a), is specified by (4.8b) ≡ (4.8c):

$$Z_e = R + L\,(i\omega - \lambda) - \frac{\lambda + i\omega}{C\,(\omega^2 + \lambda^2)}. \tag{4.8c}$$

For example, measuring the current J along an electrical circuit, and knowing the impedance Z_e, the electromotive force E being used to drive the circuit is specified.

4.4 Decomposition of Impedance into Inductance and Reactance

The mechanical (electrical) impedance, denoted by Z_m in (4.4b) [Z_e in (4.8b)], consists for real ζ, that is, an imposed frequency without decay or amplification,

$$\zeta = \omega, \lambda = 0: \quad Z_m = b + i\left(m\omega - \frac{k}{\omega}\right), \quad Z_e = R + i\left(L\omega - \frac{1}{\omega C}\right), \tag{4.9a,b}$$

of the damping coefficient b (resistance R) for the real part, plus an imaginary part involving the mass m (induction L of a self) and resilience k of a spring (inverse capacity $1/C$ of a capacitor). A circuit is said to be reactive if the impedance is real, that is, it consists only of a damper (resistance), and the mechanical (electromotive) force is in phase with the velocity (current). The opposite case is the inductive circuit, for which the impedance is imaginary, and which can include a mass m (induction L) and spring of resilience k (capacitor of capacity C), implying that the mechanical (electromotive) force is out-of-phase by $\pm\pi/2$ relative to the velocity (current). Table 4.1 summarizes the electromechanical analogy, in the general case, when the impedance can be decomposed into a reactance and inductance that are, respectively, its real and imaginary parts. If the mechanical force has a frequency and a decay/amplification (4.5a), the real (imaginary) part of the impedance (4.10a), called **reactance** X (**inductance** Y) is (i) for a mechanical circuit (4.5b):

$$Z_m \equiv X_m + iY_m: \quad X_m = b - m\lambda - \frac{k\lambda}{\omega^2 + \lambda^2}, \ \mathrm{Y}_m = m\omega - \frac{k\omega}{\omega^2 + \lambda^2}, \tag{4.10a–c}$$

(ii) for an electrical circuit (4.8c):

$$Z_e \equiv X_e + iY_e: \quad X_e = R - \lambda L - \frac{\lambda/C}{\omega^2 + \lambda^2}, \quad Y_e = \omega L - \frac{\omega/C}{\omega^2 + \lambda^2}, \tag{4.11a–c}$$

The inverse of the impedance:

$$\frac{1}{A} = Z = X + iY: \quad (\text{admittance})^{-1} \equiv \text{impedence} \equiv (\text{resistance}) + i(\text{reactance}), \tag{4.12}$$

is called **admittance**.

4.5 Activity in Terms of the Velocity, Force, and Impedance

The product of the force (4.13a) by the velocity (4.13b),

$$\frac{f\,(t)}{f_0} = \cos(\omega t) = \mathrm{Re}\left(\mathrm{e}^{i\omega t}\right) = \frac{v\,(t)}{v_0}, \tag{4.13a,b}$$

is the **activity** or work per unit time, which is specified (i) in the real representation by

$$A(t) \equiv f(t)\,v(t) = f_0 v_0 \cos^2(\omega t)\,; \tag{4.14a}$$

(ii) in the complex representation (3.22a) by

$$\begin{aligned} A(t) &= \operatorname{Re}\{f(t)\}\operatorname{Re}\{v(t)\} = \frac{1}{2}\operatorname{Re}(fv + fv^*) = \frac{1}{2}f_0 v_0 \operatorname{Re}\left(\mathrm{e}^{i2\omega t} + 1\right) \\ &= \frac{1}{2}f_0 v_0\,[1 + \cos(2\omega t)]. \end{aligned} \tag{4.14b}$$

These are equivalent (4.14a) = (4.14b) on account of (3.31d). The activity may be expressed in terms of the velocity alone using the impedance (4.4a), which is complex. Thus, the real representation is applied in two stages: (i) calculation of the force:

$$Z = X + i\mathrm{Y}: \quad \begin{aligned} f(t) &= \operatorname{Re}\{Zv(t)\} = v_0 \operatorname{Re}\left\{(X + i\mathrm{Y})\,\mathrm{e}^{i\omega t}\right\} \\ &= v_0\,[X(\cos(\omega t) - \mathrm{Y}\sin(\omega t))], \end{aligned} \tag{4.15a}$$

which shows that the resistance is in phase and the reactance out-of-phase by $\pi/2$, viz. (2.6a); (ii) the activity follows from

$$A(t) = f(t)\,v(t) = v_0^2\left[X\cos^2(\omega t) - \frac{Y}{2}\sin(2\omega t)\right]. \tag{4.15b}$$

The activity can also be calculated in terms of the velocity and impedance in the complex representation:

$$\begin{aligned} A(t) &= \operatorname{Re}\{f(t)\}\operatorname{Re}\{v(t)\} = \operatorname{Re}\{Zv(t)\}\operatorname{Re}\{v(t)\} \\ &= \frac{1}{2}\operatorname{Re}\left(Zv^2 + Zvv^*\right) = \frac{1}{2}|v|^2\operatorname{Re}(Z) + \frac{1}{2}\operatorname{Re}\left(Zv^2\right); \end{aligned} \tag{4.16a}$$

the r.h.s. of (4.16a) ≡ (4.16c) is evaluated in terms of the real and imaginary components of the velocity (4.16b):

$$\begin{aligned} \{v_x, v_y\} &\equiv v_0 \cos, \sin(\omega t): \\ A(t) &= \frac{X}{2}|v|^2 + \frac{1}{2}\operatorname{Re}\left[(X + i\mathrm{Y})\left(v_x^2 - v_y^2 + 2iv_x v_y\right)\right] \\ &= \frac{X}{2}\left(v_x^2 + v_y^2\right) + \frac{X}{2}\left(v_x^2 - v_y^2\right) - \mathrm{Y}v_x v_y \\ &= Xv_x^2 - \mathrm{Y}v_x v_y = v_0^2\left[X\cos^2(\omega t) - \frac{\mathrm{Y}}{2}\sin(2\omega t)\right], \end{aligned} \tag{4.16b,c}$$

in agreement with (4.16c) ≡ (4.15b).

The averages over a period,

$$\tau = \frac{2\pi}{\omega}, \theta = \omega t: \qquad \omega\int_0^{2\pi/\omega}\sin(2\omega t)\mathrm{d}t = \int_0^{2\pi}\sin 2\theta\,\mathrm{d}\theta = \left[\frac{-\cos(2\theta)}{2}\right]_0^{2\pi} = 0, \tag{4.17a}$$

$$\omega\int_0^{2\pi/\omega}\cos^2(\omega t)\,\mathrm{d}t = \int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\mathrm{d}\theta = \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_0^{2\pi} = \pi, \tag{4.17b}$$

applied to (4.15b) ≡ (4.16c) lead the conclusion that

$$\overline{A\,(t)} \equiv \frac{1}{\tau}\int_0^\tau A\,(t)\,\mathrm{d}t = \frac{\pi}{\omega\tau}Xv_0^2 = \frac{1}{2}Xv_0^2 = \frac{1}{2}A\,(0) = \frac{f_0 v_0}{2}, \tag{4.18}$$

the average activity over a period is one-half of the square of the amplitude of the velocity multiplied by the resistance, that is, one-half of the initial value, which is the peak value (4.14a) ≡ (4.14b) for the resistive term in (4.17b); the contribution (4.17a) of the reactance to the activity cancels over a period, and thus has zero average. Similar reasonings apply to the energy (activity) associated with the electric current $J(t)$ [velocity $v(t)$] and electromotive force $E(t)$ [mechanical force $F(t)$] with the electrical (mechanical) impedance playing the same role.

4.6 Mechanical Circuits in Parallel or Series

When N mechanical circuits are associated in parallel, for example, $N = 2$ in Figure 4.2a, the velocity v is the same for all of them (4.19a) and if the impedances z_n are generally distinct, so will be the mechanical forces F_n:

$$F_n = z_n v, \quad F_{mp} = \sum_{n=1}^{N} F_n = v\sum_{n=1}^{n} z_n, \tag{4.19a,b}$$

which add up to the total force (4.19b). The latter specifies the total impedance (4.20a):

$$Z_{mp} = \frac{F_{mp}}{v}, \quad Z_{mp} = \sum_{n=1}^{N} z_n, \tag{4.20a,b}$$

so that *the total impedance of mechanical circuits associated in parallel is the sum of the impedances (4.20b)*. For example, if a wheel is suspended from two parallel springs, the force required to move it at a given velocity increases, and so the inductance (imaginary part of the impedance, apart from sign) is greater.

In contrast, if the N mechanical circuits are associated in series, for example, $N = 2$ in Figure 4.2b, the forces exerted by each of them must be equal, in order that there may be equilibrium at the junctions. The same force F with generally different impedances z_n leads to distinct velocities v_n in (4.21a):

$$v_n = \frac{F}{z_n}, \quad v = \sum_{n=1}^{N} v_n = F\sum_{n=1}^{N}\frac{1}{z_n}, \tag{4.21a,b}$$

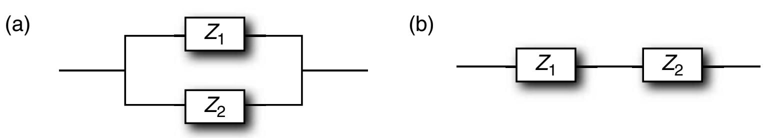

FIGURE 4.2
Two circuits can be associated in parallel (a) or in series (b). The laws of association of impedances are opposite for mechanical and electrical circuits: (i) the impedances add for mechanical (electrical) circuits in parallel (a) [series (b)]; and (ii) the inverse impedances add for mechanical (electrical) circuits in series (b) [parallel (a)].

which add up to the total velocity (4.21b). The latter specifies the total impedance (4.14a):

$$v = \frac{F}{Z_{ms}}, \qquad \frac{1}{Z_{ms}} = \sum_{n=1}^{N} \frac{1}{z_n}, \tag{4.22a,b}$$

and it follows that *for mechanical circuits associated in series, the inverses of the impedances add up to the inverse of the total impedance (4.22b).*

4.7 Electromechanical Analogy and Contrasting Laws

In the case of N electrical, instead of mechanical, circuits associated in parallel (series), for example, $N = 2$ in Figure 4.2a (4.2b), the currents add up (are the same) and the electromotive forces are the same (add up):

$$\frac{E}{Z_{ep}} = J = \sum_{n=1}^{N} J_n = E \sum_{n=1}^{N} \frac{1}{z_n}, \qquad Z_{es} = \frac{E}{J} = \frac{\sum_{n=1}^{N} E_n}{J} = \sum_{n=1}^{N} z_n. \tag{4.23a,b}$$

It follows that

$$\frac{1}{Z_{ep}} = \sum_{n=1}^{N} \frac{1}{z_n}, \qquad Z_{es} = \sum_{n=1}^{N} z_n \tag{4.24a,b}$$

for electrical circuits associated in parallel (series) the inverse impedances (impedances) are added (4.24a) [(4.24b)].

The laws of association of electrical circuits (4.24a,b) are the inverse of those (4.19b, 4.20b) applying to mechanical circuits; the electromechanical analogy (Table 4.1) is followed by a contrast: *the association of mechanical circuits in parallel is equivalent to the association of electrical circuits in series, and vice versa viz:* (i) the impedances add for mechanical (electrical) circuits in parallel (series) in (4.25a):

$$Z_{mp} = \sum_{n=1}^{N} z_n = Z_{es}; \qquad Z_{ms} = \left\{ \sum_{n=1}^{N} z_n^{-1} \right\}^{-1} = Z_{ep}, \tag{4.25a,b}$$

(ii) the inverse impedances add for mechanical (electrical) circuits in series (parallel) in (4.25b). The latter law (4.25b) can be rewritten:

$$Z_{ms} = \frac{\prod_{n=1}^{N} z_n}{\sum_{n=1}^{N} \prod_{\substack{m=1 \\ m \neq n}}^{N} z_m} = Z_{ep}, \tag{4.26}$$

where (i) the numerator is the product of all impedances; (ii) the denominator is a sum of N terms that are all the distinct products of $N - 1$ impedances. The simplest cases are

$$N = 2: \quad Z_{ms}^{-1} = z_1^{-1} + z_2^{-2} \equiv Z_{ep}^{-1}, \quad Z_{ms} = \frac{z_1 z_2}{z_1 + z_2} \equiv Z_{ep}, \tag{4.27a,b}$$

or for two impedances and

$$N = 3: \quad Z_{ms}^{-1} = z_1^{-1} + z_2^{-2} + z_3^{-2} \equiv Z_{ep}^{-1}, \quad Z_{ms} = \frac{z_1 z_2 z_3}{z_1 z_2 + z_2 z_3 + z_1 z_3} \equiv Z_{ep}, \tag{4.28a,b}$$

for three impedances.

4.8 Comparison of Two Circuits in Parallel and in Series

Considering two circuits, of impedances z_1, z_2 that can be associated either in parallel $Z_{mp}\,(Z_{ep})$ or in series $Z_{ms}\,(Z_{es})$ in a mechanical (electrical) circuit:

$$Z_{mp} = z_1 + z_2 = Z_{es}, \quad Z_{mp} = \left(z_1^{-1} + z_2^{-1}\right)^{-1} = \frac{z_1 z_2}{z_1 + z_2} = Z_{ep}. \tag{4.29a,b}$$

In the case of identical impedances (4.30a) the total impedance is (4.30b,c)

$$z_1 = z_2 \equiv z_0 : \quad Z_{mp} = 2z_0 = Z_{es}, \quad Z_{ms} = \frac{z_0}{2} = Z_{ep}. \tag{4.30a–c}$$

Generalizing to N identical subcircuits in (4.31a) in (4.25a,b) leads to (4.31b,c):

$$z_1 = z_2 = \cdots = z_N \equiv z_0 : \quad Z_{mp} = Nz_0 = Z_{es}, \quad Z_{ms} = \frac{z_0}{N} = Z_{ep}. \tag{4.31a–c}$$

Thus *association of* N *identical mechanical (electrical) circuits, multiplies the impedance by* N *if they are in parallel (series), and divides it by* N *if they are in series (parallel).* In the case of two resistances, if they are placed in parallel, there are two paths, so that a larger total current flows for the same electromotive force, implying a reduced resistance (real part of the impedance); if the resistances are placed in series, for the same current, a larger electromotive force is required, and the resistance is greater.

4.9 Hybrid Associations of Three Circuits

Starting with three circuits of impedances z_1, z_2, z_3, there are four distinct types of associations illustrated in Figure 4.3, ranging from all in parallel (4.3a) to all in series (4.3d), through two hybrid cases (4.3b,c). In each case, the total mechanical (electrical) impedances are distinct: (i) all in parallel (series) in Figure 4.3a (4.3d):

$$Z_{mA} = z_1 + z_2 + z_3 = Z_{eD}; \tag{4.32}$$

(ii) two in series (parallel) and one in parallel (series) in Figure 4.3b (4.3c):

$$Z_{mC} = \left(z_1^{-1} + z_2^{-1}\right)^{-1} + z_3 = z_3 + \frac{z_1 z_2}{z_1 + z_2} = Z_{eB}; \tag{4.33}$$

(iii) two in parallel (series) and one series in (parallel) in Figure 4.3c (4.3b):

$$Z_{mB} = \left\{(z_1 + z_2)^{-1} + z_3^{-1}\right\}^{-1} = \frac{z_3\,(z_1 + z_2)}{z_1 + z_2 + z_3} = Z_{eC}; \tag{4.34}$$

(iv) all in series (parallel) in Figure 4.3d (4.3a):

$$Z_{mD} = \left(z_1^{-1} + z_2^{-1} + z_3^{-1}\right)^{-1} = \frac{z_1 z_2 z_3}{z_1 z_2 + z_1 z_3 + z_2 z_3} = Z_{eA}. \tag{4.35}$$

In the case of three identical mechanical (electrical) circuits of impedances (4.36a):

$$z_1 = z_2 = z_3 \equiv z_0 : \quad Z_{mA}, Z_{mB}, Z_{mC}, Z_{mD} = \left\{3, \frac{3}{2}, \frac{2}{3}, \frac{1}{3}\right\} z_0 = Z_{eD}, Z_{eB}, Z_{eC}, Z_{eA}, \tag{4.36a–e}$$

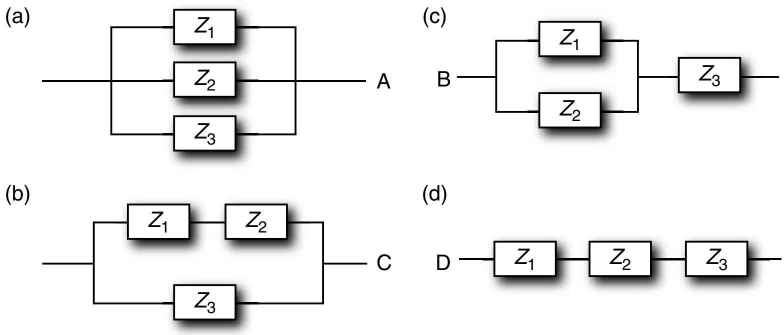

FIGURE 4.3
There are four possible associations of three circuits from all in parallel (a) to all in series (d). The intermediate cases are two in parallel and one in series (c) or two in series and one in parallel (b). The total impedances differ for mechanical (electrical) circuits, for example, are added for (a) [(d)].

it follows that *for mechanical force (electrical) circuits the impedance increases (decreases) as more are put in parallel (series),* viz. (Figure 4.3) it is (i) maximum (4.36a,b) $\equiv$ (4.32) for all impedances in parallel (series) in Figure 4.3a (4.3d); (ii) minimum (4.36e) $\equiv$ (4.35) for all impedances in series (parallel) in Figure 4.3d (4.3a); and (iii) in the intermediate cases, it is larger for (4.36c) $\equiv$ (4.33) than for (4.36d) $\equiv$ (4.34) because there is one impedance in parallel (series) in Figure 4.3b (4.3c) with the other two. There are (i) two distinct associations of two impedances (Section 4.8); (ii) four distinct associations of three impedances (Section 4.9); and (iii) ten distinct associations of four impedances (Example 10.8).

Note 4.1. *Impedances of Circuits and Waves*: The impedance is defined by the ratio between the mechanical (electromotive) force and velocity (electric) current assuming both are complex exponential functions of time, that is, have constant frequency and damping/amplification. The impedance is defined for waves, for example, sound (Subsection 22.4.1) as the ratio of pressure to velocity,

$$p(\omega) = -Z(\omega)v(\omega), \tag{4.37}$$

in the frequency domain with opposite sign. In this case, a Fourier transform (Subsection 17.5.2) leads to a convolution integral in the time domain. The main point of the preceding analysis, that the impedance is complex, remains valid, as do the laws of association of circuits.

Note 4.2. *Electromechanical Analogies and Controls*: The laws of association of impedances are extensible to the association of control systems; they relate also to the time evolution of mechanical and electrical circuits. They are an instance of electromechanical analogies that extend to potential fields and waves. These in turn open the way to interactions between electromagnetism and matter, for example, fluids and solids; also ionized fluids or plasmas. These links extend to large-scale relativistic and small-scale quantum phenomena.

Conclusion 4: Analogy between (Figure 4.1) mechanical (a) [electrical (b)] circuits consisting of mass (induction), damper (resistance) and spring (capacitor). The association of two impedances (Figure 4.2) in parallel (a) and in series (b). With three impedances (Figure 4.3) there are four possible distinct associations, ranging from all in parallel (a) to all in series (d), through two hybrid cases, with two circuits in series and one in parallel (b) or vice-versa (c).

5

Power, Root, and Logarithm

The power with positive integer order n, is an example (Section 5.1) of a single-valued function, since to each z corresponds only one z^n; the corresponding points for successive n lie on a logarithmic spiral (Section 5.2). The inverse function (Section 5.3), namely the root of order n, is ***n*-valued**, since for each z there are n distinct possible values of $\sqrt[n]{z}$; the corresponding points lie at the vertices of a regular polygon (Section 5.4) with n sides, leading to a number of trigonometric identities (Section 5.5). There are also **many-valued** functions, like the logarithm (Section 5.6), such that to each value of z corresponds an infinite number of values of $\log z$. The function logarithm appears in the definition (Section 5.7) of z^a complex power z with complex exponent a that generalizes both the ordinary nth power and nth root that correspond respectively to $a = n$ and $a = 1/n$. It can be shown that the behavior of the complex power z^a at the origin $z \to 0$ and infinity $z \to \infty$ depends only on the modulus of the base and (Section 5.8) on the real part of the exponent, and not on the imaginary part; this is a particular case of the exponential of a power having different asymptotic behavior as $z \to \infty$ in different angular sectors (Section 5.9). Selecting of the "correct" branch of a function, and taking into account "jump conditions" between branches, may be necessary to arrive at correct results; it is also an indication that the problem in question has several solutions that may or may not be equivalent for a specific purpose.

5.1 *n*-th Power as the $(n-1)$-Times Iterated Product

The ***n*-th power**, with n a positive integer, of a complex number z is defined as the product by itself iterated $(n-1)$ times, and is most conveniently calculated in the polar representation:

$$z = r\mathrm{e}^{i\varphi}: \quad z^n \equiv z \ldots z\,[(n-1)\,\text{times}] = r^n\,\mathrm{e}^{in\varphi}; \tag{5.1a,b}$$

the modulus and argument of the power are given by (1.21a,b), and the real and imaginary parts by

$$\mathrm{Re}\,(z^n) = r^n \cos(n\varphi), \quad \mathrm{Im}\,(z^n) = r^n \sin(n\varphi); \tag{5.2a,b}$$

the plane may be divided into n angular sectors, in which the real (imaginary) parts of the power alternate in sign according to Table 5.1; the sectors where each sign is constant have an angular width π/n and are rotated by $\pi/2n$ between the real and imaginary parts of z, as shown by the null-lines $k = 0, \ldots, n-1$:

$$k = 0, \ldots, 2n-1: \quad \varphi_k = \left(k + \frac{1}{2}\right)\frac{\pi}{n}, \frac{k\pi}{n}, \tag{5.3a,b}$$

respectively, for the real (5.3a) and imaginary (5.3b) parts of the power.

TABLE 5.1
Signs of Real and Imaginary Parts of Integral Power

z^n	$\mathrm{Re}(z^n)$	$\mathrm{Im}(z^n)$
Positive	$\left(k-\frac{1}{2}\right)\frac{\pi}{n} < \varphi < \left(k+\frac{1}{2}\right)\frac{\pi}{n}$	$\frac{2k\pi}{n} < \varphi < (2k+1)\frac{\pi}{n}$
Zero	$\varphi = \left(k+\frac{1}{2}\right)\frac{\pi}{n}$	$\varphi = \frac{k\pi}{n}$
Negative	$\left(k+\frac{1}{2}\right)\frac{\pi}{n} < \varphi < \left(k+\frac{3}{2}\right)\frac{\pi}{n}$	$(2k-1)\frac{\pi}{n} < \varphi < \frac{2k\pi}{n}$

$k = 0, \ldots, 2n-1$.
Note: The signs of the real and imaginary parts of z^n depend on the angular sector through the origin.

5.2 Discrete Set of Points on Logarithmic Spiral

Graphically (Figure 5.1) the power z^n of z to the order n (i) raises the modulus to the power n, that is, yields points progressively farther from (closer to) the origin for $|z| > 1$ ($|z| < 1$), and points on the unit circle for the intermediate case of modulus unity $|z| = 1$; (ii) multiplies the argument by n, and thus rotates counterclockwise (clockwise) around the origin for $\arg(z) > 0(\arg(z) < 0)$, leaving the points on a straight line only in the case of a real base $\arg(z) = 0$:

$$\frac{\log|z^n|}{\log|z|} = n = \frac{\arg(z^n)}{\arg(z)}; \tag{5.4a,b}$$

the powers of a complex number z number lie (Figure 5.1) on a logarithmic spiral:

$$\log r = k\varphi, \quad k \equiv \frac{\log|z|}{\arg(z)}, \tag{5.5a,b}$$

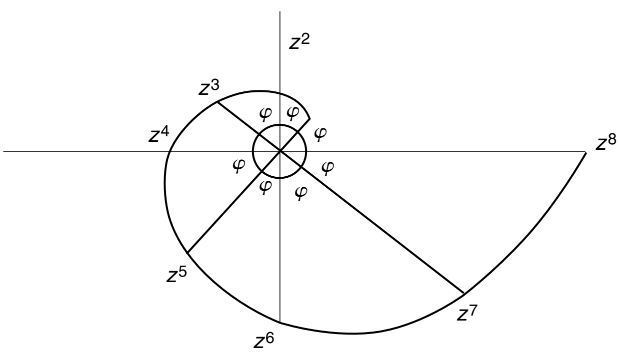

FIGURE 5.1
The successive powers z^n of a complex number lie at equal angles $\varphi = \arg(z)$ along a logarithmic spiral $\log|z^n| = n\log|z|$, and (i/ii) diverge (converge) toward infinity (the origin) as $n \to \infty$ for $|z| > 1$ ($|z| < 1$); (iii) in the intermediate case $|z| = 1$ the spiral degenerates into a circle and z^n divide it in equal angular sectors, for example, three in Figure 5.2.

that is right- (left-) handed for positive arg(z) >0 *[negative arg(z)* <0*] argument, and unfolds to infinity (curls up to the origin) for modulus* $|z|>1(|z|<1)$ *greater (smaller) than unity.* The degenerate cases $\arg(z)=0$ and $|z|=1$ the successive powers lie, respectively, along the real axis (unit circle) that have zero (unit) curvature.

5.3 Inversion of the Power: Roots of Order n

The n-th root is defined as any inverse of the n-th power:

$$w \equiv \sqrt[n]{z} \quad \Leftrightarrow \quad w^n \equiv z \equiv r\mathrm{e}^{i\varphi}. \tag{5.6}$$

Using the polar form (5.7a) of w:

$$w = \sigma \mathrm{e}^{i\psi}, \quad \sigma^n \mathrm{e}^{in\psi} = r\mathrm{e}^{i\varphi}, \tag{5.7a,b}$$

it follows that σ, ψ have to be determined satisfying (5.7b) for given r, φ. The number of distinct solutions of (5.7b) specifies the number of roots: (i) there is a single solution for σ:

$$\sigma = \sqrt[n]{r}, \quad \left|\sqrt[n]{z}\right| = \sqrt[n]{|z|}, \tag{5.8a,b}$$

that is, all roots of order n have the same modulus that is the root of order n of the modulus; (ii) the solution for ψ is

$$\mathrm{e}^{in\psi} = \mathrm{e}^{i\varphi} = \mathrm{e}^{i\varphi + i2k\pi}, \quad \psi_k = \frac{\varphi + 2k\pi}{n}, \tag{5.9a,b}$$

where k is any integer; so that the phase (5.9b) proceeds in steps of $2\pi/n = \psi_{k+1} - \psi_k$, and repeats itself, that is, adds 2π, after n steps $\psi_{k+n} = \psi_k + 2\pi$. A set of n distinct values is obtained letting

$$k = 0, 1, \ldots, n-1: \quad \sqrt[n]{z} = r^{1/n} \exp\left\{ i\left(\frac{\varphi}{n} + \frac{2k\pi}{n}\right)\right\}, \tag{5.10}$$

so that *the n roots of order n of a complex number (5.6) are specified by (5.10).* Using (5.10) it follows that:

$$\sqrt[3]{-8i} = \left(2^3 \mathrm{e}^{i3\pi/2}\right)^{1/3} = 2\left\{\mathrm{e}^{i\pi/2},\ \mathrm{e}^{i7\pi/6},\ \mathrm{e}^{i11\pi/6}\right\} = 2i, -i \pm \sqrt{3}, \tag{5.11}$$

$$\sqrt[6]{-64} = \left(2^6 \mathrm{e}^{i\pi}\right)^{1/6} = 2\left\{\mathrm{e}^{\pm i\pi/6},\ \mathrm{e}^{\pm i\pi/2},\ \mathrm{e}^{\pm i5\pi/6}\right\} = \pm 2i, \pm i \pm \sqrt{3}, \tag{5.12}$$

where the six roots of (5.12) coincide with the three roots of (5.11) and their complex conjugates, because $\sqrt[6]{-64} = \sqrt[3]{\pm 8i}$ and $\sqrt[3]{8i} = \left(\sqrt[3]{-8i}\right)^*$. Example 10.9 also uses (5.10).

5.4 Regular Polygon Contained in a Circle

The n roots of order n of a complex number can be represented graphically by the construction, illustrated for $n=3$ in Figure 5.2, by noting that: (i) all roots lie on the circle

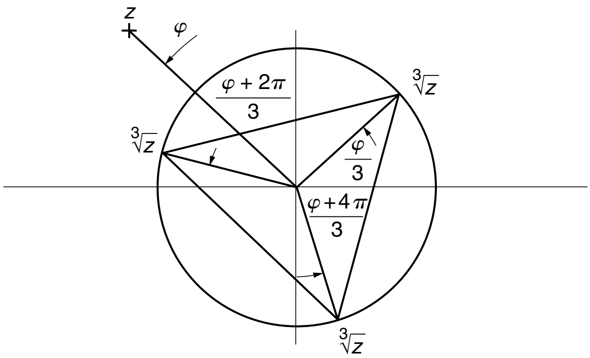

FIGURE 5.2
The roots of order n ($n = 3$ in Figure 5.2) lie on the circle of radius $|z|^{1/n}$ and center at the origin, at equal angular intervals $2\pi/n$, starting in the direction arg $(z)/n$.

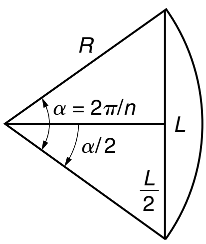

FIGURE 5.3
The n roots of a complex number lie on the vertices of a regular polygon with n sides and center at the origin, for example, an equilateral triangle for $n = 3$ in Figure 5.2. The polygon is made of n triangles, and the perimeter nL can be calculated relating the length of the side L to the radius R. The limit $n \to \infty$ is the perimeter of the circle.

with (5.9b) center at the origin and radius (5.8a) equal to the n-th root of the modulus; (ii) the first root has argument equal to $1/n$ of that of z; (iii) the remaining $n-1$ roots divide the circle into n equal angular sectors measuring $2\pi/n$ each. *Hence the roots of order n of z determine a* ***regular polygon*** *with n sides, contained within the circle of radius* $|z|^{1/n}$ *and center at the origin, with one vertex lying in the direction arg(z)/n.*

Since the polygon has n sides, the angle of two vertices is $\alpha = 2\pi/n$, and the length of one side $L = 2R\sin(\alpha/2)$, where (Figure 5.3) the radius is R. Hence, *the length of the perimeter of a regular polygon with n sides, with vertices lying on a circle of radius R, is given by:*

$$u \equiv \frac{\pi}{n}: \quad P_n = n\,2R\sin\left(\frac{\pi}{n}\right) = 2\pi R u^{-1}\sin u. \tag{5.13}$$

Since the perimeter of the polygon cannot exceed that of the circle, follows (5.14a):

$$u^{-1}\sin u \le 1, \quad \lim_{u\to 0} u^{-1}\sin u = 1; \tag{5.14a,b}$$

the equality holds only (5.14b) in the limit $u \to 0$, corresponding to a polygon with $n \to \infty$ sides that coincides with the circle. The result (5.14a) that u exceeds sin u except at the origin, where they coincide (5.14b), can be proved analytically (19.39a).

5.5 Multiple Sums of Sines or Cosines of Equal Angles

As another application consider the set of N-th order roots of $\exp(iN\alpha)$, viz.:

$$n = 0,\dots,N-1: \quad \{\exp(iN\alpha)\}^{1/N} = \exp\left(i\alpha + i\frac{2\pi n}{N}\right); \tag{5.15}$$

they are the roots of the polynomial of degree N:

$$z^N - \mathrm{e}^{iN\alpha} = \prod_{n=0}^{N-1}\left\{z - \exp\left(i\alpha + i\frac{2\pi n}{N}\right)\right\} \tag{5.16}$$

that can be expanded into the N factors indicated. The power z^{N-1} is missing from the l.h.s. of (5.16), and so the coefficient of z^{N-1} on the r.h.s. is zero:

$$N \geq 2: \quad 0 = -z^{N-1}\sum_{n=0}^{N-1}\exp\left\{i\left(\alpha + \frac{2\pi n}{N}\right)\right\}; \tag{5.17}$$

equating real and imaginary parts to zero leads to

$$N \geq 2: \quad \sum_{n=0}^{N-1}\cos\left(\alpha + \frac{2\pi n}{N}\right) = 0 = \sum_{n=0}^{N-1}\sin\left(\alpha + \frac{2\pi n}{N}\right), \tag{5.18a,b}$$

showing that*: if the sector 2π is divided in N equal parts, the sum of the cosines (sines) is zero; the result still holds if to all angles is added an arbitrary value α*. The geometrical interpretation is that if a regular polygon is contained in a circle of center at the origin, the sum of the cosines (sines) at the vertices is zero. This result also has a mechanical interpretation: (i) consider a **star of forces**, that is, n forces with the same magnitude, applied at the same point (e.g., the origin), and making equal angles $2\pi/n$ (e.g., for $n = 3$ with end points on the vertices of the equilateral triangle in Figure 5.2 as shown in Figure 5.4a); (ii) this system of forces is equivalent to zero, that is, the total or resultant force must be zero (this is most obvious when $n = 2p$ is even, because then there are p pairs of opposite forces that cancel, for example, $n = 4$ in Figure 5.4b); (iii) the x- and y-components of the total force are zero, leading to (5.18a,b). This is generalized next to (15.22a,b).

If $N \geq 3$ the same reasoning can be applied to the coefficient of z^{N-2} that is zero on the l.h.s. of (5.16), and on the r.h.s. is given by

$$N \geq 3: \quad 0 = z^{N-2}\sum_{n=0}^{N-1}\exp\left\{i\left(\alpha + \frac{2\pi n}{N}\right)\right\}\sum_{m=0}^{N-1}\exp\left\{i\left(\alpha + \frac{2\pi m}{N}\right)\right\}; \tag{5.19}$$

equating real and imaginary parts leads to

$$N \geq 3: \quad 0 = \sum_{n,m=0}^{N-1}\begin{matrix}\cos\\ \sin\end{matrix}\left\{2\alpha + 2\pi\frac{(n+m)}{N}\right\} = 0. \tag{5.20a,b}$$

This result (5.20a,b) is geometrically less obvious than (15.18a,b), and can be extended to $1 \leq p \leq N-1$ sets of numbers $n_1,\dots,n_p$, by equating in (15.16) the coefficients of z^{N-p}:

$$1 \leq p \leq N-1: \quad 0 = (-)z^{N-p}\prod_{q=1}^{p}\sum_{n_q=0}^{N-1}\exp\left\{i\left[\alpha + \left(\frac{2\pi}{N}\right)n_q\right]\right\}; \tag{5.21}$$

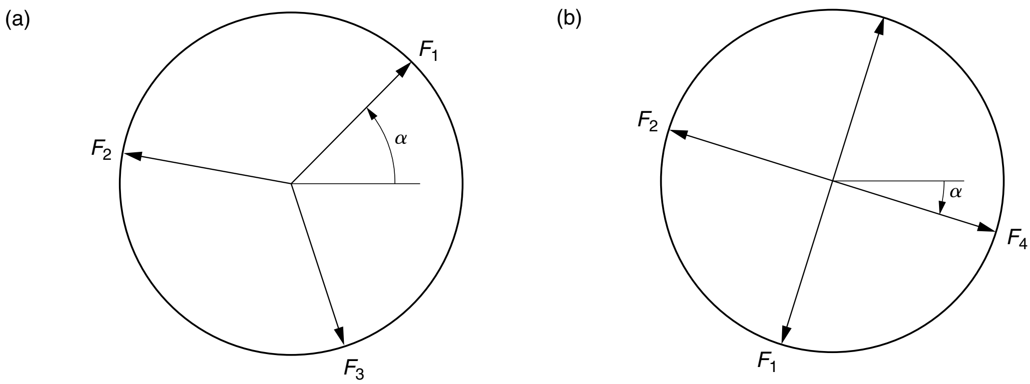

FIGURE 5.4
Dividing the circle in equal sectors, for example, via the roots of a complex number, leads to a system of forces equivalent to zero, for example, three in (a) ≡ (Figure 5.2) and four in (b). The system of forces is equivalent to zero because (i) all forces pass through the same point, the center, so there is no torque; (ii) the components of the forces in any direction cancel, regardless of rotation angle α.

The real and imaginary parts of (5.20) yield:

$$1 \le p \le N-1: \quad 0 = \sum_{n_1,\ldots,n_p=0}^{N-1} \begin{matrix}\cos\\ \sin\end{matrix} \left(p\alpha + \frac{2\pi}{N}\sum_{q=1}^{p} n_q \right), \tag{5.22a,b}$$

showing that: *if* $0 \le \alpha < 2\pi$ *is any angle, the identities* (5.*22a,b) hold for any natural number* $N \ge 1+p$, *and any* $1 \le p \le N-1$, *for example, the particular cases* $p=1$ $(p=2)$ *are (5.18a,b) [(5.20a,b)] that hold for* $N \ge 2$ $(N \ge 3)$. The only remaining results from (5.16) are two checks: (i) an obvious identity for the power z^N; (ii) for the power $z^0 \equiv 1$, viz.:

$$\begin{aligned} e^{-iN\alpha} &= (-)^N \prod_{n=0}^{N-1} \exp\left[i\left(\alpha + \frac{2\pi n}{N}\right)\right] = (-)^N e^{iN\alpha} \exp\left(\frac{2\pi}{N}\sum_{q=1}^{p} n_q\right) \\ &= (-)^N e^{iN\alpha} e^{i\pi(N-1)} = (-)^N e^{iN\alpha} (-)^{N-1} = -e^{iN\alpha}, \end{aligned} \tag{5.23}$$

another identity that proves nothing new.

5.6 Single-, Multi-, and Many-Valued Functions

The power z^n is a single-valued function, the square root $\sqrt{z}$ a two-valued function, the n-th root $\sqrt[n]{z}$ an n-valued or multivalued function, and many-valued functions, with infinitely many values, also exist, for example, the logarithm. Considering:

$$k = 0, \pm 2, \ldots: \quad z = re^{i\varphi} = re^{i\varphi + i2\pi k}, \tag{5.24}$$

where k is any integer; taking logarithms leads to

$$k \in |Z: \quad \log z = \log r + i\varphi + i2\pi k; \tag{5.25}$$

to each z there correspond infinitely many values of $\log z$, all having the same real part, and differing only in the imaginary part, by multiples of 2π. The reason is that the logarithm is the inverse of the exponential:

$$\exp(z) = \exp(i2\pi k)\exp(z) = \exp(z + i2\pi k), \tag{5.26}$$

and the latter is periodic with period 2π; when inversion is performed, the phase is undetermined to within a multiple of 2π, leading to a many-valued function, viz.:

$$\log(1 \pm i) = \log\left(\sqrt{2}e^{\pm i\pi/4}\right) = \frac{1}{2}\log 2 \pm i\frac{\pi}{4} + i2k\pi, \tag{5.27}$$

is an example.

5.7 Power with Complex Base and Exponent

The n-th power and root of order n are particular cases, respectively $a = n$ and $a = 1/n$ with n a positive integer, of the power with base z and exponent a both complex:

$$k \in |Z: \quad z^a \equiv \exp(a\log z) = \exp(a\log z + i2\pi ka), \tag{5.28}$$

and generally a many-valued function, because its definition involves the logarithm. The latter introduces the factor $\exp(i2\pi ka)$, with k an arbitrary integer: (i) for $a = p$ an integer, the factor is unity, and the function reduces to the integral power z^p that is single-valued; (ii) for $a = 1/q$ the inverse of an integer, the factor takes q distinct values, and the function reduces to the root $\sqrt[q]{z}$ of order q that is q-valued; (iii) for $a = p/q$ a rational number in its lowest terms, the factor takes q distinct values, and the function $\sqrt[q]{z^p}$ is q-valued, as should be expected from (i) and (ii); (iv) for all other values of a, viz. real irrational, imaginary, or complex, the factor takes an infinite number of distinct values, and function z^a is many-valued. An example is:

$$k \in |Z: \quad i^i \equiv \exp(i\log i) = \exp\left\{i\left(i\frac{\pi}{2} + i2\pi k\right)\right\} = \exp\left[-\left(2k + \frac{1}{2}\right)\pi\right], \tag{5.29}$$

where k is arbitrary integer.

5.8 Limiting Behavior at the Origin and Infinity

As another example, with u,v real and positive, consider:

$$\begin{aligned} u,v > 0: \quad (iu)^{-iv} &\equiv \exp\{iu\log(-iv)\} = \exp\left\{iu\log v + \frac{\pi u}{2}\right\} \\ &= \exp\left(\frac{\pi u}{2}\right)\ \{\cos(u\log v) + i\sin(u\log v)\}, \end{aligned} \tag{5.30}$$

where the principal value was taken, viz. $k = 0$ in (5.28). The function z^a, with complex $z(a)$ in the polar (5.31a) [Cartesian (5.31b)] form, is specified by (5.31c):

$$\begin{aligned} a = \alpha + i\beta: \quad z^a &= \exp\{(\alpha + i\beta)(\log r + i\varphi)\} = \exp(\alpha\log r - \beta\varphi)\exp\{i(\beta\log r + \alpha\varphi)\} \\ z = re^{i\varphi}: \quad &= r^\alpha e^{-\beta\varphi}[\cos(\beta\log r + \alpha\varphi) + i\sin(\beta\log r + \alpha\varphi)]; \end{aligned}$$

(5.31a–c)

TABLE 5.2
Limits of z^a as $z \to 0, \infty$ for All a

z^a	Re$(a) > 0$	Re$(a) < 0$
as $z \to 0$:	$z^a \to 0$	$z^a \to \infty$
as $z \to \infty$:	$z^a \to \infty$	$z^a \to 0$

Note: The limits of z^a at the origin $z \to 0$ and infinity $z \to \infty$ depend only on the sign of the real part of the exponent Re(a).

the power with complex base and exponent generally involves [e.g., (5.30) and (5.31a–c)] the real functions cosine and sine with logarithmic arguments. The second factor in (5.31c) has modulus unity and $\beta\varphi$ is bounded, so that the behavior at the origin $r = 0$ and infinity $r = \infty$ is determined solely by $\alpha \equiv \mathrm{Re}\,(a)$ viz.:

$$\lim_{z \to 0,\infty} |z^a| \sim \lim_{|z| \to 0,\infty} \exp\{\mathrm{Re}\,(a) \log |z|\}, \tag{5.32}$$

where $\sim$ means that there may be a finite factor. From (5.32) follow the results in Table 5.2, viz.: *the limit of the complex power z^a as the base z tends to the origin $z \to 0$ (to infinity $z \to \infty$) is determined solely by the real part of the exponent* a, *and is* $0\ (\infty)$ *for* Re(a) > 0, *and conversely* $\infty\,(0)$ *for* Re$(a) < 0$. The limits in Table 5.2 are isotropic, that is, independent of direction arg(z) for the function z^a.

5.9 Vanishing and Divergence on Alternate Sectors

For other functions the limit may be anisotropic, for example, depend on direction, such as:

$$\exp\left(z^2\right) = \exp\left[(x + iy)^2\right] = \exp\left(x^2 - y^2\right) \exp\left(i2xy\right), \tag{5.33}$$

that as $z \to \infty$, is oscillatory with unit modulus along $x^2 = y^2$ the diagonals of the quadrants $y = \pm x$, and in the four sectors separated by these (Figure 5.5) alternatively diverges and vanishes, according to Table 5.3. This is the particular case $n = 2$ of the exponential of an integral power (5.34a):

$$|\exp\left(z^n\right)| = \exp\{r^n \cos\left(n\varphi\right)\}, \quad \lim_{z \to 0} \exp\left(z^n\right) = 1, \tag{5.34a,b}$$

that (i) tends to unity (5.34b) as $z \to 0$ tends to the origin in any direction; (ii) as $z \to \infty$ tends to infinity it vanishes or diverges on alternate sectors of width π/n, viz. from (5.2a):

$k = 0, \ldots, 2n - 1$:

$$\lim_{z \to \infty} \exp\left(z^n\right) = \begin{cases} \infty & \text{for } (k - 1/2)\,\pi/n < \varphi < (k + 1/2)\,\pi/n, & (5.35\text{a}) \\ \exp\{i\,|z|^n\,(-)^n\} & \text{for } \varphi = (k + 1/2)\,\pi/n, & (5.35\text{b}) \\ 0 & \text{for } (k + 1/2)\,\pi/n < \varphi < (k + 3/2)\,\pi/n; & (5.35\text{c}) \end{cases}$$

(iii) along the separation lines (5.5a) it oscillates in phase with modulus unity. Note that (i) the first column of Table 5.1 coincides with the conditions (5.35a–c), viz. both specify the sign of $\cos(n\varphi)$; (ii) Table 5.3 is the particular case $n = 2$ of (5.35a–c).

Note 5.1. *Balancing of Forces*: The results (5.18a,b) hold if the star of forces is rotated by an angle α, which may vary with time $\alpha\,(t)$, leading to an angular velocity $\omega \equiv \mathrm{d}\alpha/\mathrm{d}t$. Thus

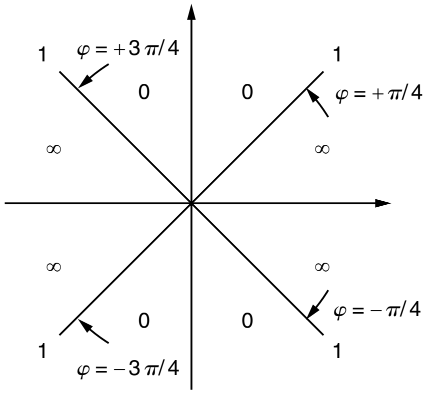

FIGURE 5.5
The function $\exp(z^2)$ in Table 5.3 diverges (vanishes) at infinity in the angular sectors between the diagonals of quadrants which contain the real (imaginary) axis. The boundary between these sectors is the diagonals of quadrants, along which the function oscillates with unit modulus.

TABLE 5.3
Behavior of $\exp(z^2)$ at Infinity in All Directions

$\exp(z^2) \to$	∞	$\exp(\pm i 2x^2)$	0
(x,y)	$x^2 > y^2$	$y = \pm x$	$y^2 > x^2$
$r \to \infty$	$-\frac{\pi}{4} < \varphi < +\frac{\pi}{4}$	$\varphi = \pm\frac{\pi}{4}$	$\frac{\pi}{4} < \varphi < \frac{3\pi}{4}$
	or		or
	$+\frac{3\pi}{4} < \varphi < +\frac{5\pi}{4}$		$-\frac{3\pi}{4} < \varphi < -\frac{\pi}{4}$

Note: The signs of the real part of z^2 change (Table 5.1) across the diagonals of quadrants. This implies that $\exp(z^2)$ diverges (tends to zero) at infinity in alternate sectors between the diagonals of quadrants in Figure 5.5. It oscillates with unit modulus along the diagonals which separate the sectors of decay and divergence.

the result (5.18a,b), and its generalizations (5.20a,b) and (5.22a,b) apply to the balancing of multicylinder piston engines; these have a single crankshaft, to which are linked at equal angles the rod and crank mechanisms of each piston, through which the forces are exerted. This is one of the many practical cases of balancing of forces in classical mechanics.

Conclusion 5: The successive integral powers of a fixed complex number lie on a logarithmic spiral (Figure 5.1), whereas the n roots of order n of a complex number specify a regular polygon with the n vertices on a circle of center at the origin (Figure 5.2); one side is represented (Figure 5.3), to calculate its length, and hence the perimeter of the polygon. The diagonals of the quadrants divide the complex z-plane into four sectors (Figure 5.5) containing the directions for which the exp (z^2) function diverges (∞) or vanishes (0) as $z \to \infty$ tends to infinity; it oscillates with unit modulus on the diagonals of quadrants, which are the boundaries of the four sectors. The divisions (Figure 5.4) in three (four) sectors correspond to a star of three (a) [four (b)] forces.

6

Electron in an Electromagnetic Field

A charged particle (Section 6.1) placed in an electric field, experiences an acceleration (Section 6.2) that increases as the electric charge increases and the mass decreases. In the presence of a magnetic field (Sections 6.3 and 6.4), the charged particle rotates around it, with an angular velocity that increases with increasing charge and decreasing mass. The superposition of these two motions (Sections 6.6 and 6.7) specifies the trajectory of the charged particle in an external electromagnetic field, uniform or not. Electromagnetic fields may be used to "focus" a beam of charged particles (Section 6.8), distinguish between positive and negative charges, and separate particles or atoms with different charges and/or masses (Section 6.9). The motion of charged particles in electromagnetic fields can be observed in laboratory experiments and occurs in astrophysics, for example, in the ionosphere of the earth and interplanetary space; some electrical devices, like cathode ray tubes and television screens, are based on deflection of the trajectories of charged particles.

6.1 Electromagnetic or Laplace–Lorentz Force

A particle, for example, an electron of **charge** e placed in an electric **field** $\vec{\mathrm{E}}$ is subject (Section 24.3) to an **electric force** (6.1a); if the charged particles moves with velocity $\vec{\mathrm{v}}$ it carries an **electric current** (6.1b):

$$\vec{\mathrm{F}}_e = e\vec{\mathrm{E}}, \quad \vec{\mathrm{J}} = e\vec{\mathrm{v}}, \quad \vec{\mathrm{F}}_m = \frac{\vec{\mathrm{J}} \wedge \vec{\mathrm{B}}}{c}, \tag{6.1a–c}$$

and in the presence of a magnetic field of induction $\vec{\mathrm{B}}$, it is acted upon (Section 26.3) by a magnetic force $\vec{\mathrm{F}}$ given by (6.1c), where c is an universal constant, namely, the speed of light in vacuo. The sum of the electric and magnetic forces is (Section 28.3) the Laplace–Lorentz or total **electromagnetic force** that balances the inertia force, equal to mass times acceleration:

$$\vec{\mathrm{F}}_e + \vec{\mathrm{F}}_m = \vec{\mathrm{F}}_{em} = e\vec{\mathrm{E}} + \frac{e}{c}\vec{\mathrm{v}} \wedge \vec{\mathrm{B}} = m\frac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t}; \quad \frac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{d}t} + \vec{\Omega} \wedge \vec{\mathrm{v}} = \vec{\mathrm{a}} \equiv \frac{\vec{\mathrm{F}}}{m}, \tag{6.2a,b}$$

the equation (6.2a) is similar to the Euler equation (6.2b), describing the combined translation and rotation of a spherical rigid body with an angular velocity $\vec{\Omega}$, under a mechanical force $\vec{\mathrm{F}}$ corresponding per unit mass m to an acceleration $\vec{\mathrm{a}}$. Comparison of (6.2a) with (6.2b) shows that

$$\vec{\mathrm{a}} = \frac{e}{m}\vec{\mathrm{E}}, \quad \vec{\Omega} = \frac{e\vec{\mathrm{B}}}{mc}, \tag{6.3a,b}$$

the electric field $\vec{\mathrm{E}}$(*magnetic induction* $\vec{\mathrm{B}}$) *causes a translation (rotation) with acceleration (6.3a) [angular velocity (6.3b)] proportional to the ratio of charge to mass.*

6.2 Uniform Fields and Larmor (1897) Frequency

The integration of the equation of motion (6.2a) specifies the velocity and the trajectory, as shown next in the case of uniform electric and magnetic fields. A Cartesian reference frame is chosen with OZ-axis aligned with the magnetic induction, and OY-axis such that the electric field lies on the YOZ-plane:

$$\vec{\mathrm{B}} = (0, 0, B), \quad \vec{\mathrm{E}} = (0, E_\perp, E_{//}); \tag{6.4a,b}$$

thus $E_\perp\,(E_{//})$ is the component of the electric field orthogonal (parallel) to the magnetic induction. The vector system (6.2a) of three differential equations takes the form:

$$\dot{v}_x = \Omega v_y, \quad \dot{v}_y = -\Omega v_x + a_\perp, \quad \dot{v}_z = a_{//}, \tag{6.5a–c}$$

where (i) dot means derivative with regard to time; (ii) $a_\perp\,(a_{//})$ is the acceleration (6.3a) due to the component of the electric parallel (6.6a) [orthogonal (6.6b)] to the induction:

$$a_{//} = \frac{eE_{//}}{m}, \quad a_\perp = \frac{eE_\perp}{m}, \quad \Omega \equiv \left|\vec{\Omega}\right| = \frac{eB}{mc}; \tag{6.6a–c}$$

(ii) Ω is associated with the magnetic induction (6.3b) $\equiv$ (6.6c), has the dimensions of inverse time, and is designated **Larmor frequency (1897)**.

6.3 Longitudinal Translation and Transverse Rotation

The system of three differential equations (6.5a–c) separates into two subsets, viz. (6.5c) that specifies alone the longitudinal motion, and (6.5a,b) that are coupled and determine the transversal motion. The motion along the magnetic induction (6.5c) is independent of the transversal motion, and is determined (6.6a) by:

$$v_z(t) = \frac{eE_{//}}{m} t + v_{//}, \quad z(t) = \frac{eE_{//}}{2m} t^2 + v_{//} t, \tag{6.7a,b}$$

where $v_{//}$ is the initial longitudinal velocity and z obtained by integrating $v_z = \dot{z}$ and is given by (6.7b), choosing the XOZ-plane so as to pass through the particle at initial time $t = 0$. From (6.7a,b) it follows that the longitudinal motion is uniformly accelerated (uniform) in the presence $E_{//} \neq 0$ (absence $E_{//} = 0$) of a longitudinal electric field.

To describe the transverse motion that is coupled between the directions OX and OY, a complex velocity is introduced (6.8a):

$$v \equiv v_x + iv_y: \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = \dot{v}_x + i\dot{v}_y = -i\Omega\,(v_x + iv_y) + ia_\perp = -i\Omega v + ia_\perp, \tag{6.8a,b}$$

that replaces the pair of coupled real equations (6.5a,b) by a single complex differential equation (6.8b); the integration of the latter is elementary:

$$-i\Omega t = -i\Omega \int_0^t \mathrm{d}t = \int_{v_0}^{v} \left(v - \frac{a_\perp}{\Omega}\right)^{-1} \mathrm{d}v = \log\left(\frac{v - a_\perp/\Omega}{v_0 - a_\perp/\Omega}\right), \tag{6.9a}$$

where v_0 denotes the initial complex velocity at time $t = 0$.

6.4 Components of the Velocity and Trajectory of the Particle

The initial complex velocity $v_0 = v_{0x} + iv_{0y}$ may be taken as real $v_0 = v_{0x} \equiv v_\perp$ provided that initial time $t = 0$ is chosen when the particle velocity lies in the XOZ-plane:

$$v(t) = \left(v_\perp - \frac{a_\perp}{\Omega}\right)\exp(-i\Omega t) + \frac{a_\perp}{\Omega}; \tag{6.9b}$$

separating real and imaginary parts of (6.9b) ≡ (6.8a) specifies (6.3b,6.6b) the two transverse components of the velocity:

$$v_x(t) = \left(v_\perp - \frac{cE_\perp}{B}\right)\cos(\Omega t) + \frac{cE_\perp}{B}, \quad v_y(t) = -\left(v_\perp - \frac{cE_\perp}{B}\right)\sin(\Omega t); \tag{6.10a,b}$$

integrating with regard to time, leads to the equations of the trajectory:

$$x(t) = \int_0^t v_x(t)\,\mathrm{d}t = \left(\frac{v_\perp}{\Omega} - \frac{cE_\perp}{B\Omega}\right)\sin(\Omega t) + \frac{cE_\perp}{B}t, \tag{6.11a}$$

$$y(t) = \int_0^t v_y(t)\,\mathrm{d}t = \left(\frac{v_\perp}{\Omega} - \frac{cE_\perp}{B\Omega}\right)\cos(\Omega t), \tag{6.11b}$$

in which the origin of the reference frame was placed at the position of the particle at time $t=0$. Thus *a particle of charge e and mass m placed in an uniform electromagnetic field, undergoes a motion specified by the velocity (6.7a; 6.10a,b) and trajectory (6.7b; 6.11a,b).* An alternative derivation of these results using only real instead of complex variables is presented in Example 10.10. The motion will be analyzed first in the cases of (Section 6.5) zero electric field $E_\perp = 0 = E_{//}$, and (Section 6.6) zero initial velocity $v_\perp = 0 = v_{//}$, before proceeding to the general case (Section 6.7) when both are nonzero; all cases are included in the Table 6.1.

6.5 Linear, Circular, and Helical Motion

In the absence of an electric field (6.12a,b), the velocity (6.7a; 6.10a,b) ≡ (6.12c–e):

$$E_{//} = 0 = E_\perp: \quad v_x(t) = v_\perp\cos(\Omega t), \quad v_y(t) = -v_\perp\sin(\Omega t), \quad v_z(t) = v_{//}, \tag{6.12a–e}$$

and coordinates (6.7b; 6.11a,b):

$$x(t) = \frac{v_\perp}{\Omega}\sin(\Omega t), \quad y(t) = \frac{v_\perp}{\Omega}\cos(\Omega t), \quad z(t) = v_{//}t \tag{6.13a–c}$$

of the electron show that, in the presence of uniform magnetic induction, the following trajectories are possible: (I) if it is initially at rest $v_{//} = 0 = v_\perp$ it remains at rest, that is, the magnetic field alone cannot start a motion since the magnetic force $(e/c)\vec{\mathrm{v}} \wedge \vec{\mathrm{B}} = 0$ is zero for zero velocity $\vec{\mathrm{v}} = 0$; (II) if the initial velocity is longitudinal $v_{//} \neq 0 = v_\perp$ the motion is uniform along the magnetic induction, and is unaffected by it because the magnetic force $(e/c)\vec{\mathrm{v}} \wedge \bar{\mathrm{B}} = 0$ is zero for $\vec{\mathrm{v}}$ parallel to $\vec{\mathrm{B}}$; (III) if the initial velocity is transverse (6.14a,b)

TABLE 6.1
Trajectories of an Electron in a Uniform Electromagnetic Field

Section	Case	Formulas (6.)	Initial velocity		Electric field		Trajectory: motion	Figure
			Longitudinal*	Transversal*	Longitudinal*	Transversal*		
6.5	I	—	$v_{//} = 0$	$v_{\perp} = 0$	$E_{//} = 0$	$E_{\perp} = 0$	At rest	—
	II	(12e, 13c)	$v_{//} \neq 0$	$v_{\perp} = 0$	$E_{//} = 0$	$E_{\perp} = 0$	Linear: uniform	—
	III	(12c,d; 13a,b; 14c)	$v_{//} = 0$	$v_{\perp} \neq 0$	$E_{//} = 0$	$E_{\perp} = 0$	Circular	6.1
	IV	(12c–e; 13a–c; 14c; 15c,d)	$v_{//} \neq 0$	$v_{\perp} \neq 0$	$E_{//} = 0$	$E_{\perp} = 0$	Cylindrical helix	6.2
6.6	V	(16c, 17c)	$v_{//} = 0$	$v_{\perp} = 0$	$E_{//} \neq 0$	$E_{\perp} = 0$	Linear uniformly accelerated	—
	VI	(16a,b; 17a,b; 18c,d)	$v_{//} = 0$	$v_{\perp} = 0$	$E_{//} = 0$	$E_{\perp} \neq 0$	Cycloid	6.3
	VII	(16a–c; 17a–c; 18c,d)	$v_{//} = 0$	$v_{\perp} = 0$	$E_{//} \neq 0$	$E_{\perp} \neq 0$	Elongated cycloid	—
6.7	VIII	(12c,d; 13a,b; 14c; 16c; 17c; 19a,b)	$v_{//} \neq 0$	$v_{\perp} \neq 0$	$E_{//} \neq 0$	$E_{\perp} = 0$	Elongated cylindrical helix	6.4
	IX	(10a,b; 11a,b; 18c,d; 20a,b)	$v_{//} \neq 0$	$v_{\perp} \neq 0$	$E_{//} = 0$	$E_{\perp} \neq 0$	Trochoid: curled elongated	6.6/6.5
	X	(16c; 17c; 10a,b; 11a,b)	$v_{//} \neq 0$	$v_{\perp} \neq 0$	$E_{//} \neq 0$	$E_{\perp} \neq 0$	Distorted trochoid	—

*Relative to the direction of magnetic induction.

Note: These ten cases depend on the nonzero components of the electric field $\vec{\mathrm{E}}$ and initial velocity $\vec{\mathrm{v}}$ in the direction parallel $E_{//}, v_{//}$ and perpendicular $(E_{\perp}, V_{\perp})$ to the magnetic field as shown in Figures 6.1–6.6.

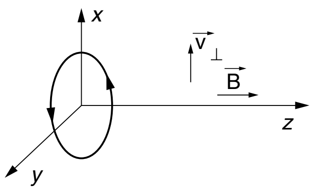

FIGURE 6.1
An electron with initial velocity transverse to an external magnetic field moves in a circle around the latter with constant velocity.

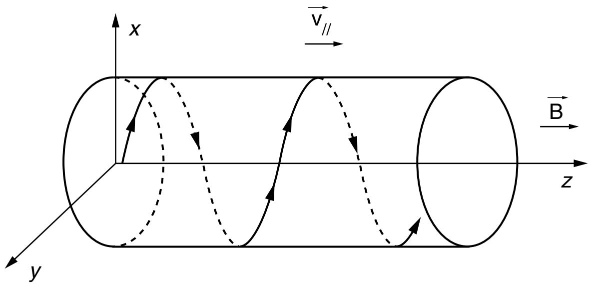

FIGURE 6.2
An electron with initial velocity oblique to an external magnetic field moves along a cylindrical helix with constant azimuthal (axial) velocity, equal to the component of the initial velocity orthogonal (parallel) to the magnetic field.

the motion takes place in the plane passing through the initial velocity and orthogonal to the magnetic induction, and the trajectory (6.12a,b; 6.13a,b) is circular (Figure 6.1):

$$v_{//} = 0 \neq v_\perp : \quad \left| x(t)^2 + y(t)^2 \right|^{1/2} = R \equiv \frac{v_\perp}{\Omega} = \frac{mev_\perp}{cB}; \tag{6.14a–c}$$

it is described with constant tangential velocity equal to the initial velocity and constant angular velocity Ω equal to the Larmor frequency (6.6c) so that the radius (6.14c) increases for larger electric charge and mass, and smaller magnetic induction; (IV) if the initial velocity has both longitudinal and transversal components nonzero (6.15a,b), the superposition of cases (II) and (III) shows that the trajectory is a **cylindrical helix** (Figure 6.2), of radius (6.14c), taken at the angular velocity (6.6c), with linear tangential velocity (6.15c):

$$v_{//} \neq 0 \neq v_\perp : \quad v_0 = \sqrt{v_\perp^2 + v_{//}^2}, \quad \alpha = \operatorname{arc}\tan\left(\frac{v_{//}}{v_\perp}\right), \tag{6.15a–d}$$

making a constant angle (6.15d) with the direction of the magnetic induction.

6.6 Linear Acceleration and Cycloid in the Plane

In the case where the particle is initially at rest $v_{//} = 0 = v_\perp$, the motion can only be started by the electric field, although it is then modified by the magnetic induction. The components of the velocity (6.7a; 6.l0a,b):

$$v_\perp = 0 : v_x(t) = \frac{cE_\perp}{B}\left[1 - \cos(\Omega t)\right], \quad v_y(t) = \frac{cE_\perp}{B}\sin(\Omega t), \quad v_z(t) = \frac{eE_{//}}{m}t, \tag{6.16a–c}$$

and the coordinates of the particle (6.7b; 6.11a,b):

$$x(t) = \frac{E_\perp c}{B\Omega}\left[\Omega t - \sin(\Omega t)\right], \quad y(t) = -\frac{E_\perp c}{B\Omega}\cos(\Omega t), \quad z(t) = \frac{E_{//}e}{2m}t^2, \tag{6.17a–c}$$

show that three trajectories (V–VII) are possible. If (V) the electric field and magnetic induction are parallel $E_{//} \neq 0 = E_\perp$, the motion is uniformly accelerated along their common direction (6.16c; 6.17c) because only the electric force and not the magnetic force acts $e\vec{\mathrm{E}} \neq 0 = (e/c)\,\vec{\mathrm{v}} \wedge \vec{\mathrm{B}}$. If (VI) the electric field and magnetic induction are orthogonal (6.18a,b), the trajectory lies in the plane (x, y) orthogonal to the magnetic field, on which lies the electric field (6.16a,b; 6.17a,b); the trajectory corresponds to an uniform circular motion with angular velocity equal to the Larmor frequency Ω in (6.6c) and linear velocity (6.18c) and radius (6.18d):

$$E_{//} = 0 \neq E_\perp : \quad U = \frac{E_\perp c}{B}, \quad S = \frac{U}{\Omega} = \frac{E_\perp c^2 m}{B^2 e}, \tag{6.18a–d}$$

upon which is superimposed an uniform motion with velocity U along the OX axis; thus the trajectory is a **cycloid** (Figure 6.3) that is also the curve described by a point on a circle rolling without sliding on a straight line. If (VII) the electric field and magnetic induction are oblique $E_{//} \neq 0 \neq E_\perp$, the superposition of cases (V) and (VI) leads to a trajectory that projects as the cycloid on the XOY-plane, and is elongated by an uniform acceleration along the OZ axis.

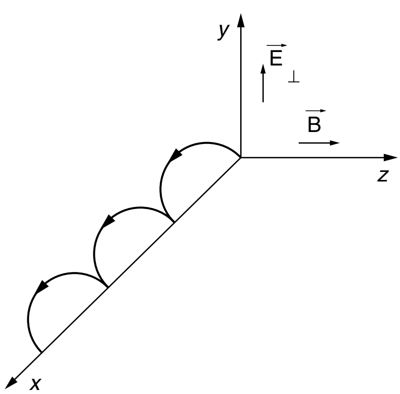

FIGURE 6.3
If the electron starts at rest it will only move in the presence of an electric field. If the latter is orthogonal to the magnetic field the trajectory is a cycloid. It corresponds to a fixed point on a circle rolling without sliding on a plane. It has a cusp where the trajectory touches the axis and inverts the direction changing the tangent between $\pm\infty$.

6.7 Elongated Helix and Plane Trochoid

In the general case of a charged particle with initial velocity in the presence of uniform electric and induction fields, three more (VIII–X) trajectories are possible, in addition to the seven particular cases considered earlier. If (VIII) the electric field and magnetic induction are parallel $E_\perp = 0 \neq E_{//}$, the superposition of the cases (III) and (V) shows that the motion is circular and uniform (6.12c,d; 6.13a,b) in the transversal plane and uniformly accelerated (6.16c; 6.17c) in the direction parallel to the induction; thus the trajectory is a **cylindrical helix progressively elongated** (Figure 6.4), with constant radius (6.14c), angular velocity (6.6c), tangential velocity (6.19a), and inclination (6.19b) increasing with time:

$$v(t) = \left| v_\perp^2 + \left(\frac{eE_{//}}{m} t + v_{//} \right)^2 \right|^{1/2}, \quad \alpha(t) = \arctan\left(\frac{eE_{//}}{mv_\perp} t + \frac{v_{//}}{v_\perp} \right). \tag{6.19a,b}$$

If (IX) the electric field and magnetic induction are orthogonal $E_{//} = 0 \neq E_\perp$, the motion along OZ is uniform, and the trajectory in the XOY-plane is the superposition (6.10a,b; 6.11a,b) of a circular motion with tangential velocity (6.20a) and radius (6.20b):

$$V = \frac{E_\perp c}{B} - v_\perp, \quad R = \frac{|V|}{\Omega} = \left| \frac{E_\perp c}{B\Omega} - \frac{v_\perp}{\Omega} \right| = \left| \frac{E_\perp mc^2}{eB^2} - \frac{v_\perp mc}{eB} \right|, \tag{6.20a,b}$$

and a translation along OX at velocity U in (6.18b); thus the trajectory would be a cycloid (Figure 6.3) if $U = V$ or $v_\perp = 0$ in case VI, and in the present case IX is an **elongated (curled) trochoid** illustrated in Figure 6.5 (6.6); it corresponds to $v_\perp > 0$ $(v_\perp < 0)$ translation velocity larger $U > V$ (smaller $U < V$) than the tangential velocity, as for a circle rolling on a plane and sliding in the direction of (direction opposite to) the translation. In the most general case (X) of oblique electric field and magnetic induction, the trajectory projects on the XOY-plane as a trochoid, and is elongated in the OZ direction; thus the particle rotates around the magnetic induction $\vec{B}$ and is simultaneously accelerated in the oblique direction of the electric field $\vec{E}$, changing its initial velocity that is generally oblique to both $\vec{B}$ and $\vec{E}$.

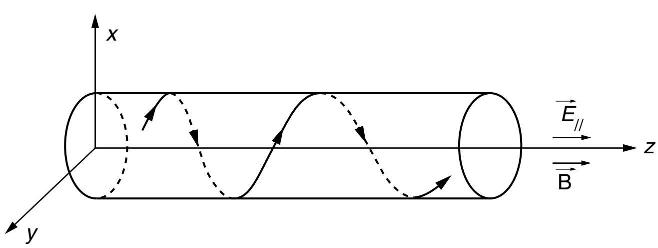

FIGURE 6.4
If the electric and magnetic fields are parallel, and the electron starts with arbitrary oblique initial velocity, the cylindrical helix in Figure 6.2 is deformed by the acceleration along the electric field, that is, increasingly elongated in the axial direction.

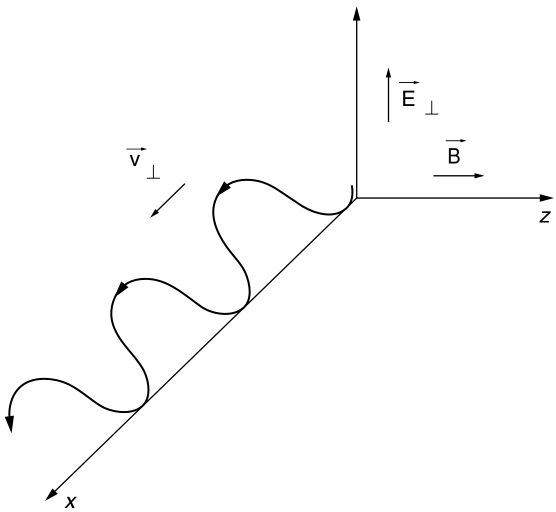

FIGURE 6.5
If the electric and magnetic fields are orthogonal, apart from a uniform motion along the magnetic field, the trajectory is a deformation of the cycloid in Figure 6.3. If the transverse component of the initial velocity is along the cycloid, the latter becomes an elongated trochoid without cusps, because a positive horizontal velocity has been added.

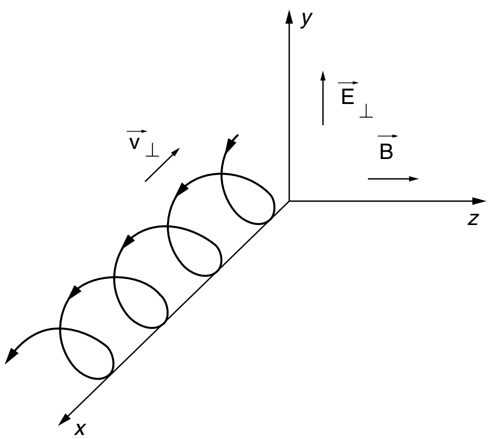

FIGURE 6.6
As in Figure 6.5: if the transverse component of the initial velocity is negative it inverts the motion at the cusps of the cycloid in Figure 6.3, leading to a curled trochoid with double points, where the electron passes again through the same point.

6.8 Oval and Conical Helices and Magnetic Focusing

In the absence of an electric field, the trajectory of the electron can be deformed if the magnetic induction is nonuniform: (XI) if the initial velocity of particle is transverse to $\vec{B}$, the trajectory is circular (Figure 6.1), provided that $\vec{B}$ be constant or varies only longitudinally;