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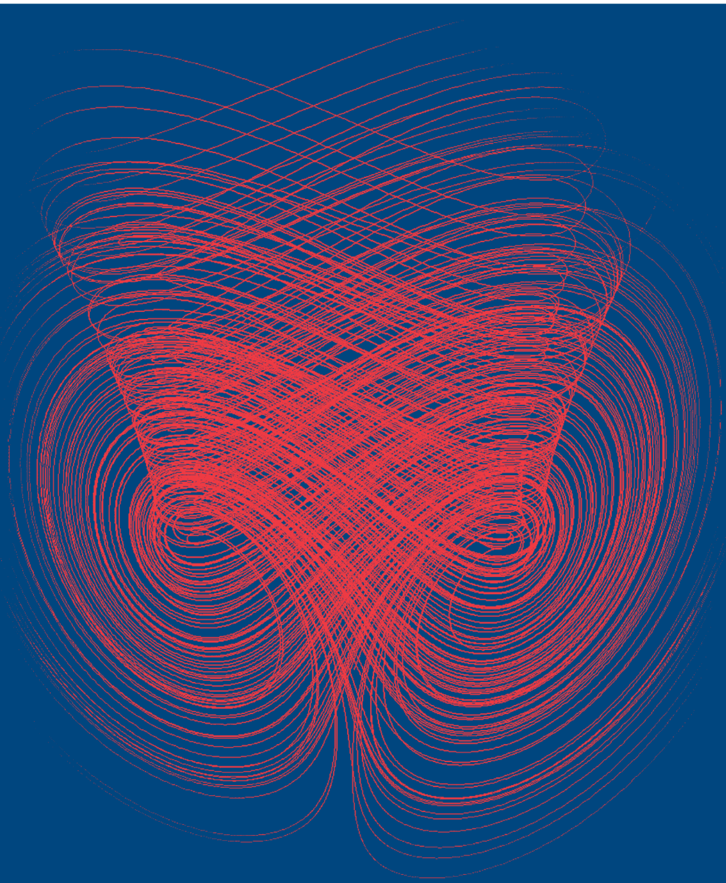
INTRODUCTION TO FUZZY SYSTEMS



Guanrong Chen
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INTRODUCTION TO FUZZY SYSTEMS

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Preface

The word "fuzzy" is no longer fuzzy to many engineers today. Introduced in the earlier 1970s, fuzzy systems and fuzzy control methodologies as an emerging technology targeting industrial applications has added a promising new dimension to the existing domain of conventional control systems engineering. It is now a common belief that when a complex physical system does not provide a set of differential or difference equations as a precise or reasonably accurate mathematical model, particularly when the system description requires certain human experience in linguistic terms, fuzzy systems and fuzzy control theories have some salient features and distinguishing merits over many other approaches.

Fuzzy control methods and algorithms, including many specialized software and hardware available on the markets, may be classified as one type of intelligent control. This is because fuzzy systems modeling, analysis, and control incorporate a certain amount of human knowledge into its components such as fuzzy sets, fuzzy logic, and fuzzy rule bases. Using human expertise in system modeling and controller design is not only advantageous but often necessary. Classical controllers design has already incorporated human knowledge and skills; for instance, what type of controller to use and how to determine the controller structure and parameters largely depend on the decision and preference of the designer, especially when multiple choices are possible. The fuzzy control technology provides just one more choice for this consideration; it has the intention to be an alternative, by no means a simple replacement, of the existing control techniques such as classical control and other intelligent control methods (e.g., neural networks, expert systems, etc.). Together, they supply systems and control engineers with a more complete toolbox to deal with the complex, dynamic, and uncertain real world. Fuzzy control technology is one of the many tools in this toolbox that is developed not only for elegant mathematical theories, but more importantly, for many practical problems with various technical challenges.

Compared with various conventional approaches, fuzzy control utilizes more information from domain experts and yet relies less on mathematical modeling about a physical system.

On the one hand, fuzzy control theory can be quite heuristic and somewhat ad hoc. This sometimes is preferable or even desirable, particularly when low-cost and easy operations are required where mathematical rigor is not the main concern. There are many examples of this kind in industrial applications, for which fuzzy sets and fuzzy logic are easy to use. Within this context, determining a fuzzy set or a fuzzy rule base seems to be somewhat subjective, where human knowledge about the underlying physical system comes into play. However, this may not require more human knowledge than selecting a suitable mathematical model in a deterministic control approach, where the following questions are often being asked beforehand: "Should one use a linear or a nonlinear model?" "If linear, what's the order or dimension; if nonlinear, what type of nonlinearity?" "What kind of optimality criterion should be used to measure the performance?" "What kind of norm should be used to measure the robustness?" It is also not much more subjective than choosing a suitable distribution function in the stochastic control approach, where the following questions are frequently being asked: "Gaussian or non-Gaussian noise?" "White noise or just unknown but bounded uncertainty?" Although some of these questions can be answered on the basis of statistical analysis of available empirical data in classical control systems, the same is true for establishing an initial fuzzy rule base in fuzzy control systems.

On the other hand, fuzzy control theory can be rigorous and fuzzy controllers can have precise formulations with analytic structures and guaranteed closed-loop system stability and performance specifications, if such characteristics are intended. In this direction, the ultimate objective of the current fuzzy systems and fuzzy control research is appealing: the fuzzy control system technology is moving toward a solid foundation as part of the modern control theory. The trend of a rigorous approach to fuzzy control, starting from the mid-1980s, has produced many exciting and promising results. For instance, some analytic structures of fuzzy controllers, particularly fuzzy PID controllers, and their relationship with corresponding conventional controllers are much better understood today. Numerous analysis and design methods have been well developed. As a consequence, the existing analytical control theory has made the fuzzy control systems practice safer, more efficient, and more cost-effective.

This textbook represents a continuing effort in the pursuit of analytic theory and rigorous design for fuzzy control systems. More specifically, the basic notion of fuzzy mathematics (Zadeh fuzzy set theory, fuzzy membership functions, interval and fuzzy number arithmetic operations) is first studied. Consequently, in a comparison with the classical two-valued logic, the fundamental concept of fuzzy logic is introduced. Some real-world applications of fuzzy logic will then be discussed, just after two chapters of studies, revealing some practical flavors of fuzzy logic. This is then followed by the basic fuzzy systems theory (Mamdani and Takagi-Sugeno modeling, along with parameter estimation and system identification) and fuzzy control theory. Here, fuzzy control theory is introduced, mainly based on the well-known classical Proportional-Integral-Derivative (PID) controllers theory and design methods. In particular, fuzzy PID controllers are studied in greater detail. These controllers have precise analytic structures, with rigorous analysis and guaranteed closed-loop system stability; they are comparable and also compatible with the classical PID controllers. To that end, a new notion of verb-based fuzzy-logic control theory is briefly described.

The primary purpose of this course is to provide some rather systematic training for systems and control majors, both senior undergraduate and first-year graduate students, and to familiarize them with some fundamental mathematical theory and design methodology in fuzzy control systems. The authors have tried to make this book self-contained, so that no preliminary knowledge of fuzzy mathematics and fuzzy control systems theory is needed to understand the materials presented in this textbook. Although it is assumed that the students are aware of some very basic classical set theory and classical control systems theory, the fundamentals of these subjects are briefly reviewed throughout the text for their convenience.

Some common terminology in the field of fuzzy control systems has become quite standard today. Therefore, as a textbook written in a classical style, the authors have taken the liberty to omit some personal and specialized names such as “TS fuzzy model” and “t-norm,” to name just a couple. One reason is that too many names have to be given to too many items in doing so, which will distract the readers’ attention in their reading. Nevertheless, closely related references are given at the end of the book for crediting and for one’s further searching. Also, an * in the Table of Contents indicates those relatively advanced materials that are beyond the basic scope of the present text; they are used only for the reader's further studies of the subject and can be omitted in regular teaching.

This textbook is a significantly modified version of the authors’ book “Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems” (CRC Press, 2001), which has been used by the authors for a graduate course in the City University of Hong Kong since its publication, prior to which the authors’ Lecture Notes had been taught for several years in the University of Houston at Texas of USA. This new book differs from the original one in many aspects. First of all, two new chapters, Chapters 3 and 7, have been added, with emphasis on some real applications of fuzzy logic and some new development of the verb-based fuzzy control theory and methodology. To keep the book within a modest size and make it more readable to new comers, some advanced topics on adaptive fuzzy control and high-level applications of fuzzy logic, presented in Chapters 6 and 7 of the original book, have been removed. Secondly, almost all chapters in the original book have been simplified, keeping the most fundamental contents and aiming at a more elementary textbook that can be easily used for teaching of the subject. Last but not least, more practical examples and review problems have been added or revised, with problem solutions provided at the end of the book, which should benefit both class-room teaching and self-studying.

In the preparation of this new textbook, the authors received some suggestions and typo-corrections on the previous book from their students each year. In particular, Dr. Tao Yang provided the materials of Chapter 7 on verb-based fuzzy control theory, which has quite significantly enhanced the contents of the book.

It is the authors' hope that students will benefit from this textbook in obtaining some relatively comprehensive knowledge about fuzzy control systems theory which, together with their mathematical foundations, can in a way better prepare them for the rapidly developing applied control technologies emerging from the modern industries.

The Authors

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CHAPTER 1

Fuzzy Set Theory

The classical set theory is built on the fundamental concept of "set." An individual is either a member or not a member of a specified set in question. A sharp, crisp, and unambiguous distinction exists between a member and a nonmember for any well-defined "set" in this classical theory of mathematics, and there is a very precise and clear boundary indicating whether or not an individual belongs to the set. When one is asked the question "Is this individual a member of that set?" The answer is either "yes" or "no."

The classical concept of "set" holds for both the deterministic and the stochastic cases. In probability and statistics, one may ask, "What is the probability of this individual being a member of that set?" Although an answer to this question could be like "The probability for this individual to be a member of that set is 90%," the final outcome (i.e., conclusion) is still either "it is" or "it is not" a member of that set. Here, the chance for one to make a correct prediction as "it is a member of that set" is 90%, which does not mean that it has 90% membership in the set and, meanwhile, it possesses 10% non-membership of the same set. In other words, in the classical set theory, it is not permissible for an individual to be partially in a set and also partially not in the same set at the same time. Thus, many real-world application problems cannot be described and, as a result, cannot be solved by the classical set theory, including all those involving elements with only partial membership of a set. On the contrary, fuzzy set theory accepts partial memberships; therefore, in a sense it generalizes the classical set theory to some extent.

In order to introduce the concept of fuzzy sets, the elementary set theory in classical mathematics is first reviewed. It will be seen that the fuzzy set theory is a very natural extension of the classical set theory, and is also a rigorous mathematical notion.

I. CLASSICAL SET THEORY

Let $\boldsymbol{S}$ be a nonempty set, called the *universe set* below, consisting of all possible elements of interest within a particular context.

Each of these elements is called a *member*, or an *element*, of $\boldsymbol{S}$. A union of some (finite or infinitely many) members of $\boldsymbol{S}$ is called a *subset* of $\boldsymbol{S}$.

To indicate that a member s of $\boldsymbol{S}$ belongs to a subset S of $\boldsymbol{S}$, one usually writes

$$s \in S,$$

but if s is not a member of S,

$$s \notin S.$$

To indicate that S is a subset of $\boldsymbol{S}$, one writes

$$S \subset \mathbf{S}.$$

Usually, this notation implies that S is a strictly proper subset of $\mathbf{S}$ in the sense that there is at least one member $x \in \mathbf{S}$ but $x \notin S$. If it can be either $S \subset \mathbf{S}$ or $S = \mathbf{S}$, then one writes

$$S \subseteq \mathbf{S}.$$

An empty subset is denoted by $\varnothing$. A subset of certain members that all have properties $P_1, \ldots, P_n$ will be denoted by a capital letter, say A, as

$$A = \{\, a \mid a \text{ has properties } P_1, \ldots, P_n \,\}.$$

A subset $A \subseteq \mathrm{R}^n$ is said to be *convex* if

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in A \qquad \text{and} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in A$$

implies

$$\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in A \qquad \text{for any } \lambda \in [0,1].$$

Let A and B be two subsets. If every member of A is also a member of B, i.e., if $a \in A$ implies $a \in B$, then A is said to be a *subset* of B, and is denoted $A \subset B$.

If both $A \subset B$ and $B \subset A$ are true, then they are *equal*, for which it is indicated by $A = B$. If it can be either $A \subset B$ or $A = B$, then it is denoted $A \subseteq B$. Therefore, $A \subset B$ is equivalent to both $A \subseteq B$ and $A \neq B$.

The *difference* of two subsets A and B is defined by

$$A - B = \{\ c \mid c \in A \text{ and } c \notin B\ \}.$$

In particular, if $A = \boldsymbol{S}$ is the universe set, then $\boldsymbol{S} - B$ is called the *complement* of B, and is denoted by $\overline{B}$, namely,

$$\overline{B} = \boldsymbol{S} - B.$$

Clearly,

$$\overline{\overline{B}} = B, \quad \overline{\boldsymbol{S}} = \varnothing, \text{ and } \overline{\varnothing} = \boldsymbol{S}.$$

Let $r \in R$ be a real number and A be a subset of R. Then, the *multiplication* of r and A is defined to be the set

$$r\,A = \{\ r\,a \mid a \in A\ \}.$$

The *union* of two subsets A and B is defined as

$$A \cup B = B \cup A = \{\ c \mid c \in A \text{ or } c \in B\ \}.$$

Thus, one always has

$$A \cup \boldsymbol{S} = \boldsymbol{S}, \quad A \cup \varnothing = A, \quad \text{and} \quad A \cup \overline{A} = \boldsymbol{S}.$$

The *intersection* of two subsets A and B is defined by

$$A \cap B = B \cap A = \{\ c \mid c \in A \text{ and } c \in B\ \}.$$

Clearly,

$$A \cap \boldsymbol{S} = A, \quad A \cap \varnothing = \varnothing, \text{ and } \quad A \cap \overline{A} = \varnothing.$$

Moreover, two subsets A and B are said to be *disjoint* if

$$A \cap B = \varnothing.$$

Basic properties of the classical set theory are summarized in Table 1.1, where $A \subseteq$ S and $B \subseteq$ S.

For any set A, the *characteristic function* of A is defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in A. \end{cases} \tag{1.1}$$

For any two sets A and B and for any element $x \in \boldsymbol{S}$, one has

$$X_{A \cup B}(x) = \max\{\ X_A(x), X_B(x)\ \},$$

$$X_{A \cap B}(x) = \min\{\ X_A(x), X_B(x)\ \},$$

$$X_{\overline{A}}(x) = 1 - X_A(x).$$

Table 1.1 Properties of Classical Set Operations

Involutive law	$\overline{\overline{A}} = A$
Commutative law	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative law	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cup A = A$ $A \cap A = A$ $A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$ $A \cup (\overline{A} \cap B) = A \cup B$ $A \cap (\overline{A} \cup B) = A \cap B$ $A \cup \boldsymbol{S} = \boldsymbol{S}$ $A \cap \varnothing = \varnothing$ $A \cup \varnothing = A$ $A \cap \boldsymbol{S} = A$ $A \cap \overline{A} = \varnothing$ $A \cup \overline{A} = \mathrm{S}$
DeMorgan’s law	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

II. FUZZY SET THEORY

The following simple example serves as an introduction to the concept of fuzzy sets.

Example 1.1. Let S be the set of all human beings and consider its subset

$$S_f = \{ s \in S \mid s \text{ is old} \}.$$

Then, S_f is a "fuzzy set" because the property "old" is not well defined in the sense of classical mathematics and cannot be precisely measured: given a person who is 40-year old, it is not clear if this person belongs to the set S_f, and even if so, it is still unclear whether or not a person of age 39 also belongs to the same set.

In classical set theory, one may draw a line at the exact age of 40. As a result, a person who is exactly 40 years old belongs to the set and is considered to be "old," but another person of one-day-less than 40 years old will not be considered "old." This distinction is mathematically correct, but practically unreasonable.

To do mathematics, one nevertheless has to precisely define the subset S_f. For this purpose, one needs to quantify the concept "old," so as to characterize the subset S_f in a precise and rigorous way.

Instead of a sharp cut at the exact age of 40, let's say, one would like to describe the concept "old" by the curve shown in Figure 1.1 (a), using common sense, where the only ones who are considered to be "absolutely old" are those 120-year old or older, and the only people who are considered to be "absolutely young" are those newborns. Meanwhile, all the other people are old as well as young at the same time, with different degrees of oldness and youngness depending on their actual ages.

For example, a person 40 years old is considered to be "old" with "degree 0.5" and at the same time also "young" with "degree 0.5" according to the measuring curve that has been chosen for use. One cannot exclude this person from the subset S_f, nor include him completely.

Thus, the curve in Figure 1.1 (a) establishes a mathematical measure for the oldness of a human being. The curve shown in Figure 1.1 (a) is called a *membership function* associated with the subset S_f. It is a generalization of the classical characteristic function X_{S_f} defined by (1.1), which can only be used to conclude that a person either "is" or "is not" a member of the subset S_f.

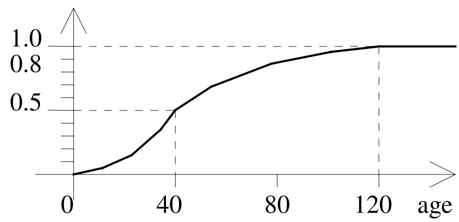

Figure 1.1 (a) A fuzzy membership function for "oldness"

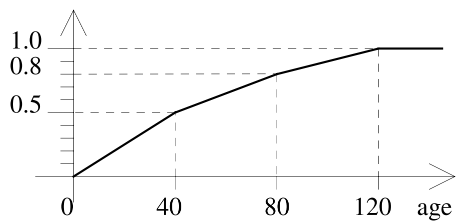

Figure 1.1 (b) Another fuzzy membership function for "oldness"

One may also use the piecewise linear membership function shown in Figure 1.1 (b) to describe the same concept of oldness for the same subset S_f, depending on whichever is more meaningful and more convenient for the application under consideration. Clearly, both membership functions are reasonable and acceptable in common sense.

In general, a fuzzy membership function can have various shapes, as shown in Figure 1.2, chosen by the user based on the nature of the application in consideration.

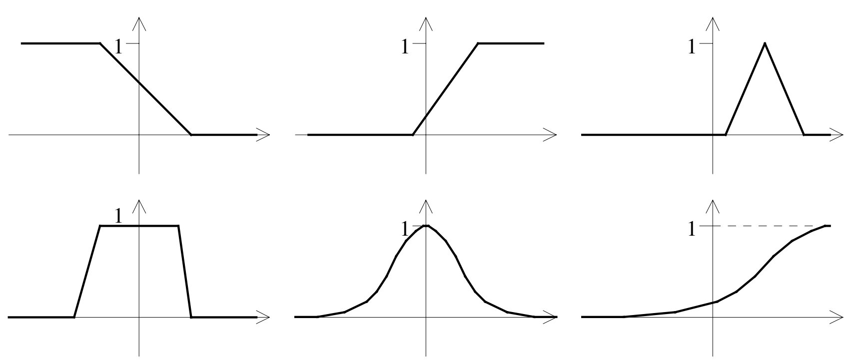

Figure 1.2 Various shapes of commonly used membership functions

Next, consider again the subset

$$S_f = \{\ s \in \mathrm{S} \mid s \text{ is old }\}.$$

Suppose that a membership function associated with it, say the one shown in Figure 1.1 (a), has been chosen for use. Then, this subset S_f , along with the chosen membership function, denoted by $\mu_{S_f}(s)$ with $s \in S_f$, is called a *fuzzy set.*

It is now clear that a fuzzy set consists of two components: a regular set and a membership function associated with it. This is different from the classical set theory, where all sets (and subsets) actually share the same (and the only) simple membership function: the two-valued characteristic function X_{S_f} defined by (1.1), which is not even being mentioned because in such simple cases it is not necessary to do so.

To familiarize this new concept, consider one more example in the following.

Example 1.2. Let S be the set of real numbers and let

$$S_f = \{\ s \in \mathrm{S} \mid s \text{ is positive and large }\}.$$

This set, S_f, is not well-defined in the sense of classical set theory because, although the statement "s is positive" is precise, the statement "s is large" is vague ("fuzzy").

However, if one introduces a membership function, reasonable and meaningful in the present discussion, to characterize or measure the property "large," then the fuzzy set S_f, associated with this membership function $\mu_{S_f}(s)$, is well defined.

Here, the membership function shown in Figure 1.3 may be chosen to use.

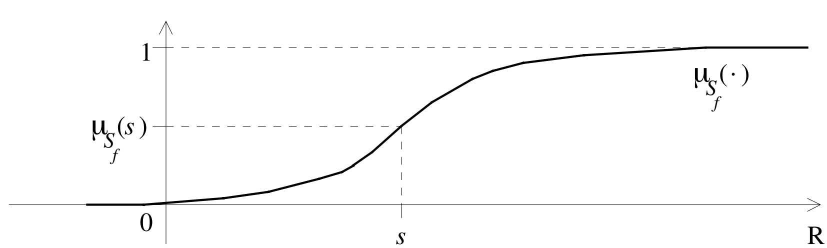

Figure 1.3 A membership function for a positive and large real number

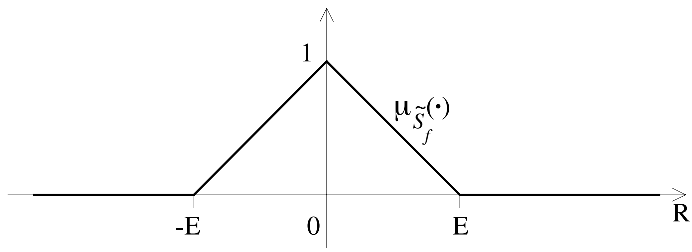

Figure 1.4 A membership function for a real number of small value

This membership function is quantified by

$$\mu_{S_f}(s) = \begin{cases} 0 & \text{if } s \le 0, \\ 1 - e^{-s} & \text{if } s > 0, \end{cases}$$

which is reasonable for describing a positive and "large" real number in common sense. Depending on how large is considered to be large in the application at hand, some other functions may be chosen instead. This is the user's choice. It is similar to the case when one is doing least-squares data fitting; if he believes a straight-line is simple and reasonable to use for a particular data set, then he selects it to use.

Similarly, a membership function for the set

$$\tilde{S}_f = \{ s \in \mathrm{S} \mid |s| \text{ is small } \}$$

may be chosen to be the one shown in Figure 1.4, where the cutting edge E is determined by the user according to his knowledge and preference about the concerned application.

Summary of general features of fuzzy membership functions:

All membership functions discussed above have been normalized to have maximum value 1, as usual, since 1 = 100% describes a full membership and is convenient to use.

Although a membership function is a nonnegative-valued function, it differs from the probability density functions in that the area under the curve of a membership function does not have to be equal to unity (in fact, it can be any value between 0 and ∞, including 0 and ∞). Moreover, a membership function does not have to be continuous, or integrable.

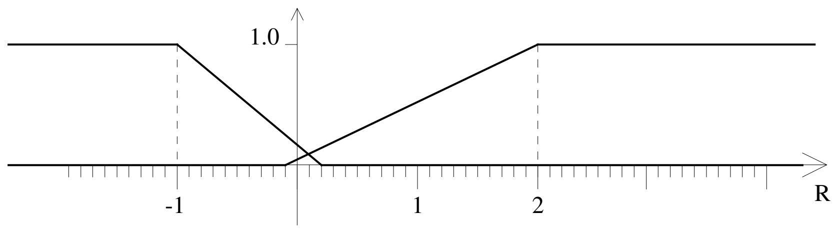

Figure 1.5 $s = 0.1$ is both "positive large" and "negative small"

Another distinction between the fuzzy set theory and the classical one is that a member of a fuzzy set may assume two or more membership values, and these membership values can even be conflicting. For example, if the two membership functions shown in Figure 1.5 are used to measure "positive and large" and "negative and small," respectively, then a member $s = 0.1$ has the first membership value 0.095 and the second 0.080: they do not sum up to 0.175, nor cancel out to be 0.015. Moreover, these two concepts are conflicting: s is positive and in the meantime also negative, with different degrees of correctness to be so. This situation is just like someone who is old and simultaneously young, which classical mathematics cannot accept. Such a vague and conflicting description of a fuzzy set is acceptable by fuzzy mathematics: in fact, it turns out to be very useful in describing and solving many real-world application problems where conflicting conditions are not uncommon. More importantly, the use of conflicting membership functions will not cause any logical or mathematical problems in the consequence, provided that a correct approach is taken carefully. Such a correct approach does exist; that is the *fuzzy set theory* to be further studied in the following sections of this chapter.

III. INTERVAL ARITHMETIC

Recall that a *fuzzy set* consists of two parts: a *set* defined in the ordinary sense and a *membership* function defined on the set, and this membership function is also defined in the ordinary sense.

Now, some fundamental properties and operation rules pertaining to a special yet important kind of sets – *intervals* – are first studied, which will be needed in the sequel.

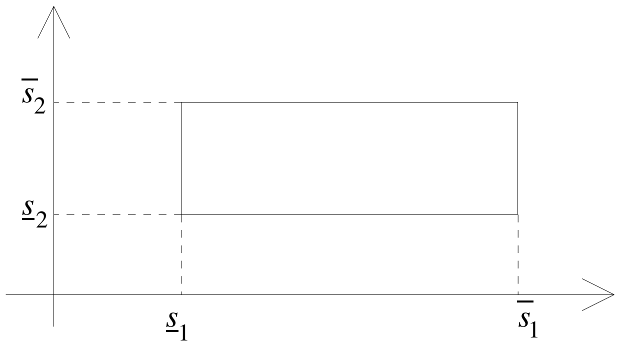

Figure 1.6 An interval of confidence in the two-dimensional case

A. Some Fundamental Concepts

The concern here is the situation where the value of a member s of a set is *uncertain* but bounded:

$$\underline{s} \leq s \leq \overline{s},$$

where $[\underline{s}, \overline{s}] \subset R$ is called the *interval of confidence* about the values of s. Only closed intervals in the form of $[\underline{s}, \overline{s}]$ are considered in this text, not those like $(\underline{s}, \overline{s}]$, $[\underline{s}, \overline{s})$, $(\underline{s}, \overline{s})$, except perhaps $[\underline{s}, \infty)$ and $(-\infty, \overline{s}]$ in some special cases.

A special case is, when $\underline{s} = \overline{s}$, it becomes $[\underline{s}, \overline{s}] = [s, s] = s$.

In the two-dimensional case, an interval of confidence is sometimes called a *region of confidence*, as shown in Figure 1.6.

Definition 1.1

(a) ***Equality***: Two intervals $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$ are said to be equal if and only if $\underline{s}_1 = \underline{s}_2$ and $\overline{s}_1 = \overline{s}_2$:

$$[\underline{s}_1, \overline{s}_1] = [\underline{s}_2, \overline{s}_2]$$

(b) ***Intersection***: The *intersection* of two intervals $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$ is

$$[\underline{s}_1, \overline{s}_1] \cap [\underline{s}_2, \overline{s}_2] = [\max\{\underline{s}_1, \underline{s}_2\}, \min\{\overline{s}_1, \overline{s}_2\}]$$

Note: $[\underline{s}_1, \overline{s}_1] \cap [\underline{s}_2, \overline{s}_2] = \varnothing$ if and only if $\underline{s}_1 > \overline{s}_2$ or $\underline{s}_2 > \overline{s}_1$.

(c) ***Union***: The *union* of two intervals $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$ is

$$[\underline{s}_1, \overline{s}_1] \cup [\underline{s}_2, \overline{s}_2] = [\min\{\underline{s}_1,\underline{s}_2\}, \max\{\overline{s}_1, \overline{s}_2\}],$$

provided that $[\underline{s}_1, \overline{s}_1] \cap [\underline{s}_2, \overline{s}_2] \neq \varnothing$. Otherwise, it is undefined (since the result is not an interval).

(d) ***Inequality***: Interval $[\underline{s}_1, \overline{s}_1]$ is said to be *less than* (resp., *greater than*) interval $[\underline{s}_2, \overline{s}_2]$, denoted by

$$[\underline{s}_1, \overline{s}_1] < [\underline{s}_2, \overline{s}_2] \qquad (\text{resp., } [\underline{s}_1, \overline{s}_1] > [\underline{s}_2, \overline{s}_2])$$

if and only if $\overline{s}_1 < \underline{s}_2$ (resp., $\underline{s}_1 > \overline{s}_2$). Otherwise, they cannot be compared. Note that the relations $\leq$ and $\geq$ are not defined for intervals.

(e) ***Inclusion***: The interval $[\underline{s}_1, \overline{s}_1]$ is *being included* in the interval $[\underline{s}_2, \overline{s}_2]$ if and only if both $\underline{s}_2 \leq \underline{s}_1$ and $\overline{s}_1 \leq \overline{s}_2$:

$$[\underline{s}_1, \overline{s}_1] \subseteq [\underline{s}_2, \overline{s}_2]$$

Example 1.3. For three intervals, $S_1 = [-1,0]$, $S_2 = [-1,2]$, and $S_3 = [2,10]$:

$$S_1 \cap S_2 = [-1,0] \cap [-1,2] = [-1,0],$$

$$S_1 \cap S_3 = [-1,0] \cap [2,10] = \varnothing,$$

$$S_2 \cap S_3 = [-1,2] \cap [2,10] = [2,2] = 2,$$

$$S_1 \cup S_2 = [-1,0] \cup [-1,2] = [-1,2],$$

$$S_1 \cup S_3 = [-1,0] \cup [2,10] = \text{undefined},$$

$$S_2 \cup S_3 = [-1,2] \cup [2,10] = [-1,10],$$

$$S_1 = [-1,0] < [2,10] = S_3,$$

$$S_1 = [-1,0] \subset [-1,2] = S_2.$$

B. Interval Arithmetic

Let $[\underline{s}, \overline{s}]$, $[\underline{s}_1, \overline{s}_1]$, and $[\underline{s}_2, \overline{s}_2]$ be intervals.

Definition 1.2 Interval Arithmetic

(1) ***Addition***:

$$[\,\underline{s}_1, \overline{s}_1\,] + [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1 + \underline{s}_2, \overline{s}_1 + \overline{s}_2\,].$$

(2) ***Subtraction***:

$$[\,\underline{s}_1, \overline{s}_1\,] - [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1 - \overline{s}_2, \overline{s}_1 - \underline{s}_2\,].$$

(3) ***Reciprocal:***

If $0 \notin [\,\underline{s}, \overline{s}\,]$ then $[\,\underline{s}, \overline{s}\,]^{-1} = [\,1/\overline{s}, 1/\underline{s}\,]$;

if $0 \in [\,\underline{s}, \overline{s}\,]$ then $[\,\underline{s}, \overline{s}\,]^{-1}$ is undefined.

(4) ***Multiplication***:

$$[\,\underline{s}_1, \overline{s}_1\,] \cdot [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{p}, \overline{p}\,],$$

where

$$\underline{p} = \min\{\, \underline{s}_1\,\underline{s}_2,\ \underline{s}_1\,\overline{s}_2,\ \overline{s}_1\,\underline{s}_2,\ \overline{s}_1\,\overline{s}_2 \,\},$$

$$\overline{p} = \max\{\, \underline{s}_1\,\underline{s}_2,\ \underline{s}_1\,\overline{s}_2,\ \overline{s}_1\,\underline{s}_2,\ \overline{s}_1\,\overline{s}_2 \,\}.$$

(5) ***Division***:

$$[\,\underline{s}_1, \overline{s}_1\,] / [\,\underline{s}_2, \overline{s}_2\,] = [\,\underline{s}_1, \overline{s}_1\,] \cdot [\,\underline{s}_2, \overline{s}_2\,]^{-1},$$

provided that $0 \notin [\,\underline{s}_2, \overline{s}_2\,]$.

(6) ***Maximum***:

$$\max\{\, [\,\underline{s}_1, \overline{s}_1\,], [\,\underline{s}_2, \overline{s}_2\,] \,\} = [\,\underline{p}, \overline{p}\,],$$

where

$$\underline{p} = \max\{\, \underline{s}_1, \underline{s}_2 \,\},$$

$$\overline{p} = \max\{\, \overline{s}_1, \overline{s}_2 \,\}.$$

(7) ***Minimum***:

$$\min\{ [\underline{s}_1, \overline{s}_1], [\underline{s}_2, \overline{s}_2]\} = [\underline{p}, \overline{p}],$$

where

$$\underline{p} = \min\{ \underline{s}_1, \underline{s}_2 \},$$

$$\overline{p} = \min\{ \overline{s}_1, \overline{s}_2 \}.$$

Remarks:

(a) Interval arithmetic intends to obtain an interval as the result of an operation such that the resulting interval contains all possible solutions. Therefore, these kinds of operational rules are defined in a conservative way, in the sense that they intend to make the resulting interval as large as necessary so as to avoid losing any true solution. For example, $[1,2]-[0,1]=[0,2]$ means that for any $a\in[1,2]$ and any $b\in[0,1]$, it is guaranteed that $a-b\in[0,2]$.

(b) This conservatism may produce some unusual results that could seem to be inconsistent with the ordinary numerical solutions. For instance, according to the subtraction rule (2), one has $[1,2] - [1,2] = [-1,1] \neq [0,0] = 0$. The result $[-1,1]$ here contains 0, but not only 0. The reason is that there can be other possible solutions: if one takes 1.5 from the first interval and 1.0 from the second, then the result is 0.5 rather than 0; and 0.5 is indeed in $[-1,1]$. Thus, an interval subtract itself is equal to zero (a point) only if this interval is itself a point (a trivial interval).

(c) Some general rules for the interval arithmetic:

For any interval Z,

$$Z - Z = 0 \qquad \text{and} \qquad Z / Z = \mathrm{I} = [1,1] \qquad (0 \notin Z)$$

only if $Z = [z,z]$ is a point.

For any intervals X, Y, and Z,

$$X + Z = Y + Z \Rightarrow X = Y.$$

For any interval Z, with $0 \in Z$,

$$Z^2 = Z \cdot Z = [\,\underline{z},\ \overline{z}\,] \cdot [\,\underline{z},\ \overline{z}\,] = [\,\underline{p},\ \overline{p}\,],$$

where

$$\underline{p} = \min\{\,\underline{z}^2,\ \underline{z}\,\overline{z},\ \overline{z}^{\,2}\,\} = \underline{z}\,\overline{z},$$

$$\overline{p} = \max\{\,\underline{z}^2,\ \underline{z}\,\overline{z},\ \overline{z}^{\,2}\,\} = \max\{\,\underline{z}^2,\ \overline{z}^{\,2}\}.$$

(d) Every computational system has restrictions (e.g., the ordinary arithmetic does not allow dividing by zero). Interval arithmetic is no exception. Fortunately, not much interval arithmetic will be involved in fuzzy systems, fuzzy-logic-based decision making, and fuzzy control applications, at least not much are needed within the context of this text, therefore generally no confusion would arise about such "unusual" phenomena and rules. Although many more details exist in the mathematical literature about special interval arithmetic rules, so that conflicts can be avoided in a way similar to the problems caused by "dividing by zero" in classical mathematics, this issue will not be further discussed here.

Note that the interval operations of addition (+), subtraction (–), multiplication (·), and division (/) are all (set-variable and set-valued) *functions*: for three intervals, $X = [\,\underline{x},\ \overline{x}\,]$, $Y = [\,\underline{y},\ \overline{y}\,]$, and $Z = [\,\underline{z},\ \overline{z}\,]$,

$$Z = f(X,Y) = X * Y, \qquad\qquad * \in \{+, -, \cdot, /\,\}$$

are continuous functions defined on intervals.

C. Algebraic Properties of Interval Arithmetic

Theorem 1.1 The addition and multiplication operations of intervals are *commutatitve* and *associative* but not *distributive*:

(1) $X + Y = Y + X$;

(2) $Z + (\,X + Y\,) = (\,Z + X\,) + Y$;

(3) $Z\,(\,X\,Y\,) = (\,Z\,X\,)\,Y$;

(4) $X\,Y = Y\,X$;

(5) $Z + \mathbf{0} = \mathbf{0} + Z = Z$ and $Z\mathbf{0} = \mathbf{0}Z = \mathbf{0}$, where $\mathbf{0} = [0,0]$;

(6) $Z\mathrm{I} = \mathrm{I}Z = Z$, where $\mathrm{I} = [1,1]$;

(7) $Z(X+Y) \neq ZX + ZY$, except when:

(a) $Z = [z,z]$ is a point; or

(b) $X = Y = \mathbf{0}$; or

(c) $xy \geq 0$ for all $x \in X$ and $y \in Y$.

In general, only the *subdistributive* law holds:

$$Z(X+Y) \subseteq ZX + ZY.$$

Example 1.4. Let

$$Z = [1,2],\ X = \mathrm{I} = [1,1],\ Y = -\mathrm{I} = [-1,-1].$$

Then

$$Z(X+Y) = [1,2](\mathrm{I} - \mathrm{I}) = [1,2]\cdot\mathbf{0} = \mathbf{0};$$

$$ZX + ZY = [1,2]\cdot[1,1] + [1,2]\cdot[-1,-1] = [-1,1] \supset \mathbf{0}.$$

A more general rule for interval arithmetic operations is the following fundamental law of *monotonic inclusion*, established in the mathematical literature.

Theorem 1.2 Let X_1, X_2, Y_1, and Y_2 be intervals such that

$$X_1 \subseteq Y_1 \quad \text{and} \quad X_2 \subseteq Y_2.$$

Then, for all operations $* \in \{+, -, \cdot, /\}$,

$$X_1 * X_2 \subseteq Y_1 * Y_2.$$

Corollary 1.1 Let X and Y be intervals with $x \in X$ and $y \in Y$. Then, for any operation $* \in \{+, -, \cdot, /\}$,

$$x * y \in X * Y$$

Finally, consider the problem of solving the *interval equation*

$$AX = B,$$

where A and B are both given intervals with $0 \notin A$, and X is to be determined.

Theorem 1.3 Let X be a solution of the interval equation

$$AX = B,\ 0 \notin A .$$

Then, $X \subseteq B / A$.

Example 1.5

Consider the interval equation $AX=B$, with

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix}.$$

Then

$$B/A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix} = \begin{bmatrix} [0,2] \\ [-1,1] \end{bmatrix}.$$

It is clear that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in B / A$ but it is not a solution of the interval

equation $AX=B$, namely,

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} [0,1] \\ [0,1] \end{bmatrix}.$$

Therefore, $X \subset B / A$ but $X \neq B / A$.

All the above basic theoretical results are standard in interval mathematics. Hence, only the conclusions, but not their proofs, are provided here, as preliminaries.

D. Interval Evaluation

An ordinary real-variable and real-valued functions f: R→R can easily be extended to an interval-variable and interval-valued function f: **I**→**I**, where **I** is the family of intervals defined on R. Such extended functions include the following arithmetic functions:

$$Z = f(X,Y) = X * Y, \qquad * \in \{ +, -, \cdot, / \},$$

where X, Y, $Z \in \mathbf{I}$.

Note that for any ordinary continuous function f: R→R and any interval $X \in \mathbf{I}$, the interval-variable and interval-valued function

$$f_{\mathbf{I}}(X) = \left[\min_{x \in \overline{X}} f(x), \max_{x \in \overline{X}} f(x) \right]$$

is also a continuous function.

Now, let A_1, ..., A_m be intervals in **I.** For any interval $X \in \mathbf{I}$, one can further define a function, $f(x;a_1,\ldots,a_m)$, $x \in X$, which depends on m parameters $a_k \in A_k$, $k = 1, 2,\ldots,m$, by

$$f_{\mathbf{I}}(X;A_1,\ldots,A_m) = \{ f(x;a_1,\ldots,a_m) \mid x \in X,\ a_k \in A_k,\ 1 \le k \le m \}$$

$$= \left[\min_{\substack{x \in \overline{X} \\ a_k \in A, 1 \le k \le m}} f(x,a_1,\ldots,a_m), \max_{\substack{x \in \overline{X} \\ a_k \in A, 1 \le k \le m}} f(x,a_1,\ldots,a_m) \right]$$

Example 1.6

Consider the real-variable and real-valued function

$$f(x;a) = \frac{ax}{1-x}, \qquad x \ne 1, \quad x \ne 0 .$$

If $X = [2, 3]$ and $A = [0, 2]$ are intervals, with $x \in X$ and $a \in A$, then the interval expression of f is given by

$$f_{\mathbf{I}}(X;A) = \left\{ \frac{ax}{1-x} \,\middle|\, 2 \le x \le 3,\ 0 \le a \le 2 \right\}$$

$$= \left[\min_{\substack{2\le x\le 3\\ 0\le a\le 2}} \frac{ax}{1-x},\ \max_{\substack{2\le x\le 3\\ 0\le a\le 2}} \frac{ax}{1-x} \right]$$

$$= [-4, 0].$$

The following result is important and useful. It states that all common interval arithmetic expressions have the ***inclusion monotonic*** *property*.

Theorem 1.4 Let $f : \mathrm{R}^{n+m} \to \mathrm{R}$ be a real-variable and real-valued continuous function with an arithmetic interval expression $f_{\mathbf{I}}(X_1,...,X_n,A_1,...,A_m)$. Then, for all

$$X_k \subseteq Y_k, \quad k = 1,2,...,n \quad \text{and} \quad A_l \subseteq B_l, \quad l = 1,2, \ldots, m,$$

one has

$$f_{\mathbf{I}}(X_1,...,X_n,A_1,...,A_m) \subseteq f_{\mathbf{I}}(Y_1,...,Y_n,B_1,...,B_m)\ .$$

Example 1.7

Let $X = [0.2, 0.4]$ and $Y = [0.1, 0.5]$. Then $X \subset Y$.

(a) $X^{-1} = \dfrac{1}{[0.2,0.4]} = [2.5, 5.0]$,

$Y^{-1} = \dfrac{1}{[0.1,0.5]} = [2.0, 10.0]$,

$X^{-1} \subset Y^{-1}$.

(b) $1 - X = [1.0, 1.0] - [0.2, 0.4] = [0.6, 0.8]$,

$1 - Y = [1.0, 1.0] - [0.1, 0.5] = [0.5, 0.9]$,

$1 - X \subset 1 - Y$.

(c) $\dfrac{1}{1-X} = \dfrac{1}{[0.6,0.8]} = [5/4, 5/3]$,

$\dfrac{1}{1-Y} = \dfrac{1}{[0.5,0.9]} = [10/9, 2.0]$,