SECOND EDITION

OPERATIONS RESEARCH CALCULATIONS HANDBOOK

DENNIS BLUMENFELD

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OPERATIONS RESEARCH CALCULATIONS HANDBOOK

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Preface to the Second Edition

The field of operations research encompasses a growing number of technical areas. The scope of the second edition has been expanded to cover several additional topics. These include new chapters on order statistics, heuristic search methods, and traffic flow and delay. Some chapters have also been updated with new material, and many new references have been added. As before, the focus is on presenting handy analytical results and formulas that allow quick calculations and provide the understanding of system models.

Dennis E. Blumenfeld

Preface to the First Edition

Operations research uses analyses and techniques from a variety of branches of mathematics, statistics, and other scientific disciplines. Certain analytical results arise repeatedly in applications of operations research to industrial and service operations. These results are scattered among many different textbooks and journal articles, sometimes in the midst of extensive derivations. The idea for a handbook of operations research results came from a need to have frequently used results to be readily available in one source of reference.

This handbook is a compilation of analytical results and formulas that have been found useful in various applications. The objective is to provide students, researchers, and practitioners with convenient access to wide range of operations research results in a concise format.

Given the extensive variety of applications of operations research, a collection of results cannot be exhaustive. The selection of results included in this handbook is based on experience in the manufacturing industry. Many of the results are basic to system modeling, and are likely to carry over to applications in other areas of operations research and management science.

This handbook focuses on areas of operations research that yield explicit analytical results and formulas. With the widespread availability of computer software for simulations and algorithms, many analyses can be easily performed numerically without knowledge of explicit formulas. However, formulas continue to play a significant role in system modeling. While software packages are useful for obtaining numerical results for given values of input parameters, formulas allow general conclusions to be drawn about system behavior as parameter values vary. Analytical results and formulas also help to provide an intuitive understanding of the underlying models for system performance. Such understanding is important in the implementation of operations research models as it allows analysts and decision makers to use models with confidence.

Dennis E. Blumenfeld

Happy is the man that findeth wisdom, and the man that getteth understanding.

—Proverbs 3:13

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1

Introduction

Operations research can be considered as the science of decision making. It encompasses many scientific disciplines, such as mathematics, statistics, computer science, physics, engineering, economics, and social sciences, and has been successful in providing a systematic approach to complex decisions in manufacturing, service, military, and financial operations.

One of the reasons for the appeal and success of operations research is that it draws on basic mathematical principles and uses them in clever and novel ways to solve all kinds of real-world problems. Many of the applications make use of handy analytical results and formulas derived from system models, and can reveal how system performance varies with model parameters.

Often, these analytical results and formulas offer insight that numerical methods do not provide. Even though numerical solutions can now be easily obtained with the greatly increased speed and power of computers in recent years, there is still a need for analytical results to highlight trade-offs between the different parameters in a model, and to make the mathematical relationships between the parameters readily apparent.

Analytical results and formulas often require minimal data and allow quick "back-of-the-envelope" calculations that are very useful for initial analyses. This is important when an approximate estimate is all that is needed, or all there is time for, in the many real-world situations where decisions must be made quickly. In situations where there is more time, and many alternatives are to be evaluated, such initial analyses can provide focus as to where more detailed numerical analyses are warranted. Since formulas are not limited to any particular programming language, computer operating system, or user interface, they can be readily used on their own for system analyses or be included as components of comprehensive decision-making tools.

The objective of this handbook is to provide a concise collection of analytical results and formulas that arise in operations research applications. The material is organized into chapters based on the following topics.

The first few chapters are devoted to results on the stochastic modeling aspects of operations research. Chapter 2 covers a range of formulas that involve the mean and the variance of random variables. Chapters 3 and 4 list the main properties of widely used discrete and continuous probability distributions. Chapter 5 contains a collection of other analytical results that frequently arise in probability. Chapters 6 and 7 present formulas that arise in stochastic processes and queueing theory.

The next four chapters cover specific applications of operations research in the areas of stochastic modeling. Chapter 8 presents some results in production systems modeling and Chapter 9 covers the basic formulas in inventory control. Chapter 10 gives distance formulas that are useful in logistics and spatial analyses. Chapter 11 presents basic results in traffic flow and delay.

Chapters 12 and 13 cover the standard linear programming formulations and heuristic search methods. These subjects deal with the development of algorithms and methodologies in optimization. In keeping with the intent of this handbook, which is to focus on analytical results and formulas, these two chapters present the mathematical formulations and basic concepts, and give references for the solution methods.

The remaining chapters contain basic mathematical results that are relevant to operations research. Chapter 14 covers key results in order statistics, Chapter 15 lists some common mathematical functions that arise in applications, Chapter 16 presents useful results from elementary and more advanced calculus, Chapter 17 lists the standard properties of matrices, Chapter 18 gives the standard formulas for combinatorial calculations, Chapter 19 lists some common results for finite and infinite sums, and, finally, Chapter 20 gives basic interest formulas that are important in economic analysis.

To supplement the various results and formulas, references are given for derivations and additional details.

2

Means and Variances

2.1 Mean (Expectation) and Variance of a Random Variable

For a discrete random variable X that takes the values x_0, x_1, x_2, …, the mean of X is given by

$$E[X] = \sum_{i=0}^{\infty} x_i \cdot \Pr\{X = x_i\} \tag{2.1}$$

where
$E[X]$ denotes the mean (expected value or expectation) of X
Pr $\{X=x_i\}$ denotes the probability that X takes the value x_i ($i=0, 1, 2, \ldots$)

If X takes nonnegative integer values only ($X=0, 1, 2, \ldots$), then the mean of X is given by

$$E[X] = \sum_{n=0}^{\infty} n \cdot \Pr\{X = n\} \tag{2.2}$$

$$= \sum_{n=0}^{\infty} \Pr\{X > n\} \tag{2.3}$$

For a continuous random variable X ($-\infty < X < \infty$), the mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx \tag{2.4}$$

$$= \int_{0}^{\infty} [1 - F(x)] dx - \int_{-\infty}^{0} F(x) dx \tag{2.5}$$

where

$E[X]$ denotes the mean (expected value) of X
$f(x)$ is the probability density function of X

and

$$F(x) = \Pr\{X \le x\} = \int_{-\infty}^{x} f(t)\,dt$$

denotes the cumulative distribution function of X.

If X is continuous and takes nonnegative values only ($0 \le X < \infty$), then the mean of X is given by

$$E[X] = \int_{0}^{\infty} x f(x)\,dx \tag{2.6}$$

$$= \int_{0}^{\infty} \left[1 - F(x)\right] dx \tag{2.7}$$

Çinlar (1975, pp. 22, 25–26); Lefebvre (2006, pp. 96–97); Mood, Graybill, and Boes (1974, pp. 64–65).

For any random variable X, the variance is given by

$$Var[X] = E\left\{(X - E[X])^2\right\} \tag{2.8}$$

$$= E\left[X^2\right] - (E[X])^2 \tag{2.9}$$

where $Var[X]$ denotes the variance of X
and

$$E\left[X^2\right] = \begin{cases} \displaystyle\sum_{x} x^2 \cdot \Pr\{X = x\} & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} x^2 f(x)\,dx & \text{if } X \text{ is continuous} \end{cases} \tag{2.10}$$

The standard deviation of X, $St\ Dev[X]$, is given by

$$St\,Dev[X] = \sqrt{Var[X]} \tag{2.11}$$

Binmore (1983, pp. 268–269); Çinlar (1975, p. 31); Feller (1964, p. 213); Lefebvre (2006, pp. 99–100); Mood, Graybill, and Boes (1974, pp. 68, 70); Ross (2003, pp. 46–47).

2.2 Covariance and Correlation Coefficient

For any random variables X and Y, the covariance $Cov[X, Y]$ is given by

$$Cov[X,Y] = E\{(X - E[X])(Y - E[Y])\} \tag{2.12}$$

$$= E[XY] - E[X]E[Y] \tag{2.13}$$

and the correlation coefficient $Corr[X, Y]$ is given by

$$Corr[X,Y] = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} \tag{2.14}$$

The correlation coefficient is dimensionless and satisfies the condition $-1 \le Corr[X, Y] \le 1$.

If X and Y are independent, then the covariance, $Cov[X, Y]$, and correlation coefficient, $Corr[X, Y]$, are zero.

Feller (1964, pp. 215, 221); Mood, Graybill, and Boes (1974, pp. 155–156, 161); Ross (2003, p. 53).

2.3 Mean and Variance of the Sum of Random Variables

For any random variables X and Y, the mean of the sum $X+Y$ is given by

$$E[X + Y] = E[X] + E[Y] \tag{2.15}$$

This result for the mean of a sum holds even if the random variables are not independent.

If the random variables X and Y are independent, then the variance of the sum $X+Y$ is given by

$$Var[X+Y]=Var[X]+Var[Y] \tag{2.16}$$

If the random variables X and Y are not independent, then the variance of the sum $X+Y$ is given by

$$Var[X+Y]=Var[X]+Var[Y]+2Cov[X,Y] \tag{2.17}$$

where $Cov[X, Y]$ is the covariance of X and Y given by Equation 2.12.

For any random variables X and Y, and any constants a and b, the mean and the variance of the linear combination $aX+bY$ are given by

$$E\left[aX+bY\right]=aE[X]+bE[Y] \tag{2.18}$$

and

$$Var\left[aX+bY\right]=a^2Var[X]+b^2Var[Y]+2ab\,Cov[X,Y] \tag{2.19}$$

respectively.

In the special case $a=1$ and $b=-1$, Equations 2.18 and 2.19 give the mean and the variance of the *difference* between the two random variables. Thus, for any random variables X and Y, the mean of the difference $X-Y$ is given by

$$E[X-Y]=E[X]-E[Y] \tag{2.20}$$

and the variance of the difference $X-Y$ is given by

$$Var[X-Y]=Var[X]+Var[Y]-2Cov[X,Y] \tag{2.21}$$

If the random variables X and Y are independent, then $Cov[X, Y]=0$ and the variance of the difference $X-Y$ is given by

$$Var[X-Y]=Var[X]+Var[Y] \tag{2.22}$$

Equation 2.20 for the mean of the difference $X-Y$ holds even if the random variables are not independent. Note that the mean of the difference is simply the difference of the means (Equation 2.20), while the variance of the difference (for the case of independent random variables) is the *sum* of the variances (Equation 2.22), i.e., the same variance as for the sum $X+Y$ (Equation 2.16).

The results in Equations 2.18 and 2.19 for a linear combination can be generalized to n random variables. For any random variables X_1, X_2, …, X_n and any constants a_1, a_2, …, a_n, the mean and the variance of the linear combination $a_1X_1+a_2X_2+\cdots+a_nX_n$ are given by

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i] \tag{2.23}$$

and

$$Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2\, Var[X_i] + \sum_{i \neq j}\sum a_i a_j Cov\left[X_i, X_j\right] \tag{2.24}$$

respectively.

Bolch, Greiner, de Meer, and Trivedi (1998, pp. 23–24); Feller (1964, pp. 208, 214, 216); Mood, Graybill, and Boes (1974, pp. 178–179); Ross (2003, pp. 49, 53–54).

2.4 Mean and Variance of the Product of Two Random Variables

If X and Y are independent random variables, then the mean and the variance of the product XY are given by

$$E\left[XY\right] = E[X]E[Y] \tag{2.25}$$

and

$$Var[XY] = \left(E[Y]\right)^2 Var[X] + \left(E[X]\right)^2 Var[Y] + Var[X]Var[Y] \tag{2.26}$$

respectively.

If the random variables X and Y are not independent, then the mean and the variance of the product XY are given by

$$E[XY] = E[X]E[Y] + Cov[X,Y] \tag{2.27}$$