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# Syzygies and Hilbert Functions

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**Irena Peeva**

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# *Introduction*

A very interesting and important numerical invariant of a graded ideal (in a polynomial ring) is its Hilbert function. It gives the sizes of the graded components of the ideal. A case of particular importance is the Hilbert function of a projective algebraic variety $V$; this function gives the dimensions of the spaces $P(i)$ of forms of degree $i$ vanishing on $V$ (for all $i$). Important invariants (e.g., dimension, multiplicity) can be read off the Hilbert function.

A homological method for studying the structure of a finitely generated module $T$ (over a commutative Noetherian ring $R$) is to describe it by a resolution, which is a sequence of maps between free modules. The idea to associate a resolution to $T$ was introduced in Hilbert's famous 1890 and 1893 papers. He proved that if $R$ is a polynomial ring, then every finitely generated $R$-module has a finite resolution. In the local and graded cases there exists a minimal free resolution; it is unique up to an isomorphism. The invariants of $T$ are closely related to the properties of its minimal free resolution.

The studies of Hilbert functions and of resolutions are closely related. For many years, Hilbert functions and resolutions have been both central objects and fruitful tools in many fields, including algebraic geometry, combinatorics, commutative algebra, and computational algebra. There has been a surge in interest and research in this direction in recent years; a variety of new ideas and techniques were introduced, and substantial progress was made.

This book contains expository chapters on Hilbert functions and resolutions. The chapters point out highlights, conjectures, unsolved problems, and helpful examples. Some of the chapters were written by participants in the conference on resolutions held in October 2005 at Cornell University.

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# Chapter 1

# Some Results and Questions on Castelnuovo–Mumford Regularity

**Marc Chardin**

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## 1.1 The Two Most Classical Definitions of Castelnuovo–Mumford Regularity

Let $R := k[X_1, \ldots, X_n]$ be a polynomial ring over a field $k$ and $M$ a finitely generated graded $R$-module.

The two most popular definitions of Castelnuovo–Mumford regularity are the one in terms of graded Betti numbers and the one using local cohomology.

*Local cohomology modules.* Set $\mathfrak{m} := (X_1, \ldots, X_n) = R_{>0}$, then $H^0_{\mathfrak{m}}(M) := \{x \in M \mid \exists N,\ \mathfrak{m}^N x = 0\}$ and the functors $H^i_{\mathfrak{m}}(—)$ can be defined as right-derived functors of $H^0_{\mathfrak{m}}(—)$ in the category of $R$-modules. A more concrete way of considering these modules is to see them as the cohomology modules of the Čech complex $\mathcal{C}^\bullet_{\mathfrak{m}}$:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bigoplus_i M_{X_i} & \xrightarrow{\psi} & \bigoplus_{i<j} M_{X_iX_j} & \longrightarrow & \cdots & \longrightarrow & M_{X_1\cdots X_n} & \longrightarrow & 0\ . \\ & & \| & & \| & & \| & & & & \| & & \\ & & \mathcal{C}^0_{\mathfrak{m}}(M) & & \mathcal{C}^1_{\mathfrak{m}}(M) & & \mathcal{C}^2_{\mathfrak{m}}(M) & & & & \mathcal{C}^n_{\mathfrak{m}}(M) & & \end{array}$$

This is how we will view them in this article.

Recall that, from a more geometric point of view, one has graded isomorphisms

$$\Gamma M := \ker(\psi) \simeq \bigoplus_{\mu} H^0(\mathrm{Proj}(R), \widetilde{M}(\mu)),$$

where $\widetilde{M}$ is the sheaf of modules associated to $M$, and

$$H^i_{\mathfrak{m}}(M) \simeq \bigoplus_{\mu} H^{i-1}(\mathrm{Proj}(R), \widetilde{M}(\mu)),\ \forall i \geq 2.$$

There are two fundamental results. First, Grothendieck's theorem asserts the vanishing of $H^i_{\mathfrak{m}}(M)$ for $i > \dim(M)$ and $i < \operatorname{depth}(M)$, as well as the nonvanishing of these modules for $i = \dim(M)$ and $i = \operatorname{depth}(M)$. Second is Serre's vanishing theorem that implies the vanishing of graded pieces $H^i_{\mathfrak{m}}(M)_\mu$ for any $i$ and $\mu$ big enough. The Castelnuovo–Mumford regularity is a measure of this vanishing degree. Set

$$a_i(M) := \max\{\mu \mid H^i_{\mathfrak{m}}(M)_\mu \neq 0\},$$

if $H^i_{\mathfrak{m}}(M) \neq 0$ and $a_i(M) := -\infty$ else. Then,

$$\mathrm{reg}(M) = \max_i \{a_i(M) + i\}.$$

The maximum over the positive $i$'s is also an interesting invariant:

$$\mathrm{greg}(M) := \max_{i>0}\{a_i(M) + i\} = \mathrm{reg}(M/H^0_{\mathfrak{m}}(M)).$$

*Graded Betti numbers.* Let $F_\bullet$ be a minimal graded free $R$-resolution of $M$,

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$$

with $F_i = \oplus_j R[-j]^{\beta_{ij}}$. Notice that $p = \mathrm{pdim}(M) = n - \operatorname{depth}(M)$. The maps of $F_\bullet \otimes_R k$ being zero maps, $\mathrm{Tor}_i^R(M, k) = H_i(F_\bullet \otimes k) = F_i \otimes k$ and therefore $\beta_{ij} = \dim_k \mathrm{Tor}_i^R(M, k)_j$. Set

$$b_i(M) := \max\{\mu \mid \mathrm{Tor}_i^R(M, k)_\mu \neq 0\}$$

if $\operatorname{Tor}_i^R(M, k) \neq 0$ and $b_i(M) := -\infty$ else. By definition, $b_i(M)$ is the maximal degree of a minimal generator of $F_i$ and therefore of the module of $i$-th syzygies of $M$.

The Castelnuovo–Mumford regularity is also a measure of the maximal degrees of generators of the modules $F_i$:

$$\operatorname{reg}(M) = \max_i \{b_i(M) - i\}.$$

---

## 1.2 A Lemma from Homological Algebra and the Equivalence of the Definitions

One way of proving the equality $\max_i\{a_i(M) + i\} = \max_i\{b_i(M) - i\}$ is to use a double complex relating the Tor modules to the local cohomology. We will now give a lemma which is useful in this approach, and which has further applications.

We will use the following notations:

- For $F^\bullet$ (resp. $F_\bullet$), a bounded graded complex of finitely generated free $R$-modules, we set $b(F^\bullet) := \max_q\{b_0(F^q) + q\}$ (resp. $b(F_\bullet) := \max_q \{b_0(F_q) - q\}$).
- If $H$ is a graded module such that $H_\mu = 0$ for $\mu \gg 0$, we set $\operatorname{end}(H) := \max\{\mu \mid H_\mu \neq 0\}$ if $H \neq 0$, and $\operatorname{end}(0) := -\infty$.

A complex $C$ of free $R$-modules is called minimal if the differentials of $C \otimes_R k$ are zero.

**Lemma 1.2.1** *Let $F^\bullet$ be a minimal graded complex of finitely generated free $R$-modules, with $F^i = 0$ for $i < 0$; let $M$ be a finitely generated graded $R$-module and $T^\bullet := \mathcal{C}^\bullet_{\mathfrak{m}} M \otimes_R F^\bullet$.*

*Then $H^\ell(T^\bullet)_\mu = 0$, for $\mu \gg 0$ and any $\ell$. Moreover, for any $\ell$,*

(i)
$$\operatorname{end}(H^\ell(T^\bullet)) \leq \max_{p+q=\ell}\{a_p(H^q(F^\bullet \otimes_R M))\}$$

*and equality holds if* $\dim H^q(F^\bullet \otimes_R M) \leq 1$ *for all $q$ or if there exists $q_0$ such that* $\dim H^q(F^\bullet \otimes_R M) \leq 1$ *for $q < q_0$ and $H^q(F^\bullet \otimes_R M) = 0$ for $q > q_0$.*

(ii)
$$\operatorname{end}(H^\ell(T^\bullet)) \leq \max_{p+q=\ell}\{a_p(M) + b_0(F^q)\}.$$

*If further,* $b(F^\bullet) = \max\{b_0(F^0), b_0(F^1)+1\}$*, then*

$$\max_{j\leq \ell}\{\mathrm{end}(H^j(T^\bullet))+j\} = \max_{p+q\leq \ell}\{a_p(M)+b_0(F^q)+p+q\}.$$

The proof of this lemma is given in the technical appendix. Let us prove the equivalence of the definitions, and a little more, as an application:

**Corollary 1.2.2** *If $M$ is a finitely generated graded $R$-module, then for any $\ell$*

$$\max_{p\leq \ell}\{a_p(M)+p\} = \max_{q\geq n-\ell}\{b_q(M)-q\}.$$

*As a consequence,*

$$\mathrm{reg}(M) = \max_q\{b_q(M)-q\} = \max_{\mathrm{pdim} M\geq q\geq \mathrm{codim} M}\{b_q(M)-q\}.$$

*Proof* Take $F^\bullet := K^\bullet(X_1,\dots,X_n;R)$. One has $b(F^q) = -q$ and dim $H^q(F^\bullet\otimes_R M)\leq 0$ for all $q$. Therefore, by the lemma,

$$\max_{q\leq \ell}\{a_0(H^q(F^\bullet\otimes_R M))+q\} = \max_{p\leq \ell}\{a_p(M)+p\}$$

but $H^q(F^\bullet\otimes_R M)\simeq H_{n-q}(K_\bullet(X_1,\dots,X_n;M)[n])\simeq \mathrm{Tor}^R_{n-q}(M,k)[n]$. It follows that $a_0(H^q(F^\bullet\otimes_R M)) = b_{n-q}(F^M_\bullet)-n$.

Setting $q' := n-q$ the max on the left-hand side can be rewritten as $\max_{q'\geq n-\ell} b_{q'}(F^M_\bullet)-n+(n-q')$.

The second claim follows from Grothendieck's vanishing theorem. ∎

## 1.3 Other Definitions and Further Applications of the Lemma

By the definition of the regularity in terms of the Betti numbers, $\mathrm{reg}(M) = \mathrm{indeg}(M)$ if and only if $M$ is generated in a single degree and the matrices of the maps in its minimal free $R$-resolution have only linear forms as entries. Such a resolution (generators in a single degree and maps given by linear forms) is called a *linear resolution*.

A first application of the equivalence of the definitions above is the following third definition:

**Proposition 1.3.1** *For a finitely generated graded $R$-module,*

$$\operatorname{reg}(M) = \min\{\mu \mid M_{\geq\mu} \text{ has a linear resolution}\}.$$

*Proof* The modules $M$ and $M' := M_{\geq\mu}$ coincide on the punctured spectrum, therefore $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(M')$ for $i > 1$ and $\Gamma M = \Gamma M'$. Moreover, the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M)_\nu \longrightarrow M_\nu \longrightarrow \Gamma M_\nu \longrightarrow H^1_{\mathfrak{m}}(M)_\nu \longrightarrow 0$$

compared to the corresponding one for $M'$ shows that:

- $a_0(M') = a_0(M)$ if $\mu \leq a_0(M)$ and $a_0(M') = -\infty$ else,
- $a_1(M') = \max\{a_1(M), \mu\}$,

which implies that $\operatorname{reg}(M') = \max\{\operatorname{reg}(M), \mu\}$ and the proposition. $\square$

We will see in Proposition 1.9.1 (v) and Proposition 1.9.5 that one also has:

**Proposition 1.3.2** *For a finitely generated graded $R$-module,*

$$\operatorname{reg}(M) = \min\{\mu \geq \min\{a_0(M), b_0(M)\} \mid H^i_{\mathfrak{m}}(M)_{\mu-i} = 0,\ \forall i\} - 1$$

*and for a graded ideal $I$ such that $\sqrt{I} \neq \mathfrak{m}$,*

$$\operatorname{reg}(I) = \min\{\mu \mid H^i_{\mathfrak{m}}(R/I)_{\mu-i} = 0,\ \forall i\}.$$

This proposition is a persistence theorem, in which the shift by $i$ in the $i$-th cohomology reveals its usefulness.

Also recall that, by local duality, $a_i(M) = -\operatorname{indeg}(\operatorname{Ext}^{n-i}_R(M, \omega_R)) = -\operatorname{indeg}(\operatorname{Ext}^{n-i}_R(M, R)) - n$ and therefore

$$\operatorname{reg}(M) = -\min_i\{\operatorname{indeg}(\operatorname{Ext}^i_R(M, R)) + i\}.$$

## 1.4 Regularity and Gröbner Bases

**Remark 1.4.1** Taking $F^\bullet = K^\bullet(f; M)$ in Lemma 1.2.1, or applying directly Theorem 1.5.1 with $N := R/(f)$, gives the well-known fact that

$$\operatorname{reg}(M) = \max\{\operatorname{reg}(0 :_M (f)), \operatorname{reg}(M/(f)M) - \deg f + 1\}$$

if $\dim(0 :_M (f)) \leq 1$. More generally, Theorem 1.5.1 shows that, if $f := (f_1, \ldots, f_s)$ is an $s$-tuple of forms and $H_i := H_i(K_\bullet(f; M))$, one has

$$\operatorname{reg}(M) = \max_i\{\operatorname{reg}(H_i)\} - \sum_j (\deg f_j - 1),$$

if $\dim H_i \leq 1$ for $i > 0$. ∎

For a rev-lex order one has $\operatorname{in}(I : (X_n)) = \operatorname{in}(I) : (X_n)$ and $\operatorname{in}(I + (X_n)) = \operatorname{in}(I) + (X_n)$, and this may be extended to graded modules (see for instance [E, §15.7]). As a consequence, the modules $I : (X_n)/I$ and $\operatorname{in}(I) : (X_n)/\operatorname{in}(I)$ have the same Hilbert function $H$. Therefore, if $I : (X_n)/I$ has dimension zero these two modules have the same regularity: the last degree where $H$ is not 0 (or $-\infty$ if $I : (X_n) = I$). By the remark above, if $\dim(I : (X_n)/I) \leq 1$

$$\operatorname{reg}(R/I) = \max\{\operatorname{reg}(I : (X_n)/I), \operatorname{reg}(R/I + (X_n))\}$$

and

$$\operatorname{reg}(R/\operatorname{in}(I)) = \max\{\operatorname{reg}(\operatorname{in}(I) : (X_n)/\operatorname{in}(I)), \operatorname{reg}(R/\operatorname{in}(I + (X_n))\}.$$

It follows that if $(I + (X_n, \ldots, X_{i+1})) : (X_i)/(I + (X_n, \ldots, X_{i+1}))$ has finite length for $i$ from $n$ to 1, then by induction (for $i = 1$, $I = \operatorname{in}(I) = \mathfrak{m}$) we have $\operatorname{reg}(R/I) = \operatorname{reg}(R/\operatorname{in}(I))$. After a general linear change of coordinates, these modules are indeed all of finite length. This proves the first part of the following theorem of Bayer and Stillman (that also extends to modules).

**Theorem 1.4.2** [BS1, 2.4 & 2.9] *In general coordinates, for a rev-lex order,*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I)),$$

*and if furthermore $k$ is of characteristic zero,* $\operatorname{reg}(\operatorname{in}(I)) = b_0(I)$.

The second part of the theorem can be deduced from the fact that in characteristic zero, the generic initial ideal $J$ of $I$ is a strongly stable monomial ideal (i.e., for any monomial $M$, $X_iM \in J \Rightarrow X_jM \in J, \ \forall j \leq i$) and strongly stable monomial ideals have regularity equal to their maximal degree of generator. More precisely, Eliahou and Kervaire provided in [EK] a minimal free $R$-resolution of strongly stable monomial ideals, from which this is easy to deduce. One should also notice that for any monomial ideal $K$, using a resolution due to Diana Taylor, one has

$$b_i(K) \leq \max_{m_0,\ldots,m_i \in S} \deg(\gcd(m_0, \ldots, m_i)) \leq (i+1)b_0(K),$$

where $S$ is a minimal set of monomial generators of $K$. In this sense, the discrepancy between $b_0(K)$ and $\operatorname{reg}(K)$ is quite under control in the case of monomial ideals, as compared with arbitrary ideals.

The equality $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$ in the theorem has been refined by Bayer, Charalambous, and Popescu, who proved in [BCP] that the so-called extremal Betti numbers of $I$ and $\operatorname{in}(I)$ coincide. It is also part of folklore that these coincide with what one can call extremal local cohomologies by analogy. We will represent on a typical picture what this means and some comparison between graded Betti numbers and dimensions of graded pieces of local cohomology modules. For a finitely generated graded module $M$, we set $\beta'_{i,j} := \dim_k \operatorname{Tor}_i^R(M,k)_{j+i}$ and $\alpha'_{i,j} := \dim_k H^i_{\mathfrak{m}}(M)_{j-i}$ and put in a table, indexed by $i$ and $j$ the numbers $\beta'_{i,j}$ and $\alpha'_{i,j}$. Then the tables have the following shapes for both $M$ and its generic initial ideal:

Betti numbers:

| | | $c-1$ | $c$ | $c+1$ | $\cdots$ | $\cdots$ | $i$ | $\cdots$ | $\cdots$ | $p$ | $p+1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | $\cdots$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\operatorname{reg}+1$ | $\cdots$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\operatorname{reg}$ | $\cdots$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\operatorname{reg}-1$ | $\cdots$ | $*$ | $*$ | $\square$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $\square$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $\boxplus$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 | 0 | 0 | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\boxplus$ | $\boxtimes$ | $\boxtimes$ | $\blacksquare$ | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\square$ | 0 |
| | $\cdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $\square$ | 0 |
| | | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

with $p := \operatorname{pdim}(M)$ and $c := \operatorname{codim}(M)$.

Local cohomology:

$$
\begin{array}{lcccccccccccc}
 & q-1 & q & \cdots & \cdots & i & \cdots & \cdots & d-1 & d & d+1 & \\
 & & & & & & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathrm{reg}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\mathrm{reg} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & \boxtimes & 0 & \cdots \\
\mathrm{reg}-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & \blacksquare & \boxtimes & \boxtimes & \boxplus & * & 0 & \cdots \\
 & 0 & \blacksquare & \boxtimes & \boxtimes & \boxplus & * & * & * & * & 0 & \cdots \\
 & 0 & \square & * & * & * & * & * & * & * & 0 & \cdots \\
 & 0 & \square & * & * & * & * & * & * & * & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

with $q := \operatorname{depth}(M) = n - p$ and $d := \dim(M) = n - c$.

The numbers at the spots marked by $\blacksquare$ are not zero and called the corners of the Betti diagram. Notice that $i, j$ is a corner of the Betti diagram if and only if $n - i, j$ is a corner of the local cohomology diagram.

By the result of Bayer, Charalambous, and Popescu, they are unchanged when passing to the initial ideal for the rev-lex order in general coordinates.

The numbers at the spots marked by $\blacksquare$, $\boxtimes$, $\boxplus$, or $\square$ in both diagrams are related by $\tilde{\alpha}'_{i,j} \leq \beta'_{i,j} \leq \tilde{\alpha}'_{i,j} + \sum_{\ell>0} \binom{n}{\ell} \alpha'_{i-\ell,j}$, where $\tilde{\alpha}'_{i,j} := \dim_k(\operatorname{Socle}(H^i_{\mathfrak{m}}(M))_{j-i})$. This shows that:

- at spots marked by $\blacksquare$: $\alpha'_{i,j} = \beta'_{i,j} \neq 0$,
- at spots marked by $\boxtimes$: $\alpha'_{i,j} \leq \beta'_{i,j} \leq \sum_{\ell \geq 0} \binom{n}{\ell} \alpha'_{i,j-\ell}$,
- at spots marked by $\square$: $\beta'_{i,j} = \tilde{\alpha}'_{i,j}$,
- at spots marked by $\boxplus$: $\tilde{\alpha}'_{i,j} \leq \beta'_{i,j} \leq \tilde{\alpha}'_{i,j} + \sum_{\ell>0} \binom{n}{\ell} \alpha'_{i-\ell,j}$.

These facts come from the study of the spectral sequence $\oplus_j \operatorname{Tor}^R_{i+j}(H^j_{\mathfrak{m}}(M), k) \Rightarrow \operatorname{Tor}^R_i(M, k)$. See the notes of Schenzel's lectures in Barcelona in [Bar] for more details.

In another direction, Bermejo and Gimenez discovered that the Castelnuovo–Mumford regularity may also be computed from the initial ideal under weaker conditions on the genericity of the coordinates (see [BG1] and [BG2]), and that this genericity condition can be checked on the initial ideal as well. An algorithm based on this idea was implemented in singular to compute the regularity following this approach. This idea also led to other developments by Caviglia and Sbarra [CS].

The study of generic initial ideals for different orders is an active subject of research. A good introduction to the subject is the notes of Green from the summer school in Barcelona [Bar]. An important recent result is the determination by Conca and Sidman of the regularity of the generic initial ideal, for the lexicographic order, of smooth complete intersection curves in $\mathbf{P}^3$:

**Theorem 1.4.3** [CSi, 1.1] *Consider a smooth complete intersection curve in $\mathbf{P}^3$, intersection of two surfaces of degrees $a$ and $b$ with $a, b > 1$. Then the regularity of its generic initial ideal for the lexicographic order is equal to $\frac{a(a-1)b(b-1)}{2} + 1$ unless $a = b = 2$, in which case it is equal to 4.*

It is interesting to compare this value with the regularity of the lex-segment ideal associated to the complete intersection ideal, which is $\frac{a(a-1)b(b-1)}{2} + ab$. The difference is (relatively) small. Their proof relies on the use of Green's partial elimination ideals. The value of the regularity is governed by the first partial elimination ideal, which defines the singular points of a generic projection of the curve. This singular loci consists of $\frac{a(a-1)b(b-1)}{2}$ nodes.

An interesting result that they prove on the way is the following proposition for points:

**Theorem 1.4.4** *Let $I$ be the defining ideal of a set of $s$ points in sufficiently general position. Then the generic initial ideal for the lexicographic order is equal to the lex-segment ideal associated to $I$.*

For a precise statement on the "general position" condition and a generalization to other orders, see [CSi, 5.6].

The study of monomial ideals and their resolutions is a very active domain of research, with many links to combinatorics and many interesting recent results. We will not go further into this field in this short note.

## 1.5 On the Regularity of Tor Modules

A first important result is a theorem of Caviglia, who proved in [Cav] that $\mathrm{reg}(M \otimes_R N) \leq \mathrm{reg}(M) + \mathrm{reg}(N)$ if $\dim \mathrm{Tor}_1^R(M, N) \leq 1$. This work was a continuation of previous results of Conca and Herzog [CH] and of Sidman [Si].

The regularity of Tor modules was subsequently studied in detail by Eisenbud, Huneke, and Ulrich in [EHU], where they prove a result [EHU, 2.3] which is quite comparable to the following one:

**Theorem 1.5.1** *Let $M, N$ be two finitely generated graded $R$-modules, set $p := \operatorname{pdim}(M)$ and $p' := \operatorname{pdim}(N)$. Assume that $\dim \operatorname{Tor}_1^R(M, N) \leq 1$. Then,*

$$\max_i\{\operatorname{reg}(\operatorname{Tor}_i^R(M, N)) - i\} \leq \operatorname{reg}(M) + \operatorname{reg}(N).$$

*Moreover, equality holds if either $\operatorname{reg}(M) = \max_{i=p,p-1}\{b_i(M)-i\}$ or $\operatorname{reg}(N) = \max_{i=p',p'-1}\{b_i(N) - i\}$. This is in particular the case if either $\operatorname{pdim} M \leq \operatorname{codim} M + 1$ or $\operatorname{pdim} N \leq \operatorname{codim} N + 1$.*

*Proof* Let $F_\bullet^N$ be a minimal graded free $R$-resolution of $N$. We apply Lemma 1.2.1 to $F^\bullet := F_{r-\bullet}^N$ and $M$. It follows from Lemma 1.2.1 (i) with $q_0 := r$ and the first claim of Lemma 1.2.1 (ii) that

$$\max_{p,q}\{a_p(\operatorname{Tor}_{r-q}(M, N)) + p + q\} \leq \max_{p,q}\{a_p(M) + b_{r-q}(N) + p + q\},$$

which is equivalent to

$$\max_{p,q}\{a_p(\operatorname{Tor}_q(M, N))) + p - q\} \leq \max_{p,q}\{a_p(M) + b_q(N) + p - q\},$$

and the right-hand side is equal to $\operatorname{reg}(M) + \operatorname{reg}(N)$.

Let us assume that $\operatorname{reg}(N) = \max_{i=p',p'-1}\{b_i(N) - i\}$, that we can rewrite $b(F^\bullet) = \max\{b(F^0), b(F^1) + 1\}$ (by Corollary 1.2.2, this equality holds if $\operatorname{pdim} N \leq \operatorname{codim} N + 1$). The second statement of Lemma 1.2.1 (ii) implies the equality we claim.

If $\operatorname{reg}(M) = \max_{i=p,p-1}\{b_i(M) - i\}$, we reverse the roles of $M$ and $N$ in the above proof. ∎

Caviglia was probably the first to give an example, in his thesis (see [EHU, 4.4]), where $\operatorname{reg}(M \otimes_R N) > \operatorname{reg}(M) + \operatorname{reg}(N)$ when $\dim(\operatorname{Tor}_1^R(M, N)) = 2$. We will explain this example in a more general context in Section 1.13.

Remark 1.4.1 applied to $M = R$ gives bounds on the regularity of all Koszul homology modules when $\dim R/(f_1, \ldots, f_s)$ is at most 1. The same kind of arguments are used in [Ch2, 3.1] to show the following:

**Theorem 1.5.2** *Let $M, M_1, \ldots, M_s$ be finitely generated graded $R$-modules, $T_i := \operatorname{Tor}_i^R(M, M_1, \ldots, M_s)$, $d := \dim M$, and $b_\ell := \max_{i_1+\cdots+i_s=\ell}\{b_{i_1}(M_1) + \cdots + b_{i_s}(M_s)\}$. If $\dim T_1 \leq 1$, then*

(i) $a_p(T_0) \leq \max_{0\leq i\leq d-p}\{a_{p+i}(M) + b_i\}$ *for* $p \geq 0$,
(ii) $a_0(T_q) \leq \max_{0\leq i\leq d}\{a_i(M) + b_{q+i}\}$ *for* $q \geq 0$,
(iii) $a_1(T_q) \leq \max_{0\leq i\leq d}\{a_i(M) + b_{q+i-1}\}$ *for* $q \geq 1$.

*In particular, if* $\dim(M_1 \otimes \cdots \otimes M_s) \leq 1$, *then*

$$\mathrm{reg}(M_1 \otimes \cdots \otimes M_s) \leq \mathrm{reg}(M_1) + \cdots + \mathrm{reg}(M_s).$$

The last inequality may be extended to all the higher multiple Tor modules, and the condition may be weakened to $\dim(\mathrm{Tor}_1^R(M_1, \ldots, M_s)) \leq 1$. The vanishing of the latter module may be controlled by the following result, which is the natural generalization for multiple Tor modules of a result of Serre (recall that the formation of Tor commutes with localization):

**Theorem 1.5.3** *Let $R$ be a regular local ring containing a field, $M_1, \ldots$; let $M_s$ be finitely generated $R$-modules. The following are equivalent,*

(i) $\mathrm{Tor}_1^R(M_1, \ldots, M_s) = 0$ *and $M_1 \otimes_R \cdots \otimes_R M_s$ is Cohen–Macaulay,*
(ii) *the codimension of $M_1 \otimes_R \cdots \otimes_R M_s$ is the sum of the projective dimensions of the $M_i$'s,*
(iii) *the intersection of the $M_i$'s is proper and every $M_i$ is Cohen–Macaulay.*

The form of the bound in Theorem 1.5.2 is also important to notice. Let us look at the case where we have two modules $M$ and $N$. We then have, for instance,

$$a_0(M \otimes_R N) \leq \max_i\{a_i(M) + b_i(N)\},$$

and this max can be reduced to the range depth $(M) \leq i \leq \min\{\dim(M), \mathrm{pdim}(N)\}$. Notice also that the roles of $M$ and $N$ may be reversed.

## 1.6 The Behavior of Regularity Relative to Sums, Products, and Intersections of Ideals

The behavior relative to these three operations are all related to the study of Tor modules. Indeed, for any pair $I$, $J$ of ideals, $R/(I + J) = R/I \otimes_R R/J = \mathrm{Tor}_0^R(R/I, R/J)$, and there are two exact sequences:

$$0 \longrightarrow R/(I \cap J) \longrightarrow R/I \oplus R/J \longrightarrow \mathrm{Tor}_0^R(R/I, R/J) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Tor}_1^R(R/I, R/J) \longrightarrow R/IJ \longrightarrow R/(I \cap J) \longrightarrow 0.$$

It follows that:

- $\operatorname{reg}(I+J) > \max\{\operatorname{reg}(I), \operatorname{reg}(J)\}$ if and only if $\operatorname{reg}(I\cap J) > \max\{\operatorname{reg}(I), \operatorname{reg}(J)\} + 1$, and in this case
$$\operatorname{reg}(I \cap J) = \operatorname{reg}(I + J) + 1 = \operatorname{reg}(\operatorname{Tor}_0^R(R/I, R/J)) + 2,$$
- one has
$$\begin{aligned}\operatorname{reg}(R/IJ) &\leq \max\{\operatorname{reg}(R/(I \cap J)), \operatorname{reg}(\operatorname{Tor}_1^R(R/I, R/J))\}\\ &\leq \max\{\operatorname{reg}(R/I), \operatorname{reg}(R/J), \operatorname{reg}(\operatorname{Tor}_0^R(R/I, R/J)) + 1,\\ &\quad \operatorname{reg}(\operatorname{Tor}_1^R(R/I, R/J))\}.\end{aligned}$$

Using the results on Tor, it immediately follows that

**Theorem 1.6.1** *If $(I \cap J)/IJ$ is a module of dimension at most 1, then*

(i) $\operatorname{reg}(I + J) \leq \operatorname{reg}(I) + \operatorname{reg}(J) - 1$,
(ii) $\operatorname{reg}(I \cap J) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$,
(iii) $\operatorname{reg}(IJ) \leq \operatorname{reg}(I) + \operatorname{reg}(J)$.

More refined results in terms of Betti numbers can be found in [EHU, 2.2].

The possibility of extending the second and third inequalities to any number of ideals is still unclear. For the first one, the previous results on multiple Tor modules gives such an extension.

Notice that the condition $\dim(I\cap J)/IJ \leq 1$ is implied by — and equivalent to in some cases, for instance if $\dim(R/I + J) = 2$ — a more geometric one: $\forall \mathfrak{p} \supseteq I+J$ such that $\dim(R/\mathfrak{p}) \geq 2$, $(R/I)_\mathfrak{p}$ and $(R/J)_\mathfrak{p}$ are Cohen–Macaulay and $\operatorname{codim}_{R_\mathfrak{p}}(I_\mathfrak{p} + J_\mathfrak{p}) = \operatorname{codim}_{R_\mathfrak{p}}(I_\mathfrak{p}) + \operatorname{codim}_{R_\mathfrak{p}}(J_\mathfrak{p})$. In other terms: locally at primes of dimension 2 containing $I + J$ it corresponds to a proper intersection of Cohen–Macaulay schemes.

**Theorem 1.6.2** *Let $I_1, \ldots, I_s$ be graded $R$-ideals and $J$ be their sum. If $\forall \mathfrak{p} \supseteq J$ such that $\dim(R/\mathfrak{p}) \geq 2$, $(R/I_i)_\mathfrak{p}$ is Cohen–Macaulay for all $i$ and $\operatorname{codim}_{R_\mathfrak{p}}(J_\mathfrak{p}) = \sum_i \operatorname{codim}_{R_\mathfrak{p}}((I_i)_\mathfrak{p})$, then*
$$\operatorname{reg}(R/J) \leq \sum_i \operatorname{reg}(R/I_i).$$

Questions of this type were also studied in particular contexts.

- Conca and Herzog proved in [CH] that if ideals $I_1, \ldots, I_s$ are generated by linear forms, then
$$\operatorname{reg}(I_1 \cdots I_s) = \sum_i \operatorname{reg}(I_i) = s,$$

- Derksen and Sidman proved in [DS] that if ideals $I_1, \dots, I_s$ are generated by linear forms, then

$$\operatorname{reg}(I_1 \cap \cdots \cap I_s) \leq \sum_i \operatorname{reg}(I_i) = s.$$

In their article, Conca and Herzog ask if their result may be extended to an inequality for complete intersection ideals. It is shown in [CMT, 3.3] that for monomial complete intersection ideals one has

$$\operatorname{reg}(I_1 \cap \cdots \cap I_s) \leq \sum_i \operatorname{reg}(I_i),$$

and the same holds for the product if $s = 2$ by [CMT, 1.1]. Also it follows from a lemma from Hoa and Trung [HT, 3.1] that

$$\operatorname{reg}(I_1 \cdots I_s) \leq \sum_i \operatorname{reg}(I_i) + \sum_i \operatorname{codim}(I_i) - \operatorname{codim}(I_1 \cdots I_s) - s + 1,$$

which is close to the expected bound in this monomial case.

These bounds do not hold for general complete intersection ideals; a geometric approach for constructing counter-examples is given by the following result [CMT, 1.2]:

**Theorem 1.6.3** *Let $\mathcal{C}$ in $\mathbf{P}^3$ be a curve which is defined by at most four equations at the generic points of its irreducible components. Consider four elements in $I_{\mathcal{C}}$, $f_1, f_2, g_1, g_2$ such that $I := (f_1, f_2)$ and $J := (g_1, g_2)$ are complete intersection ideals and $I_{\mathcal{C}}$ is the unmixed part of $I + J$. Then, if $-\eta := \min\{\mu \mid H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(\mu)) \neq 0\} < 0$, one has*

$$\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J) + \eta - 1.$$

We will see that one can choose for $\mathcal{C}$ the locally complete intersection curve with $I_{\mathcal{C}} := (x^m t - y^m z) + (z, t)^n$ for $m, n > 1$, in which case $\eta = (m-1)(n-1)$, and take, for instance, $I := (z^n, t^n)$ and $J := (x^m t - y^m z, f)$ with $f \in I_{\mathcal{C}}$ not multiple of $x^m t - y^m z$ (e.g., $f = z^n$).

---

## 1.7 The Regularity of the Ordinary Powers of an Ideal

Applying Theorem 1.5.2 with $M := R/I^m$ and $M_1 := R/I$, so that $T_1 = \operatorname{Tor}_1^R(R/I, R/I^m) \simeq I^m/I^{m+1}$ for a homogeneous ideal $I$ with $\dim(R/I) \leq 1$, one gets the following estimate (see [Ch2, 0.4], or [EHU, 7.9] for a slightly different bound) that improves the estimate $\mathrm{reg}(I^m) \leq m\,\mathrm{reg}(I)$ proved by Chandler in [Chan] and by Geramita, Gimigliano, and Pitteloud in [GGP]:

**Theorem 1.7.1** *Let $I$ be a homogeneous ideal of $R$ such that* $\dim(R/I) \leq 1$. *Then, for any $m \geq 1$,*

$$\mathrm{reg}(I^m) \leq \max\{\mathrm{reg}(I^{sat}) + b_1(I) - 1, \mathrm{reg}(I) + b_0(I)\} + (m-2)b_0(I);$$

*in particular, unless $I$ is principal,* $\mathrm{reg}(I^m) \leq \mathrm{reg}(I) + b_1(I) - 1 + (m-2)b_0(I)$.

When $\dim(R/I) \geq 2$, the inequality $\mathrm{reg}(I^m) \leq m\,\mathrm{reg}(I)$ does not hold in general. A first counter-example was given by Terai, in characteristic different from 2: the Stanley–Reisner ideal of the minimal triangulation of the real projective plane. This ideal is a monomial ideal with 10 minimal generators of degree 3. Further investigations by Sturmfels in [St] showed that any monomial ideal $M$ with at most 7 generators, that has a linear resolution, is such that $M^2$ also has a linear resolution, which is equivalent to $\mathrm{reg}(M^2) = 2\,\mathrm{reg}(M)$. On the other hand, Sturmfels exhibited a monomial ideal with 8 generators for which $\mathrm{reg}(M^2) > 2\,\mathrm{reg}(M)$, in any characteristic.

Also, reasonable bounds for the regularity of the square of an ideal of dimension 2 may be proved for generically complete intersection ideals:

**Theorem 1.7.2** [Ch2, 0.5] *Let $I$ be a homogeneous $R$-ideal such that* $\dim(R/I) = 2$. *Assume that $I_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is a complete intersection for every prime $\mathfrak{p} \supseteq I$ such that* $\dim R/\mathfrak{p} = 2$. *Then,*

$$\begin{aligned}\mathrm{reg}(I^2) &\leq \max\{\mathrm{reg}(I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}, a_2(R/I) \\ &\quad + 2b_0(I) + 1\} \\ &\leq \mathrm{reg}(I) + \max\{\mathrm{reg}(I), 2b_0(I) - 2\}.\end{aligned}$$

**Question 1.7.3** *Does the inequality*

$$\mathrm{reg}(I^2) \leq \mathrm{reg}(I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}$$

*hold under the hypotheses of the theorem?*

By [Ch2, 4.3], this is equivalent to $a_2(\mathrm{Tor}_2(R/I, R/I)) \leq \mathrm{reg}(R/I) + \max\{b_0(I), b_1(I) - 1, b_2(I) - 2\}$. Of course one may ask if the weaker bound $\mathrm{reg}(I^2) \leq 2\,\mathrm{reg}(I)$ holds, which is equivalent to $a_2(\mathrm{Tor}_2(R/I, R/I)) \leq 2\,\mathrm{reg}(R/I) + 1$.

In arbitrary dimension, Swanson first proved in [Sw1] that for any graded ideal $I$, there exists $N$ such that $\mathrm{reg}(I^m) \leq mN$ for any $m$. Later, the asymptotic behavior of $\mathrm{reg}(I^m)$ was proved to be a linear function of $m$ by Kodiyalam in [Ko] and by Cutkosky, Herzog, and Trung in [CHT]:

**Theorem 1.7.4** *Let $J_d$ be a graded $R$-ideal. There exists* $\mathrm{indeg}(J_d) \leq a \leq b_0(J_d)$*, $b$ and $c$ such that*

$$\mathrm{reg}(I^m) = am + b, \quad \forall m \geq c.$$

The key point in the proof is the fact that the Rees algebra $\mathcal{R}_I$ has bigraded finite free resolution, which encodes the regularity of all the powers of $I$. The same type of behavior also holds for the integral closures of the powers of $I$ or for the symmetric powers of an ideal.

The numbers $b$ and $c$ can be estimated from the shifts in the bigraded resolution of the Rees algebra (see [CHT, 2.4]).

On the other hand, the regularity of saturations of the powers of an ideal may have a very irregular behavior, as shown by examples in [CHT] based on previous constructions of Cutkosky and Srinivas [CSr], and examples by Cutkosky [Cu].

***Example 1.7.5*** [CHT, 4.4, Cu Thm. 10]

- For any prime $p$ congruent to 2 mod 3, there exists a field of characteristic $p$ and an ideal $I$ in $k[x, y, z]$ such that $\mathrm{greg}(R/I^{5m+1}) = 29m + 6$ if $m$ is not a power of $p$ and $\mathrm{greg}(R/I^{5m+1}) = 29m + 7$ else.
- In arbitrary characteristic, there exists a regular curve in $\mathbf{P}^3$ such that its defining ideal $I$ satisfies $\mathrm{greg}(R/I^m) = [(9 + \sqrt{2})m] + \sigma(m)$, where $\sigma(m)$ is 1 unless $m$ belongs to a sparse subsequence $q_n$ of the integers defined recursively by $q_0 = 1$, $q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$, in which case $\sigma(m) = 0$.

Further work by Cutkosky, Ein, and Lazarsfeld in [CEL] gives a geometric approach to the understanding of the asymptotic behavior of $\frac{\mathrm{greg}(R/I^m)}{m}$ and other invariants of these powers (notably their arithmetic degree).

When the symbolic blowup is finitely generated, the asymptotic behavior is given by a finite number of linear functions, each of them corresponding to a congruence of the exponent. If the symbolic blowup is finite over the Rees ring, the regularity is eventually linear.

A recent example due to Conca also shows a very interesting phenomenon:

***Example 1.7.6*** [Co, 3.1] Let $d > 1$ and $J_d := (xz^d, xt^d, yz^{d-1}t) + (z, t)^{d+1} \subset k[x, y, z, t]$, for any field $k$. Then $\mathrm{reg}(I^m) = m(d + 1)$ (i.e., $I^m$ has a linear resolution) for $m < d$ and $b_1(J^d) \geq d(d + 2)$. It follows that $\mathrm{reg}(I^d) \geq d(d + 2) - 1 > d(d + 1)$.

Therefore, even for a monomial ideal in four variables, an arbitrary number of powers may have a linear resolution without forcing all the powers to verify the same property. The article of Conca [Co] contains in its Section 2 an interesting collection of other examples from different sources.

## 1.8 Geometric Estimates on Castelnuovo–Mumford Regularity

The first estimates for the Castelnuovo–Mumford regularity are probably:

- the bound for the regularity of smooth projective curves by Castelnuovo,
- the bound for the regularity of schemes in terms of their Hilbert polynomial by Mumford.

Both results have been extended and better understood in later works, and Mumford's technique was adapted to prove regularity bounds in terms of degrees of defining equations.

In the direction of Castelnuovo's result, there is a famous conjecture that suggests the following bound for reduced and irreducible schemes:

**Conjecture 1.8.1** [Eisenbud and Goto] *If $\mathcal{S}$ is a nondegenerate reduced and irreducible projective scheme over an algebraically closed field, then*

$$\operatorname{reg}(\mathcal{S}) \leq \deg \mathcal{S} - \operatorname{codim} \mathcal{S}.$$

(Nondegenerate means $\mathcal{S} \not\subset H$ for any hyperplane $H$.)

We recall that if $\mathcal{S} = \operatorname{Proj}(R/I)$, $\operatorname{reg}(\mathcal{S}) := \operatorname{reg}(R/I^{sat}) = \operatorname{greg}(R/I)$.

This result was known for curves when the conjecture was made. It was first established for smooth curves by Castelnuovo [Cast] and then for reduced curves with no degenerate component by Gruson, Lazarsfeld, and Peskine (over a perfect field) in [GLP]. Recently, Noma improved the bound for curves of sufficiently high genus in [No1] and [No2]. There has been a lot of work on regularity of curves, in particular on monomial curves (for instance, L'vovsky's bound [Lv]) and on the cases where the bound is close to being reached (see for instance [BSc]).

There is some evidence that the Eisenbud-Goto conjecture should be true at least for smooth schemes in characteristic zero: it is true for smooth surfaces (Pinkham and Lazarsfeld) and (up to adding small constants) in dimension at most six, by the work of several people including Lazarsfeld, Ran, and Kwak (see [Kw] and the articles it refers to).

**Theorem 1.8.2** [Kw] *Let $k$ be a field of characteristic zero and $\mathcal{S}$ a nondegenerate smooth irreducible projective scheme over $k$ of dimension $D \leq 6$, then*

$$\operatorname{reg}(\mathcal{S}) \leq \deg \mathcal{S} - \operatorname{codim} \mathcal{S} + \epsilon_D,$$

*with $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = 1$, $\epsilon_4 = 4$, $\epsilon_5 = 10$ and $\epsilon_6 = 20$.*

A key point in these proofs is the understanding of the singularities of the fiber of a general projection of $\mathcal{S}$ to a linear space of dimension $D + 1$. The fundamental results of J. Mather only give enough information on the fibers up to dimension 6.

In any dimension, it was proved by Mumford ([BM]) — and it also follows easily from a theorem of Bertram, Ein, and Lazarsfeld (Theorem 1.10.1) — that in characteristic zero a smooth scheme $\mathcal{S}$ satisfies,

$$\operatorname{reg}(\mathcal{S}) \leq (\dim \mathcal{S} + 1)(\deg \mathcal{S} - 1).$$

In positive characteristic, one has by [Ch1, 4.5] $\operatorname{reg}(\mathcal{S}) \leq \dim \mathcal{S}(\dim \mathcal{S} + 1)(\deg \mathcal{S} - 1)$ (Theorem 1.10.3 gives a slightly weaker result). There are also quite reasonable results for schemes with isolated singularities (see [Ch1, §4]).

The conjecture is known for some classes of toric varieties. In codimension two, the result was proved by Peeva and Sturmfels in [PS] and there are several results in this direction for classes of toric varieties (see for instance [HS] for the case of simplicial toric rings).

## 1.9 General Bounds on the Regularity in Terms of Degrees of Defining Equations

A general bound in terms of degrees of defining equations and in terms of the Hilbert polynomial (for saturated ideals) may be derived from the following proposition that studies the behavior when adding a sufficiently general linear form. This result goes back to Mumford ([Mu, Lect. 14]) for the essential key points and has been used in several forms or variants since then. What Mumford proved is very close to points (iv) and (v) below. Set $h^i_{\mathfrak{m}}(M)_\mu := \dim_k H^i_{\mathfrak{m}}(M)_\mu$.

**Proposition 1.9.1** *Let $M$ be a finitely generated graded $R$-module. For a linear form $l$ such that $K := 0 :_M (l)$ has finite length, set $\overline{M} := M/(l)M$. Then,*

(i) $\operatorname{greg}(M) \leq \operatorname{reg}(\overline{M}) \leq \operatorname{reg}(M)$,

(ii) *for $\mu \geq \operatorname{reg}(\overline{M})$, $h^0_{\mathfrak{m}}(M)_{\mu+1} \leq h^0_{\mathfrak{m}}(M)_\mu$, and the inequality is strict if $h^0_{\mathfrak{m}}(M)_\mu \neq 0$ and $\mu \geq \max\{\operatorname{reg}(\overline{M}) + 1, b_0(M), b_1(M) - 1\}$,*

(iii) *for* $\mu \geq \operatorname{greg}(M)+2$, $(M \otimes_R k)_\mu \simeq (H^0_{\mathfrak{m}}(M) \otimes_R k)_\mu$, *in particular* $b_0(H^0_{\mathfrak{m}}(M)) \leq \max\{\operatorname{greg}(M)+1, b_0(M)\}$,

(iv) *for* $\mu \geq \max\{\operatorname{greg}(\overline{M}), a_0(M)\}$, $h^1_{\mathfrak{m}}(M)_{\mu+1} \leq h^1_{\mathfrak{m}}(M)_\mu$, *and the inequality is strict if* $h^1_{\mathfrak{m}}(M)_\mu \neq 0$ *and* $\mu \geq \max\{\operatorname{greg}(\overline{M})+1, a_0(M), b_0(\overline{M})\}$,

(v) *for* $\mu' \geq \mu \geq \min\{a_0(M), b_0(M)\}$, $\{H^i_{\mathfrak{m}}(M)_{\mu-i} = 0,\ \forall i\} \Rightarrow \{H^i_{\mathfrak{m}}(M)_{\mu'-i} = 0,\ \forall i\}$.

The proof is given in the technical appendix, Section 1.14.

**Corollary 1.9.2** *Let* $I$ *be a homogeneous ideal and* $l$ *be a linear form such that* $K := (I : (l))/I$ *has finite length, then for* $\mu \geq \max\{b_0(I)-1, \operatorname{reg}(I+(l))\}$,

$$\operatorname{reg}(I) \leq \mu + \lambda(H^0_{\mathfrak{m}}(R/I)_\mu) = \mu + \lambda(K_{\geq\mu}) \leq \mu + \lambda(K).$$

*Proof* Let $\mu \geq \max\{b_0(I)-1, \operatorname{reg}(I+(l))\}$. By (i), $\operatorname{greg}(R/I) \leq \mu-1$, and by (ii), $h^0_{\mathfrak{m}}(R/I)_{\mu+1} < h^0_{\mathfrak{m}}(R/I)_\mu$ if $h^0_{\mathfrak{m}}(R/I)_\mu \neq 0$. It follows that $h^0_{\mathfrak{m}}(R/I)_{\mu+i} \leq \max\{0, h^0_{\mathfrak{m}}(R/I)_\mu - i\}$, hence $a_0(R/I) \leq \mu + h^0_{\mathfrak{m}}(R/I)_\mu$. Finally notice that $\lambda(K_{\geq\mu}) = \sum_{\mu'\geq\mu}(h^0_{\mathfrak{m}}(R/I)_{\mu'} - h^0_{\mathfrak{m}}(R/I)_{\mu'+1}) = h^0_{\mathfrak{m}}(R/I)_\mu$. ∎

This corollary may be used directly to prove bounds on the regularity in terms of defining equations, by recursion on the dimension. The point is then to bound $\lambda(H^0_{\mathfrak{m}}(R/I)_\mu)$, for which one may use that $H^0_{\mathfrak{m}}(R/I)_\mu \subseteq (R/I)_\mu$.

A more refined way for bounding $\lambda(H^0_{\mathfrak{m}}(R/I)_\mu)$ was found by Caviglia and Sbarra [CS]. The following lemma is a key ingredient of their proof:

**Lemma 1.9.3** [CS, 2.2] *If* $(I : (l))/I$ *has finite length, for any* $j > 0$,

$$\lambda\left(\frac{I:(l)^j}{I:(l)^{j-1}}\right) - \lambda\left(\frac{I:(l)^{j+1}}{I:(l)^j}\right) = \lambda\left(\frac{I:(l)^j+(l)}{I:(l)^{j-1}+(l)}\right).$$

Let us now sketch a variant of their proof and of their result.

They first remark that $I^{sat} = I : (l)^N$ for $N \geq a_0(R/I)$ and that by the lemma above, $K$ has the same length as $(I^{sat}+(l))/(I+(l))$ (sum up the equalities in the lemma for $j$ from 1 to $N$).

The lemma also gives $\lambda\left(\frac{I:(l)^{j+1}}{I:(l)^j}\right) \leq \lambda\left(\frac{I:(l)}{I}\right)$ for any $j > 0$, and therefore

$$\lambda\left(I^{sat}/I\right) = \sum_{j=1}^{a_0(R/I)} \lambda\left(\frac{I:(l)^j}{I:(l)^{j-1}}\right) \leq a_0(R/I)\lambda\left(\frac{I:(l)}{I}\right).$$

Also, for a linear form $l'$ such that $(I+(l)):(l')/(I+(l))$ has finite length,

$$\lambda\left(\frac{I:(l)}{I}\right) = \lambda\left(\frac{I^{sat}+(l)}{I+(l)}\right) \leq \lambda\left(\frac{(I+(l))^{sat}}{I+(l)}\right) \leq a_0(R/I + (l))\lambda\left(\frac{(I+(l)):(l')}{(I+(l))}\right).$$

Therefore, setting $I_i := I+(l_1,\ldots,l_i)$ and $K_i := I_i : (l_{i+1})/I_i$ for a sequence of linear forms $l_i$ such that $\lambda(K_i) < \infty$ for every $i$, one has $\dim(R/I_i) = \max\{0, \dim(R/I) - i\}$ and:

- $\operatorname{reg}(R/I_i) \leq \max\{\operatorname{reg}(R/I_{i+1}), d-2\} + \lambda(K_i)$,
- $\lambda(K_i) \leq \operatorname{reg}(R/I_{i+1})\lambda(K_{i+1})$,

which gives a way to bound $\operatorname{reg}(R/I_i)$ by recursion on $i$ from $\delta := \dim(R/I)$ to 0. Indeed for $I$ generated in degree at most $d$:

- for $i=\delta$, $\dim(R/I_\delta) = 0$ and therefore $\operatorname{reg}(R/I_\delta) \leq (n-\delta)(d-1)$ by Theorem 1.12.4 (this is well known and goes back to Macaulay, at least) and $\lambda(K_\delta) = \lambda(R/I_{\delta+1}) \leq d^{n-\delta-1}$,
- if $\delta \geq 1$, for $i=\delta-1$, $\dim(R/I_{\delta-1}) = 1$, $\operatorname{reg}(R/I_{\delta-1}) \leq (n-\delta+1)(d-1)$ by Theorem 1.12.4 and $\lambda(K_{\delta-1}) = \lambda(R/I_\delta) - \deg(R/I_{\delta-1}) \leq d^{n-\delta} - 1$.

And then by recursion it follows:

**Theorem 1.9.4** *If $I$ is a graded $R$-ideal generated in degree at most $d$,*

(i) $\operatorname{reg}(R/I) \leq n(d-1)$ *if* $\dim(R/I) \leq 1$,

(ii) *if* $\delta := \dim(R/I) \geq 2$,

$$\operatorname{reg}(R/I) \leq ((n-\delta+1)(d-1)d^{(n-\delta)})^{2^{\delta-2}} < (3^{\frac{1}{3}}\, d)^{(n-\delta+1)2^{\delta-2}}.$$

The bound for ideals can be extended to modules, essentially along the same lines (see [CFN]) or using some other variations (see [BGö]).

Point (iv) of Proposition 1.9.1 can be used to bound $\operatorname{greg}(R/I)$ in terms of the Hilbert polynomial $P$ of $R/I$ (this was the motivation of Mumford), as for $\mu \geq \operatorname{greg}(\overline{R/I})$ one has $P(\mu) = h^1_{\mathfrak{m}}(R/I)_\mu + \dim_k(R/I)_\mu$. This last formula follows from the equality (see for instance [BH, 4.3.5]):

$$\dim_k M_\mu = P_M(\mu) + \sum_{i\geq 0}(-1)^i h^i_{\mathfrak{m}}(M)_\mu$$

which is valid for any graded $R$-module $M$ of finite type and any $\mu$.

Another way of proving bounds in terms of the Hilbert polynomial is to remark that the regularity of the Hilbert function of $R/I^{sat}$ is strictly smaller than the one of the lex-segment ideal associated to $I^{sat}$; as a corollary, the regularity of this lex-segment ideal only depends on the Hilbert polynomial. The corresponding bound can be computed from the standard writing of the Hilbert polynomial by formulas first proved by Blancafort in her thesis (see [Bl]).

A bound on the regularity of lex-segment ideals associated to complete intersection ideals was proved in [CM], and improved and extended by Hoa and Hyry in [HoHy2], as an ingredient for proving bounds for the degrees of generators of Gröbner bases of an ideal in terms of the degrees of its generators, for any admissible order (see [CM, 3.6]).

Point (v) of Proposition 1.9.1 may be used to prove that:

**Proposition 1.9.5** *If $I$ is a graded ideal which is not $\mathfrak{m}$-primary, then*

$$\operatorname{reg}(I) = \min\{\mu \mid H^i_{\mathfrak{m}}(R/I)_{\mu-i} = 0,\ \forall i\}.$$

*Proof* Recall that if $d := \dim(R/I) > 0$, one has $H^d_{\mathfrak{m}}(R/I)_i \neq 0$ for any $i \leq d$ (see [Ho] or [BH, 9.2.4 (b)]), hence the minimum on the right is positive, and the claim follows from (v). ∎

One of the motivations for looking at regularity of ideals was the theorem of Bayer and Stillman, which asserts that for the rev-lex order and in general, coordinates $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$, where $\operatorname{in}(I)$ is the initial ideal of $I$ (generated by the leading monomials of the elements in the Gröbner basis).

Bounds in terms of the maximal degree of the generators were expected to be of much smaller order than the ones known at that time, which were of the same magnitude as the bound we proved above (at least in characteristic zero).

A big surprise came with the example of Mayr and Meyer, who provided an ideal in a polynomial ring in $10n + 2$ variables generated by polynomials of degrees at most $d + 2$ with a minimal first syzygy of degree at least $d^{2^n}$ (see [BS2]).

A very interesting study of the ideals of Mayr and Meyer, and of some closely related ideals, was done by Irena Swanson in [Sw2] and [Sw3]. It shows that these ideals have minimal primes and embedded primes in different codimensions and points to some embedded ideals that might be at the origin of their high regularity.

It is interesting to notice that these types of binomial ideals are still the unique source of examples of ideals with very high regularity. We will see in Section 1.13 examples with a much more geometric construction, but not with such a high regularity. It would be very interesting to provide a geometric construction of ideals with huge multiplicity as compared to degrees of generators.