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To Martina, Andrea, Daniel, Stephan and Claudia

Contents

Preface

1. In these notes we present some introductory material on a particular class of dynamical systems, called SEMIFLOWS. This class includes, but is not restricted to, systems that are defined, or modelled, by certain types of differential equations of evolution (DEEs in short). Our purpose is to introduce, in a relatively self-contained manner, the basic results of the theory of dynamical systems that can be naturally extended and applied to study the asymptotic behavior of the solutions of the DEEs we consider. Equations of evolution include ordinary differential equations (ODEs in short), partial differential equations of evolution (PDEEs in short), and other types of equations, such as, for instance, stochastic or difference equations. As such, they provide natural examples of dynamical systems, since one of the independent variables (usually called "time") plays a different role than the other variables (which in some situations may be called "space" variables). Thus, in this context, the heat and wave equations are considered as prototypical examples of PDEEs, while elliptic equations such as Laplace's equation are not considered as evolution equations, because in these equations all the variables have the same role. Here, we make the further distinction that "time" evolves continuously; thus, we do not consider stochastic equations, nor, except for some introductory examples, discrete systems (where "time" varies along a sequence).

2. One of the major goals of the theory of dynamical systems is the study of the evolution of a system, with the purpose of predicting, as accurately as possible, the behavior of the system as time becomes large. A quite general feature of the systems we consider, which is shared with other systems, is a property called DISSIPATIVITY. Loosely speaking, this property can be described by the fact that all solutions of these systems eventually enter, and remain, in a bounded set, called ABSORBING SET. Thus, the evolution of the solutions of the system can be studied in this set; as a result, the long time behavior of the system can be described by means of certain subsets of the absorbing set. Among these, we shall consider three types of sets, called respectively ATTRACTORS, EXPONENTIAL ATTRACTORS, and INERTIAL MANIFOLDS. (Exponential attractors are sometimes also known as INERTIAL SETS.) We will present the fundamental properties of these sets, and then proceed to show the existence of some of these sets for a number of dynamical systems, generated by fairly well known physical models. In particular, we shall consider in full detail two particular PDEEs of evolution: a semilinear version of the heat equation, and a corresponding version of the dissipative wave equation. These examples allow us to illustrate the most important features of the theory of semiflows, and to provide a sort of "template" that can then be applied, in a more or less straightforward fashion, to the analysis of other models, with the help of the many specialized references that exist in the literature.

3. Even a quick survey of much of the existing literature on dynamical systems, both at the introductory and the specialized level, reveals that the notion of "dynamical system" is used with many different meanings, according to the specific point of view of the authors. At the opposite extreme, this notion may well be not defined at all. In these notes, we do not attempt to give a general definition of dynamical system; rather, we confine ourselves to a special class of systems, properly known as CONTINUOUS, SEMI-DYNAMICAL SYSTEMS, or CONTINUOUS SEMIFLOWS. Here, the term "continuous" is used to distinguish these systems from DISCRETE ones, where only a sequence of successive time values are considered, and "semi-" refers to the fact that time evolves, i.e. we only consider nonnegative values of the time variable. For brevity, we shall refer to these systems as SEMIFLOWS (their precise definition is given in section 2.2). In the introductory chapter 1, we consider more general TWO-PARAMETER SEMIFLOWS or DYNAMICAL PROCESSES, which allows us to include some nonautonomous difference or differential equations as generators of dynamical systems. However, when our presentation can proceed in a more discursive way, and rigor is not an issue, we conform to the common use and adopt the general term "dynamical system".

4. In general, we say that an ODE defines a semiflow if the corresponding CAUCHY PROBLEM is globally well posed, in the sense we define in section 1.2.1. We can extend this definition to semiflows defined by PDEEs, by interpreting the PDEE as an abstract ODE in a suitable Banach space $\mathcal{X}$ (see remark 3.2 in chapter 3). This is a generalization of the usual interpretation of a system of ODEs as a single differential equation in the Banach space $\mathcal{X} = \mathbb{R}^n$, or in more general finite dimensional vector spaces, and explains the qualification of the systems generated by PDEEs as "infinite dimensional" ones, since in this case $\mathcal{X}$ is in general no longer a finite dimensional space. Examples of PDEEs that can be put in such abstract form are: the Navier-Stokes equations, the Kuramoto-Sivashinski equations, the "original" Burger's equation, the Chafee-Infante and Cahn-Hilliard reaction-diffusion equations, the Korteweg-de Vries and the Maxwell equations (see chapter 6). Indeed, many basic notions and results in the theory of the asymptotic behavior of infinite dimensional dissipative dynamical systems trace their origin in the study of the Navier-Stokes equations of fluid dynamics, and have been inspired by a detailed analysis of both the qualitative properties of their solutions, and their behavior with respect to numerical computations.

5. Not surprisingly, much of the general terminology in the theory of dynamical systems, as well as the general spirit of its qualitative results, borrows directly from the qualitative theory of ODEs in $\mathbb{R}^n$. For example, we shall need to recall some basic results on stability, equilibrium points, periodic orbits, ω-limit sets, etc. On the other hand, in an effort to keep these notes within a reasonable length, we shall be forced to not discuss many other important topics. In particular, we regretfully do not include any result on bifurcation theory. Among the many excellent and fairly complete references on the qualitative theory of ODEs, including ODEs as dynamical systems, we refer for example to Hirsch and Smale, [HS93], Jordan and Smith, [JS87], Perko, [Per91], Amann, [Ama90], and Verhulst, [Ver90]. A few other references, specifically on dynamical systems, are listed in the bibliography. Since so many articles and books are continually being published, it is almost impossible to compile an exhaustive list of references; on the other hand, an internet search can provide all necessary updated references on any particular topic.

6. These notes have their origin in a series of graduate seminars we held at the Universities of Dresden, Wisconsin-Milwaukee and Tsukuba. Most of the material we cover is relatively well known, although some of the results we present, in particular on the existence of an exponential attractor and of an inertial manifold for semilinear dissipative wave equations, even if not entirely new, do not seem to enjoy the recognition we feel they deserve. In part, our intention in writing these notes is to be of some help to "beginners" in the area of infinite dimensional dynamical systems; that is, anyone who, having a solid background in the classical theory of ODEs and some knowledge of functional analysis in Sobolev spaces, wishes to proceed to the study of examples of semiflows arising from DEEs, but may need some "smoothing into" the subject, before turning to more general introductory texts, such as those of Temam, [Tem88], the cycle of lectures by Oleinik, [Ole96], or, most recently, Sell-You, [SY02], and Robinson, [Rob01]. We also hope that these notes may serve as a ready reference to researchers in more applied fields, who feel the need for a clear presentation of the background material and results that are necessary for the study of the specific systems they are interested in. To this end, we have tried to "build up" the material in as careful and gradual progression as possible, with the goal of presenting the main topics (in particular, the construction of the exponential attractor and the inertial manifold), with a larger degree of detail than generally found in most sources in the literature. If successful, our effort should put the reader in a better position to refer to more specific texts on global attractors, exponential attractors, and inertial manifolds, such as, respectively, the books by Babin and Vishik, [BV92], Eden, Foias, Nicolaenko and Temam, [EFNT94], and Constantin, Foias, Nicolaenko and Temam, [CFNT89].

7. These notes are organized as follows. As an introduction to the main ideas of the abstract theory of semiflows, in chapter 1 we present some well known and well studied examples of finite dimensional dynamical systems, generated by such celebrated ODEs as Duffing's equations and Lorenz' equations. In chapter 2 we introduce the general definitions of SEMIFLOWS and their GLOBAL ATTRACTORS, and we present two sufficient conditions that guarantee the existence of the attractor under different assumption on the asymptotic properties of the semiflow. We also describe an alternate construction of the attractor, based on the idea of α-contracting maps. In chapter 3 we apply these results to show that the semiflows generated by two types of semilinear dissipative evolution PDEEs (one parabolic and the other hyperbolic) admit a global attractor in a suitable space of weak solutions. In chapter 4 we briefly develop the theory of EXPONENTIAL ATTRACTORS, and apply this theory to the models of PDEEs considered in chapter 3. In chapter 5 we present Hadamard's GRAPH TRANSFORMATION METHOD for the construction of an INERTIAL MANIFOLD, and apply this method to a one-dimensional version of the PDEEs considered in chapter 3. In chapter 6, we consider a number of other dynamical systems, generated by PDEEs that model various mathematical physics problems, and briefly show how the methods developed in the previous chapters can be applied. In chapter 7 we present a result, due to Mora and Solà-Morales, on the nonexistence of inertial manifolds for the semiflow generated by a one-dimensional version of the hyperbolic model of PDEE considered in chapter 3. Finally, in the Appendix we collect, for the reader's convenience, a list of various definitions and results from the classical theory of ODEs and PDEs, functional and nonlinear analysis, semigroup theory and Lebesgue-Sobolev spaces, that we use in these notes, and provide at least one reference for each of the definitions and theorems we state.

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Chapter 1

Dynamical Processes

In this chapter we introduce the definition of DYNAMICAL PROCESS, and the main ideas of the theory of dynamical systems that we want to investigate. We illustrate these ideas by examining some simple examples of dynamical processes generated by finite systems of ODEs and by iterated maps.

1.1 Introduction

1. Roughly speaking, the theory of dynamical systems studies mathematical models of physical "systems" which evolve in time from a "state" which is known at an initial moment; more specifically, how the evolution of a system depends, or is influenced by, its initial state. The changing density of a population from a known number of individuals (e.g., sharks in a regional sea; bacteria in an infected organism; prey-predator models); the changing of weather patterns in a particular region; the spreading of a rumor; the vapor trail in the wake of an airplane; the propagation of a fire: all these would be examples of dynamical systems.

To study the evolution of a system, we assume that its state at each time t can be described generally by means of a function $t \mapsto u(t)$, where the independent "time" variable t is measured in a parameter set $\mathcal{T} \subset \mathbb{R}$, and the corresponding dependent variable is in a set $\mathcal{X}$, called STATE SPACE. We also assume that the state $u(t)$ of the system at any given time t depends not only on the value of t, but also on its initial configuration, i.e. on the value u_0 of the system at a previous time t_0, with u_0 and t_0 given or known. A natural goal of the theory is then to study the dependence of the state $u \in \mathcal{X}$ on the time $t \in \mathcal{T}$ and the INITIAL VALUES $u_0 \in \mathcal{X}$, $t_0 \in \mathcal{T}$. In particular, we can think of a dynamical system as a way of transforming an initial state u_0 into a family of subsequent states $u(t)$, parametrized by $t \in \mathcal{T}$. We shall indeed assume that there is a specified functional dependence of $u \in \mathcal{X}$ from $u_0 \in \mathcal{X}$ and t, $t_0 \in \mathcal{T}$, described by a map

$$(t, t_0, u_0) \mapsto u(t, t_0, u_0) \,. \tag{1.1}$$

By specifying certain properties of this map, we come to a definition of a special kind of dynamical systems.

DEFINITION 1.1 *Let $\mathcal{X}$ be an arbitrary set, and $\mathcal{T}$ be one of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}$, where $\mathbb{R}_{\geq 0} := [0,+\infty[$. Set*

$$\mathcal{T}^2_* := \{(t,\tau) \in \mathcal{T} \times \mathcal{T} : t \geq \tau\}.$$

A TWO-PARAMETER SEMIFLOW, *or* DYNAMICAL PROCESS *in $\mathcal{X}$ is a family $S = (S(t,\tau))_{(t,\tau)\in\mathcal{T}^2_*}$ of maps $S(t,\tau)\colon \mathcal{X} \to \mathcal{X}$, which satisfies the following conditions:*

$$\forall t \in \mathcal{T}: \quad S(t,t) = I_{\mathcal{X}} \tag{1.2}$$

(the identity in $\mathcal{X}$), and

$$\forall t_1,t_2,t_3 \in \mathcal{T}: \quad S(t_1,t_2)S(t_2,t_3) = S(t_1,t_3). \tag{1.3}$$

The following are familiar examples of dynamical processes.

Example 1.2
Let $\mathcal{X} = \mathbb{R}$ and $\mathcal{T} = \mathbb{R}$. Let f be a continuous function on $\mathbb{R}$, and $S = (S(t,\tau))_{(t,\tau)\in\mathcal{T}^2_*}$ be the family of maps $S(t,\tau)\colon \mathbb{R} \to \mathbb{R}$ defined by

$$S(t,\tau)x := \left(\exp\left(\int_\tau^t f(s)\,\mathrm{d}s\right)\right)x, \qquad x \in \mathbb{R}. \tag{1.4}$$

Then, S is a dynamical process in $\mathbb{R}$. Indeed, verification of (1.2) and (1.3) is immediate. □

Example 1.3
Let $\mathcal{X} = \mathbb{R}^n$, and A be an $n \times n$ matrix. Then, the family $T = (\mathrm{e}^{tA})_{t\in\mathbb{R}}$ of the exponentials of the matrices tA is a linear semigroup in $\mathcal{X}$ (see section A.3). Consequently, the family S defined by

$$S(t,\tau) := \mathrm{e}^{(t-\tau)A}, \qquad (t,\tau) \in \mathbb{R}^2,$$

is a dynamical process. □

Note that, in these examples, each map $S(t,\tau)$ is linear; as we shall see, this needs not be the case in general.

According to definition 1.1, a dynamical process S on a set $\mathcal{X}$ consists of a family of transformations of $\mathcal{X}$ into itself, each defined by the map (1.1), that is,

$$\mathcal{X} \ni u_0 \mapsto u(t,\tau,u_0) =: S(t,\tau)u_0 \in \mathcal{X}. \tag{1.5}$$

We are then mainly interested in the dependence of the map $t \mapsto S(t,t_0)u_0$ on the "initial values" t_0 and u_0 or, sometimes, on u_0 only, for fixed t_0. Of course, this requires $\mathcal{X}$ to have some kind of topological structure, and we shall remove the provisional nature of definition 1.1, supplementing it by a number of continuity conditions on the maps $S(t,\tau)$ on $\mathcal{X}$, and of the map $(t,\tau)\mapsto S(t,\tau)$. In particular, as the examples we cited above indicate, we are often interested in being able to describe, or determine, the evolution of a given system "in the future". This question can be clearly related to the asymptotic properties, as $t\to+\infty$ (in $\mathcal{T}$), of the map defined in (1.1). Because of (1.5), we are then naturally led to relate the asymptotic behavior of the function u to some suitable properties of the corresponding dynamical process S, defined by (1.5). For example, a possible question would be to determine all the values $(u_0,t_0)\in\mathcal{X}\times\mathcal{T}$ such that the limit

$$\lim_{t\to+\infty} S(t,t_0)u_0 =: L(u_0) \tag{1.6}$$

exists, for a fixed t_0. As an illustration, if S is the dynamical process defined in (1.4), the limit in (1.6) exists for all $u_0\in\mathbb{R}$ if f is bounded above by a negative constant. Note that, since in this case $L(u_0)=0$ for all $u_0\in\mathbb{R}$, this limit is actually independent of the initial value u_0. Another, related question would be to study the properties of the map $u_0\mapsto L(u_0)$ defined by (1.6).

2. In these notes, we assume that the state space $\mathcal{X}$ is at least a Banach space (on $\mathbb{R}$), and the underlying time parameter set $\mathcal{T}$ will be either $\mathbb{N}$ or $\mathbb{Z}$, in which case we call the system DISCRETE, or $\mathbb{R}_{\geq 0}$ or $\mathbb{R}$, in which case we call the system CONTINUOUS. In this chapter we propose to give a first idea of the nature of the questions, related to the long time behavior of dynamical processes, that we want to investigate. To do so, we consider some introductory examples of discrete dynamical processes, generated by iterated maps, and of continuous dynamical processes, generated by finite systems of ODEs. In these cases, the Banach space $\mathcal{X}$ has finite dimension, and the corresponding dynamical process is also called FINITE DIMENSIONAL. In chapter 3 we will instead consider INFINITE DIMENSIONAL dynamical processes, generated by PDEs of evolution. In this case, the space $\mathcal{X}$ is infinite dimensional; specifically, a space of functions of some "space" variables, defined on a domain of $\mathbb{R}^n$.

One can find a large amount of examples of this type of systems in specialized texts, such as Jordan-Smith, [JS87], Marsden-McCracken, [MM76], Guckenheimer-Holmes, [GH83], Moon, [Moo92], Alligood-Sauer-Yorke, [ASY96], and many others. Among the most studied examples, we recall the models known as Duffing's equation, the logistic equation, the Lorenz system, and Hénon's horseshoe map. Most of these also illustrate another major goal of the theory of the dynamical systems, which, regretfully, we cannot pursue because of the introductory character of these notes. Namely, all these systems depend on various numerical parameters, and the influence of these parameters on the long time behavior of the system exhibits some striking phenomena, and unexpected similarities among these systems. In particular, even if the parameters are allowed to vary in a continuous fashion, and even if for a certain range of the parameters the evolution of the system seems to be quite "regular", for other parameter ranges a number of other, totally new qualitative phenomena unexpectedly appear. Examples of such phenomena are BIFURCATIONS (see e.g. Marsden-McCracken, [MM76]), FEIGENBAUM CASCADES (e.g. for the logistic map described in section 1.4.4; see e.g. Moon, [Moo92], or Feigenbaum, [Fei78]), and HORSESHOE MAPS (e.g. for Hénon's map, whose iterations converge to a set of so-called FRACTAL type; see Hénon-Pomeau, [HP76]).

1.2 Ordinary Differential Equations

1. As a first example of dynamical processes, we consider continuous systems generated by an evolution equation of the form

$$\dot{u} = F(t,u)\,, \tag{1.7}$$

where $F: \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ is a continuous function on a Banach space $\mathcal{X}$. In this case, we take $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \mathbb{R}_{\geq 0}$. If $\mathcal{X}$ is finite dimensional, (1.7) is equivalent to a system of ODEs in $\mathbb{R}^n$, where n is the dimension of $\mathcal{X}$. An example is the system of m coupled pendulums on the same vertical plane: In this case, if $\theta_1, \ldots, \theta_m$ denote the angles of each pendulum with respect to the vertical, then $u = (\theta_1, \dot{\theta}_1, \ldots, \theta_m, \dot{\theta}_m)$ and $\mathcal{X} = \mathbb{R}^{2m}$. We shall, however, be more interested in the case when the dimension of $\mathcal{X}$ is infinite, and (1.7) represents a PDEE, interpreted as an abstract evolution equation in $\mathcal{X}$. An example is the semilinear heat equation

$$u_t = \Delta u + f(u) \tag{1.8}$$

in a domain $\Omega \subset \mathbb{R}^n$, with appropriate boundary conditions. In this case, the space $\mathcal{X}$ is a space of functions defined on Ω; for example, we can consider the Lebesgue space $\mathrm{L}^2(\Omega)$, or the Sobolev space $\mathrm{H}^1_0(\Omega)$, or the Hölder space $\mathrm{C}^{0,\alpha}(\overline{\Omega})$. We can then interpret PDEEs like (1.8) as abstract ODEs in $\mathcal{X}$ by means of the following natural identification. If u is a solution of (1.8), we define a function $t \mapsto \tilde{u}(t) \in \mathcal{X}$ by

$$(\tilde{u}(t))\,(x) := u(t,x)\,, \qquad x \in \Omega\,;$$

that is, we consider for each t the image $\tilde{u}(t) \in \mathcal{X}$ as a function of the space variable x. It is common practice to identify u and $\tilde{u}$, introducing the notation

$$u(t,\cdot) := \tilde{u}(t)\,,$$

which we shall often adopt.

2. We assume that, in accord with the classical (Newtonian) theory, equation (1.7) is deterministic, in the sense that the knowledge of the initial values (t_0, u_0) (and, of course, of F) uniquely determines a solution u, defined for all "future times", of the Cauchy problem corresponding to (1.7), that is

$$\begin{cases} \dot{u} = F(t,u)\,, \\ u(t_0) = u_0\,. \end{cases} \tag{1.9}$$

More precisely, we assume that under sufficient assumptions on the function F, there is a unique function $u \in \mathrm{C}([t_0,+\infty[;\mathcal{X})$, which satisfies the Cauchy problem (1.9), either in the classical sense (if e.g. u is also in $\mathrm{C}^1([t_0,+\infty[;\mathcal{X}))$, or in a generalized sense (e.g. almost everywhere in t, or in distributional sense). This solution is typically determined at first only locally in time, that is, on a neighborhood $]t_0-\alpha,t_0+\beta[$ of t_0, and then extended uniquely to a function, which is defined at least on the unbounded interval $]t_0-\alpha,+\infty[$, and solves problem (1.9) on the whole interval $[t_0,+\infty[$. We usually denote this extended function again by u. Of course, in some cases the local solution u could also be extended to the left of $t_0-\alpha$; however, since in the context of evolution problems we are mostly interested in what happens in "the future", we will generally not be too concerned about the possibility of extending u to the left of t_0. (We also note in passing that, when trying to do so, we sometimes meet additional problems, such as the lack of backward uniqueness.) Thus, when in the sequel we use the term "global solution", we always refer to solutions that are defined globally at least to the *right* of t_0, i.e. for all $t \geq t_0$.

Clearly, the possibility of extending a local solution to a global one must in general be proven for each specific problem. This can be done in different ways; a common one is to show that any local solution satisfies a number of so-called A PRIORI ESTIMATES. These are bounds on the solution which are independent of the particular time interval where the solution is defined, and therefore allow us to extend any local solution uniquely to a global one, by means of a repeated application of the local existence result.

3. Having thus established a unique solution u of the Cauchy problem (1.9) for all choices of initial values (t_0,u_0), we are then interested in the asymptotic behavior of $u(t)$ as $t \to +\infty$. More specifically, we would like to understand how this behavior is determined (if at all) by the initial values u_0 and t_0 (or, in some cases, by u_0 only). To this end, it is convenient to introduce more proper notations. To emphasize that the solution u depends not only on t, but also on the initial values (t_0,u_0), we consider u as a function defined on $\mathbb{R}\times\mathbb{R}\times\mathcal{X}$, with values in $\mathcal{X}$, and write $u(t,t_0,u_0)$ to indicate the image of the point (t,t_0,u_0) by u. Next, we realize that the solution of the Cauchy problem (1.9) defines a family

$$S := (S(t,t_0))_{(t,t_0)\in\Theta}\,, \quad \Theta := \{(t,s) \colon t \geq s\}\,, \tag{1.10}$$

of operators $S(t,t_0) \colon \mathcal{X} \to \mathcal{X}$, parametrized by the pair (t,t_0) in the half-plane Θ. Each operator $S(t,t_0)$ is defined by

$$S(t,t_0)u_0 := u(t,t_0,u_0)\,, \qquad u_0 \in \mathcal{X}\,. \tag{1.11}$$

This family S is called the family of SOLUTION OPERATORS associated to (or, defined by) equation (1.7). Standard uniqueness theorems on solutions of the Cauchy problem (1.9) can then be used to verify that S satisfies conditions (1.2) and (1.3) of definition 1.1; hence, S is a dynamical process on $\mathcal{X}$. We say that S is GENERATED by problem (1.9). We also say that the map $t \mapsto S(t,t_0)u_0$ defined in (1.11) is a MOTION of the dynamical process S, corresponding to the initial values (t_0,u_0), and the image of this motion is an ORBIT of the system (a more precise definition of motions and orbits will be given in section 1.2.4).

Example 1.4
The Cauchy problem

$$\begin{cases} \dot{y} = f(t)y, \\ y(t_0) = y_0 \end{cases} \tag{1.12}$$

generates the dynamical process S defined in (1.4). Indeed, (1.12) has the unique solution

$$y(t) = S(t,t_0)y_0 .$$

□

1.2.1 Well-Posedness

From now on, we shall consider the value of t_0 in the Cauchy problem (1.9) as fixed; in fact, unless otherwise specified, we shall always choose $t_0 = 0$. We are then interested in the question of the dependence of solutions of (1.9) on the other initial value u_0. This question is naturally related to the WELL-POSEDNESS of the Cauchy problem (1.9). This means that solutions of (1.7) should not only be uniquely determined by the choice of the initial value u_0, but they should also depend continuously on u_0, in a specified topology.

Since we are interested in the long-time behavior of the solutions, a crucial distinction has to be made between the notion of well-posedness on arbitrary, but bounded, time intervals $[0,T]$, and that of well-posedness in the whole interval $[0,+\infty[$. Explicitly, we explain the first of these notions in

DEFINITION 1.5 *The Cauchy problem (1.9) is* WELL POSED IN THE LARGE *if for all $u_0 \in \mathcal{X}$, and all T, $\varepsilon > 0$, there exists $\delta > 0$ such that for all $v_0 \in \mathcal{X}$ and all $t \in [0,T]$,*

$$\|u_0 - v_0\| < \delta \implies \|u(t) - v(t)\| < \varepsilon , \tag{1.13}$$

where u and v are the unique solutions of (1.7) with $u(0) = u_0$ and $v(0) = v_0$, and $\|\cdot\|$ denotes the norm of $\mathcal{X}$.

We remark that, in the theory of finite dimensional dynamical systems, definition 1.5 is often referred to as “continuity with respect to time and initial conditions”. Note that, in (1.13), δ depends not only on the initial value u_0, but, in general, also on T. That is, we can define a function $(\varepsilon,T) \mapsto \delta(\varepsilon,T)$ (this function may often be defined only implicitly). If δ can be chosen independently of T, the solutions of (1.7) depend continuously on the initial data on all of $[0,+\infty[$; this corresponds to the Lyapunov stability of the solutions of (1.7) (see definition A.6). In contrast, it is well known that well-posedness in the large is not sufficient to guarantee stability, since the dependence of δ on T may be "bad", in the sense that

$$\liminf_{T\to+\infty} \delta(\varepsilon, T) = 0 .$$

To show this, it is sufficient to consider the following elementary example.

Example 1.6
Consider the Cauchy problems for the ODEs

$$\dot{u} = -u , \tag{1.14}$$

$$\dot{u} = u , \tag{1.15}$$

with initial data at $t = 0$. Both problems have globally defined unique solutions for each choice of initial values, but only the first is globally well posed for $t \geq 0$. In fact, when checking (1.13) we can take $\delta = \varepsilon$ for (1.14), but for (1.15) we are forced to take $\delta = \varepsilon e^{-T}$, so in this case $\delta \to 0$ as $T \to +\infty$. We can interpret this in another way, realizing that the effect of any error in the initial value for equation (1.14) becomes negligible, up to arbitrary tolerance, if sufficient time is allowed to pass; on the contrary, even if two solutions of equation (1.15) are initially very close, after sufficient time they will be arbitrarily apart. Indeed, for (1.14), given any M and $\varepsilon > 0$, even if initially $|u_0 - v_0| \geq M$, it will be $|u(t) - v(t)| \leq \varepsilon$ for all $t \geq \ln(M/\varepsilon)$, while for (1.15), given again any M and $\varepsilon > 0$, even if initially $|u_0 - v_0| \leq \varepsilon$, it will be $|u(t) - v(t)| \geq M$ for all $t \geq \ln(M/\varepsilon)$. For instance, if we approximate $u_0 = \pi$ by $v_0 = 3.141$, the initial error is less than 10^{-3}, but for the corresponding solutions of (1.15) we have $|u(t) - v(t)| \geq 10^3$ for all $t \geq \ln(10^3/(\pi - 3.141)) \approx 14.5087$. This phenomenon is illustrated in figures 1.1 and 1.2. In terms of Lyapunov stability, the

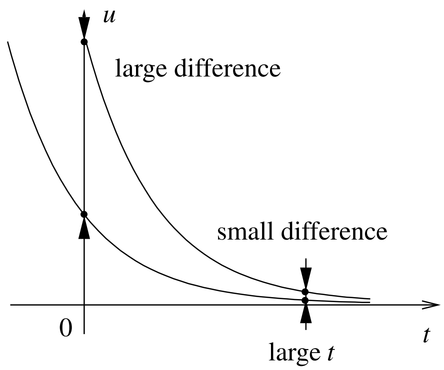

Figure 1.1: Exponential stability for $\dot{u} = -u$: A large difference in initial values still results in a small difference of the solutions after sufficient time.

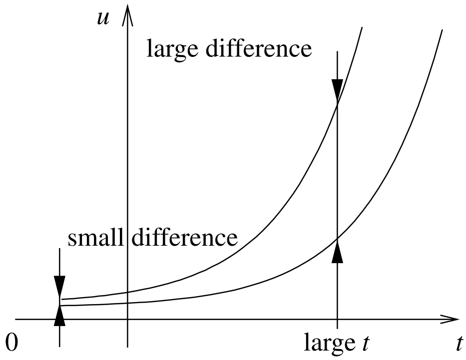

Figure 1.2: Exponential loss of information for $\dot{u} = u$: Even a small difference in initial values is drastically amplified after sufficient time.

point $u = 0$ in the phase space $\mathcal{X} = \mathbb{R}$, which corresponds to the solution $u(t) \equiv 0$ of both equations, is (uniformly) stable only for system (1.14), while system (1.15) is highly unstable under arbitrarily small perturbations of the initial value $u_0 = 0$. In fact, if in (1.15) $\pm u_0 > 0$, then as $t \to +\infty$, $u(t) \to \pm\infty$ (exponentially, of course), even if $|u_0| < \varepsilon$. Loosely speaking, this means that all control on the solution is lost if sufficient time is allowed to elapse. $\square$

1.2.2 Regular and Chaotic Systems

As we have mentioned, the theory of dynamical systems is largely concerned with the behavior of the orbits $t \mapsto u(t)$ as $t \to +\infty$, and, more specifically, with how such behavior is influenced by the choice of the initial value u_0. This explains the use of notations like (1.19) below, which emphasize the dependence of the solution, at each time t, on its initial value u_0.

With a great degree of simplification, we distinguish between two kinds of situations, which we call REGULAR and CHAOTIC. This choice of terms is rather arbitrary, and by no means universal; indeed, we find many different definitions of regularity and chaos in the literature, and even among those definitions that are mathematically rigorous, no one is universally accepted. Rather, different definitions are preferred for different applications.

Roughly speaking, regular systems are those for which perturbations in the initial values will influence the orbits only for a short period of time (called TRANSIENT). After this time, different orbits would have the same qualitative behavior, and in particular the same asymptotic behavior. This type of situation is usually described by theorems like those on the asymptotic stability of a system, or the existence of limit cycles. In some cases, the asymptotic behavior is even *independent* of the initial values, in the sense that two orbits, even if starting from two points that are arbitrarily apart, after sufficient time (i.e. the transient, whose length depends on how far apart the initial values are) they will be *and remain* arbitrarily close to each other, and so exhibit the same qualitative asymptotic behavior.

Chaotic systems are instead those for which a sort of opposite situation holds. That is, these systems are extremely sensitive to even small variations of the initial values, in the sense that “close” initial conditions eventually move arbitrarily apart. The evolution of this type of system will be “regular” for a short time only (this is in general a consequence of some result analogous to the well-posedness of ODEs on *compact* time intervals). However, if observed for sufficiently long time periods, these systems not only do not exhibit any indication of convergence towards any sort of stable or periodic configuration, but their evolution seems to be totally unpredictable. More precisely, we give the following

DEFINITION 1.7 *Let* $S = (S(t,t_0))_{(t,t_0)\in\mathcal{T}^2_*}$ *be a dynamical process on* $\mathcal{X}$*. S is said to* DEPEND SENSITIVELY ON ITS INITIAL CONDITIONS *if there is* $R > 0$ *such that for all* $t_0 \in \mathcal{T}$, $x_0 \in \mathcal{X}$, *and all* $\delta > 0$, *there are* $y_0 \in \mathcal{X}$ *and* $t_1 \in \mathcal{T}$, $t_1 \geq t_0$, *such that*

$$0 < \|x_0 - y_0\| \leq \delta \quad \text{and} \quad \|S(t_1,t_0)x_0 - S(t_1,t_0)y_0\| \geq R\,. \tag{1.16}$$

We remark that the notion of sensitive dependence of a dynamical process on its initial conditions is a natural generalization of that of uniform Lyapunov instability for ODEs (see section A.1).

Example 1.8

Consider the dynamical processes S_1 and S_2 in $\mathbb{R}$ generated, respectively, by the ODEs (1.14) and (1.15) of example 1.6. Then S_1 is regular, while S_2 depends sensitively on its initial conditions. In fact, given ε and R such that $0 < \varepsilon < R$, any two solutions x and y of (1.14) which initially differ by R will be such that $|x(t) - y(t)| \leq \varepsilon$ for all $t \geq t_0 := \ln \frac{R}{\varepsilon}$. That is, t_0 is the transient after which these solutions will always differ by at most ε. In contrast, any two solutions x and y of (1.15) which differ initially by ε will be such that $|x(t) - y(t)| \geq R$ for all $t \geq t_0 := \ln \frac{R}{\varepsilon}$ (compare to (1.16)). □

Example 1.8 shows that there are dynamical processes for which *no matter how close* two initial values may be, if sufficient time is allowed to pass the corresponding orbits will be arbitrarily apart. That is, the asymptotic behavior of these systems, which is still completely and uniquely determined by their initial values (the systems *are* deterministic), may be drastically different. To put this in another way, in this type of system all relevant information carried by the initial data is rapidly lost, and, consequently, it becomes impossible to maintain any reasonable control on the evolution of the system. Examples of this kind of situation are the smoke of a cigarette, the dynamics of large populations, of traffic patterns, economic cycles, etc. Probably, the most familiar example is that of the various meteorological models for the evolution of weather, whose prediction is in general relatively accurate only in a short time range (and the shorter the time interval, the better the prediction), but after sufficient time all predictions lose any practical value.

In section 1.4 we shall see some other simple examples of systems that exhibit chaotic behavior, as described by their being sensitive to their initial conditions. Before proceeding, we mention another possible way of describing chaotic systems, whereby "an orbit can begin roughly anywhere and end up roughly anywhere". More precisely, given any two open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{X}$, there is $x_0 \in \mathcal{U}$ such that the corresponding orbit intersect $\mathcal{V}$. For an exhaustive discussion of these, and other, possible descriptions of the chaotic behavior of a dynamical system, we refer e.g. to Robinson, [Rob99], and to Alligood, Sauer and Yorke, [ASY96].

Of course, the possibility of determining whether a given system is regular or chaotic (we should rather say, whether the system may exhibit chaotic features or is guaranteed not to) is of extreme importance in applications, for at least two reasons. First, actual initial values depend on physical measurements, and are therefore never "exact" (this is not just a problem of the "real world": Even in a simple numerical exercise in ODEs, initial values like $\sqrt{2}$ can only be introduced within approximations). Second, because in practice we cannot afford to observe the evolution of a system for very long time periods (deadlines have to be met, computer simulation time is expensive ...). Moreover, even if we could, we are still bound to observations in *finite* time intervals, and there is no guarantee that any such period of time, in which we may see "irregular" behavior, is still not part of a very long transient, after which the system may yet settle into a regular evolution.

In these notes, we are concerned with a sort of intermediate situation between the two extremes described above. There are in fact examples of systems, whose evolution may appear to be chaotic, and yet after sufficient time their solutions seem to settle into a pattern that preserves a certain degree of order, which allows for some control of the disturbances typical of a chaotic regime. This type of behavior is usually better seen in the state space $\mathcal{X}$, to which the solution curves $(u(t))_{t\geq 0}$ belong. More precisely, these systems are characterized by the existence of some subsets of $\mathcal{X}$, to which the solution curves appear to converge (in the topology of the phase space), as $t \to +\infty$. These subsets are therefore called ATTRACTING SETS, and can be thought of as a generalization of the sets, such as stationary points or limit cycles, that are known to be attracting for regular systems of ODEs. Thus, for example, if a bounded attractor exists, two solutions which started at close initial values may still be quite apart at arbitrary later times (indication of chaos), but their distance cannot be arbitrarily large, since they both converge to the same attractor. In this sense the system is still controllable. Thus, even if we cannot decide whether a given system is regular, it is clearly desirable that we be able to determine if it at least possesses an attractor. Indeed, if this is the case, we would then know that, even if the system may possibly evolve chaotically, it will nevertheless settle into some type of controlled behavior. This is of course of fundamental importance in applications.

1.2.3 Dependence on Parameters

In many physical examples, the equation (1.7) which models the evolution of a dynamical process may also depend on various numerical parameters, such as, for instance, the dielectric and permeability constants in Maxwell's equations, or the viscosity coefficient in Navier-Stokes' equations of fluid dynamics. In this case, equation (1.7) takes the more general form

$$\dot{u} = F(\lambda, t, u), \qquad \lambda \in \Lambda \subset \mathbb{R}^m, \tag{1.17}$$

and the corresponding solution operator also depends on the parameters λ. In applications, it is of course of great importance to have a good knowledge of how the evolution of a system is influenced not only by (small) variations of the initial value u_0, but also by (small) variations of these parameters. For example, if the arm of a robot has the task of repeatedly moving an object from one position to another, and its motion is governed by a differential equation like (1.17), we are interested in the choice of parameters that make such motion as smooth as possible, and to avoid those that may make it irregular or, worse, chaotic.

We will not present any theoretical results on the dependence of dynamical systems, in particular infinite dimensional ones, on numerical parameters, since this topic is too extensive and specialized, and a large quantity of the available insights and results are most often obtained by means of extensive and robust numerical simulation. Indeed, an experimental analysis of the equations modelling many physical examples indicates that various kinds of bifurcation phenomena typically occur at different, increasing values of λ. We refer to Temam, [Tem88, ch. 1], for a very general outline of various scenarios that are possible.

1.2.4 Autonomous Equations

1. Most classical results on the theory of the asymptotic behavior of dynamical systems involve systems generated by evolution equations (1.7) that are AUTONOMOUS. These systems, which occur quite frequently in applications, correspond to the case when the function F in (1.7) is independent of t, that is, when (1.7) has the form

$$\dot{u} = F(u), \tag{1.18}$$

with $F\colon \mathcal{X} \to \mathcal{X}$ continuous. For example, the heat equation (1.8) is autonomous. In this case, we can always reduce ourselves, by a shift of the time coordinate, to a fixed choice of t_0. This means that the operators of S have the form

$$S(t,\tau) = S(t-\tau, t_0) =: \tilde{S}(t-\tau),$$

where now $\tilde{S} = (\tilde{S}(t))_{t \geq t_0}$ is a one-parameter family of operators, i.e. a SEMIFLOW, on $\mathcal{X}$. In particular, we choose $t_0 = 0$ for simplicity. We use again the letter S to denote this one-parameter family; that is, we write $S = (S(t))_{t \geq 0}$, and (1.11) reads

$$S(t)u_0 = u(t, 0, u_0). \tag{1.19}$$

In particular, conditions (1.2) and (1.3) of definition 1.1 are satisfied if S is a SEMIGROUP of (not necessarily linear) operators on $\mathcal{X}$, i.e. if

$$S(0) = I_{\mathcal{X}} \tag{1.20}$$

(the identity in $\mathcal{X}$), and for all t, $s \geq 0$,

$$S(t+s) = S(t)S(s) \tag{1.21}$$

(fig. 1.3). Indeed, if S is the solution operator defined by the autonomous equation

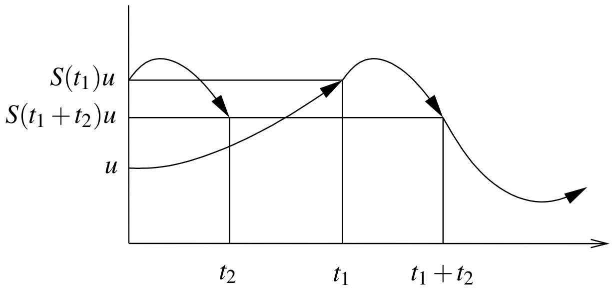

Figure 1.3: The action of the semigroup.

(1.18), (1.20) holds by virtue of the initial condition

$$S(0)u_0 = u_0 \qquad \text{for all } u_0 \in \mathcal{X}\,.$$

To show (1.21), we note that for all t, $s \geq 0$,

$$S(t)S(s)u_0 = v(t)\,,$$

where v is the solution of the Cauchy problem

$$\begin{cases} \dot{v} &= F(v)\,, \\ v(0) &= S(s)u_0 = u(s)\,. \end{cases}$$

Thus, setting $w(t) := v(t-s)$, we have

$$\dot{w}(t) = \dot{v}(t-s) = F(v(t-s)) = F(w(t))$$

and

$$w(s) = v(0) = u(s)\,.$$

By the assumed uniqueness of solutions of the differential equation, we conclude that $w(t) = u(t)$ for all $t \geq 0$. In particular,

$$u(t+s) = w(t+s) = v(t)\,; \tag{1.22}$$

and since $u(t+s) = S(t+s)u_0$ and

$$v(t) = S(t)v(0) = S(t)S(s)u_0\,,$$

(1.22) means that (1.21) holds.

Clearly, this argument may fail if the differential system is not autonomous, since then

$$\dot{v}(t-s) = F(t-s, v(t-s)),$$

and in general

$$F(t-s, w) \neq F(t, w).$$

Example 1.9

The first order autonomous system

$$\begin{cases} \dot{u} = & v \\ \dot{v} = & -u \end{cases} \tag{1.23}$$

generates the dynamical system S in $\mathbb{R}^2$, defined by

$$S(t)(x,y) := A(t)(x,y)^\top, \qquad t \in \mathbb{R},$$

where $A(t)$ is the 2×2 matrix defined by

$$A(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

and $\top$ denotes transposition. Indeed, it is immediate to verify that for all $(x,y) \in \mathbb{R}^2$, the vector function $t \mapsto U(t) := A(t)(x,y)^\top$ solves system (1.23) with initial values $U(0) = (x,y)^\top$. Furthermore, $A(0) = I$, and for all t and $s \in \mathbb{R}$,

$$\begin{aligned} A(t+s) &= \begin{pmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{pmatrix} \\ &= \begin{pmatrix} \cos t \cos s - \sin t \sin s & \sin t \cos s + \cos t \sin s \\ -\sin t \cos s - \cos t \sin s & \cos t \cos s - \sin t \sin s \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \\ &= A(t)A(s). \end{aligned}$$

□

Example 1.10

The solution operator defined by the ODE

$$\dot{u} = \cos t$$

is not a semigroup. Indeed, for arbitrary t and $s \in \mathbb{R}$ we have

$$\begin{aligned} u(t) &= u_0 + \sin t\,, \\ S(t+s)u_0 &= u_0 + \sin(t+s)\,, \\ S(t)S(s)u_0 &= u_0 + \sin s + \sin t\,. \end{aligned}$$

On the other hand, the solution operator defined by the autonomous ODE

$$\dot{u} = 1 - u$$

is indeed a semigroup. In fact, for arbitrary t and $s \in \mathbb{R}$ we have

$$\begin{aligned} u(t) &= (u_0 - 1)\mathrm{e}^{-t} + 1\,, \\ S(t+s)u_0 &= (u_0 - 1)\mathrm{e}^{-(t+s)} + 1\,, \\ S(t)S(s)u_0 &= (S(s)u_0 - 1)\,\mathrm{e}^{-t} + 1 \\ &= \big((u_0 - 1)\mathrm{e}^{-s} + 1 - 1\big)\,\mathrm{e}^{-t} + 1 \\ &= (u_0 - 1)\mathrm{e}^{-s-t} + 1\,. \end{aligned}$$

□

Except for some elementary introductory examples, in these lectures we shall only consider autonomous systems. For an extensive account on the nonautonomous case, see e.g. Haraux, [Har91].

2. When a system is autonomous, we call the corresponding family S of solution operators a SEMIFLOW on $\mathcal{X}$, and the space $\mathcal{X}$ is often called the PHASE SPACE of the dynamical system. The map $u\colon\ [0,+\infty[\, \to \mathcal{X}$ defined by

$$u(t) := S(t)u_0\,, \qquad t \geq 0\,,$$

is called a MOTION, and the image of u in $\mathcal{X}$, i.e. the subset (or curve)

$$\gamma_{u_0} := \bigcup_{t\geq 0} u(t) \subset \mathcal{X}\,,$$

is called the ORBIT of the motion u, starting at u_0. (When the system is not autonomous, we would need to consider the product $\mathbb{R} \times \mathcal{X}$ as an extended phase space.) Then, the asymptotic behavior of solutions of (1.18) is related to the evolution of the corresponding orbits, as subsets of $\mathcal{X}$. Indeed, the recourse to the notion of orbits in the phase space (as opposed to that of solution of the differential equation) quite naturally allows us to introduce, together with the appropriate instruments from analysis in metric spaces to determine limiting behaviors etc., a more geometric approach, in which we study and exploit the topological properties of the orbits, seen in their own right as subsets of the phase space $\mathcal{X}$. The example of definition of stability in the theory of ODEs is a familiar one; another example in two dimensions of space, i.e. for $\mathcal{X} = \mathbb{R}^2$, is the Poincaré-Bendixon theorem (see e.g. theorem A.32), which describes conditions under which the orbits of an autonomous system of two ODEs converge, in a suitable sense, to a limit cycle.

3. In conclusion, we have seen in what sense an autonomous differential equation (1.18) generates a continuous semiflow S, by means of the solution operator defined in (1.19). If F is sufficiently regular, S is also differentiable. It is worth to point out that the converse is also true; that is, a differentiable semiflow $S = (S(t))_{t\geq 0}$ is always generated by an autonomous ODE.

PROPOSITION 1.11
Let S be a semiflow defined on $\mathcal{X}$, and assume that for all $x_0 \in \mathcal{X}$, the map

$$[0,+\infty[\, \ni t \mapsto S(t)x_0 \in \mathcal{X}$$

is differentiable at $t=0$. Let $F\colon \mathcal{X} \to \mathcal{X}$ be defined by

$$F(x) := \frac{\mathrm{d}}{\mathrm{d}t}(S(t)x)\Big|_{t=0}, \qquad x \in \mathcal{X},$$

and, for $x_0 \in \mathcal{X}$ and $t \geq 0$, set $x(t) := S(t)x_0$. Then x is differentiable in $[0,+\infty[$, and satisfies the autonomous Cauchy problem

$$\begin{cases} \dot{x} = F(x), \\ x(0) = x_0. \end{cases} \tag{1.24}$$

PROOF Fix $t_0 \geq 0$. For $t \geq t_0$, we compute that

$$\begin{aligned} \frac{x(t)-x(t_0)}{t-t_0} &= \frac{S(t)x_0 - S(t_0)x_0}{t-t_0} = \frac{S(t-t_0+t_0)x_0 - S(t_0)x_0}{t-t_0} \\ &= \frac{S(t-t_0)S(t_0)x_0 - S(t_0)x_0}{t-t_0}. \end{aligned} \tag{1.25}$$

Let $y_0 := S(t_0)x_0$ and $\theta := t - t_0$. Then from (1.25)

$$\frac{x(t)-x(t_0)}{t-t_0} = \frac{S(\theta)y_0 - y_0}{\theta}.$$

Since the map $t \mapsto S(t)y_0$ is differentiable at $t=0$, we have that, as $\theta \to 0$

$$\frac{x(t)-x(t_0)}{t-t_0} \longrightarrow \frac{\mathrm{d}}{\mathrm{d}\theta}(S(\theta)y_0)\Big|_{\theta=0} = F(y_0) = F(S(t_0)x_0).$$

This proves that x is differentiable from the right, and

$$x'_+(t) = F(x(t)).$$

If instead $0 < t < t_0$, we compute that

$$x'_-(t_0) = \lim_{t\to t_0^-} \frac{x(t)-x(t_0)}{t-t_0} = \lim_{s\to t_0^+} \frac{x(2t_0-s)-x(t_0)}{t_0-s}$$

$$
\begin{aligned}
&= \lim_{s\to t_0^+} \frac{S(t_0 - s)y_0 - y_0}{t_0 - s} = \lim_{\theta\to 0^+} \frac{S(\theta)y_0 - y_0}{\theta} \\
&= F(y_0) = F(x(t_0))\,.
\end{aligned}
$$

This proves that x is differentiable also from the left, and

$$x'_-(t) = F(x(t))\,.$$

Hence, x is differentiable, and satisfies the equation of (1.24). The initial value of (1.24) is obviously taken. ∎

1.3 Attracting Sets

We have mentioned that in some cases, even if the evolution of a system appears to be chaotic, a certain degree of order seems to be preserved, in the sense that the orbits of the system appear to settle into a somewhat regular pattern, described by the fact that they converge, or at least remain "close", to some subset of $\mathcal{X}$. We can often describe this situation in terms of subsets that are ATTRACTING, or at least ABSORBING, in the following sense.

1. Absorbing Sets. In the theory of ODEs, a first step in the study of the asymptotic behavior of the solution of a given system is to recognize that these solutions are bounded as $t \to +\infty$. Analogously, given a dynamical system S on a Banach space $\mathcal{X}$, it may be possible, in some cases, to recognize the existence of a subset $\mathcal{B} \subset \mathcal{X}$ into which all orbits, or at least those starting from some subset $\mathcal{U} \subseteq \mathcal{X}$ containing $\mathcal{B}$, enter and, after possibly leaving $\mathcal{B}$ a finite number of times, eventually remain in $\mathcal{B}$ for ever. This set $\mathcal{B}$ is thus called an ABSORBING SET. If a *bounded* absorbing set exists, this is taken as an expression of a specific property of the system, generically called DISSIPATIVITY.

2. Attracting Sets. When an absorbing set exists, it is sometimes possible to also recognize the existence of a smaller subset $\mathcal{A} \subset \mathcal{B}$, to which all orbits starting from $\mathcal{U}$ converge as $t \to +\infty$ after having entered $\mathcal{B}$; see fig. 1.4.

(The precise definition of convergence of an orbit to a set of $\mathcal{X}$ is given in section 2.1 of chapter 2.) Such sets $\mathcal{A}$ are generally called ATTRACTING SETS. We will see that if a dynamical system admits an attractor, it necessarily has an absorbing set as well. Attracting sets may have a quite complicated geometric or topological structure (they may be self-similar sets, or FRACTALS), and the convergence of the orbits to these sets may be quite slow. However, these sets often possess some important properties, that may allow for a better understanding of the evolution of the system (in particular, if the system appears to be chaotic). For example, the set $\mathcal{A}$ may be *compact*, and (often but not always) it may have a *finite fractal dimension*

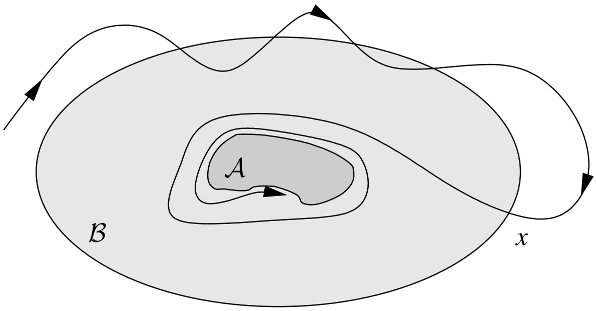

Figure 1.4: Absorbing and attracting sets. After entering the absorbing set $\mathcal{B}$ for the last time at x, the orbit remains in $\mathcal{B}$, and then converges to $\mathcal{A}$.

(the definition of which we recall in section 2.8 of chapter 2). The set $\mathcal{A}$ may also be *invariant*, which means that

$$S(t)\mathcal{A} = \mathcal{A} \qquad \text{for all } t \geq 0\,. \tag{1.26}$$

That is, if $u_0 \in \mathcal{A}$ then $u(t) = S(t)u_0 \in \mathcal{A}$ for all $t \geq 0$ and, conversely, every $u_0 \in \mathcal{A}$ is on some orbit starting from some point in $\mathcal{A}$.

3. Attractors. Bounded, positively invariant attracting sets are generally called ATTRACTORS. Of particular importance are attractors that are finite dimensional, because the corresponding dynamics is also finite dimensional. Indeed, the invariance of the attractor implies, by (1.26), that orbits which originate in the attractor remain in the attractor for all future times; consequently, the evolution of a system on a finite dimensional attractor would essentially be governed by a finite system of ODEs. In fact, a celebrated theorem of Mañé, [Mañ81], states that if a dynamical system possesses a finite dimensional attractor, this attractor can be generated by (or, as it is sometimes said, is "imbedded into") a finite system of ODEs. This result allows us to reduce, at least in principle, the study of the long time behavior of orbits which converge to a finite dimensional attractor to that of the solutions of a finite dimensional system of ODEs on $\mathcal{A}$.

This question, together with the description of the corresponding ODEs, is one of the most challenging problems in the theory of dynamical systems. Moreover, in most cases the reduction of the study of the evolution of the system on the attractor cannot be pursued in practice, because of several difficulties, which partially motivate the search for "friendlier" sets, such as the inertial manifolds discussed below. For example, we have mentioned the generally nonsmooth geometrical or topological structure of the attractor, which may cause the corresponding ODEs to only admit generalized solutions. Another problem, of special importance in applications, is that in many cases the available estimates on the dimension of the attractor, and therefore on the dimension of the system of ODEs, are simply too large for computational feasibility. For instance, in meteorology it is not uncommon to have estimates of the order of 10^m, $m \geq 20$. Also, attractors are in most cases not sufficiently stable under perturbations of the data, so that their numerical approximations, and the consequent propagation of errors, may be quite difficult to control. For example, approximations of attractors with respect to the Hausdorff distance (see section 2.1) are in general only upper semicontinuous. Finally, the rate of convergence of the orbits to the attractor may really be no better than polynomial, as the following example shows.

Example 1.12
Consider the semiflow S generated by the autonomous ODE

$$\dot{u} = f(u) := -u^3 \,. \tag{1.27}$$

The attractor of S is the set $\mathcal{A} = \{0\}$, but the convergence of the orbits to $\mathcal{A}$ is at most polynomial, as we see from the explicit solution of the Cauchy problem relative to (1.27) with initial value $u(0) = u_0$, that is,

$$u(t) = \frac{u_0}{\sqrt{1 + 2u_0^2 t}} \,.$$

$\square$

4. Inertial Manifolds. On the other hand, there are systems whose attractors do not present this type of difficulties, since they are imbedded into a finite dimensional Lipschitz manifold $\mathcal{M}$ of $\mathcal{X}$, and the orbits converge to this manifold with a uniform exponential rate. Such a set $\mathcal{M}$ is called an INERTIAL MANIFOLD of the system (fig. 1.5). When an inertial manifold exists, the evolution of the semiflow on the

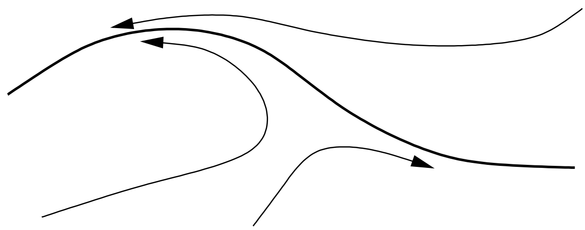

Figure 1.5: Inertial Manifolds.

manifold is governed by a finite system of ODEs, called the INERTIAL FORM of the semiflow. This finite system of ODEs will in general admit solutions with a certain degree of smoothness, depending on the smoothness of the manifold. Since orbits converge to the inertial manifold with a uniform exponential rate, we see that, in turn, the dynamics on the manifold will be a good description of the long time behavior of solutions of equation (1.7). Clearly, the possibility of imbedding the attractor into an inertial manifold provides an indirect way to obtain the above mentioned desired system of ODEs. Moreover, the uniformity of the rate of convergence of the orbits to the manifold makes these systems extremely stable under perturbations and numerical approximations. Unfortunately, there are not many examples of systems which are known to admit an inertial manifold; among these, we mention the semiflows generated by a number of reaction-diffusion equations of "parabolic" type, and by the corresponding hyperbolic (small) perturbations of these equations. A typical model is that of the so-called Chafee-Infante equations, which we present in chapter 5.

5. Exponential Attractors. An intermediate situation occurs when a system admits a so-called EXPONENTIAL ATTRACTOR. These sets, which are also sometimes called INERTIAL SETS in the literature, are somehow intermediate between attractors and inertial manifolds, in the sense that while they do not necessarily have a smooth structure, they can still be imbedded into a finite system of ODEs. In addition, these sets retain at least three of the features of the inertial manifolds that attractors do not necessarily have: the finite dimensionality, the exponential convergence of the orbits, and a high degree of stability with respect to approximations (for example, continuity with respect to the Hausdorff distance). This means that when an exponential attractor exists, after an "exponentially short" transient the dynamics of the system are essentially governed by a finite system of ODEs (the classical image is that of an airplane, landing at a "fast" speed and then "slowly" taxiing to the arrival gate).

The following is a simple example of a regular system, whose solutions converge exponentially to its attractor.

Example 1.13
Consider the function $f\colon [0,1] \to [0,1]$ defined by $f(x) = (1+x)^{-1}$, and the corresponding discrete system $(S^n)_{n\in\mathbb{N}}$ defined by the iterated sequence (1.30). This system has an attractor, which is the set $\mathcal{A} = \{\ell\}$, with $\ell := (\sqrt{5}-1)/2$. We now show that $\mathcal{A}$ is also an exponential attractor; that is, there is $\alpha > 0$ such that, for all initial values $x_0 \in [0,1]$,

$$|S^n x_0 - \ell| \le \mathrm{e}^{-\alpha n}. \tag{1.28}$$

Indeed, setting

$$S^n x_0 = f(x_n) =: x_{n+1},$$

we see that this sequence converges to the positive solution of the equation $x = f(x)$, which is precisely ℓ. Since $\ell = f(\ell)$, we compute that

$$\ell - x_{n+1} = f(\ell) - f(x_n) = \frac{x_n - \ell}{(1+\ell)(1+x_n)}. \tag{1.29}$$

Since $1+\ell > \frac{3}{2}$ and $1+x_n \ge 1$ for each n, we deduce from (1.29) that

$$|x_{n+1} - \ell| \le \tfrac{2}{3}|x_n - \ell|,$$

from which we conclude that

$$|x_n - \ell| \le (\tfrac{2}{3})^n |x_0 - \ell| \le (\tfrac{2}{3})^n .$$

This shows that (1.28) holds, with e.g. $\alpha = \ln \frac{3}{2} > 0$. We explicitly note that α is independent of the initial values: this ensures that the iterates $S^n x_0$ converge to $\mathcal{A}$ with a uniform rate. ∎

Of course, not all dynamical systems will possess attractors, exponential attractors or inertial manifolds. In the sequel, we shall try to present a theory, by now quite well established, that provides a number of sufficient conditions on the system for at least some of these sets to exist. In particular, since attractors will contain stationary and periodic solutions of (1.17), this theory is really a natural extension of the classical theory of stability for ODEs.

1.4 Iterated Sequences

Not surprisingly, many of the ideas (and difficulties) in the theory of continuous dynamical systems already surface in the context of discrete dynamical systems generated by ITERATED SEQUENCES. These are sequences $(u_n)_{n\in\mathbb{N}} \subset \mathcal{X}$, of the form

$$u_{n+1} = f(u_n) , \tag{1.30}$$

where f is a map of $\mathcal{X}$ into itself. Thus, each sequence is completely determined by its initial value u_0, assigned separately. Iterated sequences generate a DISCRETE dynamical system $S := (S^n)_{n\in\mathbb{N}}$ on $\mathcal{X}$, defined by

$$S^0 = I , \quad S^{n+1} = S \circ S^n ,$$

where S^n is the n-th iterate of S, and $\circ$ denotes the composition of maps in $\mathcal{X}$. Thus, $\mathcal{T} = \mathbb{N}$, and the orbits of S are the sequences $(S^n u_0)_{n\in\mathbb{N}}$. We are interested in how the behavior of each such sequence, as $n \to +\infty$, depends on its initial term u_0.

In this section we present some well known examples of discrete systems in $\mathbb{R}^n$, $n \le 3$, each defined by a sequence like (1.30).

For future reference, we recall the following

DEFINITION 1.14 *Let $F : \mathcal{X} \to \mathcal{X}$ be a map (not necessarily linear), and $x \in \mathcal{X}$.*

1. *x is a* FIXED POINT *of F if $x = F(x)$.*
2. *A fixed point x of F is said to be* STABLE *if, given any neighborhood $\mathcal{U}$ of x there is another neighborhood $\tilde{\mathcal{U}} \subset \mathcal{U}$ of x such that for all x_0 in $\tilde{\mathcal{U}}$, the corresponding recursive sequence $(x_n)_{n\in\mathbb{N}}$, starting at x_0 and defined by $x_{n+1} = F(x_n)$, is contained in $\mathcal{U}$. Otherwise, x is said to be* UNSTABLE.

3. *A fixed point x of F is said to be* ATTRACTIVE *if for all x_0 in a neighborhood of x, the above defined recursive sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.*

4. *A stable and attractive fixed-point is called* ASYMPTOTICALLY STABLE.

5. *A point x is said to be p-*PERIODIC *($p \in \mathbb{N}$) if $F^p(x) = x$.*

Note that not all stable fixed points are attractive, as we see by taking $F(x) = x$. For this map, each point x is a stable, but not attractive, fixed point. On the other hand, we have the following

THEOREM 1.15
Let $\mathcal{X} = \mathbb{R}$, and x_0 be a fixed point of a C^1 function F. Then x_0 is asymptotically stable if $|F'(x_0)| < 1$, while if $|F'(x_0)| > 1$, x_0 is unstable.

PROOF Without loss of generality, we can confine ourselves to symmetric neighborhoods of x_0.

1) Assume first that $|F'(x_0)| < 1$. There exists then a number $\varepsilon \in]|F'(x_0)|, 1[$, and, correspondingly, a number $\delta > 0$ such that if $|x - x_0| \leq \delta$, then

$$|F(x) - F(x_0)| \leq \varepsilon |x - x_0| . \tag{1.31}$$

Since $\varepsilon < 1$ and $F(x_0) = x_0$, (1.31) implies that

$$|F(x) - x_0| \leq |x - x_0| \leq \delta .$$

Consequently, we can repeat estimate (1.31), and obtain that for all $n \in \mathbb{N}_{\geq 1}$,

$$|F^n(x) - x_0| \leq \varepsilon^n |x - x_0| . \tag{1.32}$$

From this, it follows that x_0 is asymptotically stable: Indeed, given any neighborhood $\mathcal{U} :=]x_0 - \rho, x_0 + \rho[$ of x_0, let $\delta \in]0, \rho]$, and set $\tilde{\mathcal{U}} :=]x_0 - \delta, x_0 + \delta[$. Then, (1.32) implies that if $x \in \tilde{\mathcal{U}}$, each iterate $F^n(x)$ is in $\mathcal{U}$, because

$$|F^n(x) - x_0| \leq \varepsilon^n |x - x_0| \leq \delta \leq \rho .$$

Thus, x_0 is stable; clearly, (1.32) also implies that x_0 is also attractive.

2) Conversely, assume that $|F'(x_0)| > 1$. Then, as before, given any $a \in]1, |F'(x_0)|[$, we can determine $\gamma > 0$ such that if $|x - x_0| \leq \gamma$, then

$$|F(x) - x_0| \geq a |x - x_0| . \tag{1.33}$$

We wish to prove that there is $\bar{\rho} > 0$ such that for all $\delta \in]0, \bar{\rho}]$, there are $\bar{x}$ and $\bar{n}$ such that

$$|\bar{x} - x_0| \leq \delta \qquad \text{and} \qquad |F^n(\bar{x}) - x_0| \geq \bar{\rho} .$$

Arguing by contradiction, taking $\rho = \gamma$, we can determine $\delta \in]0,\gamma]$ such that if $|x - x_0| \leq \delta$, then for all $n \in \mathbb{N}_{>0}$,

$$|F^n(x) - x_0| \leq \rho = \gamma. \tag{1.34}$$

Now, (1.34) and (1.33) imply that for all n,

$$|F^n(x) - x_0| \geq a^{n-1}\,|F(x) - x_0|; \tag{1.35}$$

but since $|x - x_0| \leq \delta \leq \gamma$, (1.33) implies that, in fact,

$$|F^n(x) - x_0| \geq a^n\,|x - x_0| \tag{1.36}$$

for all n. Choose then, for example, $x = x_0 + \frac{1}{2}\delta$. Then, (1.36) implies that

$$\gamma \geq |F^n(x) - x_0| \geq \tfrac{1}{2}\delta a^n\,. \tag{1.37}$$

Since $a > 1$, letting $n \to +\infty$ in (1.37) we achieve the desired contradiction. ∎

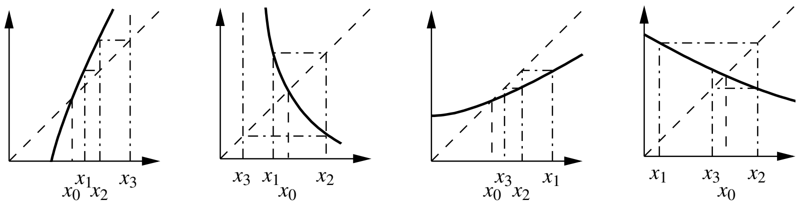

Figure 1.6: The four possibilities: $F'(x_0) > 1$, $F'(x_0) < -1$, $0 < F'(x_0) < 1$, $-1 < F'(x_0) < 0$.

We remark that when $|F'(x_0)| = 1$, x_0 can be either attractive, or unstable. This is easily seen by considering a function F which changes concavity at x_0. For example, if $F'(x_0) = 1$, and F changes from convex to concave at x_0, then x_0 is attractive, while if F changes from concave to convex at x_0, then x_0 is unstable.

1.4.1 Poincaré Maps

Given a continuous dynamical system, it is in many cases possible to construct a discrete one, whose asymptotic behavior is essentially the same as that of the continuous system. One way to do so is to choose a sequence $(t_n)_{n\in\mathbb{N}}$ of equidistant values $t_n \to \infty$ and, given a solution of the continuous autonomous system (1.18), to consider the corresponding sequence $(u_n)_{n\in\mathbb{N}}$ of points $u_n := u(t_n)$ in the phase space $\mathcal{X}$. Clearly, each of these points lies on the orbit starting at u_0. This choice defines a map $\Phi\colon \mathcal{X} \to \mathcal{X}$, by

$$u_{n+1} = \Phi(u_n)\,. \tag{1.38}$$

Maps constructed in this way are called STROBOSCOPIC MAPS. For example, the choice $t_n = n+1$ in (1.38) yields the sequence $(u_n)_{n\in\mathbb{N}}$, defined by

$$S := S(1)\,, \quad u_{n+1} = S^n u_0 \quad \text{for } n \in \mathbb{N}\,.$$

We can visualize a stroboscopic map by considering the graph of u in the product space $[0,+\infty[\,\times\mathcal{X}$; that is, the set

$$\operatorname{graph} u := \{(t,u(t))\colon t \geq 0\}\,. \tag{1.39}$$

Then, the sequence in $\mathcal{X}$ defined by the stroboscopic map (1.38) is obtained by projecting on $\mathcal{X}$ the points $(t_n, u(t_n))$.

In the case of finite dimensional systems, a remarkable construction is that of the so-called POINCARÉ MAPS. These maps are constructed by fixing a hyperplane $\Sigma \subset \mathbb{R}^n$, called a POINCARÉ SECTION, and considering on Σ the sequence of points P_n defined by the "first returns" of the (graph of the) solution on Σ, i.e. by the successive intersections of the semiorbit $\{u(t)\colon t \geq 0\}$ with Σ (figs. 1.7 and 1.8). Indeed,

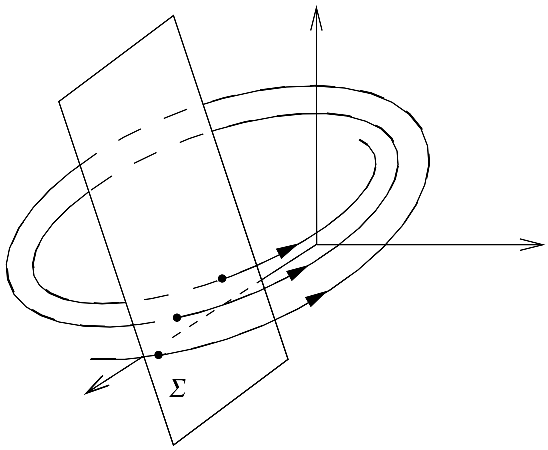

Figure 1.7: The Poincaré section.

Poincaré maps are sometimes also known as "first return" maps. More precisely, we consider again the intersection of the graph (1.39) with $\mathbb{R}\times\Sigma$ (both as subsets of $\mathbb{R}\times\mathbb{R}^n$), and construct the sequence of points $(u(t_n))_{n\in\mathbb{N}} \subseteq \Sigma$, as ordered by the first argument t_n; that is, by the time of the n-th intersection of the orbit with the hyperplane Σ. Set $u_n := u(t_n)$. The sequence $(u_n)_{n\in\mathbb{N}}$ can then be considered as a recursive sequence on Σ, defined by a map

$$u_{n+1} = \Phi_\Sigma(u_n)\,.$$

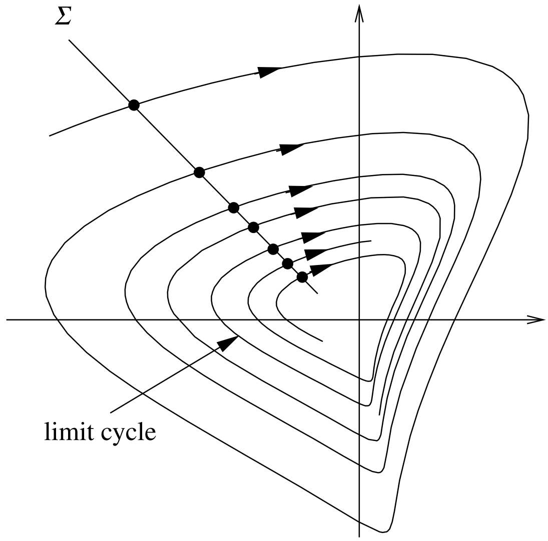

Figure 1.8: In the plane, the Poincaré section is a line.

The map Φ_Σ is called the POINCARÉ MAP associated to the semiflow defined by (1.18). Poincaré maps can thus be used to study the asymptotic behavior of a continuous semiflow, by reducing it to a discrete one. For example, if (1.7) has a periodic solution with period T, the Poincaré map with sampling synchronized with the period, i.e. with $t_n = nT$, will have a fixed point (fig. 1.9). Of course, for a given ODE, or system of ODEs, even autonomous ones, it may not be clear how to find suitable sampling sequences $(t_n)_{n\in\mathbb{N}}$, and extensive numerical experimentation may well be required.

Finally, we mention that the notion of Poincaré maps can be generalized to infinite dimensional continuous dynamical systems (see e.g. Marsden-McCracken, [MM76]).

1.4.2 Bernoulli's Sequences

We start with an example that illustrates the phenomenon of the loss of information from the initial data after sufficient time is allowed to pass.

The so-called BERNOULLI'S SEQUENCE is the recursive sequence $x_{n+1} = f(x_n)$ generated by the function $f\colon [0,1] \to [0,1]$ defined by

$$f(x) := 2x - \lfloor 2x \rfloor ,$$

where $\lfloor x \rfloor$ denotes the integer part of x (that is, the largest integer less than or equal to x). Note that f is not continuous at $x = \frac{1}{2}$ (fig. 1.10); however, f can be nicely described as a so-called "circle-doubling" map, if we identify the endpoints of the domain interval $[0,1]$ with each other. More precisely, if we define $g\colon [0,1] \to \mathbb{R}^2$ by

$$g(x) := (\cos(2\pi f(x)), \sin(2\pi f(x))) ,$$