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Integral theorems for functions and differential forms in C^m

$$\overline{D} := \begin{pmatrix} \bar{\partial} & \bar{\partial}^* \\ \bar{\partial}^* & \bar{\partial} \end{pmatrix}$$

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Contents

Introduction

I.1 The theory of holomorphic functions of several complex variables emerged as an attempt to generalize adequately onto the multidimensional situation the corresponding theory in one variable. In the course of a century long, extensive and intensive development it has proved to have beauty and profundity; many remarkable features and peculiarities have been found; new and far-reaching notions and concepts have been constructed. A multitude of applications to many areas of mathematics as well as to other sciences have been obtained.

I.2 At the same time, the deepening of the knowledge in several complex variables theory has been bringing those working in that field to the revelation of more and more paradoxical differences and distinctions between the structures of the two theories. S. Krantz, the author of many books and articles on several complex variables, writes in Preface of his book [Kr2, p.VII], that *"Chapter 0 consists of a long exposition of the differences between one and several complex variables."*

It is almost generally accepted that one of the deepest, most fundamental reasons for those differences lies in the absence of the universal and holomorphic Cauchy kernel i.e., a reproducing kernel which serves in any domain of $\mathbb{C}^m$, with reasonably smooth boundary but of any shape, and most importantly, is holomorphic. As S. Krantz writes on p.1 in [Kr2], *"there are infinitely many Cauchy integral formulas in several variables; nobody knows what the right one is, but there are several good candidates."*

In fact, what motivated us was exactly the desire to find *the right* Cauchy integral representation in several complex variables. To realize what it really is, it proved to be necessary to come to a completely new approach: the *right* Cauchy integral representation can be constructed for a *right* set of functions which does not reduce to that of holomorphic functions but must be much more ample.

I.3 To explain the origin of the above-mentioned idea, let us analyze the basic elements which underlie one-dimensional, not multidimensional, complex analysis. There are many definitions of holomorphy there; all of them are equivalent, thus one can start from any of them. We shall use the standard notation:

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{I.3.1}$$

Null solutions to those operators provide us with the two classes of functions, respectively, holomorphic and antiholomorphic. Crucial is the fact that they factorize the two-dimensional Laplace operator $\Delta_{\mathbb{R}^2}$:

$$\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \circ \frac{\partial}{\partial z} = \frac{1}{4}\Delta_{\mathbb{R}^2}. \tag{I.3.2}$$

Combining this factorization with Green's (or the two-dimensional Stokes) formula, all the main integral theorems are routinely obtained: Cauchy and Morera, Borel-Pompeiu (= Cauchy-Green), Cauchy integral, etc.

As a matter of fact (although normally it is considered to be too trivial to mention), the definitions (I.3.1) and the factorization (I.3.2) are based on the excellent algebraic structure of $\mathbb{C}$, the range of functions under consideration. In particular, complex conjugation provides the possibility to factorize a non-negative quadratic form into a product of linear forms: $z \cdot \bar{z} = |z|^2 \geq 0$, and, of course, the factorization (I.3.2) is a manifestation of this property of complex numbers.

It is worthwhile to note that the commutativity of the multiplication in $\mathbb{C}$ is useful and pleasant to work with, but just in the above-mentioned integral theorems it is not of great importance.

I.4 Let $w = f(z) = u + iv,\ z = x + iy$, then the condition $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the system of the Cauchy-Riemann equations which says that the components u, v of the holomorphic function f are not independent, but are interdependent. In other words one can say that the definition of holomorphy involves w and z *entirely*, wholly, not coordinate-wisely. This (trivial) observation will be helpful in realizing some essential aspects of what follows below.

I.5 Let now f be a holomorphic function in $\Omega \subset \mathbb{C}^m$, i.e., $\frac{\partial f}{\partial \bar{z}_1} = 0, \ldots, \frac{\partial f}{\partial \bar{z}_m} = 0$ in Ω, $m > 1$. Equivalently, there exist all complex **partial** derivates of the first order, with **no relations** between them. One sees immediately, hence, that the definition lacks the above described feature for $m = 1$: the definition includes certain conditions with respect to *each, partial* complex variable, z_k, and not with respect to the *entire* variable $z := (z_1, \ldots, z_m)$. Of course, this is related to the absence of two *mutually conjugate* operators factorizing the Laplace operator in $\mathbb{C}^m$. What is called *the Cauchy-Riemann conditions* in $\mathbb{C}^m$, should be more relevantly termed *partial Cauchy-Riemann conditions* to emphasize the difference in principle of both notions.

The idea of a holomorphic mapping loses much more from the original definition in $\mathbb{C}^1$. Indeed, if $\mathbf{F} = (f_1, \ldots, f_n)$ is a holomorphic mapping from $\Omega \subset \mathbb{C}^m$ into $\mathbb{C}^n$ then $\mathbf{F}$ keeps lacking any relation between complex partial derivatives of its components, and there are no relations, in general, between the components themselves.

I.6 Thus, looking for a *one-dimensional structure* in several complex variables we are going to depart from the following **heuristic reasonings**. Given a domain $\Omega \subset \mathbb{C}^m$, try to find the following objects:

1^0. A complex algebra $\mathcal{A}$ with unit, not necessarily commutative.

2^0. Two first-order partial differential operators with coefficients from $\mathcal{A}$, or from a wider algebra, denote them by $\mathfrak{D}$ and $\mathfrak{D}^*$, such that

$$\mathfrak{D} \circ \mathfrak{D}^* = \mathfrak{D}^* \circ \mathfrak{D} = \Delta_{\mathbb{C}^m} . \qquad \text{(I.6.1)}$$

The idea of such a factorization is very well known in partial differential equations (see, e.g., [T1], [T2] but many other sources as well), and the fine point is contained, of course, in the last condition:

3^0. Holomorphic functions and mappings should belong to $\ker\mathfrak{D}$, or to $\ker\mathfrak{D}^*$.

To show that such a program is feasible is the aim of this book.

It is meant neither that in this setting the problem has a unique solution nor that the general case of arbitrary mappings will be covered. Our algebra $\mathcal{A}$ consists of 2×2 matrices whose entries are taken from the Grassmann algebra generated by differential forms with complex-conjugate differentials only, that is, of type $(0, q)$ in conventional terminology. Notice that it is possible to consider 1×2 columns instead of matrices, but then we loose the structure of a complex algebra in the range of functions, for which reason we chose to work with matrices.

I.7 The book is organized as follows. Chapter 1 recalls some basic notation which is necessary to work with functions and differential forms in $\mathbb{C}^m$. Chapter 2 introduces the main object of the study, differential forms whose coefficients are 2×2 matrices, as well as the differential operators acting on such differential forms and possessing the basic property (I.6.1).

The latter are called *the hyperholomorphic Cauchy-Riemann operators*. The fine point here is that their (2×2)-matrix coefficients contain not only differential forms but the so-called contraction operators also; the deep reasons for that will be explained in Chapter 9: a *right* algebra should be generated not only by differential forms.

As a matter of fact, the structure of the hyperholomorphic Cauchy-Riemann operators determines a special structure of other (2×2)-matrices involved — in particular, a unit normal vector to a surface in $\mathbb{C}^m$ is represented as such a matrix, the representation itself being an operator, not a differential form with matrix coefficients. The same about *the hyperholomorphic Cauchy kernel*, which is an operator, not a differential form, and which can be considered as a kind of a fundamental solution but in a specified meaning. All this leads to the hyperholomorphic versions of both the Stokes formula and the Borel-Pompeiu integral representation of a smooth differential form (here with (2×2)-matrix coefficients, of course), i.e., those versions which are consistent with the hyperholomorphic Cauchy-Riemann operators. There is given a detailed analysis of the structure of the hyperholomorphic Borel-Pompeiu formula and of its intimate relation with the Bochner-Martinelli integral representation.

In Chapter 3, hyperholomorphic differential forms with (2×2)-matrix coefficients are introduced as null solutions of the hyperholomorphic Cauchy-Riemann operator. The class of such differential forms in a given domain includes both holomorphic and antiholomorphic functions (the latter considered as coefficients of specific differential forms), and all other *holomorphic-like* functions, i.e., those holomorphic with respect to certain variables and antiholomorphic with respect to the rest of them — all in the same domain and, again, taken as coefficients of specific differential forms. But this is not enough, and there are differential forms which do not correspond to any holomorphic-like functions. What is highly important here is the fact that just the whole class, not its more famous subclasses, preserves the deep similarity with the theory of holomorphic functions of one variable.

I.8 This similarity allows, in Chapters 4 through 7, to obtain quickly the main integral theorems. But even if, for instance, the Cauchy integral and the Morera theorems go in the usual way, anyhow certain peculiarities arise. The hyperholomorphic Cauchy-Riemann operator can be applied to a given matrix both on the left- and on the right-hand side.

There is no direct symmetry between left- and right-hand-side notions of hyperholomorphy, but we present versions of the Cauchy integral theorem and its inverse, the Morera theorem, which involves both types of hyperholomorphy.

The hyperholomorphic Cauchy integral formula (Chapter 6) represents any hyperholomorphic differential form as a surface integral with the hyperholomorphic Cauchy kernel. In particular, for holomorphic functions it reduces just to the Bochner-Martinelli integral representation of such functions which explains, in a certain sense, why the latter holds in spite of non-holomorphy of the Bochner-Martinelli kernel. One more manifestation of the above stated similarity is the solution of the non-homogeneous hyperholomorphic Cauchy-Riemann equation. In contrast to its counterpart for holomorphic Cauchy-Riemann equations, the hyperholomorphic case becomes trivial, since there exists a right inverse operator for the hyperholomorphic Cauchy-Riemann operator. All this is rigorously analyzed in Chapter 7, where many interpretations are also given, but the most remarkable applications are moved to the next Chapter.

I.9 In Chapter 8, differential forms are considered which are, simultaneously, $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. They form a subclass of hyperholomorphic differential forms, but they are of independent interest and of importance from the point of view of conventional multidimensional complex analysis. That is why we, first of all, describe the direct corollaries of the theorems which have been proved for general hyperholomorphic differential forms. What is more, there are several results here which may be viewed also as corollaries, being at the same time much less direct and evident. One of them concerns the $\overline{\partial}$-problem for functions and differential forms in an **arbitrary**, i.e., of an arbitrary shape, domain in $\mathbb{C}^m$ with a piecewise smooth boundary. There is given a necessary and sufficient condition on the given $(0,1)$-differential form g in order for the equation $\overline{\partial} f = g$ to have a solution which is a function. The condition is quite explicit and verifiable: a $(0,2)$-differential form whose coefficients are certain improper integrals of g should satisfy the complex Hodge-Dolbeault system, i.e., should be $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. A particular solution is again quite explicit, being a sum of improper integrals of the same type as above. If g is an **arbitrary** differential form (with smooth coefficients) then for the problem $\overline{\partial} f = g$ the necessary and sufficient condition obtained is not that explicit, but the particular solution has the same transparent structure as the one described above.

There exists a huge amount of literature on the $\overline{\partial}$-problem, see, e.g., [AiYu], [Ko], [Li], [Ky], [R], [Kr1], [Kr2], but in no way do we pretend that the above list is complete or even representative. It is a separate task to compare what has been obtained already on the $\overline{\partial}$-problem with the approach of this book.

I.10 In the same Chapter 8, we establish also a deep relation between solutions of the complex Hodge-Dolbeault system and the

Koppelman formula. The latter one is a representation of a smooth $(0, 2)$-differential form as a sum of a surface integral and of two volume integrals. For the case of functions, i.e., of $(0, 0)$-differential forms, the volume integrals disappear on holomorphic functions, and thus it is important to have a class of differential forms on which the volume integrals in the Koppelman formula disappear also. We show that the Koppelman formula is a particular case of the hyperholomorphic Borel-Pompeiu integral representation, which leads immediately to the conclusion that the volume integrals in the Koppelman formula are annihilated by the solutions of the complex Hodge-Dolbeault system.

We believe this will have deep repercussions for the theory of complex differential forms.

I.11 Although all the eight first chapters are written in the language of complex analysis, the underlying ideas were inspired by the authors' experience in research in Clifford and quaternionic analysis. What is the direct relation between those, at the present time, formally different areas of analysis is explained in Chapter 9. It appears that the hyperholomorphic theory restricted onto (2×2) matrices with equal rows is isomorphic to the function theory for the Dirac operator of Clifford analysis, see the books [BrDeSo], [DeSoSo], [Mit], [KrSh], [GüSp1], [GüSp2], [GiMu]. But we refer to many articles as well; other important aspects of the Dirac operators one can find in [BeGeVe] for instance. The general case of (2×2) matrices does not reduce to the theory of **one** Dirac operator but is a kind of a direct sum of the theories for **two** Dirac-like operators considered in the same domain of $\mathbb{C}^m$. The peculiarity of this relation is the necessity to use not the canonical basis of the Clifford algebra but the so-called Witt basis which fits perfectly well into the complex analysis setting. What is more, one half of the elements of the Witt basis generates the algebra of elementary differential forms while the other half generates the contraction operators. Hence the function theory using only differential forms lacks the *symmetry* of Clifford analysis, which causes new phenomena, such as, for instance, the fact that the hyperholomorphic Cauchy kernel is an operator, not a differential form.

I.12 Only small fragments of the book have been published already [RSS2], [RSS3], but the joint article by the authors [RSS1] may be considered as directly antecedent to the book; what is more, it may be seen as a direct impulse to realizing certain important ideas of it. At the same time, in their preceding separate works one can find many observations, hints, and indications on the relations between several complex variables theory and Clifford analysis ideas: F. Sommen treated those relations in [So1] (considering integral transform between monogenic functions of Clifford analysis and holomorphic functions of several complex variables), [So2] (deriving the Bochner-Martinelli formula), [So3]–[So5], see also the books [BrDeSo] and [DeSoSo]; M. Shapiro treated the applications of quaternionic analysis to holomorphic functions in $\mathbb{C}^2$ in joint papers with N. Vasilevski [VaSh1], [VaSh2], [VaSh3] and with I. Mitelman [MiSh1], [MiSh2]; see also the paper [Sh1]; the papers by M. Shapiro [Sh2] and by R. Rocha-Chávez and M. Shapiro [RoSh1], [RoSh2] do not have any direct relation to several complex variables, but they contain several important ideas which were very helpful in realizing some essential aspects of the book.

We know of not too many other papers on the topic. J. Ryan in [Ry1], [Ry2] considered a subclass of holomorphic functions for which a function theory is valid with the structure quite similar to that of Clifford analysis. V. Baikov [Ba] and V. Vinogradov [Vi] considered boundary value properties of holomorphic functions in, respectively, $\mathbb{C}^2$ and $\mathbb{C}^m$ using ideas from quaternionic and Clifford analysis. Quite recently S. Bernstein [Be] and G. Kaiser [Ka] found new connections between holomorphic functions and Clifford analysis.

I.13 In the course of the preparation of the book the Mexican authors were partially supported by CONACYT in the framework of its various projects and by the Instituto Politécnico Nacional via CGPI and COFAA programs, and they are indebted to those bodies.

Chapter 1

Differential forms

1.1 Usual notation

We shall denote by $\mathbb{C}$ the field of complex numbers, and by $\mathbb{C}^m$ the m-dimensional complex Euclidean space. If $z \in \mathbb{C}^m$, then by $z_1, \ldots,$ z_m we denote the canonical complex coordinates of z. For $z,\ z' \in \mathbb{C}^m$ we write:

$$\begin{aligned} \bar{z} &:= (\bar{z}_1,\ \ldots,\ \bar{z}_m), \\ \langle z,\ z' \rangle &:= z_1 z_1' + \cdots + z_m z_m', \\ |z| &:= \left(|z_1|^2 + \cdots + |z_m|^2 \right)^{\frac{1}{2}} = \sqrt{\langle z,\ \bar{z} \rangle}. \end{aligned}$$

$\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}^m$ denotes the m-dimensional real Euclidean space.

Topology in $\mathbb{C}^m$ is determined by the metric $d(z,\ z') := |z - z'|$.

Orientation on $\mathbb{C}^m$ is defined by the order of coordinates $(z_1,\ \ldots,$ $z_m)$, which means that the differential form of volume is

$$dV := (-1)^{\frac{m(m-1)}{2}} \frac{(-1)^m}{(2i)^m} dz \wedge d\bar{z} = (-1)^{\frac{m(m-1)}{2}} \frac{1}{(2i)^m} d\bar{z} \wedge dz,$$

where

$$\begin{aligned} dz &:= dz^1 \wedge \ldots \wedge dz^m, \\ d\bar{z} &:= d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^m. \end{aligned}$$

If $z \in \mathbb{C}^m$ then

$$\begin{aligned} x_j &:= Re\,(z_j) \in \mathbb{R}, \\ y_j &:= Im\,(z_j) \in \mathbb{R}. \end{aligned}$$

So, one can write $z = (x_1 + y_1 \cdot i,\ \ldots,\ x_m + y_m \cdot i)$. Hence $\mathbb{C}^m \cong \mathbb{R}^{2m}$ as oriented real Euclidean spaces, where the orientation in $\mathbb{R}^{2m}$ is defined by the order of coordinates $(x_1,\ y_1,\ \ldots,\ x_m,\ y_m)$, which means that the differential form of volume on $\mathbb{R}^{2m}$ is $dx^1 \wedge dy^1 \wedge \ldots \wedge dx^m \wedge dy^m$.

The word *domain* means an **arbitrary** (not necessarily connected) open set. The word *neighborhood* means an **open** neighborhood.

Some more standard notations:

1. $\mathbb{N}$ denotes the set of all positive integers,
2. $\mathbb{B}\,(z;\ \varepsilon) := \{\zeta \in \mathbb{C}^m \mid |z - \zeta| < \varepsilon\}$,
3. $\mathbb{S}\,(z;\ \varepsilon) := \{\zeta \in \mathbb{C}^m \mid |z - \zeta| = \varepsilon\}$,
4. $E_{2\times 2} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
5. $\breve{E}_{2\times 2} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Mention that $\breve{E}^2_{2\times 2} = E_{2\times 2}$.

1.2 Complex differential forms

The term **"differential form"** (or simply **"form"** and **d.f.** sometimes) will be used for differential forms with measurable complex-valued coefficients. The support of a differential form F will be denoted by $supp\,(F)$. For a fixed $k \in \mathbb{N}$, C^k-forms are those forms with k times continuously differentiable coefficients (this definition is independent of the local coordinate system of class C^{k+1}). Continuous forms will be called also C^0-forms, and $F \in C^\infty$ means that F is a form of class C^k for any $k \in \mathbb{N}$.

A form F of class C^k defined on $\mathbb{C}^m$ is called an $(r,\ s)$-form (i.e., a form of bidegree $(r,\ s)$) if, with respect to local coordinates $(z_1,\ \ldots,\ z_m)$ of class C^{k+1}, $0 \le k \le \infty$, it is represented as

$$F\left(z\right) = \sum_{|\mathbf{j}|=r,\ |\mathbf{k}|=s} F_{\mathbf{jk}}\left(z\right) dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}, \tag{1.2.1}$$

where the summation runs over all strictly increasing r-tuples $\mathbf{j} = (j_1,\ \ldots,\ j_r)$ and all strictly increasing s-tuples $\mathbf{k} = (k_1,\ \ldots,\ k_s)$ in $\{1,\ \ldots,\ m\}$, and $dz^{\mathbf{j}} := dz^{j_1} \wedge \ldots \wedge dz^{j_r}$, $d\bar{z}^{\mathbf{k}} := d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_s}$, with the coefficients $F_{\mathbf{jk}}$ being complex-valued functions of class C^k.

It is worthwhile to note that although we use the same letter z both for independent variable and for differentials dz^q, $d\bar{z}^p$, it is sometimes convenient and necessary to distinguish between them, so we will write $d\zeta^q$, $d\bar{\zeta}^p$ or dw^q, $d\bar{w}^p$, etc. This causes no abuse of notation, because these differentials do not depend on z. In that occasion, we will write $F\left(z,\ d\zeta,\ d\bar{\zeta}\right)$ instead of $F\left(z\right)$.

1.3 Operations on complex differential forms

Consider the following important differential operators. The linear contraction operators $\widehat{d\bar{z}^q}$ and $\widehat{dz^q}$ are defined as endomorphisms by their action on the generators:

1. if $q = k_p$, then

$$\widehat{d\bar{z}^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{d\bar{z}^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} :=$$

$$:= (-1)^{|\mathbf{j}|+p-1} dz^{\mathbf{j}} \wedge d\bar{z}^{k_1} \wedge \ldots \wedge d\bar{z}^{k_{p-1}} \wedge \bar{z}^{k_{p+1}} \wedge \ldots \wedge d\bar{z}^{k_s},$$

2. if $q \notin \{k_1,\ \ldots,\ k_s\}$, then

$$\widehat{d\bar{z}^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{d\bar{z}^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} := 0,$$

3. if $q = j_p$, then

$$\widehat{dz^q}\left[dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}}\right] := \widehat{dz^q} \wedge dz^{\mathbf{j}} \wedge d\bar{z}^{\mathbf{k}} :=$$

$$:= (-1)^{p-1} dz^{j_1} \wedge \ldots \wedge z^{j_{p-1}} \wedge dz^{j_{p+1}} \wedge \ldots \wedge dz^{j_r} \wedge d\bar{z}^{\mathbf{k}},$$