

# **Commutative Algebra**

Geometric, Homological,  
Combinatorial and  
Computational Aspects

Edited by

**Alberto Corso**

**Philippe Gimenez**

**Maria Vaz Pinto**

**Santiago Zarzuela**



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# **Commutative Algebra**

Geometric, Homological,  
Combinatorial and  
Computational Aspects

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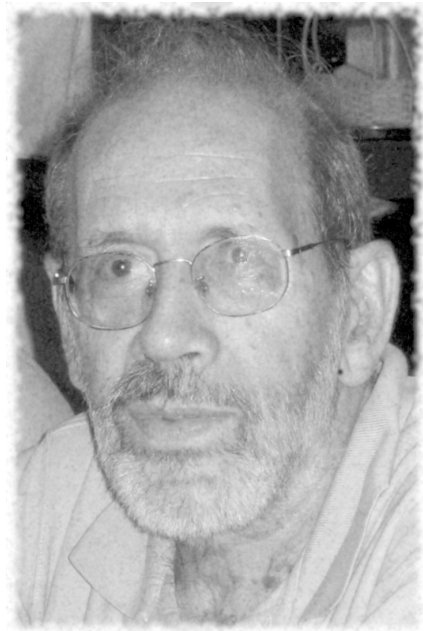
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This collective volume is dedicated to  
WOLMER V. VASCONCELOS  
for being such a wonderful mathematician, mentor, colleague and friend.

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# Preface

The last decades have seen a great deal of activities and developments in several areas at the interface of commutative algebra and algebraic geometry. Exciting new results have been achieved “borrowing” methods proper to neighboring fields such as combinatorics, homological algebra, polyhedral geometry, symbolic computation, and topology. This volume – *Commutative Algebra: Geometric, Homological, Combinatorial and Computational Aspects* – would like to highlight some of these proficuous interactions by gathering a collection of refereed research papers which encompass several of those aspects. Blowup algebras, Castelnuovo-Mumford regularity, integral closure and normality, Koszul homology, liaison theory, multiplicities, polarization, and reductions of ideals are some of the topics featured in the fifteen original research articles included in this volume. Three survey articles on topics of current interest also enrich the publication: Poincaré series of singularities, uniform Artin-Rees theorems, and Gorenstein rings. Most of the results published in this volume were presented during two meetings that were held during the month of June 2003, in Spain and Portugal. Namely,

- *Commutative Algebra: Geometric, Homological, Combinatorial and Computational Aspects*, one of the Special Sessions held during the First Joint Meeting of the American Mathematical Society and the Real Sociedad Matemática Española in Sevilla, Spain, in the period June 18-21, 2003;
- *Lisbon Conference on Commutative Algebra*, held at the Instituto Superior Técnico in Lisbon, Portugal, in the period June 23-27, 2003.

These conferences are part of a long-established tradition of Western European summer meetings in commutative algebra and algebraic geometry. Besides the traditional opportunity for experts from the U.S. and Western Europe to gather



and exchange the most recent discoveries, special attention was paid to allow for the participation of scholars from underrepresented countries. Overall, more than one hundred participants came together from numerous parts of the World: Brazil, Canada, France, Germany, Greece, India, Iran, Italy, Japan, Mexico, The Netherlands, Norway, Portugal, Romania, Spain, Switzerland, United Kingdom, United States of America, and Vietnam. This certainly enhanced the breath and scope of the two events.

Needless to say, we would like to express our gratitude to the contributors of this volume for their enthusiasm in the project. The anonymous referees, who worked very closely with us, also deserve a special credit for all their time spent in reading and correcting the original manuscripts: We are aware of the many demands on our time that the academic profession requires from each of us. Finally, we would also like to take the opportunity that this Preface offers us to personally acknowledge the many colleagues and institutions that contributed to the success of these two conferences. First, we would like to express our thanks to Joan Elias (Universitat de Barcelona), John Greenlees (University of Sheffield), Giuseppe Valla (Università di Genova), Wolmer V. Vasconcelos, and Charles Weibel (Rutgers University) for serving as members of the Scientific Committee of the Lisbon Conference. A special mention goes to Pedro Ferreira dos Santos, Gustavo Granja, Michael Paluch and the entire staff of the Instituto Superior Técnico for their endless help and support in planning and organizing the Lisbon Conference from the very first stage. Finally, we wish to express our heartfelt gratitude to the following institutions for their generous financial support: Banco BPI, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos (Instituto Superior Técnico), Fundação Calouste Gulbenkian, Fundação Luso-Americana para o Desenvolvimento, Fundação Oriente, Fundação para a Ciência e a Tecnologia, Universitat de Barcelona and Universidad de Valladolid.

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March 15, 2005

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# List of Sevilla's Talks

## **Some invariants of local rings**

JOSEP ÀLVAREZ, Universitat Politècnica de Catalunya, Spain

## **Saturation and Castelnuovo-Mumford regularity**

ISABEL BERMEJO, Universidad de la Laguna, Spain

## **Multiplier ideals in exotic settings**

ANA BRAVO, Universidad Autónoma de Madrid, Spain

## **Asymptotic behavior of cohomology**

MARKUS BRODMANN, Universität Zürich, Switzerland

## **The structure of the Rao module and the geometry of schemes**

MARTA CASANELLAS, Universitat de Barcelona, Spain

## **Liaison of varieties of small dimension and deficiency modules**

MARC CHARDIN, CNRS and Université Paris VI, France

## **Deformations of monomial ideals**

ALDO CONCA, Università di Genova, Italy

## **Poincaré series of resolution of surface singularities**

STEVEN DALE CUTKOSKY, University of Missouri

## **Linearly presented ideals**

DAVID EISENBUD, MSRI and University of California at Berkeley

**Secant varieties to Grassmann varieties**

ANTHONY GERAMITA, Queen's University at Kingston, Ontario, and Università di Genova, Italy

**Positive combinatorial formulae for quiver polynomials**

EZRA MILLER, MSRI and University of Minnesota

**Monomial ideals and their core**

CLAUDIA POLINI, University of Notre Dame

**The equality  $I^2 = QI$  in Buchsbaum rings**

HIDETO SAKURAI, Meiji University, Japan

**On birational Macaulayfications and Cohen-Macaulay canonical modules**

PETER SCHENZEL, Martin-Luther-Universität Halle-Wittenberg, Germany

**Ideals of linear type in Cremona maps**

ARON SIMIS, Universidade Federal de Pernambuco, Brazil

**A formula for the core of ideals**

BERND ULRICH, Purdue University

**Divisors of integrally closed modules**

WOLMER V. VASCONCELOS, Rutgers University

**Resolution of singularities: Computational aspects**

ORLANDO VILLAMAYOR, Universidad Autónoma de Madrid, Spain

**Monomial ideals and normality**

RAFAEL H. VILLARREAL, CINVESTAV, Instituto Politécnico Nacional, Mexico

# List of Lisbon's Talks

## **The structure of F-pure rings**

IAN ABERBACH, University of Missouri

## **Cohomological and geometric periodicity**

ALEJANDRO ADEM, University of Wisconsin-Madison

## **Homology over local homomorphisms**

LCHEZAR AVRAMOV, University of Nebraska

## **A simplicial description of Kähler differentials**

JOSÉ I. BURGOS GIL, Universitat de Barcelona, Spain

## **Gin, lex and Koszul homology**

ALDO CONCA, Università di Genova, Italy

## **Multiplicity of the special fiber of blowups**

ALBERTO CORSO, University of Kentucky

## **Initial ideal and its associated graded ring**

EMANUELA DE NEGRI, Università di Genova, Italy

## **Computing the Castelnuovo-Mumford regularity of a homogeneous ideal**

PHILIPPE GIMENEZ, Universidad de Valladolid, Spain

## **The equality $I^2 = QI$ in Buchsbaum rings**

SHIRO GOTO, Meiji University, Japan

**Detecting the depth of group cohomology**

DAVID J. GREEN, Bergische Universität Wuppertal, Germany

**Rigid resolutions and big Betti numbers**

JÜRGEN HERZOG, Universität GH-Essen, Germany

**Linear strands in minimal resolutions**

SRIKANTH IYENGAR, University of Missouri

**The module structure of a group action on a polynomial ring:****A finiteness theorem**

DIKRAN KARAGUEUZIAN, SUNY Binghamton

**Bousfield localizations in commutative algebra**

ANA JEREMÍAS LÓPEZ, Universidade de Santiago de Compostela, Spain

**Fitting ideals and finite projective dimension over complete intersections**

DANIEL KATZ, University of Kansas

**The support of top graded local cohomology modules**

MORDECHAI KATZMAN, University of Sheffield, United Kingdom

**Local cohomology and its applications**

GENNADY LYUBEZNIK, University of Minnesota

**The canonical ring of surfaces with  $p_g = 0$** 

MARGARIDA MENDES LOPES, Universidade de Lisboa, Portugal

**On quasi-simplicial toric varieties**

MARCEL MORALES, Institut Fourier, Grenoble, France

**On the uniform Artin-Rees properties**

FRANCESC PLANAS VILANOVA, Universitat Politècnica de Catalunya, Spain

**Core of monomial ideals**

CLAUDIA POLINI, University of Notre Dame

**On generic initial ideals**

SORIN POPESCU, State University of New York at Stony Brook

**Hilbert-Samuel polynomial of a Cohen-Macaulay module and applications to the fiber cone**

MARIA EVELINA ROSSI, Università di Genova, Italy

**Ideals in the perfect closure, linear growth of primary decompositions and the localization problem for tight closure**

RODNEY Y. SHARP, University of Sheffield, United Kingdom

**Algebraic free divisors and the Jacobian ideal**

ARON SIMIS, Universidade Federal de Pernambuco, Brazil

**Multiplicities of monomial ideals**

HEMA SRINIVASAN, University of Missouri

**Stanley-Reisner rings and the radical of lattice ideals**

APOSTOLOS THOMA, University of Ioannina, Greece

**Mixed multiplicities of ideals and mixed volumes of polytopes**

NGÔ VIỆT TRUNG, Institute of Mathematics, Hanoi, Vietnam

**Linearly presented ideals and a subadditivity formula for the degrees of syzygies**

BERND ULRICH, Purdue University

**Big Degr and applications**

WOLMER V. VASCONCELOS, Rutgers University

**Normality of Rees algebras of monomial ideals**

RAFAEL VILLARREAL, CINVESTAV, Instituto Politécnico Nacional, Mexico

**Building interesting commutative rings**

SYLVIA WIEGAND, National Science Foundation and University of Nebraska

**Degeneration and G-dimension of modules**

YUJI YOSHINO, Okayama University, Japan



# A theorem of Eakin and Sathaye and Green's hyperplane restriction theorem

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**Abstract:** A theorem of Eakin and Sathaye relates the number of generators of a certain power of an ideal with the existence of a distinguished reduction for that ideal. We prove how this result can be obtained as a special case of Green's hyperplane restriction theorem.

## 1 Introduction

The purpose of these notes is to show how the following Theorem 2.1, due to Eakin and Sathaye, can be viewed, after some standard reductions, as a corollary of Green's hyperplane restriction theorem.

**Theorem 2.1 (Eakin-Sathaye)** *Let  $(R, m)$  be a quasi-local ring with infinite residue field. Let  $I$  be an ideal of  $R$ . Let  $n$  and  $r$  be positive integers. If the*

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number of minimal generators of  $I^i$ , denoted by  $v(I^i)$ , satisfies

$$v(I^i) < \binom{i+r}{r},$$

then there are elements  $h_1, \dots, h_r$  in  $I$  such that  $I^i = (h_1, \dots, h_r)I^{i-1}$ .

Before proving Theorem 2.1 we recall some general facts about Macaulay representation of integer numbers: this is needed for the understanding of Green's hyperplane restriction theorem. For more details on those topics we refer the reader to [3] and [4].

## 1.1 Macaulay representation of integer numbers

Let  $d$  be a positive integer. Any positive integer  $c$  can then be uniquely expressed as

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_1}{1},$$

where the  $k_i$ 's are non-negative and strictly increasing, i.e.,  $k_d > k_{d-1} > \dots > k_1 \geq 0$ . This way of writing  $c$  is called the  $d$ 'th *Macaulay representation* of  $c$ , and the  $k_i$ 's are called the  $d$ 'th *Macaulay coefficients* of  $c$ . For instance, setting  $c = 13$  and  $d = 3$  we get  $13 = \binom{5}{3} + \binom{3}{2} + \binom{0}{1}$ .

**Remark 1.1** An important property of Macaulay representation is that the usual order on the integers corresponds to the lexicographical order on the arrays of Macaulay coefficients. In other words, given two positive integer  $c_1 = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_1}{1}$  and  $c_2 = \binom{h_d}{d} + \binom{h_{d-1}}{d-1} + \dots + \binom{h_1}{1}$  we have  $c_1 < c_2$  if and only if  $(k_d, k_{d-1}, \dots, k_1)$  is smaller lexicographically than  $(h_d, h_{d-1}, \dots, h_1)$ .

**Definition 1.2** Let  $c$  and  $d$  be positive integers. We define  $c_{<d>}$  to be

$$c_{<d>} = \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \dots + \binom{k_1-1}{1}$$

where  $k_d, \dots, k_1$  are  $d$ 'th Macaulay coefficients of  $c$ . We use the convention that  $\binom{a}{b} = 0$  whenever  $a < b$ .

**Remark 1.3** It is easy to check that if  $c_1 \leq c_2$  then  $c_{1<d>} \leq c_{2<d>}$ . This property, as we see in the following, allows us to iteratively apply Green's Theorem and prove Corollary 1.5.

## 1.2 Green's hyperplane restriction theorem

Let  $R$  be a standard graded algebra over an infinite field  $K$ . We can write  $R$  as  $K[X_1, \dots, X_n]/I$  where  $I$  is an homogeneous ideal not containing any linear forms. We say that  $r$  generic homogeneous linear forms of  $R$  satisfy a certain property if there exists a Zariski open set of  $(\mathbb{A}^n)^r$  whose points satisfy that property. Note that each point of  $(\mathbb{A}^n)^r$  can be thought of as an  $r$ -uple of homogeneous linear forms. Given a generic homogeneous linear form  $L$  we let  $R_L = K[X_1, \dots, X_{n-1}]/I_L$  denote the restriction of  $R$  to the hyperplane given by  $L$ . Note that since  $L$  is generic we can write it as  $L = l_1X_1 + \dots + l_nX_n$  where  $l_n \neq 0$ , therefore  $I_L$  is defined as

$$I_L = (P(X_1, \dots, X_{n-1}, X_n - (L/l_n)) | P \in I).$$

We will denote by  $R_d$  the  $d$ 'th graded component of  $R$ . Mark Green proved the following Theorem.

**Theorem 1.4 (Green's hyperplane restriction theorem)** *Let  $R$  be a standard graded algebra over an infinite field  $K$ , and let  $L$  be a generic homogeneous linear form of  $R$ . Setting  $S$  to be  $R_L$ , we have*

$$\dim_K S_d \leq (\dim_K R_d)_{<d>}.$$

The hyperplane restriction theorem first appeared in [4], where it was proved with no assumption on the characteristic of the base field  $K$ .

A different, and more combinatorial, proof can be found in [3] where the characteristic zero assumption is a working hypothesis. A person interested in reading this last proof can observe that the arguments in [3] also work in positive characteristic with a few minor changes.

A direct corollary of Green's Theorem is the following:

**Corollary 1.5** *Let  $R$  be a standard graded algebra over an infinite field  $K$ , and let  $L_1, \dots, L_r$  be generic homogeneous linear forms of  $R$ . Let  $\binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_1}{1}$  be the Macaulay representation of  $\dim R_d$ , and define  $S = R/(L_1, \dots, L_r)$ . Then*

$$\dim_K S_d \leq \binom{k_d - r}{d} + \binom{k_{d-1} - r}{d-1} + \dots + \binom{k_1 - r}{1}.$$

**Proof:** Note that the above inequality gives an open condition and therefore it is enough to show that there exist  $r$  homogeneous linear forms satisfying it. Let  $L$  be as in Theorem 1.4. Since  $R_L$  is isomorphic to  $R/(L)$  by Theorem 1.4 one deduces  $\dim_K(R/(L))_d \leq \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_1-1}{1}$ . On the other hand, by Remark 1.3 we can apply Green's Theorem again and obtain the result by induction.  $\square$

## 2 The Eakin-Sathaye Theorem

We now prove Theorem 2.1. First of all note that since  $v(I^i)$  is finite, without loss of generality we can assume that  $I$  is also finitely generated: in fact, if  $J \subseteq I$  is a finitely generated ideal such that  $J^i = I^i$ , the result for  $J$  implies the one for  $I$ . Moreover, by the use of Nakayama's Lemma, we can replace  $I$  by the homogeneous maximal ideal of the fiber cone  $S = \bigoplus_{i \geq 0} I^i/mI^i$ . Note that  $S$  is a standard graded algebra finitely generated over the infinite field  $R/m = K$ .

Theorem 2.1 can be rephrased as:

**Theorem 2.1 (Eakin-Sathaye)** *Let  $R$  be a standard graded algebra finitely generated over an infinite field  $K$ . Let  $i$  and  $r$  be positive integers such that*

$$\dim_K(R_i) < \binom{i+r}{r}.$$

*Then there exist homogeneous linear forms  $h_1, \dots, h_r$  such that  $(R/(h_1, \dots, h_r))_i$  is equal to zero.*

**Proof:** First of all note that  $\dim_K R_i \leq \binom{i+r}{r} - 1 = \binom{i+r}{i} - 1 = \binom{i+r-1}{i} + \binom{i+r-2}{i-1} + \cdots + \binom{i+r-j}{i-j+1} + \cdots + \binom{r}{1}$ . This well-known equality can be proven by recalling that  $\binom{a+b}{a}$  is the number of monomials of degree  $a$  in  $b+1$  variables. The binomial coefficient  $\binom{i+r}{i}$  is interpreted as monomials in  $r+1$  variables and split into parts by fixing the exponent of one of the variables from 0 up to  $i$ .

By Corollary 1.5, taking  $r$  generic homogeneous linear forms  $h_1, \dots, h_r$ , we have

$$\dim_K(R/(h_1, \dots, h_r))_i \leq \binom{i-1}{i} + \binom{i-2}{i-1} + \cdots + \binom{0}{1}.$$

The term on the right-hand side is zero and therefore the theorem is proved.  $\square$

## References

- [1] P. Eakin and A. Sathaye, Prestable ideals, *J. Algebra* **45** (1976), 439–454.
- [2] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.
- [3] M. Green, Generic initial ideals, Six lectures on commutative algebra (Bellaterra, 1996), 119–186, *Progr. Math.* **166**, Birkhäuser, Basel, 1998.
- [4] M. Green, Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann, Algebraic curves and projective geometry (Trento, 1988), 76–86, *Lecture Notes in Math.* **1389**, Springer, Berlin, 1989.
- [5] J. Sally, *Numbers of Generators of Ideals in Local Rings*, Marcel Dekker, Inc., New York-Basel, 1978.

