Volume 242

A Series of Lecture Notes in Pure and Applied Mathematics

# Control Theory of Partial Differential Equations


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# Control Theory of Partial Differential Equations

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Chapman & Hall/CRC
Taylor & Francis Group
Boca Raton London New York Singapore

Published in 2005 by
Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742


Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-10: 0-8247-2546-8 (Softcover)
International Standard Book Number-13: 978-0-8247-2546-4 (Softcover)



**Library of Congress Cataloging-in-Publication Data**

Catalog record is available from the Library of Congress

T&F informa

Taylor & Francis Group
is the Academic Division of T&F Informa plc.

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# *Preface*

The present volume contains contributions by participants in the "Conference on Control Theory for Partial Differential Equations," which was held over a two-and-a-half day period, May 30 to June 1, 2003, at Georgetown University, Washington, D.C. The conference was dedicated to the occasion of the retirement of Professor Jack Lagnese from the Mathematics Department of Georgetown University.

It seemed most appropriate to honor the productive and successful scientific career of Jack Lagnese by convening a conference that would bring together a select group of international specialists in the theory of partial differential equations and their control. Over the years, many of the invitees have enjoyed a personal and professional association with Jack. The lasting impact of Jack's contributions to control theory of partial differential equations and applied mathematics is well documented by over 80 research articles and three books. In addition, Jack served the scientific community for many years in his capacity, at various times, as a program director in the Applied Mathematics Program within the National Science Foundation, as an editor on the boards of several journals, as editor-in-chief of the *SIAM Journal on Control and Optimization*, and as president of the SIAM Activity Group on Control and Systems Theory. He was also a consultant to The National Institute for Standards and Technology for a number of years.

Control theory for distributed parameter systems, and specifically for systems governed by partial differential equations, has been a research field of its own for more than three decades. Although having a distinctive identity and philosophy within the theory of dynamical systems, this field has also contributed to the general theory of partial differential equations. Optimal interior and boundary regularity of mixed problems, global uniqueness issues for over-determined problems and related Carleman estimates, various types of *a priori* inequalities, and stability and long-time behavior are just some examples of important developments in the theory of partial differential equations arising from control theoretic considerations. In recent years, the field has broadened considerably as more realistic models have been introduced and investigated in areas such as elasticity, thermoelasticity, and aeroelasticity; in problems involving interactions between fluids and elastic structures; and in other problems of fluid dynamics, to name but a few. These new models present fresh mathematical challenges. For example, the mathematical foundations of fundamental theoretical issues have to be developed, and conceptual insights that are useful to the designer and the practitioner need to be provided. This process leads to novel numerical challenges that must also be addressed. The papers contained in this volume provide a broad range of significant recent developments, new discoveries, and mathematical tools in the field and further point to challenging open problems.

The conference was made possible through generous financial support by the National Science Foundation and Georgetown University, whose sponsorship is greatly appreciated.

We wish to thank Marcel Dekker for agreeing to include this volume in its well-known and highly regarded series "Lecture Notes in Pure and Applied Mathematics" and for its high professional standards in handling this volume.

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# *Contents*

# Chapter 1

## Asymptotic Rates of Blowup for the Minimal Energy Function for the Null Controllability of Thermoelastic Plates: The Free Case

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**Abstract** Continuing the analysis undertaken in References 8 and 9, we consider the null-controllability problem for thermoelastic plate partial differential equations (PDEs) models in the absence of rotational inertia, defined on a two-dimensional domain $\Omega$, and subject to the *free* mechanical boundary conditions of second and third order. It is now known that such uncontrolled systems generate *analytic* semigroups on finite energy spaces. Consequently, the concept of *null* controllability is indeed an appropriate question for consideration. It is shown that all finite energy states can be driven to zero by means of $L^2[(0, T) \times \Omega]$ controls in either the mechanical or thermal component. However, the main intent of the paper is to quantify the singularity, as $T \downarrow 0$, of the minimal energy function relative to null controllability. In particular we shall show that in the case of one control function acting upon the system, the singularity of minimal energy is optimal; it is in fact of order $\mathcal{O}(T^{-\frac{5}{2}})$, which is the same rate of blowup as that of any finite dimensional approximation of the problem. The PDE estimates, which are obtained in the process of deriving this sharp numerology, will have a strong bearing on regularity properties of related stochastic differential equations.

[1]The research of George Avalos was partially supported by NSF DMS-0208121.
[2]The research of Irena Lasiecka was partially supported by NSF DMS-01043 and ARO DAAD19-02-1-0179.

## 1.1 Introduction

In this chapter we address specific questions related to the null controllability of thermoelastic plates subject to *free mechanical boundary conditions*, these being represented by shear forces and moments. These particular boundary conditions are of particular interest in the control theory of plates [22, 24, 23]. As we shall see below, the model under consideration is one which corresponds to an infinite speed of propagation; accordingly, null controllability—in arbitrarily short time—is an appropriate topic for study in regard to these plates. We will give at length a full and precise description of our thermoelastic control problem; but for the benefit of the reader and in order to motivate the specific problem under study, we will first provide a few opening remarks.

### 1.1.1 Motivation

There are several ways of controlling a given plate dynamic. This control can be accomplished by using: 1. internal controls, 2. boundary controls, or 3. controls localized on an open subset of $\Omega$. In addition, one may use either one control action (be it thermal or mechanical) or simultaneous mechanical and thermal controls (i.e., controls located on *both* the mechanical and thermal components of the system). Depending on the objective to be achieved, one framework of control might be more advantageous than another. For instance, if the particular issue at hand is to guarantee the minimal support of control functions, then boundary control would be the most appropriate control situation. However, if one is concerned with the cost of control—or equivalently, with quantifying the associated "minimal energy"—then internal controls should be considered. In this connection, a question of both practical and mathematical relevance is the question of finding the optimal asymptotics that describe the singularity of the associated minimal energy, as $T \downarrow 0$. Since the work of T. Seidman in Reference 34, the optimal asymptotics are well defined and well known for *finite dimensional* control systems. In fact, these asymptotics are given by the *sharp* formula $T^{-k-\frac{1}{2}}$ where index $k$ corresponds to the Kalman rank condition and measures the defect of controllability (see below). The above formula actually gives a *lower* bound for the singularity of the minimal energy associated with any PDE system.

Given then the existence of formula in Reference 34 for controlled finite dimensional systems, we are in a position to loosely define the "optimal" singularity for any controlled PDE. In fact, for a given infinite dimensional system, *the "optimal" rate of singularity of its associated minimal energy will be the rate of singularity enjoyed by approximating (or truncated) finite dimensional systems* (assuming of course that each finite dimensional truncation has the same Kalman rank). For example, scalar first order (in time) models will have its optimal rate of blowup of minimal energy as being $\mathcal{O}(T^{-\frac{3}{2}})$; in general, the optimal singularity for vectorial coupled structures will depend on the number of controls used with respect to number of interactions. Thus, in the case of thermal plates with one control only, the optimal singularity of any finite dimensional truncation is $T^{-\frac{5}{2}}$ (this is seen below). In the case of two controls used (both thermal and mechanical) the optimal singularity is $T^{-\frac{3}{2}}$. Whether, however, the minimal energy asymptotics actually obeys the optimal rate of singularity (predicted from finite dimensions) is an altogether different matter. Indeed, in References 34 and 36 (highly nontrivial) finite-dimensional estimates are derived and can be subsequently applied to finite-dimensional truncations of infinite-dimensional systems; however, the delicate estimates are controlled by a constant $C_n$, say, where $n$ stands for the dimensionality of the respective approximation. These constants may well tend to infinity as $n$ goes to infinity. In such an event (as seen in References 14, 6, and 40) the optimal asymptotics for the original PDE are lost. This brings us to the key question asked in this chapter: *Is it possible to achieve the optimal rate of singularity for a (fully infinite dimensional) controlled PDE model*?

The answer to the above question—in the negative—has been known for many years in the case of the heat equation with either boundary or localized controls. Indeed, the rate for boundary control of the heat equation is the exponential blowup rate $e^{\mathcal{O}(\frac{1}{T})}$; see References 35 and 37. This rate is known to be sharp [20]. A similar negative answer has been provided in the case of thermoelastic systems under the influence of boundary controls—in fact, such boundary controls likewise lead to $e^{\mathcal{O}(\frac{1}{T})}$ exponential blowup [25]. Therefore, in light of the rational rates of minimal energy blowup exhibited by finite-dimensional controlled systems (as shown in Reference 34) and of the definition given above for optimal rates of minimal energy blowup for controlled PDEs, it is manifest that thermoelastic plates under the influence of boundary or localized controls will not give rise to minimal energies that exhibit an optimal (finite dimensional) singularity. Thus, in searching for PDE control situations, which will yield up the optimal algebraic singularity enjoyed by finite dimensional truncations, the only reasonable choice left is the implementation of *internal* controls. In the specific context of our thermoelastic PDE, the relevant question then becomes: *Do the minimal energies of internally controlled (fully infinite dimensional) thermoelastic plates exhibit the optimal rate of blowup* $O(T^{-\frac{5}{2}})$ *by either mechanical or thermal control*?

The relevance of this question should not be underestimated from both a practical and mathematical point of view. Indeed, from a practical point of view one would like to know whether a given finite-dimensional approximation of the system contains critical information and moreover reflects controllability properties of the original PDE model. From a mathematical point of view, the solution to the null controllability problem is not only of interest in its own right as an issue in control theory, but this solution can also give rise to deep and significant connections between the algebraic optimal singularity of minimal energy and other fields of analysis, including stochastic analysis. In point of fact, within the field of stochastic differential equations, there is an acute need to know of those PDE control environments that will yield up optimal (and algebraic) rates of singularity of minimal energy. These particular rates are critical in finding the regularity and solvability of certain stochastic differential equations [14, 15, 19], as well as in setting conditions for the hypoellipticity of certain degenerate infinite dimensional elliptic problems [32]. It is shown in Reference 32 that Hormander's hypoellipticity condition is strongly linked to the singularity of the minimal energy function. Null controllability is also related to the analysis of regularity of the Bellman's function, which is associated with the minimal time control problem. Indeed, as eloquently described in References 14 and 15, this property bears a close relation to the regularity of some Markov semigroups, including Orstein–Uhlenbeck processes and related Kolmogorov equations. For some of these semigroups (see, e.g., Reference 15—Theorem 8.3.3) the minimal energy singularity associated with null controllability describes differentiability properties and regularizing effects of the Orstein–Uhlenbeck process. Moreover, the regularity of solutions to the Kolmogorov equation depends on the singularity of the minimal energy as $T \downarrow 0$. Also, as shown in Reference 14, optimal estimates for the norms of controls are critical in being able to prove Liouville's property for harmonic functions of Markov processes (see p. 108 in Reference 15). In sum, there is an abundance of examples from the literature that clearly illustrate that, in the context of computing optimal minimal energy asymptotics as $T \downarrow 0$, the tools of controllability can potentially enable a mathematical control theorist to transcend his or her deterministic realm so as to solve fundamental problems in other areas of analysis, including stochastic PDEs.

In addition, the procurement of optimal algebraic estimates for the minimal energy allows one to clearly explain the role of the hyperbolic-parabolic coupling within the PDE structure (in Eq. (1.1) below). In particular, it has been shown recently in Reference 25 that, owing to optimal algebraic singularities of minimal energy, it is possible to offset the singularity of minimal energy by introducing a very strong coupling within the system. Thus, in some sense, the lack of a second control in the system may be quantitatively compensated for by taking large values of the coupling parameter "$\alpha$." From our remarks above, it is clear that this compensatory phenomenon will not be observed with boundary or partially supported controls, which, as we have said, lead to blowups of exponential type.

Having decribed the goal and motivation for the problem considered, we shall describe the main contribution of this chapter within the context of recent work in that area. The problem of controllability/reachability for thermal plates has attracted considerable attention in recent years with many contributions available in the literature [22, 23, 24, 1, 2, 3, 10, 18, 16, 17, 11], but we shall focus particularly on works related directly to singular behavior, as $T \downarrow 0$, of the minimal energy relative to null controllability.

The study of optimal singularity for thermoelastic plates with internal controls started in References 8, 9, and 40, where for the first time the optimal rates $T^{-\frac{5}{2}}$ were established for the "commutative" case (i.e., plates with hinged mechanical boundary conditions). The proof given in Reference 40 is based on a spectral method that exploits the commutativity in an essential way, whereas the proof given in Reference 8 is based on weighted energy estimates, thereby giving one the chance to extend this method to other noncommutative models (e.g., clamped or free mechanical boundary conditions). The "commutative" case (hinged boundary conditions) has been also treated in Reference 11, where null controllability with thermal controls of partially localized support was proved. For this commutative model under boundary control (either thermal or boundary), the exponentially blowing up and sharp asymptotics $e^{\mathcal{O}(\frac{1}{T})}$ have been shown [25]. The techniques used in these papers rely critically on spectral analysis and commutativity.

It turns out that a proper and necessarily more technical extension of the method introduced in Reference 8 will allow the consideration of *noncommutative* models. (By "noncommutative models" we mean those models wherein the domains of the respective spatial differential operators of the plate and heat dynamics do not necessarily enjoy any sort of compatibility.) In particular, the optimal singularity of the (null control) minimal energy is proved in Reference 9 for clamped plates with one control only. It should be stressed that the proof in the noncommutative case depends in an essential manner upon estimates provided by the analyticity of the underlying thermoelastic semigroup; this property of analyticity was discovered for the clamped case in References 31 and 28 and for free case in Reference 27. The most challenging case is, of course, that of the *free* mechanical boundary conditions (introduced in the context of control theory in Reference 22), in which a coupling between thermal and mechanical variables also occurs on the boundary. This additional coupling compels us to develop below a delicate string of trace estimates that measure the singularity at the boundary.

The main aim of this chapter is to provide a complete analysis of the free case. We shall show that in the case of mechanical control one still obtains the optimal singularity. Instead, in the case of thermal control the estimate is "off" by $3/4$. A question whether this estimate can be improved, thereby leading to the optimal singularity $T^{-\frac{5}{2}}$, still remains an *open question*.

### 1.1.2 Description of the PDE Model and Statement of the Problem

Having given our general remarks above, we now proceed to precisely describe the present problem under consideration; this work will continue and extend the analysis that has been previously undertaken in References 6, 7, 8, 9 through and 40. We will consider throughout the two-dimensional PDE system of thermoelasticity in the absence of rotational inertia. As we have already stated, it is now known that for all possible mechanical boundary conditions, the thermoelastic PDE model is associated with the generator of an *analytic* $C_0$-semigroup (see References 31, 28, 39, and 27). Given then that the underlying PDE dynamics are "parabolic-like," it is natural to consider the *null controllability problem* for the thermoelastic system, namely, can one find $L^2(Q)$ controls (mechanical or thermal) that steer the solution of the PDE from the initial data to the zero state? (We shall make this control theoretic notion more precise below. As usual, $Q$ here denotes the cylinder $\Omega \times (0, T)$.) Having established $L^2(Q)$-null controllability for the PDE, and moreover assuming that the controllability time is arbitrary, we can subsequently proceed to measure the rate of blowup, as $T \downarrow 0$, for the minimal energy function that is associated to null controllability. As is well known, and as we shall see below, this work is very much tied up with obtaining the *sharp* observability inequality associated with null controllability; moreover, this analysis is rather sensitive to the mechanical boundary conditions imposed. In Reference 8—as well as in Reference 40 via a very different methodology—the problem of blowup for the minimal energy function was undertaken in the canonical case of *hinged* mechanical boundary conditions; in Reference 9, we revisit this problem for the more difficult *clamped* case. In this paper, we complete the picture by analyzing the singularity of minimal energy for the case of the thermoelastic PDE under the so-called *free* boundary conditions. In general, the analyses involved in the attainment of (null *and* exact control) observability inequalities for thermoelastic systems are profoundly sensitive to the particular set of boundary conditions are being imposed. But the free case, presently under consideration, will give rise to the most problematic scenario of all. This situation is due to the high degree of coupling between the mechanical and the thermal variables, with the coupling taking place in the PDE itself and in the free mechanical boundary conditions.

We describe the problem in detail. Let $\Omega$ be a bounded open set of $\mathbb{R}^2$, with smooth boundary $\Gamma$. For the free case, following [22, 23] the corresponding model PDE is as follows: the (mechanical) variables $[\omega(t,x), \omega_t(t,x)]$ and the (thermal) variable $\theta(t,x)$ solve, for given data $\{[\omega_0, \omega_1, \theta_0], u_1, u_2\}$, the PDE system

$$\begin{cases} \begin{cases} \omega_{tt} + \Delta^2\omega + \alpha\Delta\theta = a_1 u_1 \\ \theta_t - \Delta\theta - \alpha\Delta\omega_t = a_2 u_2 \end{cases} \quad \text{on } (0,T)\times\Omega \\ \\ \begin{cases} \Delta\omega + (1-\mu)B_1\omega + \alpha\theta = 0 \\ \dfrac{\partial\Delta\omega}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\omega}{\partial\tau} - \omega + \alpha\dfrac{\partial\theta}{\partial\nu} = 0 \end{cases} \quad \text{on } (0,T)\times\Gamma \\ \\ \dfrac{\partial\theta}{\partial\nu} + \lambda\theta = 0 \ \text{ on } (0,T)\times\Gamma, \quad \text{where } \lambda > 0 \\ \\ \omega(t=0) = \omega_0;\, \omega_t(t=0) = \omega_1;\, \theta(t=0) = \theta_0 \quad \text{on } \Omega. \end{cases} \tag{1.1}$$

Here, $\alpha > 0$ is the parameter that *couples* the disparate dynamics (i.e., the heat equation vs. the Euler plate equation); the constant $\mu \in (0,1)$ is Poisson's ratio. Also, the (control) parameters $a_1$ and $a_2$ satisfy $a_1 \geq 0$, $a_2 \geq 0$ and $a_1 + a_2 > 0$ (in other words, at least one of the controls, be it thermal or mechanical, is always present.) The (free) boundary operators $B_i$ are given by

$$\begin{aligned} B_1 w &\equiv 2\nu_1\nu_2\frac{\partial^2 w}{\partial x\partial y} - \nu_1^2\frac{\partial^2 w}{\partial y^2} - \nu_2^2\frac{\partial^2 w}{\partial x^2}; \\ B_2 w &\equiv \left(\nu_1^2 - \nu_2^2\right)\frac{\partial^2 w}{\partial x\partial y} + \nu_1\nu_2\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right). \end{aligned} \tag{1.2}$$

The PDE Eq. (1.1) is the model explicitly derived and analyzed in References 24 and 22 in the "limit case." That is to say, we are considering the two-dimensional thermoelastic system in the absence of rotational forces; the small and nonnegative, classical parameter $\gamma$ is taken here to be zero. As we stated at the outset, it is now well known that the lack of rotational inertia in the model Eq. (1.1) will result in the corresponding dynamics having their evolution described by the generator of an *analytic* semigroup on the associated basic space of finite energy. In short, the present case $\gamma = 0$ corresponds to *parabolic-like* dynamics; this is in stark contrast to the case $\gamma > 0$—as analyzed in the control papers [22], [23], [3] and myriad others—for which the corresponding PDE manifests *hyperbolic-like* dynamics.

In fact, if we define

$$\mathbf{H} \equiv H^2(\Omega) \times L^2(\Omega) \times L^2(\Omega), \tag{1.3}$$

then one can proceed to show by the Lumer Phillips theorem that the thermoelastic plate model can be associated with the generator of a $C_0$-semigroup of contractions on $\mathbf{H}$. That is to say, there exists $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, and $\{e^{\mathcal{A}t}\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ such that $[\omega, \omega_t, \theta]$ satisfies the PDE (1.1) if and only if $[\omega, \omega_t, \theta]$ satisfies the abstract ODE

$$\frac{d}{dt}\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} = \mathcal{A}\begin{bmatrix} \omega(t) \\ \omega_t(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ a_1u_1(t) \\ a_2u_2(t) \end{bmatrix}; \quad \begin{bmatrix} \omega(0) \\ \omega_t(0) \\ \theta(0) \end{bmatrix} = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \theta_0 \end{bmatrix}.$$

In consequence of this relation, we have immediately from classical semigroup theory that

$$\{[\omega_0, \omega_1, \theta_0], [u_1, u_2]\} \in \mathbf{H} \times [L^2(Q)]^2 \Rightarrow [\omega, \omega_t, \theta] \in C([0, T]; \mathbf{H}). \tag{1.4}$$

Because of the underlying analyticity, which will ultimately mean that there are smoothing effects associated with the application of the semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$, the null controllability problem for the controlled PDE Eq. (1.1)—with respect to internal $L^2$-controls—is an appropriate one to study. Moreover, one might speculate that, as in the case of the canonical heat equation [12], should the PDE Eq. (1.1) in fact be null controllable, it will be so in *arbitrary* small time (because of the underlying infinite speed of propagation). It is this speculation that motivates our working definition of null controllability for the present paper.

***DEFINITION 1.1*** *The PDE* (1.1) *is said to be null controllable if, for any* $T > 0$ *and arbitrary initial data* $\mathbf{x} \equiv [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, *there exists a control function* $[u_1, u_2] \in [L^2(Q)]^2$ *such that the corresponding solution* $[\omega, \omega_t, \theta] \in C([0, T]; \mathbf{H})$ *satisfies* $[\omega(T), \omega_t(T), \theta(T)] = [0, 0, 0]$.

However, the issue of null controllability, although certainly an important part of this paper, is subordinate to our main objective, which is to measure the rate of singularity of the associated *minimal energy function*.

We develop this notion of "minimal energy." Assume for the time being that the Eq. (1.1) is null controllable within the class of $[L^2(Q)]^2$-controls, in the sense of the Definition 1.1. Subsequently, one can then speak of the associated minimal norm control, relative to given initial data $\mathbf{x} \equiv [\omega_0, \omega_1, \theta_0] \in \mathbf{H}$ and given terminal time $T$. That is to say, we can consider the problem of finding a control $\mathbf{u}_T^0(\mathbf{x})$ that steers the solution $[\omega, \omega_t, \theta]$ of Eq. (1.1) (with $[u_1, u_2] = \mathbf{u}_T^0(\mathbf{x})$ therein) from initial data $\mathbf{x}$ to zero in arbitrary time $T$ and minimizes the $L^2$ norm. In fact, by standard convex optimization arguments (see, e.g., Reference 13), given any $\mathbf{x} \in \mathbf{H}$ and fixed $T$, one can find a control $\mathbf{u}_T^0(\mathbf{x})$ which solves the problem

$$\left\|\mathbf{u}_T^0(\mathbf{x})\right\|_{[L^2(Q)]^2} = \min \|\mathbf{u}\|_{[L^2(Q)]^2},$$

where, above, the minimum is taken with respect to all possible null controllers $\mathbf{u} = [u_1, u_2] \in [L^2(Q)]^2$ of the PDE (1.1) (which steer initial data $\mathbf{x}$ to rest at time $t = T$). Subsequently, we can define the *minimal energy function* $\mathcal{E}_{\min}(T)$ as

$$\mathcal{E}_{\min}(T) \equiv \sup_{\|\mathbf{x}\|_{\mathbf{H}}=1} \left\|\mathbf{u}_T^0(\mathbf{x})\right\|_{[L^2(Q)]^2}. \tag{1.5}$$

Under the assumption of null controllability, as defined in Definition 1.1, we have that $\mathcal{E}_{\min}(T)$ is bounded away from zero. A natural follow-up question is "how does $\mathcal{E}_{\min}(T)$ behave as terminal time $T \downarrow 0$, or equivalently (by Eq. (1.5)), for given time $T$, how exactly does the quantity $\|\mathbf{u}_T^0(\mathbf{x})\|_{[L^2(Q)]^2}$ grow as $T \downarrow 0$?"

The problem of studying the rate of blowup for minimal norm controls is a classical one and has its origins from the finite dimensional setting. In fact, a very complete and satisfactory solution has been given in Reference 34 for the following controlled ODE in $\mathbb{R}^n$:

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + B\vec{u}(t), \quad \vec{y}_0 \in \mathbb{R}^n \tag{1.6}$$

where $\vec{u} \in L^2(0, T; \mathbb{R}^m)$ and $A$ (resp., $B$) is an $n \times n$ (resp. $n \times m$) matrix, with $m \leq n$ (so consequently the solution $\vec{y} \in C([0, T]; \mathbb{R}^n)$. The problem in this finite dimensional milieu, like that for our controlled PDE (1.1), is to ascertain the rate of singularity for the associated minimal energy function, which is defined in the same way as in Eq. (1.5). The solution to this problem is tied up with the classical Kalman's rank condition. Namely, a beautifully simple (though highly nontrivial) formula in Reference 34—an alternative constructive proof of this formula is given in Reference 40; see also Reference 36—yields that the minimal energy function associated to the null controllability of Eq. (1.6) is $\mathcal{O}(T^{-k-\frac{1}{2}})$, where $k$ is the Kalman's rank of the system Eq. (1.6) (that is, $k$ is the smallest integer such that *rank* $([B, AB, \ldots, A^k B]) = n$; see Reference 41).

By a formal application of Seidman's finite dimensional result, one can get an inkling of the numerology involved in the computation of the minimal energy $\mathcal{E}_{\min}(T)$ for the PDE system Eq. (1.1). For example, let us consider the thermoelastic Eq. (1.1) but with now $\omega$ satisfying the canonical *hinged* mechanical/Dirichlet thermal boundary conditions

$$\omega|_\Gamma = \Delta\omega|_\Gamma = \theta|_\Gamma = 0 \text{ on } \Sigma. \tag{1.7}$$

In this case, it is shown in Reference 27 that when, say, thermal control only is implemented (i.e., $a_1 = 0$ in Eq. (1.1)), the thermoelastic PDE under the hinged boundary conditions Eq. (1.7) may be associated with the ordinary differential equation (ODE) (1.6), with

$$A = \Delta \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ a_2 \end{bmatrix}. \tag{1.8}$$

This ODE in three space dimensions is a direct consequence of the analysis undertaken for the canonical hinged case in Reference 26. By way of obtaining the ODE (1.6), we have formally "factored out" the Laplacian from the (rearranged) infinitesimal generator of the thermoelastic semigroup, which is given in (Section 1.2.2) of Reference 27 (see also Reference 28, p. 311). Considering now finite dimensional truncations of $\Delta$ (by making use of the spectral resolution of the Laplacian under Dirichlet boundary conditions) and applying the algorithm of Seidman to the given controllability pair $[A, B]$ in Eq. (1.8), we compute readily that the minimal energy function associated with the null controllability of the finite dimensional Eq. (1.6)—an approximation in some sense of the thermoelastic system under the hinged boundary conditions—blows up at a rate on the order of $T^{-\frac{5}{2}}$. These numerics lead to the following question: Does the minimal energy Eq. (1.5) (i.e., the minimal energy for the full-fledged *infinite dimensional* system) obey the law $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$? Of course, Seidman's formula for matrices gives no conclusive proof as to what is actually happening for the fully infinite dimensional model. In fact, it is well known that the minimal energy of a given infinite dimensional system may bear no relation to the limit of minimal energies of any given sequence of finite dimensional approximations. For example, it was shown in Reference 14 that the growth of the minimal energy function for a given infinite dimensional system may be arbitrarily large, even when Kalman's rank $k = 1$ and spectral diagonal systems are being considered. Moreover, in Reference 35 it is shown that for the case of the boundary controlled heat equation, the sharp observability inequality corresponding to the (null) minimal energy of a given heat operator's finite dimensional truncation obeys *rational* rates of singularity. On the other hand, the asymptotics of the minimal energy, which are obtained for the (infinite dimensional) heat equation, are of *exponential* type. A similar phenomenon is observed in References 40 and 6, wherein strongly damped wave equations under internal control are considered. In this situation, with the damping operator given by $A^\beta$, the asymptotics of minimal energy behave as $T^{-\frac{\beta}{2(1-\beta)}}$ for any $\beta > \frac{3}{4}$. Thus, when the damping operator approaches Kelvin's Voight damping, the singularity loses its algebraic character with $\frac{\beta}{2(1-\beta)} \uparrow \infty$. Instead, for $\beta \leq \frac{3}{4}$, the singularity is optimal (i.e., the same as that for finite dimensional truncations) and is equal to $T^{-\frac{3}{2}}$.

But as formal as the application of Seidman's finite dimensional algorithm may seem in the present context, there is in fact a relevance here to the thermoelastic PDE, which is approximately described by controllability pair $[A, B]$. The minimal energy function with respect to null controllability of the thermoelastic PDE, under the hinged boundary conditions Eq. (1.7), does indeed obey the singular rate $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. This minimal energy analysis for the hinged case was shown independently in References 8 and 40 (and most recently in Reference 25 where the asymptotics with respect to the coupling $\alpha$ are also provided). In Reference 40, it is of prime importance that the hinged boundary conditions Eq. (1.7) be in play, for these mechanical boundary conditions allow a fortuitous spectral resolution of the underlying thermoelastic generator. With the eigenfunctions of the thermoelastic dynamics in hand, it is shown in Reference 40 via a constructive class of suboptimal steering controls that the delicate observability estimates for solutions for the spectrally truncated adjoint problem—adjoint with respect to null controllability—are preserved; as a consequence, a rational rate of singularity for the infinite dimensional null minimal energy is obtained in the limit. However, for other sets of mechanical boundary conditions, including the physically relevant clamped and (above all) free boundary conditions under consideration at present, there will be no such available spectral decomposition.

On the other hand, the methodology employed in References 8 and 9, and the present work, is "eigenfunction independent"; in particular, we blend a weighted multiplier method of Carleman's type with boundary trace estimates exhibiting singular behavior of the boundary traces. This rather special behavior is a consequence of the underlying analyticity. In principle, our work in Reference 8 to estimate the blowup of the "minimal norm control" as $T \downarrow 0$ is applicable to a variety of dynamics. (In fact, our method of proof in Reference 8 and in the present work is used in Reference 7 to estimate the minimal norm control of the abstract wave equation under Kelvin–Voight damping.) Moreover, the robustness of our method allows us in Reference 9 to analyze the rate of singularity of the minimal energy function for the null controllability of thermoelastic plates in the case of *clamped* boundary conditions. As we said above, there is no spectral decomposition or factorization of the thermoelastic generator in the case of mechanical boundary conditions other than the canonical hinged case and thus no rigorous association with the abstract ODE (1.8). Still, we show in Reference 9 that for the clamped case, the minimal energy obeys the singular rate "predicted" in Reference 34, namely, $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. Our intent in this paper is to bring the story to a close by investigating the minimal energy function for the null controllability of thermoelastic systems under the high-order free boundary conditions that are present in Eq. (1.1).

### 1.1.3 Main Result

In regards to our stated problem, the main result is as follows:

***THEOREM 1.1***

*Let terminal time $T > 0$ be arbitrary and $a_1$, $a_2 \geq 0$ with $a_1 + a_2 > 0$. Then, given initial data $[\omega_0, \omega_1, \theta_0] \in \mathbf{H}$, there exist control(s) $[u_1, u_2] \in [L^2(Q)]^2$ such that the corresponding solution $[\omega, \omega_t, \theta]$ of* (1.1) *satisfies $[\omega(T), \omega_t(T), \theta(T)] = [0, 0, 0]$. (That is to say, the PDE model Eq.* (1.1) *is null controllable within the class of* $[L^2(Q)]^2$*—controls in arbitrary short time.) Moreover, We have the following rates of blowup for the minimal energy function:*

1. *(thermal control) If* $a_1 = 0$*, then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{13}{4}-\epsilon})$ *for all* $\epsilon > 0$;

2. *(mechanical control) If* $a_2 = 0$*, then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$;

3. *If* $a_1 > 0$ *and* $a_2 > 0$*, then* $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{3}{2}})$.

**REMARK 1.1** The null controllability of thermoelastic plates with free boundary conditions and under one internal control (be it mechanical or thermal) appears to be, as far as we know, a new result in the literature. The Theorem 1.1 above, in addition to asserting the said null controllability property, provides the asymptotics for the singularity of the associated minimal energy function. These asymptotics are optimal in the case of a single mechanical control and in the case of two controls acting upon the system. In the case of a single thermal control the estimate is "off" by 3/4 with respect to the desired "finite dimensional prediction" in Reference 34. Whether this estimate can be improved upon is an open question.

Our method of proof of Theorem 1.1 is based on weighted energy estimates that are flexible enough to accomodate analytic estimates and the resulting singularity. The proof has the following main technical ingredients:

1. special weighted *nonlocal multipliers* introduced in Reference 4 and subsequently invoked in References 3, 5, 29, and 6, and elsewhere;

2. the analyticity of semigroups associated with thermoelastic PDE models in the absence of rotational forces, as demonstrated in References 31, 26, 27, and 28;

3. new singular estimates for boundary traces of solutions of Eq. (1.9), which are of their own intrinsic interest and which are needed to handle the boundary terms resulting from the weighted estimates employed.

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## 1.2 The Necessary Observability Inequality

The proof of Theorem 1.1 is based on the derivation of the observability inequality associated with the null controllability of the PDE (1.1) with respect to thermal or mechanical control or both. This inequality is formulated in terms of the solution of the homogeneous PDE, which is "dual" or "adjoint" to that in Eq. (1.1). Namely, we shall consider solutions $[\phi, \phi_t, \vartheta]$ to the following system:

$$\begin{cases} \begin{cases} \phi_{tt} + \Delta^2\phi + \alpha\Delta\vartheta = 0 & \text{on } (0,T)\times\Omega \\ \vartheta_t - \Delta\vartheta - \alpha\Delta\phi_t = 0 & \text{on } (0,T)\times\Omega \end{cases} \\ \begin{cases} \Delta\phi + (1-\mu)B_1\phi + \alpha\vartheta = 0 \\ \dfrac{\partial\Delta\phi}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\phi}{\partial\tau} - \phi + \alpha\dfrac{\partial\vartheta}{\partial\nu} = 0 \end{cases} \quad \text{on } \Sigma \\ \dfrac{\partial\vartheta}{\partial\nu} + \lambda\vartheta = 0 \ \text{ on } \Sigma, \quad \lambda > 0 \\ [\phi(0), -\phi_t(0), \vartheta(0)] = [\phi_0, \phi_1, \vartheta_0] \in \mathbf{H}. \end{cases} \tag{1.9}$$

If we define the bilinear form $a(\cdot,\cdot) : H^2(\Omega) \times H^2(\Omega) \to \mathbb{R}$ by

$$a(w,\tilde{w}) \equiv \int_\Omega [w_{xx}\tilde{w}_{xx} + w_{yy}\tilde{w}_{yy} + \mu(w_{xx}\tilde{w}_{yy} + w_{yy}\tilde{w}_{xx}) + 2(1-\mu)w_{xy}\tilde{w}_{xy}]\, d\Omega,$$

then we can state the "Green's formula," which involves this bilinear form (see Reference 22) and which is valid for functions $w, \tilde{w}$ (smooth enough):

$$\begin{aligned} \int_\Omega (\Delta^2 w)\tilde{w}\, d\Omega = a(w,\tilde{w}) &+ \int_\Gamma \left[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau}\right]\tilde{w}\, d\Gamma \\ &- \int_\Gamma [\Delta w + (1-\mu)B_1 w]\frac{\partial \tilde{w}}{\partial \nu} d\Gamma. \end{aligned} \tag{1.10}$$

Let $\mathcal{E}(t)$ denote the energy of the adjoint system Eq. (1.9), where

$$\mathcal{E}(t) \equiv \frac{1}{2}a(\phi(t),\phi(t))d + \frac{1}{2}\int_\Gamma |\phi(t)|^2\, d\Gamma + \frac{1}{2}\int_\Omega |\vartheta(t)|^2\, d\Omega. \tag{1.11}$$

In terms of this energy, then one can show by classical functional analytical arguments (see, e.g., References 41 and 3) that the PDE (1.1) is null controllable, in the sense of 1, if and only if the adjoint variables $[\phi, \phi_t, \vartheta]$ of Eq. (1.9) satisfy the following *continuous observability inequality*, for some constant $C_T$:

$$\|[\phi(T), \phi_t(T), \vartheta(T)]\|_{\mathbf{H}} \le C_T(a_1\|\phi_t\|_{L^2(Q)} + a_2\|\vartheta\|_{L^2(Q)}). \tag{1.12}$$

Having worked to establish the sharp constant $C_T$ in the observability inequality Eq. (1.12), one can proceed through an algorithmic procedure—using an explicit representation of the minimal norm control, by convex optimization—so as to have that for all terminal time $T > 0$,

$$\mathcal{E}_{\min}(T) = \mathcal{O}(C_T).$$

Because the details of this argument are known and have been previously spelled out (see, e.g., References 9 and 8), we defer from repeating them here.

Because of this characterization of the behavior of $\mathcal{E}_{\min}(T)$ with the constant $C_T$ in Eq. (1.12), our work will accordingly be geared toward establishing this inequality (where, again, control parameters $a_i$ satisfy $a_1, a_2 \ge 0$, and $a_1 + a_2 > 0$).

## 1.3 Some Preliminary Machinery

In this section, we explicitly define the underlying generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, which describes the thermoelastic flow. Subsequently, a proposition is derived with which to associate powers of this generator with specific Sobolev spaces. This characterization of the powers will be critical in work.

- To start, we define the linear operator $A_D : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} A_D &\equiv -\Delta; \\ D(A_D) &= H^2(\Omega) \cap H_0^1(\Omega). \end{aligned} \tag{1.13}$$

- We will also need the following (Dirichlet) map $D : L^2(\Gamma) \to L^2(\Omega)$:

$$Df = g \Leftrightarrow \Delta g = 0 \text{ on } \Omega \quad \text{and} \quad g|_\Gamma = f \text{ on } \Gamma. \tag{1.14}$$

  By the classical elliptic regularity, we have that $D \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{1}{2}}(\Omega))$ for all $s$ (see Reference 30).

- We also define the linear operator $\mathring{\mathbf{A}} : D(A_D) \subset L^2(\Omega) \to L^2(\Omega)$ by setting $\mathring{\mathbf{A}}\varpi = \Delta^2\varpi$, for $\varpi \in D(\mathring{\mathbf{A}})$, where

$$D(\mathring{\mathbf{A}}) = \Bigg\{\varpi \in H^4(\Omega) : [\Delta\varpi + (1-\mu)B_1\varpi]_\Gamma = 0 \\ \text{and} \quad \left[\frac{\partial\Delta\varpi}{\partial\nu} + (1-\mu)\frac{\partial B_2\varpi}{\partial\tau} - \varpi\right]_\Gamma = 0\Bigg\},$$

where the boundary operators $B_i$ are as defined in Eq. 1.3.

This operator is densely defined, positive definite, and self-adjoint. Consequently by Reference 21, one has the characterization

$$D(\mathring{\mathbf{A}}^{\frac{1}{2}}) \approx H^2(\Omega); \text{ with moreover } \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} = a(\phi,\phi) + \int_\Gamma \phi^2 d\Gamma.$$

- Moreover, we define the elliptic operators $G_i$ by

$$G_1 h = v \Leftrightarrow \left\{\begin{array}{l} \Delta^2 v = 0 \quad \text{on } \Omega \\ \left\{\begin{array}{l} \Delta v + (1-\mu)B_1 v = h \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = 0 \end{array}\right. \quad \text{on } \Gamma \end{array}\right. ;$$

$$G_2 h = v \Leftrightarrow \left\{\begin{array}{l} \Delta^2 v = 0 \quad \text{on } \Omega \\ \left\{\begin{array}{l} \Delta v + (1-\mu)B_1 v = 0 \\ \dfrac{\partial\Delta v}{\partial\nu} + (1-\mu)\dfrac{\partial B_2 v}{\partial\tau} - v = h \end{array}\right. \quad \text{on } \Gamma \end{array}\right. . \tag{1.15}$$

By elliptic regularity (see, e.g., Reference 30) one has that for all real $s$,

$$G_1 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{5}{2}}(\Omega)); \; G_2 \in \mathcal{L}(H^s(\Gamma), H^{s+\frac{7}{2}}(\Omega)). \tag{1.16}$$

With these operators defined above, we have that the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ of the thermoelastic semigroup may be given the explicit representation

$$\mathcal{A} = \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\mathring{\mathbf{A}} & 0 & \alpha(A_D(I-D\gamma_0) - \mathring{\mathbf{A}}G_1\gamma_0 + \lambda\mathring{\mathbf{A}}G_2\gamma_0) \\ 0 & -\alpha A_D(I-D\gamma_0) & -\alpha A_D(I-D\gamma_0) \end{bmatrix};$$

$$D(\mathcal{A}) = \Bigg\{[\omega_0,\omega_1,\theta_0] \in H^2(\Omega)\times H^2(\Omega)\times H^2(\Omega) : \mathring{\mathbf{A}}\,[\omega_0 + \alpha\,(G_1\gamma_0 - \lambda G_2\gamma_0)\,\theta_0] \in L^2(\Omega) \\ \text{and} \quad \left[\frac{\partial\theta_0}{\partial\nu} + \lambda\theta_0\right]_\Gamma = 0\Bigg\} \tag{1.17}$$

(here, $\gamma_0 \in \mathcal{L}(H^1(\Omega), H^{\frac{1}{2}}(\Gamma))$ is the classical Sobolev trace map; i.e., $\gamma_0 f = f|_\Gamma$ for $f \in C^\infty(\overline{\Omega})$).

As we have said, it is now known that the generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ for the thermoelastic plate, with free mechanical boundary conditions, is associated with an *analytic* $C_0$-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ of contractions on $\mathbf{H}$ (see Reference 27 and references therein), with moreover $\mathcal{A}^{-1}$ being bounded on $\mathbf{H}$.

For this realization of the generator, we now proceed to show the following:

***PROPOSITION 1.1***
*Let integer $k = 1, 2, \dots$. Then $D(\mathcal{A}^k) \subset H^{2k+2}(\Omega) \times H^{2k}(\Omega) \times H^{2k}(\Omega)$.*

**PROOF OF PROPOSITION 1.1** Let first $[\omega_0, \omega_1, \theta_0] \in D(\mathcal{A})$. Then by definition, $\omega_1, \theta_0 \in H^2(\Omega)$. Moreover, from the abstract representation in Eq. (1.17), we have

$$\mathring{\mathbf{A}}\omega_0 + \alpha\mathring{\mathbf{A}}G_1\theta_0|_\Gamma - \alpha\lambda\mathring{\mathbf{A}}G_2\theta_0|_\Gamma + \alpha\Delta\theta_0 = f \in L^2(\Omega).$$

Because $\theta_0|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$, then consequently from the elliptic regularity results posted in Eq. (1.16),

$$\omega_0 = \mathring{\mathbf{A}}^{-1}f - \alpha G_1\theta_0|_\Gamma + \alpha\lambda G_2\theta_0|_\Gamma - \alpha\mathring{\mathbf{A}}^{-1}\Delta\theta_0 \in H^4(\Omega).$$

So the assertion is true for $k = 1$. □

Proceeding now by induction, suppose that the result holds true for integer $k-1$, $k \geq 2$, and let $[\omega_0, \omega_1, \theta_0] \in D(\mathcal{A}^k)$. Then, because

$$\mathcal{A}\begin{bmatrix}\omega_0\\ \omega_1\\ \theta_0\end{bmatrix} \in D(\mathcal{A}^{k-1}),$$

we have

$$\begin{aligned} \omega_1 &\in H^{2k}(\Omega);\\ \mathring{\mathbf{A}}\omega_0 + \alpha\mathring{\mathbf{A}}G_1\theta_0|_\Gamma - \alpha\lambda\mathring{\mathbf{A}}G_2\theta_0|_\Gamma + \alpha\Delta\theta_0 &= f \in H^{2k-2}(\Omega);\\ A_R\theta_0 - \alpha\Delta\omega_1 &= g \in H^{2k-2}(\Omega). \end{aligned} \tag{1.18}$$

Here, $A_R : D(A_R) \subset L^2(\Omega) \to L^2(\Omega)$ is the elliptic operator defined by

$$A_R f = -\Delta f; \qquad D(A_R) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} + \lambda f = 0,\ \lambda > 0 \right\}. \tag{1.19}$$

Reading off the third equation in Eq. (1.18), we obtain, after using elliptic regularity,

$$\theta_0 = A_R^{-1}\left(g + D\gamma_0\theta_0 - \alpha\Delta\omega_1\right) \in H^{2k}(\Omega).$$

In turn, we can use again the result in Eq. (1.16) to have that

$$\omega_0 = \mathring{\mathbf{A}}^{-1}f - \alpha G_1\gamma_0\theta_0 + \alpha\lambda G_2\gamma_0\theta_0 - \alpha\mathring{\mathbf{A}}^{-1}\Delta\theta_0 \in H^{2k+2}(\Omega).$$

This concludes the proof of Proposition 1.1.

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## 1.4 A Singular Trace Estimate

In this section, we exploit the underlying analyticity of the thermoelastic semigroup so as to generate pointwise (in time) estimate of boundary traces of the adjoint variables $\phi_t(t)$ and $\vartheta(t)$ of Eq. (1.9). These estimates will be of use to us in the proof of Theorem 1.1, inasmuch as they each reflect a proper "distribution" between the measurement $\mathcal{E}(t)$ of the energy and the observation term—be it $\phi_t$ or $\vartheta$. The price to pay for these benefical estimates is the appearance therein of singular weights of the form $\frac{1}{t^s}$, where parameter $s$ will depend on the order of derivatives present.

***LEMMA 1.1***
*Let $\vec{x}(t) \equiv [\phi(t), \phi_t(t), \vartheta(t)]$ denote the solution of the adjoint system Eq.* (1.9), *subject to the initial condition $\vec{x}(0) = [\phi_0, -\phi_1, \vartheta_0] \in \mathbf{H}$. Let,moreover, $D_m$ be a differential operator of order $m \geq 0$ with respect to the interior variables. Then for integers $k = 1, 2, \ldots$, and all $t > 0$ we have*

1. $\|D_m\vartheta(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{m}{2}+\frac{1}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\vartheta(t)\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}$;
2. $\|D_m\phi_t(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{m}{2}+\frac{1}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\phi_t(t)\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}$;
3. $\|D_1\phi_{tt}(t)\|_{L(\Gamma)} \leq \dfrac{C_k}{t^{\frac{7}{4}}} \left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}$.

**PROOF OF LEMMA 1.1** By a trace interpolation result (see, e.g., Reference 38) and the iterative use of a classical PDE moment inequality, we have the following string of estimates, which is valid for any $g \in H^{2^{k+1}(m+1)}(\Omega)$:

$$\begin{aligned}
\|D_m g\|_{L(\Gamma)} &\leq C\,\|D_m g\|_{L(\Omega)}^{\frac{1}{2}}\,\|D_m g\|_{H^1(\Omega)}^{\frac{1}{2}} \leq C\,\|g\|_{H^m(\Omega)}^{\frac{1}{2}}\,\|g\|_{H^{m+1}(\Omega)}^{\frac{1}{2}} \\
&\leq C\,\|g\|_{L(\Omega)}^{\frac{1}{2}}\,\|g\|_{H^{2m}(\Omega)}^{\frac{1}{4}}\,\|g\|_{H^{2(m+1)}(\Omega)}^{\frac{1}{4}} \leq C\,\|g\|_{L(\Omega)}^{\frac{3}{4}}\,\|g\|_{H^{4m}(\Omega)}^{\frac{1}{8}}\,\|g\|_{H^{4(m+1)}(\Omega)}^{\frac{1}{8}} \\
&\leq \ldots \leq C\,\|g\|_{L(\Omega)}^{1-\frac{1}{2^k}}\,\|g\|_{H^{2^k m}(\Omega)}^{\frac{1}{2^{k+1}}}\,\|g\|_{H^{2^k(m+1)}(\Omega)}^{\frac{1}{2^{k+1}}}.
\end{aligned} \tag{1.20}$$

Now by virtue of the analyticity of the thermoelastic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ and Proposition 1.1, we have for all $t > 0$,

$$[\phi(t), \phi_t(t), \vartheta(t)] \in D(\mathcal{A}^{2^{k-1}m}) \Rightarrow [\phi_t(t), \vartheta(t)] \in [H^{2km}(\Omega)]^2. \tag{1.21}$$

Setting now $g \equiv \vartheta(t)$ (resp., $\phi_t(t)$) in Eq. (1.20), we obtain

$$\begin{aligned}
\|D_m\vartheta(t)\|_{L(\Gamma)} &\leq C\,\|\vartheta(t)\|_{L(\Omega)}^{1-\frac{1}{2^k}}\,\|\vartheta(t)\|_{H^{2^k m}(\Omega)}^{\frac{1}{2^{k+1}}}\,\|\vartheta(t)\|_{H^{2^k(m+1)}(\Omega)}^{\frac{1}{2^{k+1}}} \\
&\leq C\left\|\mathcal{A}^{2^{k-1}m}\vec{x}(t)\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}}\left\|\mathcal{A}^{2^{k-1}(m+1)}\vec{x}(t)\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}}\left\|\vartheta(t)\right\|_{L(\Omega)}^{1-\frac{1}{2^k}} \\
&= C\left\|\mathcal{A}^{2^{k-1}m}e^{\mathcal{A}\frac{t}{2}}e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}}\left\|\mathcal{A}^{2^{k-1}(m+1)}e^{\mathcal{A}\frac{t}{2}}e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^{k+1}}}\left\|\vartheta(t)\right\|_{L(\Omega)}^{1-\frac{1}{2^k}}.
\end{aligned} \tag{1.22}$$

∎

At this point, we can invoke the well known pointwise estimate that is valid for any generator of an analytic semigroup: for all time $t > 0$ and integer $m = 1, 2, \ldots$,

$$\|\mathcal{A}^m e^{\mathcal{A}t}\|_{\mathcal{L}(\mathbf{H})} \leq \frac{C^m}{t^m}, \tag{1.23}$$

where constant $C$ is independent of $m$ (see, e.g., Reference 33, p. 70). Applying this estimate to the chain Eq. (1.22), we have

$$\|D_m\vartheta(t)\|_{L(\Gamma)} \leq \frac{C}{t^{\frac{m}{2}+\frac{1}{4}}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{\frac{1}{2^k}}\,\|\vartheta(t)\|_{L(\Omega)}^{1-\frac{1}{2^k}}.$$

This gives (Lemma 1.1, Step 1) (Step 2) is obtained in the very same way, by setting $g = \phi_t$ in Eq. (1.22) and then invoking the containment Eq. (1.21). For (Step 3), we have along the same lines, by means of the trace interpolation inequality in Reference 38 and the containment Eq. (1.21),

$$\begin{aligned} \|D_1\phi_{tt}(t)\|_{L^2(\Gamma)} &\leq C\,\|\phi_{tt}(t)\|^{\frac{1}{2}}_{H^1(\Omega)}\,\|\phi_{tt}(t)\|^{\frac{1}{2}}_{H^2(\Omega)} \leq C\left\|\mathcal{A}^{\frac{1}{2}}\vec{x}_t(t)\right\|^{\frac{1}{2}}_{\mathbf{H}}\|\mathcal{A}\vec{x}_t(t)\|^{\frac{1}{2}}_{\mathbf{H}} \\ &\leq C\|\mathcal{A}^2\vec{x}(t)\|^{\frac{1}{2}}_{\mathbf{H}} \leq \frac{C}{t^{\frac{7}{4}}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac{1}{2}}_{\mathbf{H}}, \end{aligned}$$

which completes the proof.

---

## 1.5 Proof of Theorem 1.1(1)

### 1.5.1 Estimating the Mechanical Velocity

In what follows, we will have need of the polynomial weight $h(t)$, defined by

$$h(t) \equiv t^s(T-t)^s. \tag{1.24}$$

For the proof of Theorem 1.1(1), we will take $s \equiv 6$.

In terms of the the solution $[\phi, \phi_t, \vartheta]$ of Eq. (1.9) and its corresponding energy $\mathcal{E}(t)$, the necessary inequality for the case of thermal control is

$$\sqrt{\mathcal{E}(T)} \leq C_T\,\|\vartheta\|_{L^2(Q)}\,. \tag{1.25}$$

It is the derivation of this inequality that will drive the proof of Theorem 1.1.

We will start by applying the Laplacian to both sides of the heat equation in Eq. (1.9). This gives

$$\Delta\vartheta_t - \Delta^2\vartheta - \alpha\Delta^2\phi_t = 0 \quad \text{ in } \Omega.$$

From this expression and the free boundary conditions in Eq. (1.9), we have that the velocity term $\phi_t$ satisfies the following elliptic problem for all $t > 0$:

$$\begin{cases} \Delta^2\phi_t(t) = \dfrac{1}{\alpha}\Delta\vartheta_t(t) - \dfrac{1}{\alpha}\Delta^2\vartheta(t) \quad \text{ in } \Omega \\ \left\{\begin{array}{l} \Delta\phi_t(t) + (1-\mu)B_1\phi_t(t) = -\alpha\vartheta_t \\ \dfrac{\partial\Delta\phi_t(t)}{\partial\nu} + (1-\mu)\dfrac{\partial B_2\phi_t(t)}{\partial\tau} - \phi_t(t) = \alpha\lambda\vartheta_t \end{array}\right. \quad \text{on } \Gamma. \end{cases} \tag{1.26}$$

Using this Green's map defined in Eq. (1.15), we have from Eq. (1.26) that the velocity $\phi_t$ may be written explicitly as

$$\phi_t(t) = \frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}\left[\Delta\vartheta_t(t) - \Delta^2\vartheta(t)\right] - \alpha G_1\gamma_0(\vartheta_t(t)) + \alpha\lambda G_2\gamma_0(\vartheta_t(t)). \tag{1.27}$$

From this, we have

$$\begin{aligned} &\int_0^T h\,\|\phi_t\|^2_{L^2(\Omega)}\,dt \\ &= \int_0^T h\left(\frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}[\Delta\vartheta_t(t) - \Delta^2\vartheta(t)] - \alpha G_1\gamma_0(\vartheta_t(t)) + \alpha\lambda G_2\gamma_0(\vartheta_t(t)), \phi_t\right)_{L^2(\Omega)} dt, \end{aligned} \tag{1.28}$$

where $h(t)$ is the polynomial weight described in Eq. (1.24).

*Analysis of the right-hand side of Eq.* (1.28).

1.

$$\int_0^T h\left([G_1-\lambda G_2]\,\gamma_0\vartheta_t,\phi_t\right)_{L^2(\Omega)}dt = -\int_0^T h'([G_1-\lambda G_2]\,\gamma_0\vartheta,\phi_t)_{L^2(\Omega)}\,dt - \int_0^T h([G_1-\lambda G_2]\,\gamma_0\vartheta,\phi_{tt})_{L^2(\Omega)}\,dt. \tag{1.29}$$

a. By the regularity posted in Eq. (1.16) and an application of Lemma 1.1 (with $m=0$ and $k=2$, say) we have

$$\left|\int_0^T h'([G_1-\lambda G_2]\,\gamma_0\vartheta,\phi_t)_{L^2(\Omega)}\,dt\right| \le C\int_0^T |h'|\,\|\vartheta\|_{L^2(\Gamma)}\,\|\phi_t\|_{L^2(\Omega)}\,dt$$
$$\le C\int_0^T \frac{|h'|}{t^{\frac14}}\left(\frac{h(t)}{h(t)}\right)^{\frac58}\|\vartheta(t)\|^{\frac34}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac54}_{\mathbf{H}}dt.$$

Invoking Hölder's inequality to this right hand side, with Hölder conjugates $(\frac83,\frac85)$, we obtain now the estimate

$$\left|\int_0^T h'([G_1-\lambda G_2]\,\gamma_0\vartheta,\phi_t)_{L^2(\Omega)}\,dt\right|$$
$$\le C_\epsilon T^{\frac{26}{3}}\int_0^T h(t)\,\|\vartheta\|^2_{L^2(\Omega)}\,dt+\epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.30}$$

b. Proceeding as above, with $m=0$ and $k=2$ in Lemma 1.1, we have

$$\left|\int_0^T h([G_1-\lambda G_2]\,\gamma_0\vartheta,\phi_{tt})_{L^2(\Omega)}\,dt\right|$$
$$\le C\int_0^T \frac{h(t)}{t^{\frac14}}\|\vartheta(t)\|^{\frac34}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac14}_{\mathbf{H}}\|\mathcal{A}\vec{x}_t(t)\|_{L^2(\Omega)}\,dt$$
$$\le C\int_0^T \frac{h(t)}{t^{\frac54}}\|\vartheta(t)\|^{\frac34}_{L^2(\Omega)}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|^{\frac54}_{\mathbf{H}}dt$$
$$\le C_\epsilon T^{\frac{26}{3}}\int_0^T h(t)\,\|\vartheta\|^2_{L^2(\Omega)}\,dt+\epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.31}$$

Combining Eq. (1.30) and Eq. (1.31) now gives

$$\left|\int_0^T h([G_1-\lambda G_2]\,\gamma_0\vartheta_t,\phi_t)_{L^2(\Omega)}\,dt\right| \le C_\epsilon T^{\frac{20}{3}}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt+\epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \tag{1.32}$$

2. Next,

$$\int_0^T h\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta_t),\phi_t\right)_{L^2(\Omega)}dt = -\int_0^T h'\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta),\phi_t\right)_{L^2(\Omega)}dt - \int_0^T h\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta),\phi_{tt}\right)_{L^2(\Omega)}dt. \tag{1.33}$$

a. By Green's theorem and the Lemma 1.1, with $m = 0$ and $k = 2$, we have

$$\left|\int_0^T h'\left(\mathring{\mathbf{A}}^{-1}(\Delta\vartheta), \phi_t\right)_{L^2(\Omega)} dt\right| = \left|\int_0^T h'\left(\vartheta, \left(\frac{\partial}{\partial\nu} + I\right)\mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)} dt - \int_0^T h'\left(\vartheta, \Delta\,\mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt\right| \leq C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.34}$$

b. Likewise, by Green's Theorem, the analyticity of the semigroup and Lemma 1.1, with $m = 0$ and $k = 2$, we have

$$\left|\int_0^T h(t)\left(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_{tt}\right)_{L^2(\Omega)} dt\right| \leq C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.35}$$

Applying the estimates Eq. (1.34) and Eq. (1.35) to Eq. (1.33) now yields

$$\left|\int_0^T h(\Delta\vartheta_t, \phi_t)_{L^2(\Omega)}\, dt\right| \leq C_\epsilon T^{\frac{26}{3}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.36}$$

3. By the Green's identity posted in Eq. (1.10), we have

$$\begin{aligned}\int_0^T h\left(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt &= \int_0^T h(t)(\vartheta, \phi_t)_{L^2(\Omega)}\, dt \\ &= \int_0^T h(t)\Bigg[\left(\left[\frac{\partial\Delta}{\partial\nu} + (1-\mu)\frac{\partial B_2}{\partial\tau}\right]\vartheta,\ \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)} \\ &\quad - \left(\left[\Delta + (1-\mu)B_1\right]\vartheta, \frac{\partial}{\partial\nu}\,\mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Gamma)}\Bigg] dt.\end{aligned}$$

Applying once more the Lemma 1.1 (e.g., with $m = 3$, $k = 3$) we have

$$\left|\int_0^T h\left(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt\right| \leq C_\epsilon T^8 \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.37}$$

Combining the expression Eq. (1.28) with the estimates of Eqs. (1.32 ), (1.36), and (1.37) gives us the following estimate for the mechanical velocity:

***LEMMA 1.2***
*With $s = 6$ in Eq. (1.24), the solution $[\phi, \phi_t, \vartheta]$ of (1.9) satisfies the following estimate for all $\epsilon > 0$:*

$$\int_0^T h(t)\,\|\phi_t\|^2_{L^2(\Omega)}\, dt \leq C_\epsilon T^8 \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T \mathcal{E}\left(\frac{t}{2}\right) dt.$$

### 1.5.2 Estimating the Mechanical Displacement

Here, we shall show the following:

***LEMMA 1.3***
*The solution* $[\phi, \phi_t, \vartheta]$ *of Eq.* (1.9) *satisfies the following estimate for all* $\epsilon, \delta > 0$*:*

$$\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)}dt \le CT^{\frac{13}{2}-\delta}\,\|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt + \epsilon\int_0^T h(t)\mathcal{E}(t)dt.$$

**PROOF OF LEMMA 1.3** We start by applying the multiplier $h(t)\phi(t)$ to the mechanical component in Eq. (1.9). We arrive at the relation

$$\begin{aligned}
&\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)}dt \\
&= \int_0^T h'(t)(\phi_t,\phi)_{L^2(\Omega)}\,dt + \int_0^T h(t)\,\|\phi_t\|^2_{L^2(\Omega)}\,dt \\
&\quad + \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta,\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt - \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta,\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt \\
&\quad - \alpha\int_0^T h(t)(\vartheta,\Delta\phi)_{L^2(\Omega)}dt + \alpha\int_0^T h(t)\left((\vartheta,\lambda\phi + \frac{\partial\phi}{\partial\nu}\right)_{L^2(\Gamma)}dt.
\end{aligned} \tag{1.38}$$

∎

Now, using the elliptic regularity posted in Eq. (1.16) and the usage of Lemma 1.1, with $m = 0$ and $k = 3$, we obtain

$$\begin{aligned}
&\left|\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta,\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt - \alpha\lambda\int_0^T h(t)\big(\mathring{\mathbf{A}}^{\frac{1}{2}}G_2\gamma_0\vartheta,\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\big)_{L^2(\Omega)}\,dt\right. \\
&\qquad\left. - \alpha\int_0^T h(t)(\vartheta,\Delta\phi)_{L^2(\Omega)}\,dt + \alpha\int_0^T h(t)\left((\vartheta,\lambda\phi + \frac{\partial\phi}{\partial\nu}\right)_{L^2(\Gamma)}dt\right| \\
&\le C_\epsilon T^{\frac{26}{3}}\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.
\end{aligned} \tag{1.39}$$

Combining this estimate with that in Lemma 1.2 then gives the preliminary estimate

$$\begin{aligned}
\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)}dt \le{}& \left|\int_0^T h'(t)(\phi_t,\phi)_{L^2(\Omega)}\,dt\right| \\
&+ CT^8\int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.
\end{aligned} \tag{1.40}$$

Apparently, we must estimate the first term on the right-hand side of Eq. (1.40). To this end, we use the pointwise expression for $\phi_t$ in Eq. (1.27):

$$\int_0^T h'(\phi_t,\phi)_{L^2(\Omega)}\,dt = \int_0^T h'\left(\frac{1}{\alpha}\mathring{\mathbf{A}}^{-1}(\Delta\vartheta_t - \Delta^2\vartheta) - \alpha G_1\gamma_0\vartheta_t + \alpha\lambda G_2\gamma_0\vartheta_t,\phi\right)_{L^2(\Omega)}dt \tag{1.41}$$

1. The abstract Green's Theorem gives

$$\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt = \int_0^T h'\Bigg[(\vartheta,\phi)_{L^2(\Omega)} + \left(\left[\frac{\partial\Delta}{\partial\nu} + (1-\mu)\frac{\partial B_2}{\partial\tau}\right]\vartheta, \mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Gamma)} - \left([\Delta + (1-\mu)B_1]\,\vartheta, \frac{\partial}{\partial\nu}\mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Gamma)}\Bigg]dt.$$

Applying now the Lemma 1.1 with $m = 3, 2$ yields

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le C\int_0^T \frac{|h'|}{t^{\frac{7}{4}}}\|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt.$$

Let $k \ge 4$. Then, because $h'(t) = 6t^5(T-2t)(T-t)^5$, we can apply now Hölder's inequality with Hölder conjugates $(2\frac{2^k}{2^k-1}, \frac{1}{\frac{1}{2}+2^{-k-1}})$ so as to have

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le C_\epsilon\int_0^T t^{\frac{2^{k-1}-6}{2^k-1}}\,|T-2t|^{\frac{2^{k+1}}{2^k-1}}\,(T-t)^{\frac{2^{k+2}-6}{2^k-1}}\,\|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt \le CT^{\frac{13\times 2^{k-1}-12}{2^k-1}}\int_0^T\|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt$$

(again this inequality being valid for $k \ge 4$). Now for any $\delta > 0$, we can rechoose integer $k$ large enough so as to have $\frac{13\times 2^{k-1}-12}{2^k-1} \ge \frac{13}{2} - \delta$. This gives, then, for $T < 1$,

$$\left|\int_0^T h'(\Delta^2\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}dt\right| \le CT^{\frac{13}{2}-\delta}\int_0^T\|\vartheta\|_{L^2(\Omega)}^2\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt. \quad (1.42)$$

(This is the term which ultimately dictates the singularity.)

2. Next,

$$\int_0^T h'[(G_1-\lambda G_2)\gamma_0\vartheta_t, \phi]_{L^2(\Omega)}dt = -\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\,dt - \int_0^T h'[(G_1-\lambda G_2)\gamma_0\vartheta, \phi_t]_{L^2(\Omega)}\,dt. \quad (1.43)$$

   a. By the regularity posted in Eq. (1.16) and Lemma 1.1,

$$\left|\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\,dt\right| \le C\int_0^T\frac{|h''|}{t^{\frac{1}{4}}}\,\|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\,\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt.$$

   Applying Hölder's inequality to the right-hand side, with Hölder conjugates $(2\frac{2^k}{2^k-1}, \frac{1}{\frac{1}{2}+2^{-k-1}})$ now yields

$$\left|\int_0^T h''[(G_1-\lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}dt\right| \le C\int_0^T\frac{|h''|}{t^{\frac{1}{4}}}\left(\frac{h(t)}{h(t)}\right)^{\frac{1}{2}+2^{-k-1}} \times\|\vartheta\|_{L^2(\Omega)}^{1-\frac{1}{2^k}}\left\|e^{\mathcal{A}\frac{t}{2}}\vec{x}_0\right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}dt \le C_\epsilon T^{3\frac{5\times 2^{k-1}-4}{2^k-1}}\int_0^T\|\vartheta\|_{L^2(\Omega)}^2 + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.$$

Because for any $\delta > 0$, we can choose integer $k$ large enough so that $3\frac{5\times 2^{k-1}-4}{2^k-1} \geq \frac{15}{2} - \delta$, we then get

$$\left|\int_0^T h''[(G_1 - \lambda G_2)\gamma_0\vartheta, \phi]_{L^2(\Omega)}\, dt\right| \leq C_\epsilon T^{\frac{15}{2}-\epsilon} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.44}$$

b. In the same way as above, we have for integer $k$ large enough in Lemma 1.1,

$$\left|\int_0^T h'[(G_1 - \lambda G_2)\gamma_0\vartheta, \phi_t]_{L^2(\Omega)}\, dt\right| \leq C_\epsilon T^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.45}$$

The estimates Eqs. (1.44) and (1.45), applied to the relation Eq. (1.43) now give

$$\left|\int_0^T h'[(G_1 - \lambda G_2)\gamma_0\vartheta_t, \phi]_{L^2(\Omega)}\, dt\right| \leq C_\epsilon T^{\frac{15}{2}-\epsilon} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.46}$$

for integer $k$ large enough.

3.

$$\int_0^T h'\left(\mathring{\mathbf{A}}^{-1}\Delta\vartheta_t, \phi\right)_{L^2(\Omega)} dt = -\int_0^T h''(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}\, dt - \int_0^T h'\left(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt \tag{1.47}$$

a. By Green's Theorem and Lemma 1.5, we have in a fashon similar to that in (*1.a.*),

$$\begin{aligned}\left|\int_0^T h''(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)}\, dt\right| &= \left| -\int_0^T h''(\theta, \Delta\mathring{\mathbf{A}}^{-1}\phi)_{L^2(\Omega)} \right. \\ &\left. + \int_0^T h''\left[\theta, \left(\frac{\partial}{\partial\nu} + \lambda\right)\mathring{\mathbf{A}}^{-1}\phi\right]_{L^2(\Gamma)} \right| \leq C_\epsilon T^{3\frac{5\times 2^{k-1}-4}{2^k-1}} \int_0^T \|\vartheta\|^2_{L^2(\Omega)} \\ &+ \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt,\end{aligned} \tag{1.48}$$

for integer $k$ large enough.

b. In the same way,

$$\left|\int_0^T h'\left(\Delta\vartheta, \mathring{\mathbf{A}}^{-1}\phi_t\right)_{L^2(\Omega)} dt\right| \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.49}$$

Eqs. (1.47), (1.48), and (1.49) together give the estimate

$$\left|\int_0^T h'\left(\Delta\vartheta_t, \mathring{\mathbf{A}}^{-1}\phi\right)_{L^2(\Omega)} dt\right| \leq CT^{\frac{15}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.50}$$

Combining Eqs. (1.40), (1.41), (1.46), (1.50), and (1.42) will complete the proof of Lemma 1.3.

### 1.5.3 Conclusion of the Proof of Theorem 1.1(1)

Combining Lemmas 1.2 and 1.3 gives the following estimate for the energy:

$$\int_0^T \mathcal{E}(t)\,dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt;$$

or after changing limits of integration,

$$\int_0^{\frac{T}{2}} [(1-\epsilon)h(t) - 2\epsilon h(2t)]\,\mathcal{E}(t)\,dt + (1-\epsilon)\int_{\frac{T}{2}}^T h(t)\mathcal{E}(t)dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

For $\epsilon > 0$ small enough, this yields then

$$\int_{\frac{T}{2}}^T h(t)\mathcal{E}(t)\,dt \le C_\epsilon T^{\frac{13}{2}-\delta} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

Using the dissipation inherent in the thermoelastic system (i.e., $\mathcal{E}(t) \le \mathcal{E}(s)$ for $s \le t$), we finally obtain

$$\mathcal{E}(T) \le CT^{\frac{13}{2}-\delta-13} \int_0^T \|\vartheta\|^2_{L^2(\Omega)}\,dt.$$

This establishes the inequality Eq. (1.25), with $C_T = CT^{-q}$, where $q = \frac{13}{4} - \frac{\delta}{2}$, for any $\delta > 0$. This concludes the proof of Theorem 1.1(1).

## 1.6 Proof of Theorem 1.1(2)

### 1.6.1 A First Supporting Estimate

In what follows, we will again make use of the polynomial weight $h(t)$ in Eq. (1.24), with $s = 4$ therein.

In the present case of mechanical control, the necessary inequality (Eq. (1.12)) becomes

$$\sqrt{\mathcal{E}(T)} \le C_T \|\phi_t\|_{L^2(Q)}\,. \tag{1.51}$$

to be valid for all finite energy solutions to Eq. (1.9).

We start by establishing the following estimate:

***PROPOSITION 1.2***
*The solution* $[\phi, \phi_t, \vartheta]$ *of Eq.* (1.9) *satisfies the relation*

$$\left|\int_0^T h(t)\left(A_R^{-1}\vartheta_t, \vartheta\right)_{L^2(\Omega)} dt\right| \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\,dt + \epsilon\int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right)dt.$$

**PROOF OF PROPOSITION 1.2** From the mechanical component of Eq. (1.9) we have, after an extra differentiation in time, the expression $-\alpha\Delta\vartheta_t = \frac{\partial^3}{\partial t^3}\phi + \Delta^2\phi_t$; whence we obtain

$$A_R^{-1}\vartheta_t = \frac{1}{\alpha}A_R^{-2}\frac{\partial^3}{\partial t^3}\phi + \frac{1}{\alpha}A_R^{-2}\Delta^2\phi_t,$$

where the positive definite, self-adjoint operator $A_R : D(A_R) : L^2(\Omega) \to L^2(\Omega)$ is as defined in Eq. (1.19). Subsequently, we will have the following relation:

$$\int_0^T h(t)\big(A_R^{-1}\vartheta_t, \vartheta\big)_{L^2(\Omega)}\,dt = \frac{1}{\alpha}\int_0^T h(t)\left(\frac{\partial^3}{\partial t^3}\phi, A_R^{-2}\vartheta\right)_{L^2(\Omega)} dt + \frac{1}{\alpha}\int_0^T h(t)\big(\Delta^2\phi_t, A_R^{-2}\vartheta\big)_{L^2(\Omega)}\,dt. \tag{1.52}$$

□

We need to estimate the right-hand side of this expression.

1. For the first term on the right-hand side of Eq. (1.52), integration by parts gives

$$\int_0^T h(t)\left(\frac{\partial^3}{\partial t^3}\phi, A_R^{-2}\vartheta\right)_{L^2(\Omega)} dt = \int_0^T h(t)\big(\phi_t, A_R^{-2}\vartheta_{tt}\big)_{L^2(\Omega)}\,dt + 2\int_0^T h'(t)\big(\phi_t, A_R^{-2}\vartheta_t\big)_{L^2(\Omega)}\,dt + \int_0^T h''(t)\big(\phi_t, A_R^{-2}\vartheta\big)_{L^2(\Omega)}\,dt. \tag{1.53}$$

We proceed to scrutinize each term on the right-hand side. To this end, we introduce the (Robin) map $R \in \mathcal{L}[L^2(\Gamma), L^2(\Omega)]$, defined by

$$Rf = g \Leftrightarrow \Delta g = 0 \text{ on } \Omega \quad \text{and} \quad \frac{\partial g}{\partial \nu} + \lambda g = f \text{ on } \Gamma \tag{1.54}$$

(by elliptic regularity, we have in fact that $R \in \mathcal{L}[H^s(\Gamma), H^{s+\frac{3}{2}}(\Omega)]$ for all real $s$). Using this quantity with the heat equation in Eq. (1.9), we will then have the relations

$$\begin{aligned} A_R^{-2}\vartheta_t &= -A_R^{-1}\vartheta - \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t; \\ A_R^{-2}\vartheta_{tt} &= \vartheta + \alpha\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t - \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_{tt}. \end{aligned} \tag{1.55}$$

a. From Eq. (1.56), we have

$$\begin{aligned} \left|\int_0^T h(t)\big(\phi_t, A_R^{-2}\vartheta_{tt}\big)_{L^2(\Omega)}\,dt\right| &\le \int_0^T h(t) \\ &\times \left|\left\{\phi_t, \vartheta + \alpha\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_t\right\}_{L^2(\Omega)}\right| dt \\ &+ \int_0^T h(t)\left|\left\{\phi_t, \alpha A_R^{-1}\left[I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right)\right]\phi_{tt}\right\}_{L^2(\Omega)}\right| dt. \end{aligned} \tag{1.56}$$

To handle the most problematic term on the right-hand side of this expression (with again $\vec{x}(t) = [\phi(t), \phi_t(t), \vartheta(t)]$), we use the singular trace estimate in Lemma 1.1(3):

$$\begin{aligned}
\int_0^T h(t) \left| \left( \phi_t, A_R^{-1} R \frac{\partial}{\partial \nu} \phi_{tt} \right)_{L^2(\Omega)} \right| dt &\le C \int_0^T h(t) \, \|\phi_t\|_{L^2(\Omega)} \left\| \frac{\partial}{\partial \nu} \phi_{tt} \right\|_{L^2(\Gamma)} dt \\
&\le C \int_0^T \frac{h(t)}{t^{\frac{3}{4}} t} \|\phi_t\|_{L^2(\Omega)} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}(0) \right\|_{\mathbf{H}} dt \\
&\le C_\epsilon \int_0^T \frac{h(t)}{t^{\frac{7}{2}}} \|\phi_t\|^2_{L^2(\Omega)} \, dt \\
&\quad + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt \\
&\le C_\epsilon T^{\frac{9}{2}} \int_0^T \|\phi_t\|^2_{L^2(\Omega)} \, dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt.
\end{aligned}$$

Applying this estimate to Eq. (1.56) and treating in like fashion the other terms on the right-hand side thereof, we have

$$\left| \int_0^T h(t) \big(\phi_t, A_R^{-2} \vartheta_{tt}\big)_{L^2(\Omega)} \, dt \right| \le C_\epsilon T^{\frac{9}{2}} \int_0^T \|\phi_t\|^2_{L^2(\Omega)} \, dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.57}$$

b. Using the first relation in Eq. (1.56), we have, analogously to what was obtained in (*1.a*),

$$\begin{aligned}
&\left| \int_0^T h''(t) \big(\phi_t, A_R^{-2} \vartheta\big)_{L^2(\Omega)} \, dt + 2 \int_0^T h'(t) \big(\phi_t, A_R^{-2} \vartheta_t\big)_{L^2(\Omega)} \, dt \right| \\
&\le C \int_0^T \left[ |h''(t)| + \frac{|h'(t)|}{t^{\frac{3}{4}}} \right] \|\phi_t\|_{L^2(\Omega)} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}(0) \right\|_{\mathbf{H}} dt \\
&\le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)} \, dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt.
\end{aligned} \tag{1.58}$$

Combining Eqs. (1.56) and (1.58) now gives

$$\left| \int_0^T h(t) \left( \frac{\partial^3}{\partial t^3} \phi, A_R^{-2} \vartheta \right)_{L^2(\Omega)} dt \right| \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)} \, dt + \epsilon \int_0^T h(t) \mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.59}$$

2. By the "Green's" formula in Eq. (1.10), we have

$$\begin{aligned}
\int_0^T h(t) \big(\Delta^2 \phi_t, A_R^{-2} \vartheta\big)_{L^2(\Omega)} \, dt &= \int_0^T h(t) a\big(\phi_t, A_R^{-2} \vartheta\big)_{L^2(\Omega)} \, dt \\
&\quad + \int_0^T h(t) \big(\alpha \lambda \vartheta_t + \phi_t, A_R^{-2} \vartheta\big)_{L^2(\Gamma)} \, dt - \int_0^T h(t) \left[ \Delta \phi_t + (1-\mu) B_1 \phi_t, \frac{\partial}{\partial \nu} A_R^{-2} \vartheta \right]_{L^2(\Gamma)} dt \\
&= - \int_0^T h(t) \left[ \Delta \phi_t + (1-\mu) B_1 \phi_t, \left( \lambda I + \frac{\partial}{\partial \nu} \right) A_R^{-2} \vartheta \right]_{L^2(\Gamma)} dt \\
&\quad + \int_0^T h(t) \big(\phi_t, \Delta^2 A_R^{-2} \vartheta\big)_{L^2(\Omega)} \, dt + \int_0^T h(t) \left\{ \frac{\partial}{\partial \nu} \phi_t, [\Delta + (1-\mu) B_1] \, A_R^{-2} \vartheta \right\}_{L^2(\Gamma)} dt \\
&\quad - \int_0^T h(t) \left\{ \phi_t, \left[ \frac{\partial \Delta}{\partial \nu} + (1-\mu) \frac{\partial B_2}{\partial \tau} - I \right] A_R^{-2} \vartheta \right\}_{L^2(\Gamma)} dt.
\end{aligned} \tag{1.60}$$

For the first term on the right-hand side of Eq. (1.60), we apply the Lemma 1.1(1) (with $m = 2$ and $D_2 \equiv \Delta + (1-\mu)B_1$ therein) so as to have

$$\left| \int_0^T h(t) \left[ \Delta\phi_t + (1-\mu)B_1\phi_t, \left(\lambda I + \frac{\partial}{\partial\nu}\right) A_R^{-2}\vartheta \right]_{L^2(\Gamma)} dt \right|$$
$$\leq C \int_0^T \frac{h(t)}{t^{\frac{5}{4}}} \|\phi_t\|_{H^2(\Omega)}^{1-\frac{1}{2^k}} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}_0 \right\|_{\mathbf{H}}^{\frac{1}{2^k}} \|\vartheta\|_{L^2(\Omega)}\, dt \leq C \int_0^T \frac{h(t)}{t^{\frac{5}{4}}} \|\phi_t\|_{H^2(\Omega)}^{1-\frac{1}{2^k}} \left\| e^{\mathcal{A}\frac{t}{2}} \vec{x}_0 \right\|_{\mathbf{H}}^{1+\frac{1}{2^k}}.$$

Now letting $k = 2$, say, we can invoke Hölder's inequality, with Hölder conjugates $\left(\frac{8}{3}, \frac{5}{3}\right)$, to obtain the estimate

$$\left| \int_0^T h(t) \left( \Delta\phi_t + (1-\mu)B_1\phi_t, \left[\lambda I + \frac{\partial}{\partial\nu}\right] A_R^{-2}\vartheta \right)_{L^2(\Gamma)} dt \right|$$
$$\leq C_\epsilon T^{\frac{14}{3}} \int_0^T h(t) \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}(t/2)\, dt. \tag{1.61}$$

Applying this estimate to the right-hand side of Eq. (1.60) and subsequently handling the other terms thereof in a similar way—via the use of Lemma 1.1—we will have

$$\left| \int_0^T h(t) \left( \Delta^2\phi_t, A_R^{-1}\vartheta \right)_{L^2(\Omega)} dt \right|$$
$$\leq C_\epsilon T^{\frac{14}{3}} \int_0^T h(t) \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}(t)\, dt. \tag{1.62}$$

Combining Eqs. (1.52), (1.59), and (1.62) concludes the proof of Proposition 1.2.

### 1.6.2 Conclusion of the Proof of Theorem 1.1(2)

1. *Estimating the Thermal Component.* Applying the multiplier $h(t)A_R^{-1}\vartheta(t)$ to the heat component of the system Eq. (1.9) and subsequently invoking Proposition 1.2, we have

$$\begin{aligned}
\int_0^T h(t) \|\vartheta\|_{L^2(\Omega)}^2 &= -\int_0^T h(t)\left(A_R^{-1}\vartheta_t, \vartheta\right) dt \\
&\quad -\alpha \int_0^T h(t) \left\{ \left[ I - R\left(\lambda\gamma_0 + \frac{\partial}{\partial\nu}\right) \right] \phi_t, \vartheta \right\} dt \\
&\leq C_\epsilon T^4 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt \\
&\quad + \int_0^T h(t) \left\| \lambda\phi_t + \frac{\partial}{\partial\nu}\phi_t \right\|_{L^2(\Gamma)} \|\vartheta\|_{L^2(\Omega)}\, dt.
\end{aligned} \tag{1.63}$$

Via the Lemma 1.1 (with $m = 1$, $D_1 = \lambda I + \frac{\partial}{\partial\nu}$, and $k = 1$, say), we can estimate the third term on the right-hand side of Eq. (1.63) as

$$\int_0^T h(t) \left\| \lambda\phi_t + \frac{\partial}{\partial\nu}\phi_t \right\|_{L^2(\Gamma)} \|\vartheta\|_{L^2(\Omega)}\, dt \leq CT^5 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt.$$

Combining this estimate with Eq. (1.63), we now obtain

$$\int_0^T h(t) \|\vartheta\|_{L^2(\Omega)}^2 \leq C_\epsilon T^4 \int_0^T \|\phi_t\|_{L^2(\Omega)}^2\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.64}$$

2. *Estimating the Mechanical Component.* Here, we apply the multiplier intrinsic to uncoupled plates and beams. To wit, from the mechanical component of Eq. (1.9), we have via $h(t)\phi(t)$ and an invocation of the Green's Theorem Eq. (1.10) the expression

$$\int_0^T \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt = -\alpha \int_0^T h(t)(\vartheta, \Delta\phi)\, dt + \int_0^T h'(t)(\phi_t, \phi)\, dt + \int_0^T h(t)\, \|\phi_t\|^2_{L^2(\Omega)}\, dt. \tag{1.65}$$

Applying the estimate of Eq. (1.64) (available for the thermal component) now gives

$$\int_0^T \left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt. \tag{1.66}$$

Combining the estimates of Eqs. (1.64) and (1.66) now give the estimate for the energy

$$\int_0^T h(t)\mathcal{E}(t)\, dt \le C_\epsilon T^4 \int_0^T \|\phi_t\|^2_{L^2(\Omega)}\, dt + \epsilon \int_0^T h(t)\mathcal{E}\left(\frac{t}{2}\right) dt.$$

With this in hand, we can proceed as in the previous case so as to have the observability inequality Eq. (1.51), with $C_T = T^{-\frac{5}{2}}$. Subsequently, we will determine that in the present case of mechanical control, one has $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{5}{2}})$. This concludes the proof of Theorem 1.1(2) with free boundary conditions and one control.

## 1.7 Proof of Theorem 1.1(3)

Here we set the index $s = 2$ in Eq. (1.24). In this present case of dual—mechanical and thermal—control, the necessary inequality is

$$\sqrt{\mathcal{E}(T)} \le C_T(\|\phi_t\|_{L^2(Q)} + \|\vartheta\|_{L^2(Q)}), \tag{1.67}$$

where again $[\phi, \phi_t, \vartheta]$ solve the homogeneous system Eq. (1.9). Using the relation Eq. (1.65), we have

$$\begin{aligned}(1-\epsilon)\int_0^T h(t)\left\|\mathring{\mathbf{A}}^{\frac{1}{2}}\phi\right\|^2_{L^2(\Omega)} dt &\le C_\epsilon \int_0^T \left(h(t) + \frac{[h'(t)]^2}{h(t)}\right)\left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt \\ &\le CT^2 \int_0^T \left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt.\end{aligned}$$

This then gives

$$\int_0^T h(t)\mathcal{E}(t)\, dt \le CT^2 \int_0^T \left[\|\phi_t\|^2_{L^2(\Omega)} + \|\vartheta\|^2_{L^2(\Omega)}\right] dt,$$

whence we obtain the inequality Eq. (1.67). From here, we can use the usual algorithmic argument so as to have $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{3}{2}})$. This concludes the proof of Theorem 1.1(3).

## References

[1] P. Albano and D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system, *Electron. J. Differential Equations,* 2000, No. 22 (2000), 1–15.

[2] G. Avalos, Exact controllability of a thermoelastic system with control in the thermal component only, *Differential Integral Equations,* 13, (2000), 613–630.

[3] G. Avalos and I. Lasiecka, Boundary controllability of thermoelastic plates via the free boundary conditions, *SIAM J. Control Optim.,* 38, No. 2 (2000), 337–383.

[4] G. Avalos and I. Lasiecka, *Exponential Stability of a Thermoelastic System without Mechanical Dissipation,* Rendiconti dell'Istituto di Matematica dll'Università di Trieste, Vol. XXVIII, Supplemento (1996), 1–28.

[5] G. Avalos and I. Lasiecka, Exponential stability of a thermoelastic system with free boundary conditions without mechanical dissipation, *SIAM J. Math. Anal.,* 29, No. 1 (1998), 155–182.

[6] G. Avalos and I. Lasiecka, *A Note on the Null Controllability of a Thermoelastic Plates and Singularity of the Associated Minimal Energy Function,* Scuola Normale Superiore (Pisa), Preprints di Matematica, n. 10 (Giugno 2002).

[7] G. Avalos and I. Lasiecka, Optimal blowup rates for the minimal energy null control of the strongly damped abstract wave equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (5), II (2003), 601–616.

[8] G. Avalos and I. Lasiecka, Mechanical and thermal null controllability of thermoelastic plates and singularity of the associated minimal energy function, *Control Cybern.* special volume dedicated to K. Malanowski. 32, No. 3 (2003), 473–491.

[9] G. Avalos and I. Lasiecka, The null controllability of thermoelastic plates and singularity of the associated minimal energy function, *J. Math. Anal. Appl.,* 294, (2004), 34–61.

[10] G. Avalos and I. Lasiecka, Boundary controllability of thermoelastic plates via the free boundary conditions, *SIAM J. Control Optim.,* 38, No. 2 (2000), 337–383.

[11] A. Benabdallah and M. G. Naso, Null controllability of a thermoelastic plate, *Abstr. Appl. Anal.,* in press.

[12] A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinte Dimensional Systems,* Vol. II, Birkhäuser, Boston, (1993).

[13] J. Cea, *Lectures on Optimization—Theory and Algorithms* Published for the Tata Institute of Fundamental Research, Springer-Verlag, New York, (1978).

[14] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions,* Cambridge University Press, New York, New York, (1992).

[15] G. Da Prato, *An Introduction to Infinite Dimensional Analysis,* Scuola Normale Superiore, (2001).

[16] M. Eller, I. Lasiecka, and R. Triggiani. Simultaneous exact-approximate boundary controllability of thermoelastic plates with variabale thermal coefficients and moment control. *J. Math. Anal. Appl.,* 251, (2000), 452–478.

[17] M. Eller, I. Lasiecka and R. Triggiabi. Simultaneous exact controllability of thermoelastic plates with variable thermal coefficients and clamped/Dirichlet boundary controls, *Discrete Continuous Dynamical Syst.,* 7, No. 2 (2001), 283–302.

[18] S. Hansen and B. Zhang, Boundary control of a thermoelastic beam, *J. Math. Anal. Appl.,* 210, (1997), 182–205.

[19] F. Gozzi and P. Loreti, Regularity of the minimum time function and minimum energy problems, *SIAM J. Control Optim.,* in press.

[20] B. Guichal, A lower bound of the norm of the control operator for the heat equation. *J. Math. Anal. Appl.* 110, No. 2 (1985), 519–527.

[21] P. Grisvard, Caracterization de quelques espaces d'interpolation, *Arch. Rational Mech. Anal.,* 25, (1967), 40–63.

[22] J. Lagnese, *Boundary Stabilization of Thin Plates, SIAM Stud. Appl. Math., 10,* SIAM, Philadelphia, (1989).

[23] J. Lagnese, The reachability problem for thermoelastic plates, *Arch. Rational Mech. Anal.,* 112, (1990), 223–267.

[24] J. Lagnese and J.L. Lions, *Modelling, Analysis and Control of Thin Plates,* Masson Paris, (1988).

[25] I. Lasiecka and T. Seidman, *Thermal boundary control of a thermoelastic system,* preprint (2004).

[26] I. Lasiecka and R. Triggiani, Analyticity of thermo-elastic semigroups with clamped/hinged B.C., *Abst. Appl. Anal.,* 3, No. 2 (1998), 153–169.

[27] I. Lasiecka and R. Triggiani, *Analyticity of thermo-elastic semigroups with coupled BC. Part II: The case of free BC,* Annali Scuola Normale di Pisa, Classes Scienze (Serie IV, Fasicolo 3–4), Vol. XXVII (1998[c]), 457–497.

[28] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I: Abstract Parabolic Systems,* Cambridge University Press, New York, (2000).

[29] I. Lasiecka, Uniform decay rates for full von Karman system of dynamic thermoelasticity with free boundary conditions and partial boundary dissipation, *Commun. Partial Differential Equations,* 24, 9&10 (1999), 1801–1847.

[30] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications,* Vol. 1, Springer-Verlag, New York, (1972).

[31] Z. Liu and M. Renardy, A note on the equations of a thermoelastic plate, *Appl. Math. Lett.,* 8, No. 3 (1995), 1–6.

[32] A. Lunardi, Schauder's estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$, *Ann. Scuola Sup.* Pisa (IV), XXIV (1997), 133–164.

[33] T.I. Seidman, How fast are violent controls? *Math. Controls Signals Syst.,* (1988), 89–95.

[34] T.I. Seidman, Two results on exact boundary control of parabolic equations, *Appl. Math. Optim.,* 11, (1984), 145–152.

[35] T.I. Seidman and J. Yong, How fast are violent controls?, II, *Math. Controls Signals Syst.,* 9, (1997), 327–340.

[36] T. Seidman, S. Avdonian, and S. Ivanov, The window problem for series of complex exponentials, *J. Fourier's Anal.,* 6, (2000), 235–254.

[37] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems,* Springer-Verlag, New York, (1984).

[38] R. Triggiani, Analyticity, and lack thereof, of semigroups arising from thermo-elastic plates, in the special volume entitled *Computational Science for the 21st Century,* Chapter on Control: Theory and Numerics, Wiley 1997; Proceedings in honor of R. Glowinski, (May, 1997).

[39] R. Triggiani, Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems, *Adv. Differential Equations,* 8, No. 2 (2003), 189–229.

[40] J. Zabczyk, *Mathematical Control Theory,* Birkhäuser, Boston, (1992).