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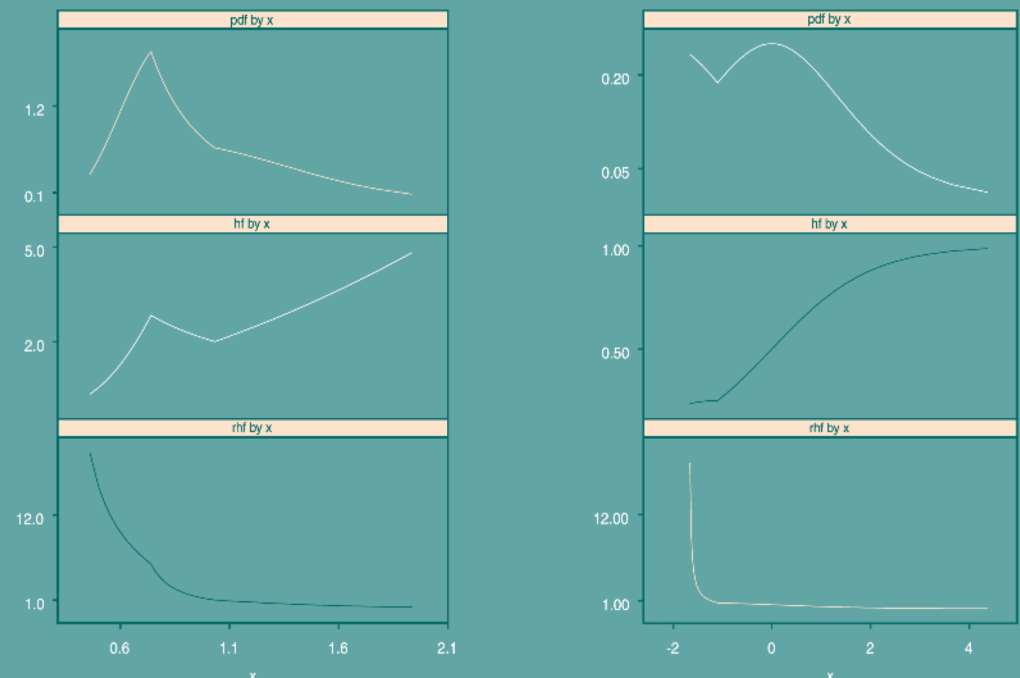

N. Balakrishnan , I.G. Bairamov,
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Preface

Statistical distributions and models play a vital role in many different applied fields in science, engineering, humanities, health and social sciences. Data obtained from real-life situations are often modeled by appropriate statistical distributions and then inferential procedures are developed under the assumption of that particular distribution. For this reason, it becomes very important that properties of statistical distributions are studied so that they can be utilized to develop optimal inferential methods for analyzing the data under the considered statistical model, and also to check the validity of that model assumption for the data at hand.

Many authoritative and encyclopedic volumes on statistical distribution theory exist in the literature. This list includes:

- Johnson, Kotz and Kemp (1992), describing discrete univariate distributions
- Stuart and Ord (1993), discussing general distribution theory
- Johnson, Kotz and Balakrishnan (1994, 1995), describing continuous univariate distributions

- Johnson, Kotz and Balakrishnan (1997), describing discrete multivariate distributions
- Wimmer and Altmann (1999), presenting a thesaurus on discrete univariate distributions
- Evans, Peacock and Hastings (2000), describing discrete and continuous distributions
- Kotz, Balakrishnan and Johnson (2000), discussing continuous multivariate distributions
- Balakrishnan and Nevzorov (2003), providing an introductory exposition to distribution theory
- Zelterman (2004), discussing discrete distributions and their applications in health sciences

All these books/volumes provide ample evidence of the importance of this area of research.

In this volume, we present 14 chapters written by internationally renowned experts. These chapters discuss characterizations and other important properties of several statistical distributions and models, inferential procedures for these distributions and models, and finally some applications to real-life problems. Each chapter has been written in a self-contained expository manner with a comprehensive list of pertinent references. These chapters are based on some selected papers that were presented at the *International Conference on Advances on Models, Characterizations and Applications* that was held in Antalya, Turkey, in December 2001.

It is our sincere hope that readers of this volume will get up-to-date information on some recent developments on characterizations and other important properties of several distributions, on some inferential issues relating to these models, and finally on some applications of these models to real-life problems.

We thank all the authors for presenting their work in this volume and also for their support and cooperation. We gratefully acknowledge the help of the referees. Our final special thanks go to Ms. Maria Allegra and Mr. Kevin Sequeira of Marcel Dekker for their support and encouragement, Ms. Preethi Cholmondeley of CRC Press – Taylor & Francis for

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Chapter 1

The Shapes of the Probability Density, Hazard, and Reverse Hazard Functions

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CONTENTS

ABSTRACT

The shapes of a probability density function, its hazard function, and its reverse hazard function restrict each other. Here, "shape" means six types of the graphs of a continuous univariate function: increasing (i, for short), decreasing (d), unimodal (id), anti-unimodal (di), increasing–decreasing–increasing (idi), and decreasing–increasing–decreasing (did). It is proved that among 216 combinations of the shapes of the three functions, 44 cases are possible.

This result is a nonparametric characterization of a triplet, the probability density, hazard and reverse hazard functions, by their shapes.

KEYWORDS AND PHRASES: Characterization, dual failure function, failure rate, hazard rate, logistic distribution, Pareto distribution, reversed hazard rate, Weibull distribution

1.1 INTRODUCTION

The reverse hazard function (r.h.f.), or the reversed hazard rate function, is the ratio of a probability density function (p.d.f.) to its distribution function (d.f.); it is used for the retrospective analysis of survival data. It was introduced by Keilson and Sumita (1982) and called the dual failure function. It has the properties dual to the hazard function (h.f.); see Shaked and Shanthikumar (1994).

Lagakos et al. (1988) used the r.h.f. for a retrospective analysis of epidemiological data on individuals in a group, who are identified by some event and the random time of an initiating event is recorded. Kalbfleisch and Lawless (1989) studied the same kind of data and suggested using the r.h.f. Block et al. (1998) proved, among others, that if a r.h.f. is increasing, its h.f. is also increasing, and its distribution range is limited to $(-\infty, \omega)$, $\omega < \infty$. Hence, the lifetime never has an increasing r.h.f. Based on this fact, they cautioned misuses of the r.h.f.

The shapes of a triplet of a probability density, its hazard, and its reverse hazard functions restrict each other. A "shape" means here a class of piecewise monotone continuous positive functions. In this paper six shapes, on at most three consecutive common intervals (see Subsection 1.3.3), are examined: increasing (i, for short), decreasing (d), unimodal (id), antiunimodal (di), increasing–decreasing–increasing (idi), and decreasing–increasing–decreasing (did). The main result is that among $6 \times 6 \times 6 = 216$ combinations of the shapes of the triplet, 44 cases in Tables 1.3.3 and 1.3.4, Section 1.3, are possible. This result is a nonparametric characterization of a triplet, the probability density, hazard and reverse hazard functions, by their shapes, and can be used as a reference for the modeling based on hazard and reverse hazard functions.

Figure 1.1.1 shows two examples of triplets with different combinations of shapes.

This paper extends a previous one on the relation between the shapes of a probability density and its hazard functions (Sibuya, 1996) and completely solves the problem raised by Block et al. (1998).

The shapes of a h.f. were studied by Aalen and Gjessing (2001) from a different point of view. Monotone h.f. and r.h.f.

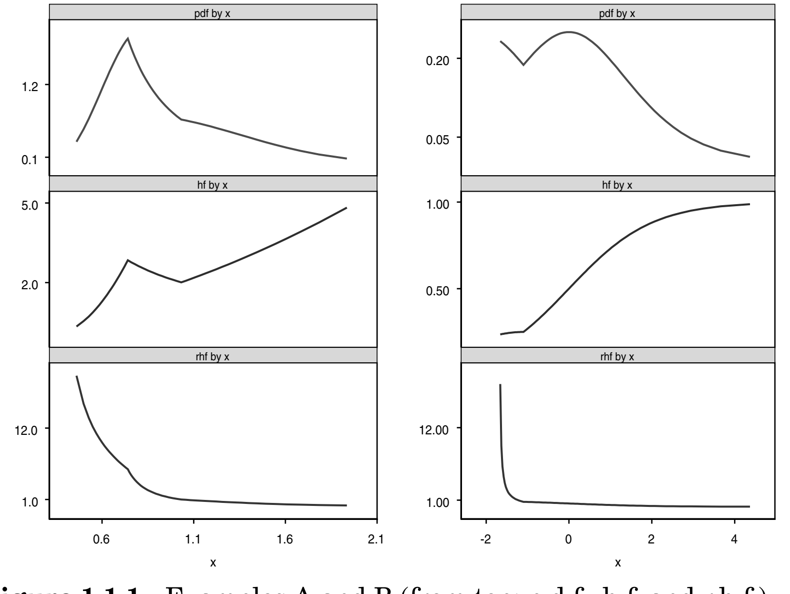

Figure 1.1.1 Examples A and B (from top: p.d.f., h.f. and r.h.f.).

were discussed in Sengupta and Nanda (1999) and Chandra and Roy (2001). An ordering of lifetime distributions by the increasing ratio of a pair of h.f.'s, or a pair of r.h.f.'s, was discussed by Sibuya and Suzuki (2001).

1.2 MONOTONE SHAPES

The h.f. is used mainly in the lifetime analysis and for positive random variables. Here, random variables are not restricted to be positive. The distribution limits of a d.f. F are defined by $\alpha := \inf\{x; F(x) > 0\}$, and $\omega := \sup\{x; F(x) < 1\}$, $-\infty \le \alpha < \omega \le \infty$. It is assumed that the p.d.f. satisfies $f(x) > 0, \alpha < x < \omega$. The shapes are invariant with respect to the piecewise change of location and scale. Hence, whether the limits are finite or infinite is the concern. The symbols α and ω are overused within tables to mean finite limits.

1.2.1 Definitions and duality

Let $\bar{F}(x) = 1 - F(x)$ be a survival function (s.f.); the h.f. is defined by

$$h(x) := \frac{d}{dx}(-\log(\bar{F}(x))) = f(x)/\bar{F}(x) \ge 0, \quad \alpha < x < \omega. \tag{1.2.1}$$

Conversely, the cumulative h.f. determines its d.f.:

$$\bar{F}(x) = \exp(-H(x)), \qquad H(x) = \int_{-\infty}^{x} h(t)\,dt.$$

$H(x)$ is increasing, $H(\alpha) = 0$, and $H(\omega) = \infty$.

The r.h.f. is a dual of h.f.:

$$\begin{aligned} \tilde{h}(x) &= \frac{d}{dx}(\log F(x)) = f(x)/F(x) \ge 0, \quad \alpha < x < \omega, \\ F(x) &= \exp(-\tilde{H}(x)) \qquad \tilde{H}(x) = \int_{x}^{\infty} \tilde{h}(t)\,dt. \end{aligned} \tag{1.2.2}$$

$\tilde{H}$ is decreasing, $\tilde{H}(\alpha) = \infty$, and $\tilde{H}(\omega) = 0$. Note that if $\alpha > -\infty$ it is possible that $H(\alpha + 0) > 0$, and if $\omega < \infty$ it is possible that $\tilde{H}(\omega - 0) > 0$. The cumulative h.f. and r.h.f. are directly related:

$$H(x) = \Omega(\tilde{H}(x)) \quad \text{and} \quad \tilde{H}(x) = \Omega(H(x)), \text{where}$$
$$\Omega(t) = -\log(1 - \exp(-t)), \quad 0 < t < \infty.$$

PROPOSITION 1.2.1 (reflection or duality) *If the p.d.f., d.f., s.f., h.f., r.h.f., cumulative h.f., and cumulative r.h.f. of a r.v. X are $f(x)$, $F(x)$, $\bar{F}(x)$, $h(x)$, $\tilde{h}(x)$, $H(x)$, and $\tilde{H}(x)$, respectively, those of the negated $-X$ are $f(-x)$, $\bar{F}(-x)$, $F(-x)$, $\tilde{h}(-x)$, $h(-x)$, $\tilde{H}(-x)$, and $H(-x)$, respectively. (Note the change of order.)*

This simple fact will be repeatedly used in this paper.

1.2.2 Monotone h.f. and r.h.f.

Before discussing monotone shapes, note that, for a constant $\lambda > 0$,

$$\begin{aligned}
&\text{If } h(x) = \lambda, \quad \bar{F}(x) = e^{-\lambda x}/\bar{F}(a) \quad \text{and} \quad \tilde{h}(x) \downarrow,\ x > a.\\
&\text{If } \tilde{h}(x) = \lambda, \quad F(x) = e^{\lambda x}/F(b) \quad \text{and} \quad h(x) \uparrow,\ x < b.\\
&\text{If } f(x) = \lambda, \quad F(x) = F(a) + \lambda(x - a), \quad \bar{F}(x) = \bar{F}(b) + \lambda(b - x)\\
&\qquad\qquad \text{and} \quad h(x) \uparrow,\ \tilde{h}(x) \downarrow,\ a < x < b.
\end{aligned}$$

Throughout the paper the terms *increasing* and *decreasing* are used in a weak sense.

From the definitions and Proposition 1.2.1,

$$\begin{aligned}
&h \uparrow (\downarrow) \Leftrightarrow H : \text{convex(concave)}, \quad \text{and}\\
&\tilde{h} \downarrow (\uparrow) \Leftrightarrow \tilde{H} : \text{convex(concave)}.\\
&H : \text{concave} \Rightarrow \tilde{H} : \text{convex}, \quad \text{and}\\
&\tilde{H} : \text{concave} \Rightarrow H : \text{convex}.
\end{aligned}$$

PROPOSITION 1.2.2

(i) If a h.f. is decreasing, its p.d.f. is decreasing, which implies its r.h.f. is decreasing.

(ii) Similarly, if a r.h.f. is increasing, its p.d.f. is increasing, which implies its h.f. is increasing.

PROOF. From the definition of h and $\tilde{h}$, a pair of fundamental inequalities are obtained:

$$\begin{aligned}(\log f)' &= (\log h)' - h \leq (\log h)',\\ (\log f)' &= (\log \tilde{h})' + \tilde{h} \geq (\log \tilde{h})'.\end{aligned} \tag{1.2.3}$$

The first one implies $f \uparrow \Rightarrow h \uparrow$ and $h \downarrow \Rightarrow f \downarrow$, and the second implies $f \downarrow \Rightarrow \tilde{h} \downarrow$ and $\tilde{h} \uparrow \Rightarrow f \uparrow$. Combine these to show the proposition. ∎

REMARK 1.2.1

1. The inequalities hold at any $x \in (\alpha, \omega)$; that is, the proposition states local properties.
2. The second half (ii) of the proposition is dual to (i).

PROPOSITION 1.2.3 (restriction of distribution range) *Let t be any number such that $\alpha < t < \omega$.*

(i) If a h.f. is decreasing in (α, t), $\alpha > -\infty$. If a h.f. is decreasing in (t, ω), $\omega = \infty$.
(ii) If a r.h.f. is increasing in (α, t), $\alpha = -\infty$. If a r.h.f. is increasing in (t, ω), $\omega < \infty$.

PROOF. If a p.d.f. is decreasing in the lower tail, *a fortiori* if a h.f. is decreasing in the lower tail, its lower distribution limit is finite, because that $f(t) \downarrow$ on $(-\infty, t)$ is impossible. Since $H(\omega) = \infty$, a decreasing h.f. cannot end at a finite point. The second half (ii) is dual to (i). ∎

REMARK 1.2.2 The facts of (i) can be confirmed by observing the cumulative h.f., H. If h is decreasing in (α, t), H is concave and increasing, hence $\alpha > -\infty$. If H is concave in (t, ω), $H(\omega)$ cannot be infinite unless $\omega = \infty$.

1.2.3 Truncation

PROPOSITION 1.2.4 (truncation)

(i) If a p.d.f. is left-truncated at a, $\alpha < a < \omega$, its h.f. does not change in (a, ω). Further, if the original r.h.f. is decreasing, the new r.h.f. is also decreasing in (a, ω).

(ii) *If a p.d.f. is right-truncated at b, $\alpha < b < \omega$, its r.h.f. does not change in (α, b). Further, if the original h.f. is increasing, the new h.f. is also increasing in (α, b).*

PROOF. For the original $f, F, \bar{F}, h$, and $\tilde{h}$, let the left-truncated be denoted with an asterisk:

$$f^*(x) = f(x)/\bar{F}(a), \qquad \bar{F}^*(x) = \bar{F}(x)/\bar{F}(a),$$
$$F^*(x) = (F(x) - F(a))/(1 - F(a)), \quad x > a.$$

Hence,

$$h^*(x) = h(x), \qquad \tilde{h}^*(x) = \frac{1 - F(a)}{1 - F(a)/F(x)}\tilde{h}(x), \quad x > a,$$

and the latter is decreasing. The second half (ii) is dual to (i). ■

1.3 NECESSITY OF RESTRICTIONS

1.3.1 The shapes of a doublet of p.d.f. and h.f. (or r.h.f.)

Before discussing the shapes of a triplet, combinations of the shapes of a doublet of p.d.f. and h.f., or a doublet of p.d.f. and r.h.f., are examined. Propositions 1.2.2 and 1.2.3 restrict possible combinations and the range.

Impossible combinations of the shapes of p.d.f. and h.f. are shown in Table 1.3.1. For other combinations, the necessary restrictions on the distribution range are shown. Similarly, impossible combinations of the shapes of p.d.f. and r.h.f., as well as restrictions on the range, are shown in Table 1.3.2. Table 1.3.1 is a modification of a previous table for the lifetime distributions (Sibuya, 1996).

Table 1.3.1 is constructed along the following rules because of Theorems 1.3.3 and 1.3.4.:

(i) If h is decreasing in some interval (symbol d) and f is increasing (symbol i) in this interval, the doublet is impossible (symbol †).

(ii) If h is decreasing in the upper tail (f is also decreasing), $\omega = \infty$ (symbol +).