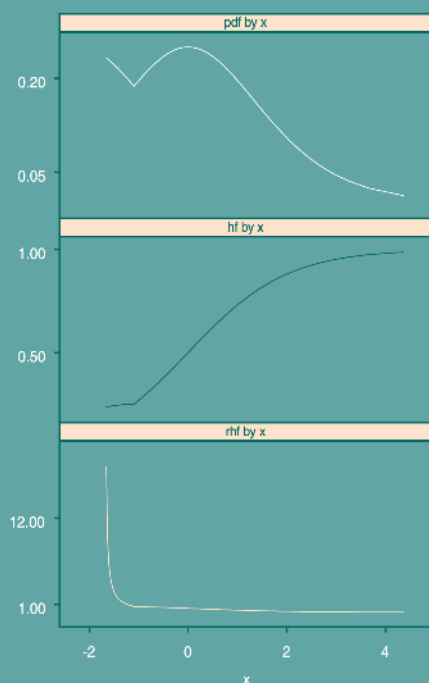
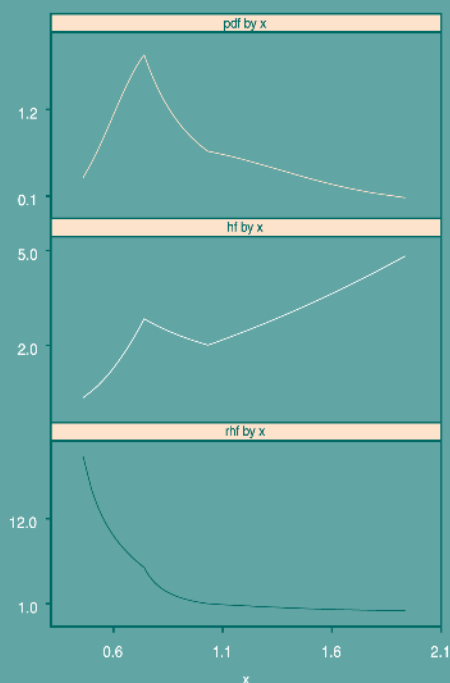


STATISTICS:
a series of **TEXTBOOKS** and **MONOGRAPHS**

Advances on Models, Characterizations and Applications



**N. Balakrishnan , I.G. Bairamov,
and O.L. Gebizlioglu**



Chapman & Hall/CRC
Taylor & Francis Group

Advances on Models, Characterizations and Applications

STATISTICS: Textbooks and Monographs

D. B. Owen

Founding Editor, 1972–1991

Associate Editors

**Statistical Computing/
Nonparametric Statistics**
Professor William R. Schucany
Southern Methodist University

Probability
Professor Marcel F. Neuts
University of Arizona

Multivariate Analysis
Professor Anant M. Kshirsagar
University of Michigan

Quality Control/Reliability
Professor Edward G. Schilling
Rochester Institute of Technology

Editorial Board

Applied Probability
Dr. Paul R. Garvey
The MITRE Corporation

Economic Statistics
Professor David E. A. Giles
University of Victoria

Experimental Designs
Mr. Thomas B. Barker
Rochester Institute of Technology

Multivariate Analysis
Professor Subir Ghosh
*University of
California–Riverside*

Statistical Distributions
Professor N. Balakrishnan
McMaster University

Statistical Process Improvement
Professor G. Geoffrey Vining
Virginia Polytechnic Institute

Stochastic Processes
Professor V. Lakshmikantham
Florida Institute of Technology

Survey Sampling
Professor Lynne Stokes
Southern Methodist University

Time Series
Sastry G. Pantula
North Carolina State University

STATISTICS: Textbooks and Monographs

Recent Titles

Handbook of Applied Economic Statistics, *Aman Ullah and David E. A. Giles*

Improving Efficiency by Shrinkage: The James-Stein and Ridge Regression Estimators, *Marvin H. J. Gruber*

Nonparametric Regression and Spline Smoothing: Second Edition, *Randall L. Eubank*
Asymptotics, Nonparametrics, and Time Series, *edited by Subir Ghosh*

Multivariate Analysis, Design of Experiments, and Survey Sampling, *edited by Subir Ghosh*

Statistical Process Monitoring and Control, *edited by Sung H. Park and G. Geoffrey Vining*

Statistics for the 21st Century: Methodologies for Applications of the Future, *edited by C. R. Rao and Gábor J. Székely*

Probability and Statistical Inference, *Nitis Mukhopadhyay*

Handbook of Stochastic Analysis and Applications, *edited by D. Kannan and V. Lakshmikantham*

Testing for Normality, *Henry C. Thode, Jr.*

Handbook of Applied Econometrics and Statistical Inference, *edited by Aman Ullah, Alan T. K. Wan, and Anoop Chaturvedi*

Visualizing Statistical Models and Concepts, *R. W. Farebrother and Michael Schyns*

Financial and Actuarial Statistics, *Dale Borowiak*

Nonparametric Statistical Inference, Fourth Edition, Revised and Expanded, *edited by Jean Dickinson Gibbons and Subhabrata Chakraborti*

Computer-Aided Econometrics, *edited by David EA. Giles*

The EM Algorithm and Related Statistical Models, *edited by Michiko Watanabe and Kazunori Yamaguchi*

Multivariate Statistical Analysis, Second Edition, Revised and Expanded, *Narayan C. Giri*

Computational Methods in Statistics and Econometrics, *Hisashi Tanizaki*

Applied Sequential Methodologies: Real-World Examples with Data Analysis, *edited by Nitis Mukhopadhyay, Sujay Datta, and Saibal Chattopadhyay*

Handbook of Beta Distribution and Its Applications, *edited by Richard Guarino and Saralees Nadarajah*

Item Response Theory: Parameter Estimation Techniques, Second Edition, *edited by Frank B. Baker and Seock-Ho Kim*

Statistical Methods in Computer Security, *William W. S. Chen*

Elementary Statistical Quality Control, Second Edition, *John T. Burr*

Data Analysis of Asymmetric Structures, *edited by Takayuki Saito and Hiroshi Yadohisa*

Mathematical Statistics with Applications, *Asha Seth Kapadia, Wenyaw Chan, and Lemuel Moyé*

Advances on Models, Characterizations and Applications, *N. Balakrishnan, I. G. Bairamov, and O. L. Gebizlioglu*

Survey Sampling: Theory and Methods, Second Edition, *Arijit Chaudhuri and Horst Stenger*

Statistical Design of Experiments with Engineering Applications, *Kamel Rekab and Muzaffar Shaikh*

Advances on Models, Characterizations and Applications

N. Balakrishnan

McMaster University
Hamilton, ON, Canada

I.G. Bairamov

Izmir University of Economics
Balcova, Izmir, Turkey

O.L. Gebizlioglu

Ankara University
Tandogan, Ankara, Turkey



Chapman & Hall/CRC

Taylor & Francis Group

Boca Raton London New York Singapore

Published in 2005 by
CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

© 2005 by Taylor & Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group

No claim to original U.S. Government works
Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2 1

International Standard Book Number-10: 0-8247-4022-X (Hardcover)
International Standard Book Number-13: 978-0-8247-4022-1 (Hardcover)
Library of Congress Card Number 2005041422

This book contains information obtained from authentic and highly regarded sources. Reprinted material is quoted with permission, and sources are indicated. A wide variety of references are listed. Reasonable efforts have been made to publish reliable data and information, but the author and the publisher cannot assume responsibility for the validity of all materials or for the consequences of their use.

No part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (<http://www.copyright.com/>) or contact the Copyright Clearance Center, Inc. (CCC) 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Library of Congress Cataloging-in-Publication Data

Advances on models, characterizations, and applications / [edited by] N. Balakrishnan, I.G. Bairamov,
and O.L. Gebizlioglu.

p. cm.

Includes bibliographical references and index.

ISBN 0-8247-4022-X (alk. paper)

1. Distribution (Probability theory) 2. Statistical hypothesis testing--Mathematical models. I.
Balakrishnan, N. II. Bairamov, I. G. III. Gebizlioglu, O. L.

QA273.6.A39 2005

519.2'4--dc22

2005041422

T&F informa

Taylor & Francis Group
is the Academic Division of T&F Informa plc.

Visit the Taylor & Francis Web site at
<http://www.taylorandfrancis.com>

and the CRC Press Web site at
<http://www.crcpress.com>

Preface

Statistical distributions and models play a vital role in many different applied fields in science, engineering, humanities, health and social sciences. Data obtained from real-life situations are often modeled by appropriate statistical distributions and then inferential procedures are developed under the assumption of that particular distribution. For this reason, it becomes very important that properties of statistical distributions are studied so that they can be utilized to develop optimal inferential methods for analyzing the data under the considered statistical model, and also to check the validity of that model assumption for the data at hand.

Many authoritative and encyclopedic volumes on statistical distribution theory exist in the literature. This list includes:

- Johnson, Kotz and Kemp (1992), describing discrete univariate distributions
- Stuart and Ord (1993), discussing general distribution theory
- Johnson, Kotz and Balakrishnan (1994, 1995), describing continuous univariate distributions

- Johnson, Kotz and Balakrishnan (1997), describing discrete multivariate distributions
- Wimmer and Altmann (1999), presenting a thesaurus on discrete univariate distributions
- Evans, Peacock and Hastings (2000), describing discrete and continuous distributions
- Kotz, Balakrishnan and Johnson (2000), discussing continuous multivariate distributions
- Balakrishnan and Nevzorov (2003), providing an introductory exposition to distribution theory
- Zelterman (2004), discussing discrete distributions and their applications in health sciences

All these books/volumes provide ample evidence of the importance of this area of research.

In this volume, we present 14 chapters written by internationally renowned experts. These chapters discuss characterizations and other important properties of several statistical distributions and models, inferential procedures for these distributions and models, and finally some applications to real-life problems. Each chapter has been written in a self-contained expository manner with a comprehensive list of pertinent references. These chapters are based on some selected papers that were presented at the *International Conference on Advances on Models, Characterizations and Applications* that was held in Antalya, Turkey, in December 2001.

It is our sincere hope that readers of this volume will get up-to-date information on some recent developments on characterizations and other important properties of several distributions, on some inferential issues relating to these models, and finally on some applications of these models to real-life problems.

We thank all the authors for presenting their work in this volume and also for their support and cooperation. We gratefully acknowledge the help of the referees. Our final special thanks go to Ms. Maria Allegra and Mr. Kevin Sequeira of Marcel Dekker for their support and encouragement, Ms. Preethi Cholmondeley of CRC Press – Taylor & Francis for

helping us with the production of the volume, and Mrs. Debbie Iscoe for her fine work in typesetting the entire volume.

N. Balakrishnan
HAMILTON, CANADA

I. Bairamov
İZMİR, TURKEY

O. Gebizlioglu
ANKARA, TURKEY

October 2004

Contents

Preface	<i>vii</i>
Contributors	<i>xv</i>

Chapter 1. The Shapes of the Probability Density, Hazard and Reverse Hazard Functions	<i>1</i>
--	----------

Masaaki Sibuya

Chapter 2. Stochastic Ordering of Risks, Influence of Dependence, and A.S. Constructions	<i>19</i>
---	-----------

Ludger Rüschendorf

Chapter 3. The q-Factorial Moments of Discrete q-Distributions and a Characterization of the Euler Distribution	<i>57</i>
--	-----------

Ch. A. Charalambides and N. Papadatos

**Chapter 4. On the Characterization of Distributions
Through the Properties of Conditional
Expectations of Order Statistics 73**

I. Bairamov and O. Gebizlioglu

**Chapter 5. Characterization of the Exponential
Distribution by Conditional Expectations
of Generalized Spacings 83**

Erhard Cramer and Udo Kamps

**Chapter 6. Some Characterizations of Exponential
Distribution Based on Progressively
Censored Order Statistics 97**

N. Balakrishnan and S. V. Malov

**Chapter 7. A Note On Regressing Order Statistics
and Record Values 111**

I. Bairamov and N. Balakrishnan

**Chapter 8. Generalized Pareto Distributions and
Their Characterizations 121**

Majid Asadi

**Chapter 9. On Some Characteristic Properties
of the Uniform Distribution 153**

G. Arslan, M. Ahsanullah and I. G. Bairamov

**Chapter 10. Characterizations of Multivariate Distri-
butions Involving Conditional Specifica-
tion and/or Hidden Truncation 161**

Barry C. Arnold

**Chapter 11. Bivariate Matsumoto-Yor Property
and Related Characterizations 177**

Konstancja Bobecka and Jacek Wesolowski

Chapter 12. First Principal Component Characterization of a Continuous Random Variable	189
---	------------

Carles M. Cuadras

Chapter 13. The Lawless-Wang's Operational Ridge Regression Estimator Under the LINEX Loss Function	201
--	------------

Esra Akdeniz and Fikri Akdeniz

Chapter 14. On the Distribution of the Reference Dose and Its Application in Health Risk Assessment	215
--	------------

Mehdi Razzaghi

Subject Index	233
----------------------------	------------

Contributors

M. Ahsanullah

Department of Management
Sciences
Rider University
Lawrenceville, NJ 08648, U.S.A.
ahsan@rider.edu

Barry C. Arnold

Department of Statistics,
University of California
Riverside
CA 92521, U.S.A.
barry.arnold@ucr.edu

Esra Akdeniz

Department of Statistics, Faculty
of Arts and Sciences
University of Cukurova
01330, Adana, Turkey
esraakdeniz1@yahoo.com

G. Arslan

Department of Statistics and
Computer Sciences
Başkent, University
06530 Ankara, Turkey
guvenca@baskent.edu.tr

Fikri Akdeniz

Department of Statistics, Faculty
of Arts and Sciences
University of Cukurova
01330, Adana, Turkey
akdeniz@mail.cu.edu.tr

Majid Asadi

Department of Statistics
University of Isfahan
Isfahan, 81744, Iran
M.Asadi@sci.ui.ac.ir

I. G. Bairamov

Department of Mathematics
İzmir University of
Economics, 35330, Balcova, İzmir,
Turkey
bayramov@science.ankara.
edu.tr

N. Balakrishnan

Department of Mathematics and
Statistics
McMaster University
Hamilton, Ontario L8S 4K1,
Canada
bala@univmail.cis.
mcmaster.ca

Konstancja Bobecka

Wydział Matematyki i Nauk
Informacyjnych
Politechnika Warszawska
Plac Politechniki 1, 00-661
Warszawa, Poland
bobecka@mini.pw.edu.pl

Ch. A. Charalambides

Department of Mathematics
University of Athens
GR-15784 Athens, Greece
ccharal@math.uoa.gr

Erhard Cramer

Institute of Statistics
RWTH Aachen University
52056 Aachen, Germany
erhard.cramer@
rwth-aachen.de

Carles M. Cuadras

Department of Statistics
University of Barcelona
08028 Barcelona, Spain
carlesm@porthos.bio.ub.es

O. Gebizlioglu

Department of Statistics
Ankara University
06100, Tandogan, Ankara, Turkey
gebizli@science.ankara.
edu.tr

Udo Kamps

Institute of Statistics
RWTH Aachen University
52056 Aachen, Germany
udo.kamps@rwth-aachen.de

S. V. Malov

Department of Mathematics and
Mechanics
St. Petersburg State University
198504, Bibliotchnaya Sq. 2,
St. Petersburg, Russia
Sergey.Malov@pobox.spbu.ru

N. Papadatos

Department of Mathematics
University of Athens
GR-15784 Athens, Greece
npapada@cc.uoa.gr

Ludger Rüschendorf

Department of Mathematical
Stochastics
University of Freiburg
Eckerstr. 1, D-79104 Freiburg,
Germany
ruschen@stochastik.uni-
freiburg.de

Mehdi Razzaghi

Department of Mathematics,
Computer Science and
Statistics, Bloomsburg University
Bloomsburg, PA 17815, U.S.A.
razzaghi@bloomu.edu

Masaaki Sibuya

Business Administration,
Takachiho University
Ohmiya, Suginami-ku
168-8508 Tokyo, Japan
sibuyam@takachiho.ac.jp

Jacek Wesołowski

Wydział Matematyki i Nauk
Informacyjnych
Politechnika Warszawska
Plac Politechniki 1, 00-661
Warszawa, Poland
wesolo@mini.pw.edu.pl

Chapter 1

The Shapes of the Probability Density, Hazard, and Reverse Hazard Functions

MASAAKI SIBUYA

Takachiho University, Tokyo

CONTENTS

1.1	Introduction	2
1.2	Monotone Shapes	4
1.2.1	Definitions and duality	4
1.2.2	Monotone h.f. and r.h.f.	5
1.2.3	Truncation	6
1.3	Necessity of Restrictions	7
1.3.1	The shapes of a doublet of p.d.f. and h.f. (or r.h.f.)	7
1.3.2	The shapes of a triplet	9
1.3.3	Four-interval description	11
1.4	Sufficiency of Restrictions	12
1.4.1	Cut and paste of a triplet	13
1.4.2	Examples of monotone shapes	13
1.4.3	General shapes	15
1.5	Additional Notes	16
	References	17

ABSTRACT

The shapes of a probability density function, its hazard function, and its reverse hazard function restrict each other. Here, “shape” means six types of the graphs of a continuous univariate function: increasing (i, for short), decreasing (d), unimodal (id), anti-unimodal (di), increasing–decreasing–increasing (idi), and decreasing–increasing–decreasing (did). It is proved that among 216 combinations of the shapes of the three functions, 44 cases are possible.

This result is a nonparametric characterization of a triplet, the probability density, hazard and reverse hazard functions, by their shapes.

KEYWORDS AND PHRASES: Characterization, dual failure function, failure rate, hazard rate, logistic distribution, Pareto distribution, reversed hazard rate, Weibull distribution

1.1 INTRODUCTION

The reverse hazard function (r.h.f.), or the reversed hazard rate function, is the ratio of a probability density function (p.d.f.) to its distribution function (d.f.); it is used for the retrospective analysis of survival data. It was introduced by Keilson and Sumita (1982) and called the dual failure function. It has the properties dual to the hazard function (h.f.); see Shaked and Shanthikumar (1994).

Lagakos et al. (1988) used the r.h.f. for a retrospective analysis of epidemiological data on individuals in a group, who are identified by some event and the random time of an initiating event is recorded. Kalbfleisch and Lawless (1989) studied the same kind of data and suggested using the r.h.f. Block et al. (1998) proved, among others, that if a r.h.f. is increasing, its h.f. is also increasing, and its distribution range is limited to $(-\infty, \omega)$, $\omega < \infty$. Hence, the lifetime never has an increasing r.h.f. Based on this fact, they cautioned misuses of the r.h.f.

The shapes of a triplet of a probability density, its hazard, and its reverse hazard functions restrict each other. A “shape” means here a class of piecewise monotone continuous

positive functions. In this paper six shapes, on at most three consecutive common intervals (see Subsection 1.3.3), are examined: increasing (i, for short), decreasing (d), unimodal (id), antiunimodal (di), increasing–decreasing–increasing (idi), and decreasing–increasing–decreasing (did). The main result is that among $6 \times 6 \times 6 = 216$ combinations of the shapes of the triplet, 44 cases in Tables 1.3.3 and 1.3.4, Section 1.3, are possible. This result is a nonparametric characterization of a triplet, the probability density, hazard and reverse hazard functions, by their shapes, and can be used as a reference for the modeling based on hazard and reverse hazard functions.

Figure 1.1.1 shows two examples of triplets with different combinations of shapes.

This paper extends a previous one on the relation between the shapes of a probability density and its hazard functions (Sibuya, 1996) and completely solves the problem raised by Block et al. (1998).

The shapes of a h.f. were studied by Aalen and Gjessing (2001) from a different point of view. Monotone h.f. and r.h.f.

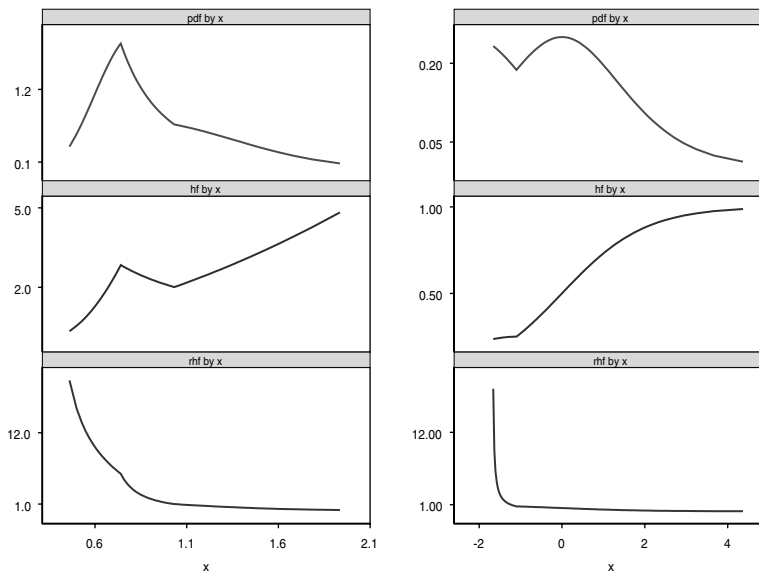


Figure 1.1.1 Examples A and B (from top: p.d.f., h.f. and r.h.f.).

were discussed in Sengupta and Nanda (1999) and Chandra and Roy (2001). An ordering of lifetime distributions by the increasing ratio of a pair of h.f.'s, or a pair of r.h.f.'s, was discussed by Sibuya and Suzuki (2001).

1.2 MONOTONE SHAPES

The h.f. is used mainly in the lifetime analysis and for positive random variables. Here, random variables are not restricted to be positive. The distribution limits of a d.f. F are defined by $\alpha := \inf\{x; F(x) > 0\}$, and $\omega := \sup\{x; F(x) < 1\}$, $-\infty \leq \alpha < \omega \leq \infty$. It is assumed that the p.d.f. satisfies $f(x) > 0$, $\alpha < x < \omega$. The shapes are invariant with respect to the piecewise change of location and scale. Hence, whether the limits are finite or infinite is the concern. The symbols α and ω are overused within tables to mean finite limits.

1.2.1 Definitions and duality

Let $\bar{F}(x) = 1 - F(x)$ be a survival function (s.f.); the h.f. is defined by

$$h(x) := \frac{d}{dx}(-\log(\bar{F}(x))) = f(x)/\bar{F}(x) \geq 0, \quad \alpha < x < \omega. \quad (1.2.1)$$

Conversely, the cumulative h.f. determines its d.f.:

$$\bar{F}(x) = \exp(-H(x)), \quad H(x) = \int_{-\infty}^x h(t) dt.$$

$H(x)$ is increasing, $H(\alpha) = 0$, and $H(\omega) = \infty$.

The r.h.f. is a dual of h.f.:

$$\begin{aligned} \tilde{h}(x) &= \frac{d}{dx}(\log F(x)) = f(x)/F(x) \geq 0, \quad \alpha < x < \omega, \\ F(x) &= \exp(-\tilde{H}(x)) \quad \tilde{H}(x) = \int_x^{\infty} \tilde{h}(t) dt. \end{aligned} \quad (1.2.2)$$

$\tilde{H}$ is decreasing, $\tilde{H}(\alpha) = \infty$, and $\tilde{H}(\omega) = 0$. Note that if $\alpha > -\infty$ it is possible that $H(\alpha + 0) > 0$, and if $\omega < \infty$ it is possible

that $\tilde{H}(\omega - 0) > 0$. The cumulative h.f. and r.h.f. are directly related:

$$H(x) = \Omega(\tilde{H}(x)) \quad \text{and} \quad \tilde{H}(x) = \Omega(H(x)), \text{ where} \\ \Omega(t) = -\log(1 - \exp(-t)), \quad 0 < t < \infty.$$

PROPOSITION 1.2.1 (reflection or duality) *If the p.d.f., d.f., s.f., h.f., r.h.f., cumulative h.f., and cumulative r.h.f. of a r.v. X are $f(x)$, $F(x)$, $\bar{F}(x)$, $h(x)$, $\tilde{h}(x)$, $H(x)$, and $\tilde{H}(x)$, respectively, those of the negated $-X$ are $f(-x)$, $\bar{F}(-x)$, $F(-x)$, $\tilde{h}(-x)$, $h(-x)$, $\tilde{H}(-x)$, and $H(-x)$, respectively. (Note the change of order.)*

This simple fact will be repeatedly used in this paper.

1.2.2 Monotone h.f. and r.h.f.

Before discussing monotone shapes, note that, for a constant $\lambda > 0$,

$$\text{If } h(x) = \lambda, \quad \bar{F}(x) = e^{-\lambda x} / \bar{F}(a) \quad \text{and} \quad \tilde{h}(x) \downarrow, \quad x > a.$$

$$\text{If } \tilde{h}(x) = \lambda, \quad F(x) = e^{\lambda x} / F(b) \quad \text{and} \quad h(x) \uparrow, \quad x < b.$$

$$\text{If } f(x) = \lambda, \quad F(x) = F(a) + \lambda(x - a), \quad \bar{F}(x) = \bar{F}(b) + \lambda(b - x) \\ \text{and} \quad h(x) \uparrow, \quad \tilde{h}(x) \downarrow, \quad a < x < b.$$

Throughout the paper the terms *increasing* and *decreasing* are used in a weak sense.

From the definitions and Proposition 1.2.1,

$$h \uparrow (\downarrow) \Leftrightarrow H : \text{convex}(\text{concave}), \quad \text{and}$$

$$\tilde{h} \downarrow (\uparrow) \Leftrightarrow \tilde{H} : \text{convex}(\text{concave}).$$

$$H : \text{concave} \Rightarrow \tilde{H} : \text{convex}, \quad \text{and}$$

$$\tilde{H} : \text{concave} \Rightarrow H : \text{convex}.$$

PROPOSITION 1.2.2

- (i) *If a h.f. is decreasing, its p.d.f. is decreasing, which implies its r.h.f. is decreasing.*
- (ii) *Similarly, if a r.h.f. is increasing, its p.d.f. is increasing, which implies its h.f. is increasing.*

PROOF. From the definition of h and $\tilde{h}$, a pair of fundamental inequalities are obtained:

$$\begin{aligned} (\log f)' &= (\log h)' - h \leq (\log h)', \\ (\log f)' &= (\log \tilde{h})' + \tilde{h} \geq (\log \tilde{h})'. \end{aligned} \tag{1.2.3}$$

The first one implies $f \uparrow \Rightarrow h \uparrow$ and $h \downarrow \Rightarrow f \downarrow$, and the second implies $f \downarrow \Rightarrow \tilde{h} \downarrow$ and $\tilde{h} \uparrow \Rightarrow f \uparrow$. Combine these to show the proposition. ■

REMARK 1.2.1

1. The inequalities hold at any $x \in (\alpha, \omega)$; that is, the proposition states local properties.
2. The second half (ii) of the proposition is dual to (i).

PROPOSITION 1.2.3 (restriction of distribution range) *Let t be any number such that $\alpha < t < \omega$.*

- (i) *If a h.f. is decreasing in (α, t) , $\alpha > -\infty$. If a h.f. is decreasing in (t, ω) , $\omega = \infty$.*
- (ii) *If a r.h.f. is increasing in (α, t) , $\alpha = -\infty$. If a r.h.f. is increasing in (t, ω) , $\omega < \infty$.*

PROOF. If a p.d.f. is decreasing in the lower tail, *a fortiori* if a h.f. is decreasing in the lower tail, its lower distribution limit is finite, because that $f(t) \downarrow$ on $(-\infty, t)$ is impossible. Since $H(\omega) = \infty$, a decreasing h.f. cannot end at a finite point. The second half (ii) is dual to (i). ■

REMARK 1.2.2 The facts of (i) can be confirmed by observing the cumulative h.f., H . If h is decreasing in (α, t) , H is concave and increasing, hence $\alpha > -\infty$. If H is concave in (t, ω) , $H(\omega)$ cannot be infinite unless $\omega = \infty$.

1.2.3 Truncation

PROPOSITION 1.2.4 (truncation)

- (i) *If a p.d.f. is left-truncated at a , $\alpha < a < \omega$, its h.f. does not change in (a, ω) . Further, if the original r.h.f. is decreasing, the new r.h.f. is also decreasing in (a, ω) .*

- (ii) *If a p.d.f. is right-truncated at b , $\alpha < b < \omega$, its r.h.f. does not change in (α, b) . Further, if the original h.f. is increasing, the new h.f. is also increasing in (α, b) .*

PROOF. For the original f , F , $\bar{F}$, h , and $\tilde{h}$, let the left-truncated be denoted with an asterisk:

$$\begin{aligned} f^*(x) &= f(x)/\bar{F}(a), & \bar{F}^*(x) &= \bar{F}(x)/\bar{F}(a), \\ F^*(x) &= (F(x) - F(a))/(1 - F(a)), & x &> a. \end{aligned}$$

Hence,

$$h^*(x) = h(x), \quad \tilde{h}^*(x) = \frac{1 - F(a)}{1 - F(a)/\bar{F}(x)} \tilde{h}(x), \quad x > a,$$

and the latter is decreasing. The second half (ii) is dual to (i). ■

1.3 NECESSITY OF RESTRICTIONS

1.3.1 The shapes of a doublet of p.d.f. and h.f. (or r.h.f.)

Before discussing the shapes of a triplet, combinations of the shapes of a doublet of p.d.f. and h.f., or a doublet of p.d.f. and r.h.f., are examined. Propositions 1.2.2 and 1.2.3 restrict possible combinations and the range.

Impossible combinations of the shapes of p.d.f. and h.f. are shown in Table 1.3.1. For other combinations, the necessary restrictions on the distribution range are shown. Similarly, impossible combinations of the shapes of p.d.f. and r.h.f., as well as restrictions on the range, are shown in Table 1.3.2. Table 1.3.1 is a modification of a previous table for the lifetime distributions (Sibuya, 1996).

Table 1.3.1 is constructed along the following rules because of Theorems 1.3.3 and 1.3.4.:

- (i) If h is decreasing in some interval (symbol d) and f is increasing (symbol i) in this interval, the doublet is impossible (symbol †).
- (ii) If h is decreasing in the upper tail (f is also decreasing), $\omega = \infty$ (symbol +).