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Differential Equations

Inverse and Direct Problems

Edited by

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Differential Equations

Inverse and Direct Problems

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Differential Equations
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Preface

The meeting on *Differential Equations: Inverse and Direct Problems* was held in Cortona, June 21-25, 2004. The topics discussed by well-known specialists in the various disciplinary fields during the Meeting included, among others: differential and integrodifferential equations in Banach spaces, linear and nonlinear theory of semigroups, direct and inverse problems for regular and singular elliptic and parabolic differential and/or integrodifferential equations, blow up of solutions, elliptic equations with Wentzell boundary conditions, models in superconductivity, phase transition models, theory of attractors, Ginzburg-Landau and Schrödinger equations and, more generally, applications to partial differential and integrodifferential equations from Mathematical Physics.

The reports by the lecturers highlighted very recent, interesting and original research results in the quoted fields contributing to make the Meeting very attractive and stimulating also to younger participants.

After a lot of discussions related to the reports, some of the senior lecturers were asked by the organizers to provide a paper on their contribution or some developments of them.

The present volume is the result of all this. In this connection we want to emphasize that almost all the contributions are original and are not expositive papers of results published elsewhere. Moreover, a few of the contributions started from the discussions in Cortona and were completed in the very end of 2005.

So, we can say that the main purpose of the editors of this volume has consisted in stimulating the preparation of new research results. As a consequence, the editors want to thank in a particular way the authors that have accepted this suggestion.

Of course, we warmly thank the Italian Istituto Nazionale di Alta Matematica that made the Meeting in Cortona possible and also the Universitá degli Studi di Milano for additional support.

Finally, the editors thank the staff of Taylor & Francis for their help and useful suggestions they supplied during the preparation of this volume.

Angelo Favini and Alfredo Lorenzi

Bologna and Milan, December 2005

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Degenerate first order identification problems in Banach spaces [1]

Mohammed Al-Horani and Angelo Favini

Abstract We study a first order identification problem in a Banach space. We discuss both the nondegenerate and (mainly) the degenerate case. As a first step, suitable hypotheses on the involved closed linear operators are made in order to obtain unique solvability after reduction to a nondegenerate case; the general case is then handled with the help of new results on convolutions. Various applications to partial differential equations motivate this abstract approach.

1 Introduction

In this article we are concerned with an identification problem for first order linear systems extending the theory and methods discussed in [7] and [1]. See also [2] and [9]. Related nonsingular results were obtained in [11] under different additional conditions even in the regular case. There is a wide literature on inverse problems motivated by applied sciences. We refer to [11] for an extended list of references. Inverse problems for degenerate differential and integrodifferential equations are a new branch of research. Very recent results have been obtained in [7], [5] and [6] relative to identification problems for degenerate integrodifferential equations. Here we treat similar equations without the integral term and this allows us to lower the required regularity in time of the data by one. The singular case for infinitely differentiable semigroups and second order equations in time will be treated in some forthcoming papers.

The contents of the paper are as follows. In Section 2 we present the nonsingular case, precisely, we consider the problem

$$\begin{aligned} &u'(t) + Au(t) = f(t)z\,, \qquad 0 \le t \le \tau\,,\\ &u(0) = u_0\,,\\ &\Phi[u(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \end{aligned}$$

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where $-A$ generates an analytic semigroup in X, X being a Banach space, $\Phi \in X^*$, $g \in C^1([0,\tau],\mathbb{R})$, $\tau > 0$ fixed, u_0, $z \in D(A)$ and the pair $(u, f) \in C^{1+\theta}([0,\tau]; X) \times C^\theta([0,\tau];\mathbb{R})$, $\theta \in (0,1)$, is to be found. Here $C^\theta([0,\tau]; X)$ denotes the space of all X-valued Hölder-continuous functions on $[0,\tau]$ with exponent θ, and

$$C^{1+\theta}([0,\tau]; X) = \{u \in C^1([0,\tau]; X);\; u' \in C^\theta([0,\tau]; X)\}.$$

In Section 3 we consider the possibly degenerate problem

$$\begin{aligned}
&\frac{d}{dt}((Mu)(t)) + Lu(t) = f(t)z\,, \qquad 0 \le t \le \tau\,,\\
&(Mu)(0) = Mu_0\,,\\
&\Phi[Mu(t)] = g(t)\,, \qquad 0 \le t \le \tau\,,
\end{aligned}$$

where L, M are two closed linear operators in X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, for some $\theta \in (0,1)$. In this possibly degenerate problem, M may have no bounded inverse and the pair $(u, f) \in C^\theta([0,\tau]; D(L)) \times C^\theta([0,\tau];\mathbb{R})$ is to be found. This problem was solved (see [1]) when $\lambda = 0$ is a simple pole for the resolvent $(\lambda L + M)^{-1}$. Here we consider this problem under the assumption that M and L act in a reflexive Banach space X with the resolvent estimate

$$\|\lambda M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \le C, \qquad \operatorname{Re}\lambda \ge 0\,,$$

or the equivalent one

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T + I)^{-1}\|_{\mathcal{L}(X)} \le C, \qquad \operatorname{Re}\lambda \ge 0\,,$$

where $T = ML^{-1}$. Reflexivity of X allows to use the representation of X as a direct sum of the null space $N(T)$ and the closure of its range $R(T)$, a consequence of the ergodic theorem (see [13], pp. 216-217). Here, a basic role is played by real interpolation space, see [12].

In Section 4 we give some examples from partial differential equations describing the range of applications of the previous abstract results.

---

2 The nonsingular case

Let X be a Banach space with norm $\|\cdot\|_X$ (sometimes, $\|\cdot\|$ will be used for the sake of brevity), $\tau > 0$ fixed, u_0, $z \in D(A)$, where $-A$ is the generator of an analytic semigroup in X, $\Phi \in X^*$ and $g \in C^1([0,\tau],\mathbb{R})$. We want to find a pair $(u, f) \in C^{1+\theta}([0,\tau]; X) \times C^{\theta}([0,\tau]; \mathbb{R})$, $\theta \in (0,1)$, such that

$$u'(t) + Au(t) = f(t)z\,, \qquad 0 \le t \le \tau\,, \tag{2.1}$$

$$u(0) = u_0\,, \tag{2.2}$$

$$\Phi[u(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{2.3}$$

under the compatibility relation

$$\Phi[u_0] = g(0)\,. \tag{2.4}$$

Let us remark that the compatibility relation (2.4) follows from (2.2)-(2.3).

To solve our problem we first apply Φ to (2.1) and take equation (2.3) into account; we obtain the following equation in the unknown $f(t)$:

$$g'(t) + \Phi[Au(t)] = f(t)\Phi[z]\,. \tag{2.5}$$

Suppose the condition

$$\Phi[z] \neq 0 \tag{2.6}$$

to be satisfied. Then we can write (2.5) under the form:

$$f(t) = \frac{1}{\Phi[z]}\left\{g'(t) + \Phi[Au(t)]\right\}\,, \qquad 0 \le t \le \tau\,, \tag{2.7}$$

and the solution u of (2.1)-(2.3) is assigned by the formula

$$\begin{aligned} u(t) &= e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\,\frac{\{g'(s) + \Phi[Au(s)]\}}{\Phi[z]}\,z\,ds \\ &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[Au(s)]}{\Phi[z]}\,z\,ds + e^{-tA}u_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)z\,ds\,. \end{aligned} \tag{2.8}$$

Apply the operator A to (2.8) and obtain

$$\begin{aligned} Au(t) &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[Au(s)]}{\Phi[z]}\,Az\,ds + e^{-tA}Au_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)Az\,ds\,. \end{aligned} \tag{2.9}$$

Let $Au(t) = v(t)$; then (2.7) and (2.9) can be written, respectively, as follows:

$$f(t) = \frac{1}{\Phi[z]}\left\{g'(t) + \Phi[v(t)]\right\}\,, \qquad 0 \le t \le \tau\,, \tag{2.10}$$

$$\begin{aligned} v(t) &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[v(s)]}{\Phi[z]}\,Az\,ds + e^{-tA}Au_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)Az\,ds\,. \end{aligned} \tag{2.11}$$

Let us introduce the operator S

$$Sw(t) = \int_0^t e^{-(t-s)A} \, \frac{\Phi[w(s)]}{\Phi[z]} \, Az \, ds \, .$$

Then (2.11) can be written in the form

$$v - Sv = h \tag{2.12}$$

where

$$h(t) = e^{-tA} A u_0 + \frac{1}{\Phi[z]} \int_0^t e^{-(t-s)A} g'(s) Az \, ds \, .$$

It is easy to notice that $h \in C([0,\tau]; X)$.

To prove that (2.12) has a unique solution in $C([0,\tau]; X)$, it is sufficient to show that S^n is a contraction for some $n \in \mathbb{N}$. For this, we note

$$\begin{aligned}
\|Sv(t)\| &\le \frac{M \, \|\Phi\|_{X^*}}{|\Phi(z)|} \int_0^t \|v(s)\| \, \|Az\| \, ds \\
\|S^2 v(t)\| &\le \frac{M \, \|\Phi\|_{X^*}}{|\Phi(z)|} \int_0^t \|Tv(s)\| \, \|Az\| \, ds \\
&\le \left(\frac{M \, \|\Phi\|_{X^*} \, \|Az\|}{|\Phi(z)|} \right)^2 \int_0^t \left(\int_0^s \|v(\sigma)\| \, d\sigma \right) \, ds \\
&\le \left(\frac{M \, \|\Phi\|_{X^*} \, \|Az\|}{|\Phi(z)|} \right)^2 \int_0^t (t-\sigma) \|v(\sigma)\| \, d\sigma \\
&\le \left(\frac{M \, \|\Phi\|_{X^*} \, \|Az\|}{|\Phi(z)|} \right)^2 \|v\|_\infty \, \frac{t^2}{2} \, ,
\end{aligned}$$

where $\|v\|_\infty = \|v\|_{C([0,\tau];X)}$.

Proceeding by induction, we can find the estimate

$$\|S^n v(t)\| \le \left(\frac{M \, \|\Phi\|_{X^*} \|Az\|}{|\Phi(z)|} \right)^n \frac{t^n}{n!} \, \|v\|_\infty \, ,$$

which implies that

$$\|S^n v\|_\infty \le \left(\frac{M \, \|\Phi\|_{X^*} \|Az\|}{|\Phi(z)|} \, \tau \right)^n \frac{1}{n!} \, \|v\|_\infty \, .$$

Consequently, S^n is a contraction for sufficiently large n. At last notice that $f(t)\, z$ is then a continuous $D(A)$-valued function on $[0,\tau]$, so that (2.1), (2.2) has in fact a unique strict solution. However, we want to discuss the maximal regularity for the solution $v = Au$, and for this we need some additional conditions. We now recall that if $-A$ generates a bounded analytic semigroup in X, then the real interpolation space $(X, D(A))_{\theta,\infty} = D_A(\theta,\infty)$ coincides with $\{x \in X; \sup_{t>0} t^{1-\theta} \|Ae^{-tA}x\| < \infty\}$, (see [3]).

Consider formula (2.11) and notice that (see [10])

$$e^{-tA}Au_0 \in C^\theta([0,\tau];X) \ \text{ if and only if } \ Au_0 \in D_A(\theta,\infty)\,.$$

Moreover, if $g \in C^{1+\theta}([0,\tau];\mathbb{R})$ and $Az \in D_A(\theta,\infty)$, then

$$\int_0^t e^{-(t-s)A}g'(s)Az\,ds \in C^\theta([0,\tau];X)$$

and

$$\int_0^t e^{-(t-s)A}Az\,\Phi[v(s)]\,ds = \left(e^{-tA}Az * \Phi[v]\right)(t) \in C^\theta([0,\tau];X)\,.$$

See [7] and [6].

Therefore, if we assume

$$Au_0,\ Az \in D_A(\theta,\infty)\,, \tag{2.13}$$

then $v(t) \in C^\theta([0,\tau];X)$, i.e., $Au(t) \in C^\theta([0,\tau];X)$ which implies that $f(t) \in C^\theta([0,\tau];\mathbb{R})$. Then there exists a unique solution $(u,f) \in C^{1+\theta}([0,\tau];X) \times C^\theta([0,\tau];\mathbb{R})$.

We summarize our discussion in the following theorem.

THEOREM 2.1 *Let $-A$ be the generator of an analytic semigroup, $\Phi \in X^*$, u_0, $z \in D_A(\theta+1,\infty)$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. If $\Phi[z] \neq 0$ and (2.4) holds, then problem (2.1)-(2.3) admits a unique solution $(u,f) \in [C^{1+\theta}([0,\tau];X) \cap C^\theta([0,\tau];D(A))] \times C^\theta([0,\tau];\mathbb{R})$.*

3 The singular case

Consider the possibly degenerate problem

$$D_t(Mu) + Lu = f(t)z\,, \qquad 0 \le t \le \tau\,, \tag{3.1}$$

$$(Mu)(0) = Mu_0\,, \tag{3.2}$$

$$\Phi[Mu(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{3.3}$$

where L, M are two closed linear operators with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$ for $\theta \in (0,1)$. Here M may have no bounded inverse and the pair $(u,f) \in C([0,\tau];D(L)) \times C^\theta([0,\tau];\mathbb{R})$, with $Mu \in C^{1+\theta}([0,\tau];X)$, is to be determined so that the following compatibility condition must hold:

$$\Phi[Mu(0)] = \Phi[Mu_0] = g(0)\,. \tag{3.4}$$

Let us assume that the pair (M, L) satisfies the estimate

$$\|\lambda M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \leq C, \qquad \operatorname{Re}\lambda \geq 0\,, \tag{3.5}$$

or the equivalent one

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T + I)^{-1}\|_{\mathcal{L}(X)} \leq C, \qquad \operatorname{Re}\lambda \geq 0\,, \tag{3.6}$$

where $T = ML^{-1}$.

Various concrete examples of this relation can be found in [8]. One may note that $\lambda = 0$ is not necessarily a simple pole for $(\lambda + T)^{-1}$, $T = ML^{-1}$. Let $Lu = v$ and observe that $T = ML^{-1} \in \mathcal{L}(X)$. Then (3.1)-(3.3) can be written as

$$D_t(Tv) + v = f(t)z\,, \qquad 0 \leq t \leq \tau\,, \tag{3.7}$$

$$(Tv)(0) = Tv_0 = ML^{-1}v_0\,, \tag{3.8}$$

$$\Phi[Tv(t)] = g(t)\,, \qquad 0 \leq t \leq \tau\,, \tag{3.9}$$

where $v_0 = Lu_0$.

Since X is a reflexive Banach space and (3.5) holds, we can represent X as a direct sum (cfr. [8, p. 153], see also [13], pp. 216-217)

$$X = N(T) \oplus \overline{R(T)}$$

where $N(T)$ is the null space of T and $R(T)$ is the range of T. Let $\tilde{T} = T_{\overline{R(T)}} : \overline{R(T)} \to T_{\overline{R(T)}}$ be the restriction of T to $\overline{R(T)}$. Clearly $\tilde{T}$ is a one to one map from $\overline{R(T)}$ onto $R(T)$ ($\tilde{T}$ is an abstract potential operator in $\overline{R(T)}$. Indeed, in view of the assumptions, $-\tilde{T}^{-1}$ generates an analytic semigroup on $\overline{R(T)}$, (see [8, p. 154]).

Finally, let P be the corresponding projection onto $N(T)$ along $\overline{R(T)}$.

We can now prove the following theorem:

THEOREM 3.1 *Let L, M be two closed linear operators in the reflexive Banach space X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. Suppose the condition (3.5) to hold with (3.4), too. Then problem (3.1)-(3.3) admits a unique solution $(u, f) \in C^{\theta}([0,\tau]; D(L)) \times C^{\theta}([0,\tau];\mathbb{R})$ provided that*

$$\Phi[(I-P)z] \neq 0\,, \qquad \sup_{t>0} t^{\theta}\|(t\tilde{T} + 1)^{-1}y_i\|_X < +\infty\,, \quad i = 1,\, 2$$

where $y_1 = (I-P)Lu_0$ and $y_2 = \tilde{T}^{-1}(I-P)z$.

Proof. Since P is the projection onto $N(T)$ along $\overline{R(T)}$, it is easy to check that problem (3.7)-(3.9) is equivalent to the couple of problems

$$D_t\tilde{T}(I-P)v + (I-P)v = f(t)(I-P)z\,, \qquad 0 \le t \le \tau\,, \tag{3.10}$$

$$\tilde{T}(I-P)v(0) = \tilde{T}(I-P)v_0\,, \tag{3.11}$$

$$\Phi[\tilde{T}(I-P)v(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{3.12}$$

and

$$Pv(t) = f(t)Pz\,. \tag{3.13}$$

Let $w = \tilde{T}(I-P)v$, so that $(I-P)v = \tilde{T}^{-1}w$, and hence system (3.10)-(3.12) becomes

$$w'(t) + \tilde{T}^{-1}w = f(t)(I-P)z\,, \qquad 0 \le t \le \tau\,, \tag{3.14}$$

$$w(0) = w_0 = \tilde{T}(I-P)v_0 = Tv_0\,, \tag{3.15}$$

$$\Phi[w(t)] = g(t)\,, \qquad 0 \le t \le \tau\,. \tag{3.16}$$

Then, according to Theorem 2.1, there exists a unique solution $(w,f) \in C^{1+\theta}([0,\tau];\overline{R(T)}) \times C^{\theta}([0,\tau];\mathbb{R})$ with $\tilde{T}^{-1}w \in C^{\theta}([0,\tau];\overline{R(T)})$ to problem (3.14)-(3.16) provided that

$$\Phi[(I-P)z] \ne 0\,, \qquad (I-P)Lu_0\,,\ \tilde{T}^{-1}(I-P)z \in D_{\tilde{T}^{-1}}(\theta,\infty)\,.$$

Therefore, $(I-P)v \in C^{\theta}([0,\tau];\overline{R(T)}),\ Pv \in C^{\theta}([0,\tau];N(T))$ and hence there exists a unique solution $(u,f) \in C^{\theta}([0,\tau];D(L)) \times C^{\theta}([0,\tau];\mathbb{R})$ with $Mu \in C^{1+\theta}([0,\tau];X)$ to problem (3.1)-(3.3). □

Our next goal is to weaken the assumptions on the data in the Theorems 1 and 2. To this end we again suppose $-A$ to be the generator of an analytic semigroup in X of negative type, i.e., $\|e^{-tA}\| \le ce^{-\omega t},\ \ t \ge 0$, where $c,\ \omega > 0$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, but we take $u_0 \in D_A(\theta+1;X)$, $z \in D_A(\theta_0,\infty)$, where $0 < \theta < \theta_0 < 1$. Our goal is to find a pair $(u,f) \in C^1([0,\tau];X) \times C([0,\tau];\mathbb{R})$, $Au \in C^{\theta}([0,\tau];X)$ such that equations (2.1)-(2.3) hold under the compatibility relation (2.4).

THEOREM 3.2 *Let $-A$ be a generator of an analytic semigroup in X of positive type, $0 < \theta < \theta_0 < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $u_0 \in D_A(\theta+1,\infty)$, $z \in D_A(\theta_0,\infty)$. If, in addition, (2.4), (2.6) hold, then problem (2.1)-(2.3) has a unique solution $(u,f) \in C^{\theta}([0,\tau],D(A)) \times C^{\theta}([0,\tau];\mathbb{R})$.*

Proof. Recall (see [10, p. 145]) that if $u_0 \in D(A)$, $f \in C([0,\tau];\mathbb{R})$, $z \in D_A(\theta_0,\infty)$, then problem (2.1)-(2.2) has a unique strict solution. Moreover, if $u_0 \in D_A(\theta+1;X)$, then the solution u to (2.1)-(2.2) has the maximal regularity u', $Au \in C([0,\tau];X) \cap B([0,\tau];D_A(\theta_0,\infty))$, where $B([0,\tau];Y)$ denotes the space of all bounded functions from $[0,\tau]$ into the Banach space Y. In addition $Au \in C^\theta([0,\tau];X)$.

In order to prove our statement, we need to study suitably the properties of the function u and to use carefully some properties of the convolution operator and real interpolation spaces.

One readily sees that u satisfies

$$Au(t) = \int_0^t \frac{\Phi[Au(s)]}{\Phi[z]}\, Ae^{-(t-s)A}\, z\, ds + e^{-tA}Au_0 + \frac{1}{\Phi[z]} \int_0^t A\, e^{-(t-s)A}\, z\, g'(s)\, ds$$

so that $v(t) = Au(t)$ must satisfy

$$v(t) = \int_0^t Ae^{-(t-s)A}\, z\, \frac{\Phi[v(s)]}{\Phi[z]}\, ds + e^{-tA}Au_0 + \frac{1}{\Phi[z]} \int_0^t A\, e^{-(t-s)A}\, z\, g'(s)\, ds\,.$$

Let us introduce the operator $S : C([0,\tau];X) \to C([0,\tau];X)$ by

$$(Sw)(t) = \int_0^t Ae^{-(t-s)A}\, z\, \frac{\Phi[w(s)]}{\Phi[z]}\, ds\,.$$

Since $z \in D_A(\theta_0,\infty)$, i.e.,

$$\|Ae^{-tA}z\| \le \frac{c}{t^{1-\theta_0}}\,, \quad t > 0\,,$$

we deduce

$$\|Sw(t)\| \le c\int_0^t \|\Phi\|_{X^*}\, \|z\|_{\theta_0,\infty}\, \frac{\|w(s)\|}{(t-s)^{1-\theta_0}}\, ds\,,$$

$$\begin{aligned}
\|S^2w(t)\| &\le [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0,\infty}] \int_0^t \frac{\|Sw(s)\|}{(t-s)^{1-\theta_0}}\, ds \\
&\le [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0,\infty}]^2 \int_0^t \frac{ds}{(t-s)^{1-\theta_0}} \int_0^s \frac{\|w(\sigma)\|}{(s-\sigma)^{1-\theta_0}}\, d\sigma \\
&= [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0,\infty}]^2 \int_0^t \left(\int_\sigma^t \frac{ds}{(t-s)^{1-\theta_0}(s-\sigma)^{1-\theta_0}} \right) \|w(\sigma)\|\, d\sigma \\
&= c_1^2 \left[\int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0}\eta^{1-\theta_0}} \right] (t-\sigma)^{1-2(1-\theta_0)} \|w(\sigma)\|\, d\sigma\,,
\end{aligned}$$

where $c_1 = c\|\Phi\|_{X^*}\, \|z\|_{\theta_0,\infty}$, $\|\cdot\|_{D_A(\theta_0,\infty)}$ denoting the norm in $D_A(\theta_0,\infty)$.

Recall that

$$B(p,q) = \int_0^1 (1-\eta)^{p-1}\, \eta^{q-1}\, d\eta = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,.$$

Then

$$\|S^3 w(t)\| \le c_1^3 \int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0}\eta^{1-\theta_0}} \int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0}\eta^{2(1-\theta_0)-1}}$$
$$\times \int_0^1 (t-\sigma)^{2-3(1-\theta_0)} \, \|w(\sigma)\| \, d\sigma$$
$$\le c_1^3 \, B(\theta_0,\theta_0)\, B(\theta_0, 2\theta_0) \int_0^1 (t-\sigma)^{2-3(1-\theta_0)} \, \|w(\sigma)\| \, d\sigma$$
$$\le c_1^3 \frac{\Gamma(\theta_0)^3}{\Gamma(3\theta_0)} \frac{t^{3\theta_0}}{3\theta_0} \, \|w\|_{C([0,t];X)} \, .$$

By induction, we easily verify that

$$\|S^n w(t)\| \le c_1^n \frac{\Gamma(\theta_0)^n}{\Gamma(n\theta_0)} \frac{t^{n\theta_0}}{n\theta_0} \, \|w\|_{C([0,t];X)} \, .$$

Since $\sqrt[n]{\Gamma(n\theta_0)} \to \infty$ as $n \to \infty$, we conclude that the operator S has spectral radius equal to 0. On the other hand, since $z \in D_A(\theta_0,\infty)$, $\theta_0 > \theta$, and $g' \in C^\theta([0,\tau];\mathbb{R})$, we deduce by [6] (Lemma 3.3) that the convolution

$$\int_0^t g'(s) A e^{-(t-s)A} \, z \, ds$$

belongs to $C^\theta([0,\tau];X)$.
Moreover, since $Au_0 \in D_A(\theta,\infty)$, $e^{-tA}Au_0 \in C^\theta([0,\tau];X)$. It follows that equation (2.12), i.e.,

$$v - Sv = h \, ,$$

with

$$h(t) = e^{-tA}Au_0 + \frac{1}{\Phi[z]} \int_0^t A e^{-(t-s)A} \, z \, g'(s) \, ds$$

has a unique solution $v \in C([0,\tau];X)$. In order to obtain more regularity for v, we use Lemma 3.3 in [6] (see also [7]) again. To this end, we introduce the following L^p-spaces related to any positive constant δ:

$$L^p_\delta((0,\tau);X) = \left\{ u : (0,\tau) \to X : \quad e^{-t\delta} u \in L^p((0,\tau);X) \right\} \, ,$$

endowed with the norms $\|u\|_{\delta,0,p} = \|e^{-t\delta}u\|_{L^p((0,\tau);X)}$. Moreover,

$$\|g\|_{\delta,\theta,\infty} = \|e^{-t\delta} g\|_{C^\theta([0,\tau];X)} \, .$$

Lemma 3.3 in [6] establishes that, in fact, if $z \in D_A(\theta_0,\infty))$, $0 < \theta < \theta_0 < 1$, then

$$\Big\| \int_0^t A e^{-(t-s)A} \, z \, \Phi[v(s)] \, ds \Big\|_{\delta,\theta,\infty} \le c \, \delta^{-\theta_0+\theta+1/p} \, \|\Phi[v(.)]\|_{\delta,0,p}$$

provided that $(\theta_0 - \theta)^{-1} < p$. Now,

$$\int_0^t |\Phi[v(t)]|^p \, e^{-\delta p t} \, dt \le \|\Phi\|_{X^*}^p \, \|v\|_{L^p_\delta((0,\tau);X)}^p \le \tau \, \|\Phi\|_{X^*}^p \, \|v\|_{\delta,\theta,\infty}^p \, .$$

Choose δ suitably large and recall that $h \in C^\theta([0,\tau];X)$. Then the norm of S as an operator from $C^\theta([0,\tau];X)$ (with norm $\|\cdot\|_{\delta,\theta,\infty}$) into itself is less than 1, so that we can deduce that the solution $v = Au$ has the regularity $C^\theta([0,\tau];X)$, as desired. □

As a consequence, Theorem 3.1 has the following improvement.

THEOREM 3.3 *Let L, M be two closed linear operators in the reflexive Banach space X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. Suppose (3.4), (3.5) to hold.*
If $0 < \theta < \theta_0 < 1$ and $\Phi[(I-P)z] \neq 0$, $\sup_{t>0} t^{\theta_0} \|(tT+1)^{-1}(I-P)z\|_X < +\infty$, $\sup_{t>0} t^{\theta} \|(tT+1)^{-1}(I-P)Lu_0\|_X < +\infty$, then problem (3.1)-(3.3) admits a unique solution $(u,f) \in C^\theta([0,\tau];D(L)) \times C^\theta([0,\tau];\mathbb{R})$ with $Mu \in C^{1+\theta}([0,\tau]; X)$.

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4 Applications

In this section we show that our abstract results can be applied to some concrete identification problems. For further examples for which the theory works we refer to [8].

Problem 1. Consider the following identification problem related to a bounded region Ω in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$

$$\begin{aligned}
&D_t u(x,t) = \sum_{i,j=1}^n D_{x_i}(a_{ij}(x) D_{x_j} u(x,t)) + f(t) v(x) \,, \quad (x,t) \in \Omega \times [0,\tau] \,, \\
&u(x,t) = 0 \,, \quad \forall \, (x,t) \in \partial\Omega \times [0,\tau] \,, \\
&u(x,0) = u_0(x) \,, \quad x \in \Omega \,, \\
&\Phi[u(x,t)] = \int_\Omega \eta(x) u(x,t) \, dx = g(t) \,, \quad \forall \, t \in [0,\tau] \,,
\end{aligned}$$

where the coefficients a_{ij} enjoy the properties

$$a_{ij} \in C(\overline{\Omega}) \,, \quad a_{ij} = a_{ji} \,, \ i,j = 1,2,...,n$$

$$\sum_{i,j=1}^n a_{ij}(x) \, \xi_i \, \xi_j \ge c_0 |\xi|^2 \quad \forall \, x \in \overline{\Omega} \,, \ \forall \, \xi \in \mathbb{R}^n \,,$$

c_0 being a positive constant. Moreover, $g \in C^1([0,\tau];\mathbb{R})$. We take

$$Au = -\sum_{i,j=1}^{n} D_{x_i}(a_{ij}D_{x_j}u)\,, \quad D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,,$$

where $1 < p < +\infty$ is assumed. Concerning η, we suppose $\eta \in L^q(\Omega)$, where $1/p + 1/q = 1$. As it is well known, $-A$ generates an analytic semigroup in $L^p(\Omega)$ and thus we can apply Theorem 3.2 provided that $u_0 \in D_A(\theta+1;\infty)$, i.e., $Au_0 \in D_A(\theta,\infty)$, $v \in D_A(\theta_0;\infty)$, $0 < \theta < \theta_0 < 1$. On the other hand, the interpolation spaces $D_A(\theta,\infty)$ are well characterized. Then our problem admits a unique solution

$$(u,f) \in C^\theta([0,\tau]; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \times C^\theta([0,\tau];\mathbb{R}),$$

if $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $g(0) = \int_\Omega \eta(x)\, u_0(x)\, dx$ and $\int_\Omega \eta(x)\, v(x)\, dx \neq 0$.

Problem 2. Let Ω be a bounded region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let us consider the identification problem

$$\begin{aligned}
&D_t u(x,t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u(x,t)) + f(t)v(x)\,, \quad (x,t) \in \overline{\Omega} \times [0,\tau]\,,\\
&u(x,t) = 0\,, \quad (x,t) \in \partial\Omega \times [0,\tau]\,,\\
&u(x,0) = u_0(x)\,, \quad x \in \overline{\Omega}\,,\\
&\Phi[u(x,t)] = u(\overline{x},t) = g(t)\,, \quad t \in [0,\tau]\,,
\end{aligned}$$

where $\overline{x} \in \Omega$ is fixed, and the pair (f,u) is the unknown.
Here we take

$$X = C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega}), u(x) = 0 \ \forall\, x \in \partial\Omega\right\},$$

endowed with the sup norm $\|u\|_X = \|u\|_\infty$.
If the coefficients a_{ij} are assumed as in Problem 1, and

$$Au = -\sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}u(x))\,, \quad D(A) = \left\{u \in C_0(\overline{\Omega})\,;\, Au \in C_0(\overline{\Omega})\right\},$$

then $-A$ generates an analytic semigroup in X. The interpolation spaces $D_A(\theta;\infty)$ have no simple characterization, in view of the boundary conditions imposed to Au. Hence we notice that Theorem 3.2 applies provided that $u_0 \in D(A^2)$ and $v_0 \in D(A)$, $0 < \theta < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $u_0(\overline{x}) = g(0)$ and $v(\overline{x}) \neq 0$.

Notice that we could develop a corresponding result to Theorem 3.2 related to operators A with a nondense domain, but this is not so simple and the problem will be handled elsewhere.

Problem 3. Let us consider the following identification problem on a bounded region Ω in $\mathbb{R}$, $n \geq 1$, with a smooth boundary $\partial\Omega$:

$$D_t[m(x)u] = \Delta u + f(t)w(x), \quad (x,t) \in \Omega \times [0,\tau]\,, \tag{4.1}$$

$$u = 0 \text{ on } \partial\Omega \times [0,\tau]\,, \tag{4.2}$$

$$(mu)(x,0) = m(x)u_0(x)\,, \quad x \in \Omega\,, \tag{4.3}$$

$$\int_\Omega \eta(x)\,(mu)(x,t)\,dx = g(t)\,, \quad \forall t \in [0,\tau]\,, \tag{4.4}$$

where $m \in L^\infty(\Omega)$, $\Delta : H_0^1(\Omega) :\to H^{-1}(\Omega)$ is the Laplacian, $u_0 \in H_0^1(\Omega)$, $w \in H^{-1}(\Omega)$, $\eta \in H_0^1(\Omega)$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $0 < \theta < 1$, and the pair $(u,f) \in C^\theta([0,\tau];H_0^1(\Omega)) \times C^\theta([0,\tau];\mathbb{R})$ is the unknown. Of course, the integral in (4.4) stands for the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Theorem 3.3 applies with $X = H^{-1}(\Omega)$, see [8, p. 75]. We deduce that if $g(0) = \int_\Omega \eta(x)\,m(x)u_0(x)\,dx$, $w(x) = m(x)\zeta(x)$ for some $\zeta \in H_0^1(\Omega)$, $\int_\Omega \eta(x)\,m(x)\zeta(x)\,dx \neq 0$ and $(\Delta u_0)(x) = m(x)\zeta_1(x)$ for some $\zeta_1 \in H_0^1(\Omega)$, then problem (4.1)-(4.4) has a unique solution $(u,f) \in C^\theta([0,\tau];H_0^1(\Omega)) \times C^\theta([0,\tau];\mathbb{R})$, $mu \in C^{1+\theta}([0,\tau];H^{-1}(\Omega))$.

Problem 4. Consider the degenerate parabolic equation

$$D_t v = \Delta[a(x)v] + f(t)w(x)\,, \quad (x,t) \in \Omega \times [0,\tau]\,, \tag{4.5}$$

together with the initial-boundary conditions

$$a(x)v(x,t) = 0\,, \quad (x,t) \in \partial\Omega \times [0,\tau]\,, \tag{4.6}$$

$$v(x,0) = v_0(x)\,, \quad x \in \Omega\,, \tag{4.7}$$

and the additional information

$$\int_\Omega \eta(x)v(x,t)\,dx = g(t)\,, \quad t \in [0,\tau]\,. \tag{4.8}$$

Here Ω is a bounded region in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $a(x) \geq 0$ on $\overline{\Omega}$ and $a(x) > 0$ almost everywhere in Ω is a given function in $L^\infty(\Omega)$, $w \in H^{-1}(\Omega)$, $v_0 \in H_0^1(\Omega)$, $\eta \in H_0^1(\Omega)$, g is a real valued-function on $[0,\tau]$, at least continuous, and the pair (v,f) is the unknown. Of course, we shall see that functions w, v_0 and g need much more regularity. Call $a(x)v = u$. Then, if $m(x) = a(x)^{-1}$ and $u_0(x) = a(x)v_0(x)$ we obtain a system like (4.1)-(4.4). Let M be the multiplication operator by m from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ and let $L = -\Delta$ be endowed with Dirichlet condition, that is, $L : H_0^1(\Omega) \to H^{-1}(\Omega)$, as previously. Take $X = H^{-1}(\Omega)$. Then it is seen in [8, p. 81] that (3.5) holds if *i)* $a^{-1} \in L^1(\Omega)$, when $n = 1$,

ii) $a^{-1} \in L^r(\Omega)$ with some $r > 1$, when $n = 2$,

iii) $a^{-1} \in L^{\frac{n}{2}}(\Omega)$, when $n \geq 3$.

In order to apply Theorem 3.3 we suppose $u_0(x) = a(x)v_0(x) \in H_0^1(\Omega)$. Assumption (3.4) reads

$$\int_\Omega \eta(x)v_0(x)\,dx = \int_\Omega \eta(x)\frac{u_0(x)}{a(x)}\,dx = g(0)\,.$$

Take $g \in C^{1+\theta}([0,\tau];\mathbb{R}),\ 0 < \theta < 1$. Since $R(T) = R((1/a)\Delta^{-1})$, let $aw = \zeta \in H_0^1(\Omega)$, $a\Delta u_0 = a\Delta(av_0) = \zeta_1 \in H_0^1(\Omega)$, $\int_\Omega \eta(x)\frac{\zeta(x)}{a(x)}\,dx \neq 0$.
Then we conclude that there exists a unique pair (v, f) satisfying (4.5)-(4.8) with regularity

$$\Delta(av) \in C^\theta([0,\tau]; H^{-1}(\Omega))\,, \quad v \in C^{1+\theta}([0,\tau]; H^{-1}(\Omega))\,.$$

In many applications $a(x)$ is comparable with some power of the distance of x to the boundary $\partial\Omega$ and hence the assumptions depend heavily from the geometrical properties of the domain Ω. For example, if $\Omega = (-1,1)$, $a(x) = (1-x^2)^\alpha$ or $a(x) = (1-x)^\alpha(1+x)^\beta,\ 0 < \alpha, \beta < 1$ are allowed.
More generally, in $\mathbb{R}^n$, one can handle $a(x) = (1-\|x\|^2)^\alpha$ for some $\alpha > 0$ with $\Omega = \{x \in \mathbb{R}^n : \|x\| < r\}$, $r > 0$. Precisely, if $n = 2$, then $0 < \alpha < 1$, if $n \geq 3$ then $0 < \alpha < 2/n$.

Problem 5. Let us consider another degenerate parabolic equation, precisely

$$D_t v = x(1-x)D_x^2 v + f(t)w(x), \quad (x,t) \in (0,1) \times (0,\tau), \tag{4.9}$$

with the initial condition

$$v(x,0) = v_0(x), \quad x \in (0,1), \tag{4.10}$$

but with a Wentzell boundary condition (basic in probability theory and in applied sciences)

$$\lim_{x \to 0} x(1-x)D_x^2 v(x,t) = 0, \quad t \in (0,1).$$

We add the additional information:

$$\Phi[v(\cdot,t)] = v(\bar{x},t) = g(t), \quad t \in [0,\tau], \tag{4.11}$$

where $\bar{x} \in (0,1)$ is fixed. Here we take $X = H^1(0,1)$, with the norm

$$\|u\|_X^2 := \|u\|_{L^2(0,1)}^2 + \|u'\|_{L^2(0,1)}^2 + |u(0)|^2 + |u(1)|^2.$$

Introduce operator $(A, D(A))$ defined by

$$D(A) := \left\{u \in H^1(0,1);\ u'' \in L^1_{loc}(0,1) \text{ and } x(1-x)u'' \in H^1_0(0,1)\right\},$$

$$Au = -x(1-x)u'', \quad u \in D(A).$$

Then $-A$ generates an analytic semigroup in $H^1(0,1)$, see [8, pp. 249-250], [4]. So, we can apply Theorem 3.2; therefore, if $0 < \theta < \theta_0 < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $v_0 \in D_A(\theta+1,\infty)$, $w \in D_A(\theta_0,\infty)$ (in particular, $v_0 \in D(A^2)$, $w \in D(A)$), $g(0) = v_0(\bar{x})$, $w(\bar{x}) \neq 0$, then there exists a unique pair $(v, f) \in C^\theta([0,\tau]; D(A)) \times C^\theta([0,\tau];\mathbb{R})$ satisfying (4.9)–(4.11) and $D_t v \in C^\theta([0,\tau]; H^1(0,1))$. Of course, general functionals Φ in the dual space $H(0,1)^*$ could be treated.

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