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Dedicated to Funke

Contents

Introduction

A representation of a discrete group G in the category $\mathcal{P}(F)$ of finite dimensional vector spaces over a field F could be defined as a pair $(V, \rho : G \to Aut(V))$ where $V \in \mathcal{P}(F)$ and ρ is a group homomorphism from G to the group $Aut(V)$ of bijective linear operators on V. This definition makes sense if we replace $\mathcal{P}(F)$ by more general linear structures like $\mathcal{P}(R)$, the category of finitely generated projective modules over any ring R with identity.

More generally, one could define a representation of G in an arbitrary category $\mathcal{C}$ as a pair $(X, \rho : G \to Aut(X))$ where $X \in ob(\mathcal{C})$ and ρ is a group homomorphism from G to the group of $\mathcal{C}$-automorphisms of X. The representations of G in $\mathcal{C}$ also form a category $\mathcal{C}_G$ which can be identified with the category $[\underline{G/G}, \mathcal{C}]$ of covariant functors from the translation category $\underline{G/G}$ of the G-set $\overline{G/G}$ (where $\underline{G/G}$ is the final object in the category of G-sets (see 1.1). The foregoing considerations also apply if G is a topological group and $\mathcal{C}$ is a topological category, i.e. a category whose objects X and $Hom_{\mathcal{C}}(X, Y)$ are endowed with a topology such that the morphisms are continuous. Here, we have an additional requirement that $\rho : G \to Aut(X)$ be continuous. For example, G could be a Lie group and $\mathcal{C}$ the category of Hilbert spaces over $\mathbb{C}$, in which case we have unitary representations of G.

It is the aim of this book to explore connections between $\mathcal{C}_G$ and higher algebraic K-theory of $\mathcal{C}$ for suitable categories (e.g. exact, symmetric monoidal and Waldhausen categories) when G could be a finite, discrete, profinite or compact Lie group.

When $\mathcal{C} = \mathcal{P}(\mathbb{C})$, ($\mathbb{C}$ the field of complex numbers) and G is a finite or compact Lie group, the Grothendieck group $K_0(\mathcal{C}_G)$ can be identified with the group of generalized characters of G and thus provides the initial contact between representation theory and K-theory. If F is an arbitrary field, G a finite group, $\mathcal{P}(F)_G$ can be identified with the category $\mathcal{M}(FG)$ of finitely generated FG-modules and so, $K_0(\mathcal{P}(F)_G) \cong K_0(\mathcal{M}(FG)) \cong G_0(FG)$ yields K-theory of the group algebra FG, thus providing initial contact between K-theory of $\mathcal{P}(F)_G$ and K-theory of group algebras (see 1.2). This situation extends to higher dimensional K-theoretic groups i.e. for all $n \geq 0$ we have $K_n(\mathcal{P}(F)_G) \cong K_n(\mathcal{M}(FG)) \cong G_n(FG)$ (see 5.2).

More generally, if R is any commutative ring with identity and G is a finite group, then the category $\mathcal{P}(R)_G$ can be identified with the category $\mathcal{P}_R(RG)$ of RG-lattices (i.e. RG-modules that are finitely generated and projective over R) and so, for all $n \geq 0$, $K_n(\mathcal{P}(R)_G)$ can be identified with $K_n(\mathcal{P}_R(RG))$ which, when R is regular, coincides with $K_n(\mathcal{M}(RG))$ usually denoted by $G_n(RG)$ (see $(5.2)^B$).

When R is the ring of integers in a number field or p-adic field F or more generally R a Dedekind domain with quotient field F or more generally still R a regular ring, the notion of a groupring RG(G finite) generalizes to the notion of R-orders Λ in a semi-simple F-algebra Σ when char(F) does not divide the order of G and so, studying K-theory of the category $\mathcal{P}_R(\Lambda)$ of Λ-

lattices automatically yields results on the computations of K-theory of the category $\mathcal{P}_R(RG)$ of RG-lattices and so, K-theory of orders is appropriately classified as belonging to Integral representation theory.

Now the classical K-theory (K_0, K_1, K_2, K_{-n}) of orders and grouprings (especially K_0 and K_1) have been well studied via classical methods and documented in several books [20, 39, 159, 168, 211, 213] and so, we only carefully review the classical situation in Part I of this book (chapters 1-4), with clear definitions, examples, statements of important results (mostly without proofs) and refer the reader to one of the books or other literature for proofs. We include, in particular, classical results which have higher dimensional versions for which we supply proofs once and for all in the context of higher K-theory. Needless to say that some results proved for higher K-theory with no classical analogues invariably apply to the classical cases also. For example, there was no classical result that $K_2(\Lambda), G_2(\Lambda)$, are finite for arbitrary orders Λ in semi-simple algebras over number fields, but we prove in this book that $K_{2n}(\Lambda), G_{2n}(\Lambda)$, are finite for all $n \geq 1$, thus making this result also available for $K_2(\Lambda)$.

Some of the impetus for the growth of Algebraic K-theory from the beginning had to do with the fact that the classical K-group of grouprings housed interesting topological/geometric invariants, e.g.

(1) Class groups of orders and grouprings (which also constitute natural generalizations to number theoretic class groups of integers in number fields) also house Swan-Wall invariants (see $(2.3)^C$ and [214, 216]) etc.

(2) Computations of the groups $G_0(RG)$, R Noetherian, G Abelian is connected with the calculations of the group 'SSF' (see $(2.4)^B$ or [19]) which houses obstructions constructed by Shub and Francs in their study of Morse-Smale diffeomorphisms (see [19]).

(3) Whitehead groups of integral grouprings house Whitehead torsion which is also useful in the classification of manifolds (see [153, 195]).

(4) If G is a finite group and $Orb(G)$ the orbit category of G (an 'EI' category (see 7.6)). X a G-CW-complex with round structure (see [137]), then the equivariant Riedemester torsion takes values in $Wh(Q\ orb(G))$ where $Wh(Q\ orb(G))$ is the quotient of $K_1(QOrb(G))$ by subgroups of "trivial units" see [137].

(5) K_2 of integral grouprings helps in the understanding of the pseudo-isotopy of manifolds (see [80]).

(6) The negative K-theory of grouprings can also be interpreted in terms of bounded h-corbordisms (see $(4.5)^E$ or [138]).

It is also noteworthy that several far-reaching generalizations of classical concepts have been done via higher K-theory. For example, the K-theoretic definition of higher dimensional class groups $C\ell_n(\Lambda)(n \geq 0)$ of orders Λ generalize to higher dimensions the notion of class group $C\ell(\Lambda)$ of orders and grouprings which in turn generalizes the number-theoretic notion of class groups of Dedekind domains and integers R in number fields (see 7.4). Note that $C\ell_1(\Lambda)$ for $\Lambda = RG$ is intimately connected with Whitehead torsion (see 7.4 or [159]) and as already observed $C\ell(\Lambda) = C\ell_0(\Lambda)$ houses some topological/geometric invariants (see $(2.3)^C$).

Moreover, the profinite higher K-theory for exact categories discussed in chapter 8 is a cohomology theory which generalizes classical profinite topological K-theory (see [199]) as well as K-theory analogues of classical continuous cohomology of schemes rooted in Arithmetic algebraic geometry.

Part II (chapters 5 to 8) is devoted to a systematic exposition of higher algebraic K-theory of orders and grouprings. Again, because the basic higher K-theoretic constructions have already appeared with proofs in several books (e.g. [25, 88, 198]), the presentation in chapters 5 and 6 is restricted to a review of important results (with examples) relevant to our context. Topics reviewed in chapter 5 include the 'plus' construction as well as higher K-theory of exact, symmetric monoidal and Waldhausen categories. We try as much as possible to emphasize the utility value of the usually abstract topological constructions.

In chapter 7, we prove quite a number of results on higher K-theory of orders and grouprings. In $(7.1)^A$ we set the stage for arbitrary orders by first proving several finiteness results for higher K-theory of maximal orders in semi-simple algebras over p-adic fields and number fields as well as higher K-theory of associated division and semi-simple algebras.

In $(7.1)^B$, we prove among other results that if R is the ring of integers in a number field F, Λ as R-order in a semi-simple F-algebra Σ, then for all $n \geq 0$, $K_n(\Lambda), G_n(\Lambda)$ are finitely generated Abelian groups, $SK_n(\Lambda), SG_n(\Lambda), SK_n(\hat{\Lambda}_p)$ and $SG_n(\hat{\Lambda}_p)$ are finite groups (see [108, 110, 112, 113]) and $SG_n(RG)$ are trivial (see [131]) where G is a finite group. In 7.2 we prove that rank $K_n(\Lambda)$ = rank $G_n(\Lambda)$ = rank $K_n(\Gamma)$ if Γ is a maximal R-order containing Λ, see [115]. We consequently prove that for all $n \geq 1$, $K_{2n}(\Lambda), G_{2n}(\Lambda)$ are actually finite groups. Hence for any finite group G, $K_{2n}(RG), G_{2n}(RG)$ are finite (see $(7.2)^B$ or [121]).

Next, we obtain in 7.3 a decomposition (for G Abelian)
$G_n(RG) \cong \oplus G_n(R < C >)$ for all $n \geq 0$ where R is a Noetherian ring, and C ranges over all cyclic quotients of G and $R < C >= R\zeta_{|C|}(\frac{1}{|C|}), \zeta_{|C|}$ being a primitive $|C|^{\text{th}}$ root of unity (see [232]). (This decomposition is a higher dimensional version of that of $G_0(RG)$ (see $(2.4)^A$.) The decomposition of $G_n(RG)$ is extended to some non-Abelian groups e.g. dihedral, quaternion and nilpotent groups (see [231, 233]). We conclude 7.3 with a discussion of a conjecture due to Hambleton, Taylor and Williams on the decomposition for $G_n(RG)$, G any finite groups (see [76]), and the counter-example provided for this conjecture by D. Webb and D. Yao (see [235].

Next, in 7.4 we define and study higher-dimensional class groups $C\ell_n(\Lambda)$ of R-orders Λ which generalize the classical notion of class groups $C\ell(\Lambda)(= C\ell_0(\Lambda))$ of orders. We prove that $\forall\ n \geq 0$, $C\ell_n(\Lambda)$ is a finite group and identify p-torsion in $C\ell_{2n-1}(\Lambda)$ for arbitrary orders Λ (see [102]) while we identify p-torsion for all $C\ell_{2n}(\Lambda)$ when Λ is an Eichler or hereditary order (see [74, 75]).

In 7.5, we study higher K-theory of grouprings of virtually infinite cyclic groups V in the two cases when $V = G \rtimes_\alpha T$, the semi-direct product of a finite group G (of order r, say) and an infinite cyclic group $T =< t >$ with respect to the automorphism $\alpha : G \to G\ g \to tgt^{-1}$ and when $V = G_0 *_H G_1$ where the groups $G_i = 0, 1$ and H are finite and $[G_i : H] = 2$. These groups V are conjectured by Farrell and Jones (see [54]) to constitute building blocks for the understanding of K-theory of grouprings of an arbitrary discrete group G - hence their importance. We prove that when $V = G \rtimes_\alpha T$, then for all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group and that $NK_n(RV)$ is r-torsion. For $V = G_0 *_H G_1$ we prove that the nil groups of V are $|H|$-torsion (see [123]).

The next section of chapter 7 is devoted to the study of higher K and G-theory of modules over 'EI'-categories. Modules over 'EI'-categories constitute natural generalizations for the notion of modules over grouprings and K-theory of such modules are known to house topological and geometric invariants and are also replete with applications in the theory of transformation groups (see [137]). Here, we obtain several finiteness and other results which are extension of results earlier obtained for higher K-theory of grouprings of finite groups.

In 7.7 we obtain several finiteness results on the higher K-theory of the category of representations of a finite group G in the category of $\mathcal{P}(\Gamma)$ where Γ is a maximal order in central division algebra over number fields and p-adic fields. These results translate into computations of $G_n(\Gamma G)$ as well as lead to showing via topological and representation theoretic techniques that a non-commutative analogue of a fundamental result of R.G. Swan at the zero-dimensional level does not hold (see [110]).

In chapter 8, we define and study profinite higher K and G-theory of exact categories, orders and grouprings. This theory is an extraordinary cohomology theory inspired by continuous cohomology theory in algebraic topology and arithmetic algebraic geometry. The theory yields several ℓ-completeness theorems for profinite K and G-theory of orders and grouprings as well as yields some interesting computations of higher K-theory of p-adic orders otherwise inaccessible. For example we use this theory to show that if Λ is a p-adic order in a p-adic semi-simple algebra Σ, then for all $n \geq 1$, $K_n(\Lambda)_\ell, G_n(\Lambda)_\ell, K_n(\Sigma)_\ell$ are finite groups provided ℓ is a prime $\neq p$. We also define and study continuous K-theory of p-adic orders and obtain a relationship between profinite and continuous K-theory of such orders (see [117]).

Now if $\underline{S}$ is the translation category of any G-set S, and $\mathcal{C}$ is a small category, then the category $[\underline{S}, \mathcal{C}]$ of covariant functors from $\underline{S}$ to $\mathcal{C}$ is also called the category of G-equivariant $\mathcal{C}$-bundles on S because if $\mathcal{C} = \mathcal{P}(\mathbb{C})$, then $[\underline{S}, \mathcal{P}(\mathbb{C})]$ is just the category of G-equivariant $\mathbb{C}$-bundles on the discrete G-space S so that $K_0[\underline{S}, \mathcal{P}(\mathbb{C})] = K_0^G(S, \mathcal{P}(\mathbb{C}))$ is the zero-dimensional G-equivariant K-theory of S. Note that if S is a G-space, then the translation category $\underline{S}$ of S as well as the category $[\underline{S}, \mathcal{C}]$ are defined similarly. Indeed, if S is a compact G-space then $K_0^G(S, \mathcal{P}(\mathbb{C}))$ is exactly the Atyah-Segal equivariant K-theory of S (see [184]).

One of the goals of this book is to exploit representation theoretic techniques (especially induction theory) to define and study equivariant higher algebraic K-theory and their relative generalizations for finite, profinite and compact Lie group actions, as well as equivariant homology theories for discrete group actions in the context of category theory and homological algebra with the aim of providing new insights into classical results as well as open avenues for further applications. We devote Part III (chapters 9 - 14) of this book to this endeavour.

Induction theory has always aimed at computing various invariants of a given group G in terms of corresponding invariants of certain classes of subgroups of G. For example if G is a finite group, it is well know by Artin induction theorem that two G-representations in $\mathcal{P}(\mathbb{C})$ are equivalent if their restrictions to cyclic subgroups of G are isomorphic. In other words, given the exact category $\mathcal{P}(\mathbb{C})$, and a finite group G, we have found a collection $D(\mathcal{P}(\mathbb{C}), G)$ of subgroups (in this case cyclic subgroups) of G such that two G-representations in $\mathcal{P}(\mathbb{C})$ are equivalent iff their restrictions to subgroups in $D(\mathcal{P}(\mathbb{C}), G)$ are equivalent. One could then ask the following general question: Given a category $\mathcal{A}$ and a group G, does there exist a collection $D(\mathcal{A}, G)$ of proper subgroups of G such that two G-representations in $\mathcal{A}$ are equivalent if their restrictions to subgroups in $D(\mathcal{A}, G)$ are equivalent?

As we shall see in this book, Algebraic K-theory is used copiously to answer these questions. For example, if G is a finite group, T any G-set, $\mathcal{C}$ an exact category, we construct in 10.2 for all $n \geq 0$, equivariant higher K-functors.

$$K_n^G(-, \mathcal{C}, T),\ K_n^G(-, \mathcal{C}, T), K_n^G(-, \mathcal{C})$$

as Mackey functors from the category $GSet$ of G-sets to the category $\mathbb{Z}$-$\mathcal{M}$od of Abelian groups (i.e. functors satisfying certain functorial properties, in particular, categorical version of Mackey subgroup theorem in representation theory) in such a way that for any subgroup H of G we identify $K_n^G(G/H, \mathcal{M}(R))$ with $K_n(\mathcal{M}(RH)) := G_n(RH), K_n^G(G/H, \mathcal{P}(R))$ with $K_n(\mathcal{P}_R(RH)) := G_n(R, H)$ and $P_n^G(G/H, \mathcal{P}(R), G/e)$ with $K_n(RH)$ for all $n \geq 0$ (see [52, 53]). Analogous constructions are done for profinite group actions (chapter 11) and compact Lie group actions (chapter 12), finite group actions in the context of Waldhausen categories, chapter 13, as well as equivariant homology theories for the actions of discrete groups (see chapter 14).

For such Mackey functors M, one can always find a canonical smallest class U_M of subgroups of G such that the values of M on any G-set can be computed from their restrictions to the full subcategory of G-sets of the form G/H with $H \in U_M$. The computability of the values of M from its restriction to G-sets of the form $G/H, H \in U_M$ for finite and profinite groups is expressed in terms of vanishing theorems for a certain cohomology theory associated with M (U_M) - a cohomology theory which generalizes group cohomology. In 9.2, we discuss the cohomology theory (Amitsur cohomology) of Mackey functors, defined on an arbitrary category with finite coproducts, finite pullbacks and final objects in 9.1 and then specialize as the needs arise for the cases of interest-category of G-sets for G finite (in chapter 9 and chapter 10), G profinite (chapter 11) - yielding vanishing theorems for the cohomology of the K-functors as well as cohomology of profinite groups (11.2) (see [109]).

The equivariant K-theory discussed in this book yields various computations of higher K-theory of grouprings. For example apart from the result that higher K-theory of RG (G finite or compact Lie group) can be computed by restricting to hyper elementary subgroups of G (see 10.4 and 12.3.3) (see [108, 116]), we also show that if R is a field k of characteristic p and G a finite or profinite group, then the Cartan map $K_n(kG) \to G_n(kG)$ induces an isomorphism $\mathbb{Z}(\frac{1}{p}) \otimes K_n(kG) \cong \mathbb{Z}(\frac{1}{p}) \otimes G_n(kG)$ leading to the result that for all $n \geq 1$, $K_{2n}(kG)$ is a p-group for finite groups G. We also have an interesting result that if R is the ring of integers in a number field, G a finite group then the Waldhausen K-groups of the category $(Ch_b(\mathcal{M}(RG), \omega)$ of bounded complexes of finitely generated RG-modules with stable quasi-isomorphisms as weak equivalences are finite Abelian groups.

The last chapter (chapter 14) which is devoted to Equivariant homology theories, also aims at computations of higher algebraic K-groups for grouprings of discrete groups via induction techniques also using Mackey functors. In fact, an important criteria for a G-homology theory $\mathcal{H}_n^G : G\mathcal{S}et \to \mathbb{Z}\text{-}\mathcal{M}\text{od}$ is that it is isomorphic to some Mackey functor: $G\mathcal{S}et \to \mathbb{Z}\text{-}\mathcal{M}\text{od}$. The chapter is focussed on a unified treatment of Farrell and Baum-Connes isomorphism conjectures through Davis-Lück assembly maps (see 14.2 or [40]) as well as some specific induction results due to W. Lück, A. Bartels and H. Reich (see 14.3 or [14]). One other justification for including Baum-Connes conjecture in this unified treatment is that it is well known by now that Algebraic K-theory and Topological K-theory of stable C^*-algebras do coincide (see [205]). We review the state of knowledge of both conjectures (see 14.3, 14.4) and in the case of Baum-Connes conjecture also discuss its various formulations including the most recent in terms of quantum group actions.

Time, space and the heavy stable homotopy theoretic machinery involved (see [147]) (for which we could not prepare the reader) has prevented us from including a G-spectrum formulation of the equivariant K-theory developed in chapters 10, 11, 12, 13. In [192, 193, 194], K. Shimakawa provided, (for G a finite group) a G-spectrum formulation of part of the (absolute) equivariant theory discussed in 10.1. It will be nice to have a G-spectrum formulation of the relative theory discussed in 10.2 as well as a G-spectrum formulation for the equivariant theory discussed in 11.1 and 12.2 for G profinite and G compact Lie group. However, P. May informs me that equivariant infinite loop space theory itself is only well understood for finite groups. He thinks that profinite groups may be within reach but compact Lie groups are a complete mystery since no progress has been made towards a recognition principle in that case. Hence, there is currently no idea about how to go from the type of equivariant Algebraic K-theory categories defined in this book to a G-spectrum when G is a compact Lie group.

Appendix A contains some known computations while Appendix B consists of some open problems.

The need for this book

1) So far, there is no book on higher Algebraic K-theory of orders and grouprings. The results presented in the book are only available in scattered form in journals and other scientific literature, and there is a need for a coordinated presentation of these ideas in book form.

2) Computations of higher K-theory even of commutative rings (e.g. $\mathbb{Z}$) have been notoriously difficult and up till now the higher K-theory of $\mathbb{Z}$ is yet to be fully understood. Orders and grouprings are usually non-commutative rings that also involve non-commutative arithmetic and computations of higher K-theory of such rings are even more difficult, since methods of etále cohomology etc. do not work. So it is desirable to collect together in book form methods that have been known to work for computations of higher K-theory of such non-commutative rings as orders and grouprings.

3) This is the first book to expose the characterization of all higher algebraic K-theory as Mackey functors leading to equivariant higher algebraic K-theory and their relative generalization, also making computations of higher K-theory of grouprings more accessible. The translation of the abstract topological constructions into representation theoretic language of Mackey functors has simplified the theory some what and it is desirable to have these techniques in book form.

4) Interestingly, obtaining results on higher K-theory of orders Λ (and hence grouprings) for all $n \geq 0$ have made these results available for the first time for some classical K-groups. For instance, it was not known classically that if R is the ring of integers in a number field F, and Λ any R-order in a semi-simple F-algebra, then $K_2(\Lambda), G_2(\Lambda)$ (or even $SK_s(\Lambda), SG_2(\Lambda)$) are finite groups. Having these results for $K_{2n}(\Lambda), G_{2n}(\Lambda)$ and hence $SK_{2n}(\Lambda), SG_{2n}(\Lambda)$ for all $n \geq 1$ makes these results available now for $n = 1$.

5) Also computations of higher K-theory of orders which automatically yield results on higher K-theory of $RG(G$ finite) also extends to results on higher K-theory of some infinite groups e.g. computations of higher K-groups of virtually infinite cyclic groups that are fundamental to the subject.

Who can use this book?

It is expected that readers would already have some working knowledge of algebra in a broad sense including category theory and homological algebra, as well as working knowledge of basic algebraic topology, representation theory, algebraic number theory, some algebraic geometry and operator algebras. Nevertheless, we have tried to make the book as self-contained as possible by defining the most essential ideas.

As such, the book will be useful for graduate students who have completed at least one year of graduate study, professional mathematicians and researchers of diverse backgrounds who want to learn about this subject as well as specialists in other aspects of K-theory who want to learn about this approach to the subject. Topologists will find the book very useful in updating their knowledge of K-theory of orders and grouprings for possible applications and representation theorists will find this innovative approach to and applications of their subject very enlightening and refreshing while number theorists and arithmetic algebraic geometers who want to know more about non-commutative arithmetics will find the book very useful.

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Notes on Notations

Notes on Notation

- $mor_{\mathcal{C}}(A, B), Hom_{\mathcal{C}}(A, B) :=$ set of $\mathcal{C}$-morphisms from A to B ($\mathcal{C}$ a category
- $\bar{A} = K(A) =$ Gröthendieck group associated to a semi-group A
- $VB_F(X) =$ category of finite dimensional vector bundles on X ($F = \mathbb{R}$ or $\mathbb{C}$)
- $X(G) =$ set of cyclic quotients of a finite group G
- $\mathcal{A}[z] =$ polynomial extension of an additive category $\mathcal{A}$
- $\mathcal{A}[z, z^{-1}]$, Laurent polynomal extension of $\mathcal{A}$
- $\mathcal{O}_{r_{\mathcal{F}}}(G) = \{G/H | H \in \mathcal{F}\}$, $\mathcal{F}$ a family of subgroups of G
- $A_{R,\mathcal{F}} =$ assembly map
- $\mathcal{H}_G(S, B) := \{G\text{-maps } f : S \to B\}$, S a G-set, B a $\mathbb{Z}G$-module
- $B(S) =$ set of all set-theoretic maps $S \to B.$ Note that
- $\mathcal{H}_G(S, B)$ is the subgroup of G-invariant elements in $B(S)$
- $\mathcal{A}(G) =$ category of homogeneous G-spaces (G a compact Lie group)
- $\Omega(\mathcal{B}) :=$ Burnside ring of a based category $\mathcal{B}$
- $\Omega(G) :=$ Burnside ring of a group G
- $\|\mathcal{B}\| :=$ Artin index of a based category $\mathcal{B}$
  $:=$ exponent of $\bar{\Omega}(\mathcal{B})/\Omega(\mathcal{B})$
- $M^n_m := n$-dimensional mod-m Moore space
- $H_n(X, \underline{E}) = \underline{E}_n(X) :=$ homology of a space X with coefficient in a spectrum $\underline{E}$
- $H^n(X, \underline{E}) = \underline{E}^n(X) :=$ cohomology of a space X with coefficients in a spectrum $\underline{E}$
- $\mathcal{P}(A) :=$ category of finitely generated projective A-modules (A a ring with identity)

- $\mathcal{M}(A)$:= category of finitely generated A-modules
- $\mathcal{M}'(A)$:= category of finitely presented A-modules
- $A - \mathcal{M}$od := category of left A-modules
- $\mathcal{P}(X)$:= category of locally free sheaves of O_X-modules (X a scheme)
- $\mathcal{M}(X)$:= category of coherent sheaves of O_X-modules (X a Noetherian scheme)
- $\mathcal{P}_R(A)$:= category of A-modules finitely generated and projective as R-modules (A an R-algebra)
- $\underline{S}$:= translation category of a G-set S (see 1.1.3) (G a group)
  $G\mathcal{S}et$:= category of G-sets
  $[\underline{S}, \mathcal{C}]$:= category of covariant functors $\underline{S} \to \mathcal{C}$ ($\mathcal{C}$ any category)
- For any Abelian group G, and a rational prime ℓ
  G_ℓ or $G(\ell)$:= ℓ-primary subgroup of G
  $G[\ell^s] = \{g \in G | \ell^s g = 0\}$
  Note. $G(\ell) = UG[\ell^s] = \varinjlim G[\ell^s]$
- If $\mathcal{G} : G\mathcal{S}et \to \mathbb{Z}$-$\mathcal{M}$od is a Green functor and A a commutative ring with identity then $\mathcal{G}^A := A \otimes_{\mathbb{Z}} \mathcal{G} : G\mathcal{S}et \to A$-$\mathcal{M}$od is a Green functor given by $(A \otimes_{\mathbb{Z}} \mathcal{G})(S) = A \otimes \mathcal{G}(S)$ If $\mathcal{D}_{\mathcal{G}}$ is a defect basis for $\mathcal{G}$ (see 9.6.1) write $\mathcal{D}^A_{\mathcal{G}}$ for the defect basis of $\mathcal{G}^A$ If $\mathcal{P}$ is a set of primes and $A = \mathbb{Z}_{\mathcal{P}} = \mathbb{Z}[\frac{1}{q} | q \notin \mathcal{P})$ write $\mathcal{D}^{\mathcal{P}}_{\mathcal{G}}$ for $\mathcal{D}^A_{\mathcal{G}}$
- $Ch_b(\mathcal{C})$ = category of bounded chain complexes in an exact category $\mathcal{C}$
- Let C be a cyclic group of order t
  $\mathbb{Z}(C) = \mathbb{Z}[\zeta]$ where ζ is a primitive t^{th} root of 1
  $\mathbb{Z} < C >= \mathbb{Z}(C)(\frac{1}{t}) = \mathbb{Z}[\zeta_t, \frac{1}{t}]$
  For any ring R, $R(C) = R \otimes \mathbb{Z}(C)$, $R < C >= R \otimes \mathbb{Z} < C >$
- For any finite group G, p a rational prime
  $G(p)$ or $S_p(G)$:= Sylow p-subgroup of G
  For G Abelian, $G(p') = \bigoplus_{\substack{q \;\; prime \\ q \neq p}} G(q)$. So $G = G(p) \times G(p')$
  If $\mathcal{P}$ is a set of primes, $G(\mathcal{P}) = \bigoplus_{p \in \mathcal{P}} G(p)$ i.e. $\mathcal{P}$-torsion part of G.
- If R is the ring of integers in a number field F, Λ an R-order in a semi-simple F-algebra, then
  $\mathcal{P}(\Lambda)$:= finite set of prime ideals $\underline{p}$ of R for which $\hat{\Lambda}_{\underline{p}}$ is not maximal
  $\check{\mathcal{P}}(\Lambda)$:= set of rational primes lying below the prime ideals in $\mathcal{P}(\Lambda)$
- For a category $\mathcal{C}$, $\mathbb{P}(\mathcal{C})$:= idempotent completion of $\mathcal{C}$

- For a discrete group G
  $All :=$ all subgroups of G
  $Fin :=$ all finite subgroups of G
  $VCy :=$ all virtually cyclic subgroups of G
  $Triv :=$ trivial family consisting of only one element i.e. the identity element of G
  $FCy :=$ all finite cyclic subgroups of G

- $\mathcal{M}\mathrm{od}^R_{\mathcal{F}}(G) :=$ category of contravariant functors

$$Or_{\mathcal{F}}(G) \longrightarrow R - \mathcal{M}\mathrm{od}$$

  $G - \mathcal{M}\mathrm{od}^R_{\mathcal{F}} :=$ category of covariant functors

$$Or_{\mathcal{F}}(G) \longrightarrow R - \mathcal{M}\mathrm{od}$$

- $E_{\mathcal{F}}(G) :=$ classifying space for a family $\mathcal{F}$ of subgroups of a discrete group G
  $:=$ universal G-space with stabilizers in $\mathcal{F}$
  $\underline{E}G = E_{Fin}(G) =$ universal space for proper actions of G

Part I

Review of Classical Algebraic K-Theory and Representation Theory

Chapter 1

Category of representations and constructions of Grothendieck groups and rings

1.1 Category of representations and G-equivariant categories

1.1.1 Let G be a discrete group, V a vector space over a field F. A representation of G on V is a group homomorphism $\rho : G \to \mathrm{Aut(V)}$ where $\mathrm{Aut(V)}$ is the group of invertible linear operators on V. Call V a representation space of ρ.

An action of G on V is a map $\rho : G \times V \to V$ $\rho(g \centerdot v) := gv$ such that $ev = v, (gh)v = g(h(v))$. Note that an action $\rho : G \times V \to V$ gives rise to a representation $\rho_V : G \to \mathrm{Aut(V)}$ where $\rho_V(g) := \rho_g : v \to \rho(g, v)$, and conversely, any representation $\rho_V : G \to \mathrm{Aut(V)}$ defines an action $\rho : G \times V \to V : (g, v) \to \rho_g(v)$.

Two representations ρ, ρ' with representation spaces V, V' are said to be equivalent if there exists an F-isomorphism β of V onto V' such that

$$\rho'(g) = \beta\rho(g).$$

The dimension of V over F is called the degree of ρ.

Remarks 1.1.1 (i) For the applications, one restricts V to finite-dimensional vector spaces. We shall be interested in representations on V ranging from such classical spaces as vector spaces over complex numbers to more general linear structures like finitely generated projective modules over such rings as Dedekind domains, integers in number fields, and p-adic fields, etc.

(ii) When G, V have topologies, we have an additional requirement that ρ be continuous.

(iii) More generally, G could act on a finite set S, i.e., we have a permutation $s \to gs$ of S satisfying the identities $1s = s, g(hs) = (gh)s$ for $g, h \in G, s \in S$. Let V be the vector space having a basis $(e_s)_{s \in S}$ indexed by

$s \in S$. So, for $g \in G$, let ρ_g be the linear map $V \to V$ sending e_x to e_{gx}. Then $\rho : G \to \mathrm{Aut(V)}$ becomes a linear representation of G called permutation representation associated to S.

(iv) Let A be a finite-dimensional algebra over a field F, and V a finite-dimensional vector space over F. A representation of A on V is an algebra homomorphism $\overline{\rho} : A \to \mathrm{Hom_F}(V, V) = \mathrm{End_F}(V)$, i.e., a mapping $\overline{\rho}$, which satisfies:

$$\begin{aligned} \overline{\rho}(a+b) &= \overline{\rho}(a) + \overline{\rho}(b)\,, & \overline{\rho}(ab) &= \overline{\rho}(a)\overline{\rho}(b) \\ \overline{\rho}(\alpha a) &= \alpha\overline{\rho}(a)\,, & \overline{\rho}(e) &= 1\,, a, b \in A, \alpha \in F\,, \end{aligned}$$

where e is the identity element of A.

Now, if $\rho : G \to \mathrm{Aut(V)}$ is a representation of G with representation space V, then there is a unique way to extend ρ to a representation $\overline{\rho}$ of FG with representation space V, i.e., $\overline{\rho}\,(\Sigma a_g \rho(g)) = \Sigma a_g \rho(g)$. Conversely, every representation of FG, when restricted to G, yields a representation of G. Hence there is a one-one correspondence between F-representations of G with representation space V and FG-modules.

Definition 1.1.1 means that we have a representation of G in the category $\mathcal{P}(F)$ of finite-dimensional vector spaces over F. This definition could be generalized to any category as follows.

1.1.2 Let $\mathcal{C}$ be a category and G a group. A G-object in $\mathcal{C}$ (or a representation of G in $\mathcal{C}$) is a pair $(X, \rho), X \in ob\,\mathcal{C}, \rho : G \to \mathrm{Aut(X)}$ a group homomorphism. We shall write ρ_g for $\rho(g)$.

The G-objects in $\mathcal{C}$ form a category $\mathcal{C}_G$ where for $(X, \rho), (X', \rho') \in ob\,\mathcal{C}_G$, $mor_{\mathcal{C}_G}((X, \rho), (X', \rho'))$ is the set of all $\mathcal{C}$-morphisms $\varphi : X \to X'$ such that for each $g \in G$, the diagram

$$\begin{tikzcd} X \arrow[r, "\rho_g"'] \arrow[d, "\varphi"] & X \arrow[d, "\varphi"] \\ X' \arrow[r, "\rho'_g"'] & X' \end{tikzcd}$$

commutes

Examples 1.1.1 (i) When $\mathcal{C} = F\mathcal{S}et$, the category of finite sets, $\mathcal{C}_G = G\mathcal{S}et$, the category of finite G-sets.

(ii) When F is a field, and $\mathcal{C} = \mathcal{P}(F)$, then $\mathcal{P}(F)_G$ is the category $\mathcal{M}(FG)$ or finitely generated FG-modules.

(iii) When R is a commutative ring with identity and $\mathcal{C} = \mathcal{M}(R)$ the category of finitely generated R-modules, then $\mathcal{C}_G = \mathcal{M}(RG)$, the category of finitely generated RG-modules.

(iv) If R is a commutative ring with identity, and $\mathcal{C} = \mathcal{P}(R)$, then $\mathcal{P}(R)_G = \mathcal{P}_R(RG)$, the category of RG-lattices, i.e., RG-modules that are finitely generated and projective over R.
Note that for a field F, every $M \in \mathcal{M}(FG)$ is an FG-lattice and so $\mathcal{M}(FG) = \mathcal{P}_F(FG)$.

1.1.3 Let S be a $G\mathcal{S}et$ (G a discrete group). We can associate to S a category $\underline{S}$ as follows:

$$ob\ \underline{S} = \text{elements of } S\,; \qquad \text{mor}_{\underline{S}}(s,t) = \{(g,s)|g \in G, gs = t\}\,.$$

Composition of morphisms is defined by $(h,t) \circ (g,s) = (hg,s)$, and the identity morphism $s \to s$ is (e,s) where e is the identity of G. $\underline{S}$ is called the translation category of S.

- For any category $\mathcal{C}$, let $[\underline{S},\mathcal{C}]$ be a category of (covariant) functors $\zeta : \underline{S} \to \mathcal{C}$, which associates to an element $s \in S$ a $\mathcal{C}$-object ζ_s and to a morphism (g,s) a $\mathcal{C}$-map $\zeta_{(g,s)} : \zeta_s \to \zeta_{gs}$, $s \in S$ $\zeta_{(e,s)} = id_{\zeta_s}$ and $\zeta_{(g,hs)} \circ \zeta_{(h,s)} = \zeta_{(gh,s)}$ for all $g,h \in G, s \in S$. Call such a functor a G-equivariant $\mathcal{C}$-bundle on S.

  The motivation for this terminology is that if $\mathcal{C}$ is the category of finite-dimensional vector spaces over the field $\mathbb{C}$ of complex numbers, then ζ is indeed easily identified with a G-equivariant $\mathbb{C}$-vector bundle over the finite discrete G-sets S.

1.1.4 Note that the category $\underline{S}$ defined above is a groupoid, i.e., a category in which every morphism is an isomorphism. More generally, for any small groupoid $\mathcal{G}$, and any small category $\mathcal{C}$, we shall write $[\mathcal{G},\mathcal{C}]$ for the category of covariant functors $\mathcal{G} \to \mathcal{C}$ and $[\mathcal{G},\mathcal{C}]'$ for the category of contravariant functors $\mathcal{G} \to \mathcal{C}$. We shall extend the ideas of this section from $[\underline{S},\mathcal{C}]$ to $[\mathcal{G},\mathcal{C}]$ for suitable $\mathcal{C}$ in chapter 14 when we study equivariant homology theories vis-a-vis induction techniques.

1.1.5 Examples and some properties of $[\underline{S},\mathcal{C}]$

(i) For any category $\mathcal{C}$, there exists an equivalence of categories $[\underline{G/G},\mathcal{C}] \to \mathcal{C}_{\mathcal{G}}$ given by $\zeta \mapsto (\zeta_*, \rho : G \to \text{Aut}(\zeta_*); g \to \zeta_{(g,*)})$ where $\zeta_{g,*} \in \text{Aut}(\zeta_*)$, since $\zeta^{-1}_{(g,*)} = \zeta_{(g^{-1},*)}$.
Hence if G is a finite group, we have

- $[\underline{G/G},\mathcal{M}(R)] \backsimeq \mathcal{M}(R)_G \backsimeq \mathcal{M}(RG)$, if R is a commutative ring with identity
- $[\mathrm{G/G},\mathcal{P}(R)] \backsimeq \mathcal{P}_R(RG)$.

(ii) (a) Let $\mathcal{C}$ be a category and X a fixed $\mathcal{C}$-object. Define a new category $\mathcal{C}/X$ (resp. $X/\mathcal{C}$) called the category of $\mathcal{C}$-objects over X (resp. under X) as follows:

The objects of $\mathcal{C}/X$ (resp. $X/\mathcal{C}$) are pairs $(A, \varphi : A \to X)$ (resp. $(B, \delta : X \to B)$) where A (resp. B) runs through the objects of $\mathcal{C}$ and φ through $\mathrm{mor}_{\mathcal{C}}(A, X)$ (resp. δ through $\mathcal{C}(X, B)$). If (A, φ), $(A', \varphi') \in \mathcal{C}/X$ (resp. $(B, \delta), (B', \delta') \in X/\mathcal{C}$), then $\mathcal{C}/X((A, \varphi), (A', \varphi')) = \{\psi \in \mathcal{C}(A, A') | \varphi = \varphi'\psi\}$ $\big($resp. $X/\mathcal{C}((B, \delta), (B', \delta')) = \{\rho \in \mathcal{C}(B, B') | \delta' = \rho\gamma\}\big)$, i.e., a morphism from (A, φ) to (A', φ') (resp. (B, δ) to (B', δ')) is a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & A' \arrow[dl, "\varphi'"] \\ & X & \end{tikzcd}$$

$$\text{(resp.}\quad \begin{tikzcd} & X \arrow[dl, "\delta"'] \arrow[dr, "\delta'"] & \\ B \arrow[rr, "\rho"'] & & B' \end{tikzcd} \quad \text{of } \mathcal{C}\text{-morphisms)}$$

(b) If in (a) $\mathcal{C} = G\mathcal{S}et, S \in G\mathcal{S}et$, we have $G\mathcal{S}et/S$ (resp. $S/G\mathcal{S}et$). Note that if $S \in G\mathcal{S}et$, the category $\underline{S}$ can be realized as a full subcategory of $G\mathcal{S}et/S$ whose objects are all maps $G/e \to S$ where $s \in S$ is identified with $f_s : G/e \to S$ $g \mapsto gs$) and (g, s) is identified with

$$(\rho(g))\,, \qquad \begin{tikzcd} G/e \arrow[rr] \arrow[dr, "f_s"'] & & G/e \arrow[dl, "f_{gs}"] \\ & S & \end{tikzcd} : x \mapsto xg^{-1}, x \in G/e$$

(c) If $\mathcal{C} = F\mathcal{S}et$ in (i), then we have an equivalence of categories $[S, F\mathcal{S}et] \backsimeq G\mathcal{S}et/S$ defined as follows.

For $\zeta \in [S, F\mathcal{S}et]$, the set $|\zeta| = \{(s, x) | s \in S, x \in \zeta_s\}$ is a G-set w.r.t. $G \times |\zeta| \to |\zeta| : (g, (s, x)) \mapsto (gs, \zeta_{(g,s)}x)$ and $|\zeta| \to S : (s, x) \mapsto s$ is a G-map (note $|\zeta|$ could be described as the disjoint union of fibres of ζ.

Conversely, if $\varphi : S' \to S$ is a G-map over S, then $\varphi : S' \to S$ gives rise to a $\mathcal{C}$-bundle ζ over S with fibres $\zeta_s = \varphi^{-1}(s)$ and maps $\zeta_{(g,s)} : \zeta_s \to \zeta_{gs} : x \mapsto gx$. It is easily checked that a $\mathcal{C}$-bundle morphism $\mu : \zeta \to \zeta'$ between two $\mathcal{C}$-bundles corresponds to a G-map between the corresponding G-sets over S and vice versa and that this way we get, indeed, an equivalence of categories.

(iii) (a) Let $\underline{\mathrm{G/H}}$ (resp. $\underline{\mathrm{H/H}}$) be the category associated with the G-set G/H (resp. H-set H/H). Then the functor $\underline{\mathrm{H/H}} \to \underline{\mathrm{G/H}}$ given by $*_H \to H \in G/H$ and $(u, *_H) \to (u, H)$ for $u \in H$ is an equivalence of categories.

Proof. The proof follows from the fact that if $\mathcal{C}_1$ is a full subcategory of a category $\mathcal{C}_2$ and if for any $\mathcal{C}_2$-object X there exists $X' \in \mathcal{C}_1$ such that $X' \cong X$, then $\mathcal{C}_1 \to \mathcal{C}_2$ is an equivalence of categories (see [141]). Now, since $\underline{\mathrm{H/H}}$ is full in $\underline{\mathrm{G/H}}$ and any object in $\underline{\mathrm{G/H}}$ is isomorphic to $*_H = H \in G/H$, we may apply this fact to $\mathcal{C}_1 = \underline{\mathrm{H/H}}$ and $\mathcal{C}_2 = \underline{\mathrm{G/H}}$.

(b) The equivalence $\underline{\mathrm{H/H}} \to \underline{\mathrm{G/H}}$ of categories in (a) defines an equivalence of categories $[\underline{\mathrm{G/H}}, \mathcal{C}] \to [\underline{\mathrm{H/H}}, \mathcal{C}]$ for any category $\mathcal{C}$. Hence $\mathcal{C}_H$, which has been shown to be isomorphic to $[\underline{\mathrm{H/H}}, \mathcal{C}]$ already, is equivalent to $[\underline{\mathrm{G/H}}, \mathcal{C}]$.

Remark. Note that it follows from (ii) and (iii)(b) that we now have an equivalence of categories between $G\mathcal{S}et/(G/H)$ and $H\mathcal{S}et \simeq H\mathcal{S}et/(H/H)$ defined by associating to any G-set S over G/H the pre-image of $*_H$ in G/H considered as an H-set.

(c) Let G be a finite group, $H \leq G$, T a H-set, and $G \underset{H}{\times} T$ the induced G-set defined as a set of H-orbits $\overline{(g,t)} \subseteq G \times T$ with respect to the H-action $h(g,t) = (gh^{-1}, ht), h \in H, g \in G, t \in T$. Then the functor $\underline{T} \to \underline{G \underset{H}{\times} T}$ given by $t \mapsto \overline{(e,t)}, (g,t) \mapsto (g, \overline{(e,t)})$ is an equivalence of categories, and so, for any category $\mathcal{C}$, $[\underline{G \underset{H}{\times} T}, \mathcal{C}] \to [\underline{T}, \mathcal{C}]$ is an equivalence of categories. Note that $h \in G$ acts on $\overline{(g,t)} \in G \underset{H}{\times} T$ by $h\overline{(g,t)} = \overline{(hg,t)}$.

Proof. Again, with $\mathcal{C}_1 = \underline{T}$, $\mathcal{C}_2 = \underline{G \underset{H}{\times} T}$, the embedding $\mathcal{C}_1 \to \mathcal{C}_2$ defined above makes $\mathcal{C}_1$ a full subcategory of $\mathcal{C}_2$ such that any object in $\mathcal{C}_2$ is isomorphic to some object in (the image of) $\mathcal{C}_1$. (Details are left to the reader as an exercise.)

(iv) If $\varphi : H \to G$ is a group homomorphism, then we have a functor $G\mathcal{S}et \to H\mathcal{S}et$ given by $S \mapsto S|_H$. Now, we can associate to any $\zeta \in [\underline{S}, \mathcal{C}]$ the H-equivariant $\mathcal{C}$-bundle $\zeta|_H$ over $S|_H$, which has the same fibres as ζ with the H-action defined by restricting the G-action to H via φ. We thus get a functor $[\underline{S}, \mathcal{C}] \to [\underline{S}|_H, \mathcal{C}]$. Note that this functor can also be derived from the canonical functor $\underline{S}|_H \to \underline{S}$ given by $s \mapsto s$, $(g,s) \mapsto (\varphi(g), s)$.

(v) If $\mathcal{C}$ is any category and S_1, S_2 are two G-sets, then

$$\left[\underline{S_1 \dot{\cup} S_2}, \mathcal{C}\right] \simeq \left[\underline{S_1}, \mathcal{C}\right] \times \left[\underline{S_2}, \mathcal{C}\right].$$

(vi) If $\varphi : S \to T$ is a G-map, then we have a functor $\underline{\varphi} : \underline{S} \to \underline{T}$ given by $s \mapsto \varphi(s), (g,s) \mapsto (g, \varphi(s))$ and hence a functor $\underline{\varphi}^* : [\underline{T}, \mathcal{C}] \to [\underline{S}, \mathcal{C}]$ given by $\zeta \mapsto \zeta\underline{\varphi} = \varphi^*(\zeta)$, the bundle ζ restricted to S via φ.

Now, if $\mathcal{C}$ has finite sums, then φ also induces a functor $\varphi_* : [\underline{S}, \mathcal{C}] \to [T, \mathcal{C}]$ defined as follows: if $\zeta \in [\underline{S}, \mathcal{C}]$, define $\varphi_*(\zeta) = \zeta_*$ where $\zeta_{*t} = \coprod_{s \in \varphi^{-1}(t)} \zeta_s$ and

$$\zeta_{*(g,t)} = \coprod_{s \in \varphi^{-1}(t)} \zeta_{(g,s)} : \zeta_{*t} = \coprod_{s \in \varphi^{-1}(t)} \zeta_s \to \coprod_{s \in \varphi^{-1}(t)} \zeta_{gs} = (\zeta_*)_{gt}$$

and for a morphism $\mu : \zeta \to \eta$ in $[\underline{S}, \mathcal{C}]$ define $\varphi_*(\mu) = \mu_* : \zeta_* \mapsto \eta_*$ in $[\underline{T}, \mathcal{C}]$ by $\mu_*(t) = \coprod_{s \in \varphi^{-1}(t)} \mu(s) : \zeta_{*t} \mapsto \eta_{*t}$.

Similarly, if $\mathcal{C}$ has finite products, then φ induces a functor $\check{\varphi}_* : [\underline{S}, \mathcal{C}] \to [T, \mathcal{C}]$ where the fibres of $\check{\varphi}_*(\zeta)$ are defined by $\check{\varphi}_*(\zeta)_t = \prod_{s \in \varphi^{-1}(t)} \zeta_s (t \in T)$ and the G-action is defined accordingly.

(vii) In (vi) above, we saw that if $\mathcal{C}$ is a category with finite sums (resp. products), G a finite group, and $\varphi : S \to T$ a G-map, then we have functors $\varphi^* : \mathcal{C}^{\underline{T}} =: [\underline{T}, \mathcal{C}] \to \mathcal{C}^{\underline{S}} =: [\underline{S}, \mathcal{C}]$ and φ_* (resp. $\check{\varphi}_*$):$[\underline{S}, \mathcal{C}] \to [\underline{T}, \mathcal{C}]$.
We now realize that φ_* ($\check{\varphi}_*$) is the left (right) adjoint of φ^*, i.e., that $\mathcal{C}^{\underline{S}}(\zeta, \varphi^*(\eta)) \cong \mathcal{C}^{\underline{T}}(\varphi_*(\zeta), \eta)$ (resp. $\mathcal{C}^{\underline{S}}(\varphi^*(\eta), \zeta) \cong \mathcal{C}^{\underline{T}}(\eta, \check{\varphi}_*(\zeta))$.
This isomorphism is given by associating to each $\mu : \zeta \to \varphi^*\eta$ (resp. $\mu : \varphi^*(\eta) \to \zeta$) (i.e., to any family of maps $\mu(s) : \zeta_s \to \eta_{\varphi(s)}$ (resp. $\mu(s) : \eta_{\varphi(s)} \to \zeta_s$) compatible with the G-action) the morphism

$$\mu' : \varphi_*(\zeta) \to \eta \qquad (\mu' : \eta \to \check{\varphi}_*(\zeta))$$

given by

$$\mu'(t) = \coprod_{s \in \varphi^{-1}(t)} \mu(s) : \coprod_{s \in \varphi^{-1}(t)} \zeta_s \to \eta_t \quad /$$

$$\left(\text{resp.} \mu'(t) = \prod_{s \in \varphi^{-1}(t)} \mu(s) : \eta_t \to \prod_{s \in \varphi^{-1}(t)} \zeta_s\right).$$

1.2 Grothendieck group associated with a semi-group

1.2.1 Let $(A, +)$ be an Abelian semi-group. Define a relation '$\sim$'on $A \times A$ by $(a,b) \sim (c,d)$ if there exists $u \in A$ such that $a + d + u = b + c + u$. One can easily check that '$\sim$' is an equivalence relation. Let $\overline{A}$ denote the set of equivalence classes of '$\sim$', and write $[a, b]$ for the class of (a, b) under '$\sim$'. We define addition $(\dot{+})$ on $\overline{A}$ by $[a, b]\dot{+}[c, d] = [a + c, b + d]$. Then $(\overline{A}, \dot{+})$ is an Abelian group in which the identity element is $[a, a]$ and the inverse of $[a, b]$ is $[b, a]$.

Moreover, there is a well-defined additive map $f : A \to \overline{A} : a \to [a + a, a]$ which is, in general, neither injective nor surjective. However, f is injective iff A is a cancellation semi-group, i.e., iff $a + c = b + c$ implies that $a = b$ for all $a, b \in A$ (see [95]).

1.2.2 It can be easily checked that $\overline{A}$ possesses the following universal property with respect to the map $f : A \to \overline{A}$. Given any additive map $h : A \to B$ from A to an Abelian group B, then there exists a unique map $g : \overline{A} \to B$ such that $h = gf$.

Definition 1.2.1 *$\overline{A}$ is usually called the Grothendieck group of A or the group completion of A and denoted by $K(A)$.*

Remarks 1.2.1 (i) The construction of $K(A) = \overline{A}$ above can be shown to be equivalent to the following:
Let $(F(A), \dot{+})$ be the free Abelian group freely generated by the element of A, and $R(A)$ the subgroup of $F(A)$ generated by all elements of the form $a\dot{+}b - (a + b)$, $a, b \in A$. Then $K(A) \simeq F(A)/R(A)$.

(ii) If A, B, C are Abelian semi-groups together with bi-additive map $f : A \times B \to C$, then f extends to a unique bi-additive map $\overline{f} : \overline{A} \times \overline{B} \to \overline{C}$ of the associated Grothendieck groups. If A is a semi-ring, i.e., an additive Abelian group together with a bi-additive multiplication $A \times A \to A\,(a, b) \to ab$, then the multiplication extends uniquely to a multiplication $\overline{A} \times \overline{A} \to \overline{A}$, which makes $\overline{A}$ into a ring (commutative if A is commutative) with identity $1 = [1 + 1, 1]$ in $\overline{A}$ if $1 \in A$.

(iii) If B is a semi-module over a semi-ring A, i.e., if B is an Abelian semi-group together with a bi-additive map $A \times B \to B : (a, b) \to a \cdot b$ satisfying $a'(ab) = (a'a)b$ for $a, a' \in A$, $b \in B$, then the associated Grothendieck group $\overline{B}$ is an $\overline{A}$-module.

(iv) If $A = \{1, 2, 3, \ldots\}$, $\overline{A} = K(A) = \mathbb{Z}$. Hence the construction in 1.2.1 is just a generalization of the standard procedure of constructing integers from the natural numbers.

(v) A sub-semi-group A of an Abelian semi-group B is said to be cofinal in B if for any $b \in B$, there exists $b' \in B$ such that $b + b' \in A$. It can be easily checked that $K(A)$ is a subgroup of $K(B)$ if A is cofinal in B.

1.2.3 K_0 **of a ring.** For any ring A with identity, let $\mathcal{P}(\Lambda)$ be the category of finitely generated projective Λ-modules. Then the isomorphism classes $I\mathcal{P}(\Lambda)$

of objects of $\mathcal{P}(\Lambda)$ form an Abelian semi-group under direct sum '$\oplus$'. We write $K_0(\Lambda)$ for $K(I\mathcal{P}(\Lambda))$ and call $K_0(\Lambda)$ the Grothendieck group of Λ. For any $P \in \mathcal{P}(\Lambda)$, we write (P) for the isomorphism class of P (i.e., an element of $I\mathcal{P}(\Lambda)$) and $[P]$ for the class of (P) in $K_0(\Lambda)$.

If Λ is commutative, then $I\mathcal{P}(\Lambda)$ is a semi-ring with tensor product $\otimes_\Lambda$ as multiplication, which distributes over '$\oplus$'. Hence $K_0(\Lambda)$ is a ring by remarks 1.2.1(ii).

Remarks 1.2.2 (i) $K_0 : \mathcal{R}ings \to \mathcal{A} : \Lambda \to K_0(\Lambda)$ is a functor — since any ring homomorphism $f : \Lambda \to \Lambda'$ induces a semi-group homomorphism $I\mathcal{P}(\Lambda) \to I\mathcal{P}(\Lambda') : P \to P \otimes \Lambda'$ and hence a group homomorphism $K_0(\Lambda) \to K_0(\Lambda')$.

(ii) K_0 is also a functor: $\mathcal{CR}ings \to \mathcal{CR}ings$.

(iii) $[P] = [Q]$ in $K_0(\Lambda)$ iff P is stably isomorphic to Q in $\mathcal{P}(\Lambda)$, i.e., iff $P \oplus \Lambda^n \simeq Q \oplus \Lambda^n$ for some integer n. In particular $[P] = \Lambda^n]$ for some n iff P is stably free (see [17, 20]).

Examples 1.2.1 (i) If Λ is a field or a division ring or a local ring or a principal ideal domain, then $K_0(\Lambda) \simeq \mathbb{Z}$.

Note. The proof in each case is based on the fact that any finitely generated Λ-module is free and Λ satisfies the invariant basis property (i.e., $\Lambda^r \simeq \Lambda^s \Rightarrow r = s$). So, $I\mathcal{P}(\Lambda) \simeq \{1, 2, 3, \ldots\}$, and so, $K_0(\Lambda) \simeq \mathbb{Z}$ by remarks 1.2.1(iv) (see [17] or [181]).

(ii) Any element of $K_0(\Lambda)$ can be written as $[P]-r[\Lambda]$ for some integer $r > 0$, $P \in \mathcal{P}(\Lambda)$, or as $s[\Lambda] - [Q]$ for some $s > 0$, $Q \in \mathcal{P}(\Lambda)$ (see [20, 211]). If we write $\tilde{K}_0(\Lambda)$ for the quotient of $K_0(\Lambda)$ by the subgroup generated by $[\Lambda]$, then every element of $\tilde{K}_0(\Lambda)$ can be written as $[P]$ for some $P \in \mathcal{P}[\Lambda]$ (see [20] or [224]).

(iii) If $\Lambda \simeq \Lambda_1 \times \Lambda_2$ is a direct product of two rings Λ_1, Λ_2, then $K_0(\Lambda) \simeq K_0(\Lambda_1) \times K_0(\Lambda_2)$ (see [17] for a proof).

(iv) Let G be a semi-simple connected affine algebraic group over an algebraically closed field. Let A be the coordinate ring of G. Then $K_0(A) \simeq \mathbb{Z}$.

Remarks. See [188] for a proof of this result, which says that all algebraic vector bundles on G are stably trivial. The result is due to A. Grothendieck.

(v) $K_0(k[x_0, x_1, \ldots, x_n]) \simeq \mathbb{Z}$. This result is due to J.P. Serre (see [188]).

Remarks 1.2.3 Before providing more examples of Grothendieck group constructions, we present in the next section 1.3 a generalization of 1.2.1 in the context of K_0 of symmetric monoidal categories.

1.3 K_0 of symmetric monoidal categories

Definition 1.3.1 *A symmetric monoidal category is a category $\mathcal{C}$ equipped with a functor $\perp: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a distinguished object "0" such that $\perp$ is "coherently associative and commutative" in the sense of Maclane (see [139]), that is,*

(i) $A \perp 0 \simeq A \simeq 0 \perp A\,.$

(ii) $A \perp (B \perp C) \simeq (A \perp B) \perp C\,.$

(iii) $A \perp B \simeq B \perp A$ *for all* $A, B, C \in \mathcal{C}\,.$

Moreover, the following diagrams commute.

(i)
$$\begin{tikzcd}
(A \perp (0 \perp B)) \arrow[r, "\sim"] \arrow[d, "\wr"] & (A \perp 0) \perp B \arrow[d, "\wr"] \\
A \perp B \arrow[r, "\sim"] & B \perp A
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
A \perp 0 \arrow[rr, "\sim"] \arrow[dr, "\sim"] & & 0 \perp A \arrow[dl, "\sim"'] \\
& A &
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
A \perp (B \perp (C \perp D)) \arrow[r, "\sim"] \arrow[d, "\wr"] & (A \perp B) \perp (C \perp D) \arrow[d, "\wr"] \\
A \perp ((B \perp (C \perp D) \arrow[d, "\wr"] & ((A \perp B) \perp C) \perp D \arrow[dl, "\sim"'] \\
(A \perp (B \perp C)) \perp D &
\end{tikzcd}$$

Let $I\mathcal{C}$ be the set of isomorphism classes of objects of $\mathcal{C}$. Clearly, if $\mathcal{C}$ is small, then $(I\mathcal{C}, \perp)$ is an Abelian semi-group (in fact a monoid), and we write $K_0^{\perp}(\mathcal{C})$ for $K(I\mathcal{C}, \perp)$ or simply $K_0(\mathcal{C})$ when the context is clear.

In other words, $K_0^{\perp}(\mathcal{C}) = F(\mathcal{C})/R(\mathcal{C})$ where $F(C)$ is the free Abelian group on the isomorphism classes $(\mathcal{C})$ of $\mathcal{C}$-objects, and $R(\mathcal{C})$ the subgroup of $F(\mathcal{C})$ generated by $(C' \perp C'') - (C') - (C'')$ for all C', C'' in $\mathrm{ob}(\mathcal{C})$.

Remarks 1.3.1 (i) $K_0^{\perp}(\mathcal{C})$ satisfies a universal property as in 1.2.2.

(ii) If $\mathcal{C}$ has another composition 'o' that is associative and distributive with respect to $\perp$, then $K_0^{\perp}(\mathcal{C})$ can be given a ring structure through 'o' as multiplication and we shall sometimes denote this ring by $K_0^{\perp}(\mathcal{C}, \perp, o)$ or $K_0(\mathcal{C}, \perp, o)$ or just $K_0(\mathcal{C})$ if the context is clear.

Examples 1.3.1 (i) If Λ is any ring with identity, then $(\mathcal{P}(\Lambda), \oplus)$ is a symmetric monoidal category (s.m.c.) and $K_0^{\oplus}(\Lambda) = K_0(\Lambda)$ as in 1.2.5. If Λ is commutative, then $K_0^{\oplus}(\Lambda)$ is a ring where $(\mathcal{P}(\Lambda), \oplus)$ has the further composition '$\otimes$'.

(ii) Let $F\mathcal{S}et$ be the category of finite sets, $\dot{\cup}$ the disjoint union. Then $(F\mathcal{S}et, \dot{\cup})$ is a symmetric monoidal category and $K_0^{\dot{\cup}}(F\mathcal{S}et) \simeq \mathbb{Z}$.

(iii) Let G be a finite group, $\mathcal{C}$ any small category. Let $\mathcal{C}_G$ be the category of G-objects in $\mathcal{C}$, or equivalently, the category of G-representations in $\mathcal{C}$, i.e., objects of $\mathcal{C}_G$ are pairs $(X, U : G \to \mathrm{Aut(X)})$ where $X \in \mathrm{ob}(\mathcal{C})$ and U is a group homomorphism from G to the group of $\mathcal{C}$-automorphism of X. If $(\mathcal{C}, \perp)$ is a symmetric monoidal category, so is $(\mathcal{C}_G, \dot{\perp})$ where for

$$(X, U : G \to \mathrm{Aut}(X)), \qquad (X', U' : G \to \mathrm{Aut}(X'))$$

in $\mathcal{C}_G$, we define

$$(X, U)\dot{\perp}(X', U') := (X \perp X', U \perp U' : G \to \mathrm{Aut}(X \perp X')),$$

where $U \perp U'$ is defined by the composition

$$G \xrightarrow{U \perp U'} \mathrm{Aut}(X) \times \mathrm{Aut}(X') \to \mathrm{Aut}(X \perp X').$$

So we obtain the Grothendieck group $K_0^{\dot{\perp}}(\mathcal{C}_G)$.

If $\mathcal{C}$ possesses a further associative composition 'o' such that $\mathcal{C}$ is distributive with respect to $\perp$ and 'o', then so is $\mathcal{C}_G$, and hence $K_0^{\dot{\perp}}(\mathcal{C}_G)$ is a ring.

Examples 1.3.2 (a) If $\mathcal{C} = \mathcal{P}(R)$, $\perp = \oplus$, 'o'$= \otimes_R$ where R is a commutative ring with identity, then $\mathcal{P}(R)_G$ is the category of RG-lattices (see [39]), and $K_0(\mathcal{P}(R)_G)$ is a ring usually denoted by $G_0(R, G)$. Observe that when $R = \mathbb{C}$, $G_0(\mathbb{C}, G)$ is the usual representation ring of G denoted in the literature by $R(G)$.

(b) If $\mathcal{C} = F\mathcal{S}ets$, '$\perp$'= disjoint union, '$o$'= Cartesian product. Then $K_0(\mathcal{C}_G)$ is the Burnside ring of G usually denoted by $\Omega(G)$. See 9.3 and 9.4 for a detailed discussion of Burnside rings.

(iv) Let G be a finite group, S a G-set. As discussed in 1.1.3, we can associate with S a category $\mathbf{S}$ as follows: $\mathrm{ob}(\mathbf{S}) = \{s | s \in S\}$. For $s, t \in S$, $\mathrm{Hom}_{\mathbf{S}}(s,t) = \{(g,s) | s \in G, gs = t\}$ where the composition is defined for $t = gs$ by $(h,t)\cdot(g,s) = (hg,s)$, and the identity morphism $s \to s$ is given by (e,s) where e is the identity element of G. Now let $(\mathcal{C}, \perp)$ be a symmetric monoidal category and let $[\mathbf{S}, \mathcal{C}]$ be the category of covariant functors $\zeta : \mathbf{S} \to \mathcal{C}$. The $([\mathbf{S}, \mathcal{C}], \dot{\perp})$ is also a symmetric monoidal category where $(\zeta \dot{\perp} \eta)_{(g,s)} : \zeta_s \perp \eta_s \to \zeta_{gs} \perp \eta_{gs}$. We write $K_0^G(S, \mathcal{C})$ for the Grothendieck group of $[\mathbf{S}, \mathcal{C}]$.

If $(\mathcal{C}, \perp)$ possesses an additional composition 'o' that is associative and distributive with respect to '$\perp$', then $K_0(\mathbf{S}, \mathcal{C})$ can be given a ring structure (see [111]). As we shall see in chapter 9, for any symmetric monoidal category $(\mathcal{C}, \perp)$, $K_0^G(-, \mathcal{C}) : G\mathcal{S}et \to \mathcal{A}b$ is a 'Mackey functor' (see example 9.1.1(iv)), and when $\mathcal{C}$ possesses an additional composition 'o' discussed before, then $K_0^G(-, \mathcal{C}) : G\mathcal{S}et \to \mathcal{A}b$ is a 'Green functor' (see example 9.1.1(iv)).

(v) Suppose that G, H are finite groups, and $\theta : H \to G$ a group homomorphism. By restricting the action of G on a G-set S to H via θ, one defines a functor $\widehat{\theta} : G\mathcal{S}et \to H\mathcal{S}et$, which can easily be checked to commute with finite sums, products, and pullbacks (and more generally, with limit and colimits). Moreover, by restricting the action of G on G-equivariant $\mathcal{C}$-bundle ζ over S to H through θ, we have a natural transformation of functors from $K_0^G(-, \mathcal{C}) : G\mathcal{S}et \to \mathbb{Z}\text{-}\mathcal{M}od$ and $K_0^H \circ \theta : G\mathcal{S}et \to H\mathcal{S}et \to \mathbb{Z}\text{-}\mathcal{M}od$. In particular, if $H \leq G$, T an H-set, we have a homomorphism $K_0^G(G \underset{H}{\times} T, \mathcal{C}) \to K_0^H(T, \mathcal{C})$ where the second map is induced by $T \to G \underset{H}{\times} T : t \to (e,t)$. We now observe that

$$K_0^G\left(G \underset{H}{\times} T, \mathcal{C}\right) \to K_0^H(T, \mathcal{C}) \tag{I}$$

is an isomorphism since by 1.1.5 iii(c) $[\underline{G \underset{H}{\times} T}, \mathcal{C}] \to [\underline{T}, \mathcal{C}]$ is an equivalence of categories.

Note that if $T = H/K$ for some subgroup $K \leq H$, we have $G \underset{H}{\times} T = G/K$, and the above isomorphism (I) is the map $K_0^G(G/K, \mathcal{C}) \to K_0(H/K, \mathcal{C})$ defined by restricting a $\mathcal{C}$-bundle over G/K to H/K and the action of G to H at the same time, i.e., the map $K_0^G(G/K, \mathcal{C}) \to K_0(H/K, \mathcal{C})$ is defined by the functor $\underline{H/K} \to \underline{G/K}$.

(vi) Let X be a compact topological space and for $= \mathbb{R}$ or $\mathbb{C}$, let $\mathbf{VB}_F(X)$ be the (symmetric monoidal) category of (finite-dimensional) vector bundles on X. Then $I\!\mathbf{VB}_F(X)$ is an Abelian monoid under Whitney sum '$\oplus$'. It is usual to write $KO(X)$ for $K_0^{\oplus}(\mathbf{VB}_{\mathbb{R}}(X))$ and $KU(X)$ for $K_0^{\oplus}(\mathbf{VB}_{\mathbb{C}}(X))$. Note that if X, Y are homotopy equivalent, then $KO(X) = KO(Y)$ and $KU(X) = KU(Y)$. Moreover, if X is contractible, we have $KO(X) = KU(X) = \mathbb{Z}$ (see [10] or [95]).

Let X be a compact space, $\mathbb{C}(X)$ the ring of $\mathbb{C}$-valued functions on X. By a theorem of R.G. Swan [208, 214], there exists an equivalence of categories $\Gamma : VB_{\mathbb{C}}(X) \to \mathcal{P}(\mathbb{C}(X))$ taking a vector bundle $E \xrightarrow{p} X$ to $\Gamma(E)$, where $\Gamma(E) = \{\text{sections } s : X \to E | ps = 1\}$. This equivalence induces a group isomorphism $KU(X) \simeq K_0(\mathbb{C}(X))$ (I).

The isomorphism (I) provides the basic initial connection between algebraic K-theory (right-hand side of (I)) and topological K-theory (left-hand side of (I)) since the K-theory of $\mathcal{P}(\Lambda)$ for an arbitrary ring Λ could be studied instead of the K-theory of $\mathcal{P}(\mathbb{C}(X))$.

Now, $\mathcal{C}(X)$ is a commutative C^*-algebra, and the Gelfand–Naimark theorem [35] says that any commutative C^* algebra Λ has the form $\Lambda = \mathbb{C}(X)$ for some locally compact space X. Indeed, for any commutative C^*-algebra Λ, we could take X as the spectrum of Λ, i.e., the set of all nonzero homomorphisms from Λ to $\mathbb{C}$ with the topology of pointwise convergence. Noncommutative geometry is concerned with the study of noncommutative C^*-algebras associated with "noncommutative" spaces and K-theory (algebraic and topological) of such C^*-algebras has been extensively studied and connected to some (co)homology theories (see, e.g., Hochschild and cyclic (co)homology theories) of such algebras through Chern characters (see, e.g., [35, 43, 44, 136]).

(vii) Let G be a group acting continuously on a topological space X. The category $VB_G(X)$ of complex G-vector bundles on X is symmetric monoidal under Whitney sum '$\oplus$', and we write $K_G^0(X)$ for the Grothendieck group $K_0(VB_G(X))$. If X is a point, $VB_G(X)$ is the category of representations of G in $\mathcal{P}(\mathbb{C})$ and $K_G^0(X) = R(G)$, the representation ring of G.

If G acts trivially on X, then $K_G^0(X) \simeq KU(X) \otimes_{\mathbb{Z}} R(G)$ (see [184, 186]).

(viii) Let R be a commutative ring with identity. Then $\mathrm{Pic}(R)$, the category of finitely generated projective R-modules of rank one (or equivalently the category of algebraic line bundles L over R) is a symmetric monoidal category, and $K_0^{\otimes}(\mathrm{Pic}(R)) = \mathrm{Pic}(R)$, the Picard group of R.

(ix) The category $\mathrm{Pic}(X)$ of line bundles on a locally ringed space is a symmetric monoidal category under '$\otimes$', and $K_0^{\otimes}(\mathrm{Pic}(X)) := \mathrm{Pic}(X)$ is called the Picard group of X. Observe that when $X = \mathrm{Spec}(R)$, we recover $\mathrm{Pic}(R)$ in (viii). It is well known that $\mathrm{Pic}(X) \simeq H^1(X, O_X^*)$ (see [79] or [198]).

The significance of discussing ringed spaces and schemes in this book lies in the fact that results on affine schemes are results on commutative rings and hence apply to commutative orders and grouprings.

(x) Let R be a commutative ring with identity. An R-algebra Λ is called an Azumaya algebra if there exists another R-algebra Λ' such that $\Lambda \otimes_R \Lambda' \simeq M_n(R)$ for some positive integer n. Let $\mathrm{Az}(R)$ be the category of Azumaya algebras. Then $(\mathrm{Az}(R), \otimes_R)$ is a symmetric monoidal category. Moreover, the category $F\mathcal{P}(R)$ of faithfully projective R-modules is symmetric monoidal with respect to $\perp = \otimes_R$ if the morphisms in $F\mathcal{P}(R)$ are restricted to isomorphisms. There is a monoidal functor $F\mathcal{P}(R) \to \mathrm{Az}(R) : P \to \mathrm{End}(P)$ inducing a group homomorphism $K_0(F\mathcal{P}(R)) \xrightarrow{\varphi} K_0(\mathrm{Az}(R))$. The cokernel of φ is called the Brauer group of R and is denoted by $\mathrm{Br}(R)$. Hence $\mathrm{Br}(R)$ is the Abelian group generated by isomorphism classes of central simple F-algebras with relations $[\Lambda \otimes \Lambda'] = [\Lambda] + [\Lambda']$ and $[M_n(F)] = 0$ (see [181]).

(xi) Let A be an involutive Banach algebra and $\mathrm{Witt}(A)$ the group generated by isomorphism classes $[Q]$ of invertible Hermitian forms Q on $P \in \mathcal{P}(A)$ with relations $[Q_1 \oplus Q_2] = [Q_1] + [Q_2]$ and $[Q] + [-Q] = 0$. Define a map $\varphi : K_0(A) \to \mathrm{Witt}(A)$ by $[P] \mapsto$ class of (P, Q) with Q positive. If A is a C^*-algebra with 1, then there exists on any $P \in \mathcal{P}(A)$ an invertible form Q satisfying $Q(x, x) \geq 0$ for all $x \in P$ and in this case $\varphi : K_0(A) \to \mathrm{Witt}(A)$ is an isomorphism. However, φ is not an isomorphism in general for arbitrary involutive Banach algebras (see [35]).

(xii) Let F be a field and $\mathrm{Sym}B(F)$ the category of symmetric inner product spaces (V, β) – V a finite-dimensional vector space over F and $\beta : V \otimes V \to F$ a symmetric bilinear form. Then $(\mathrm{Sym}B(F), \perp)$ is a symmetric monoidal category where $(V, \beta) \perp (V', \beta')$ is the orthogonal sum of (V, β) and (V', β') is defined as the vector space $V \oplus V'$ together with a bilinear form $\beta^* : (V \oplus V', V \oplus V') \to F$ given by $\beta^*(v \oplus v', v_1 \oplus v_1') = \beta(v, v_1) + \beta'(v', v_1')$.

If we define composition $(V, \beta) \odot (V', \beta')$ as the tensor product $V \otimes V'$ together with a bilinear form $\beta^*(v \otimes v', v_1 \otimes v_1') = \beta(v, v_1)\beta'(v', v_1')$, then $K_0(\mathrm{Sym}B(F), \perp, \odot)$ is a commutative ring with identity.

The Witt ring $W(F)$ is defined as the quotient of $K_0(\mathrm{Sym}B(F))$ by the subgroup $\{n, H\}$ generated by the hyperbolic plane $H = \left(F^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$.

For more details about $W(F)$ see [182].

(xiii) Let A be a ring not necessarily unital (i.e., not necessarily with identity). The ring A_+ obtained by adjoining a unit to A is defined as follows. As an Abelian group, $A_+ = A \oplus \mathbb{Z}$ with multiplication defined by $(a, r)(b, s) = (ab + rb + sa, rs)$ where $a, b \in A$, $r, s \in \mathbb{Z}$. Here the unit of A_+ is $(0, 1)$.

If A already has unit e, say, there is a unital isomorphism $\varphi : A_+ \to A \times \mathbb{Z}$ given by $\varphi(a, r) = (a + re, r)$.

If A is not-unital, there is a split exact sequence $0 \to A \to A_+ \to \mathbb{Z} \to 0$.

Define. $K_0(A) := \mathrm{Ker}(K_0(A_+) \to K_0(\mathbb{Z}) \simeq \mathbb{Z}$.

(xiv) For any Λ with identity, let $M_n(\Lambda)$ be the set of $n \times n$ matrices over Λ, and write $M(\Lambda) = \bigcup_{n=1}^{\infty} M_n(\Lambda)$. Also $GL_n(\Lambda)$ be the group of invertible $n \times n$ matrices over Λ and write $GL(\Lambda) = \bigcup_{n=1}^{\infty} GL_n(\Lambda)$. For $P \in \mathcal{P}(\Lambda)$ there exists $Q \in \mathcal{P}(\Lambda)$ such that $P \oplus Q \simeq \Lambda^n$ for some n. So, we can identify with each $P \in \mathcal{P}(\Lambda)$ an idempotent matrix $p \in M_n(\Lambda)$ (i.e., $p : \Lambda^n \to \Lambda^n$), which is the identity on P and '0' on Q.

Note that if p, q are idempotent matrices in $M(\Lambda)$, say $p \in M_r(\Lambda)$, $q \in M_s(\Lambda)$, corresponding to $P, Q \in \mathcal{P}(\Lambda)$, then $P \simeq Q$ iff it is possible to enlarge the sizes of p, q (by possibly adding zeros in the lower right-hand corners) such that p, q have the same size ($t \times t$, say) and are conjugate under the action of $GL_t(\Lambda)$ (see [181]).

Let $\mathrm{Idem}(\Lambda)$ be the set of idempotent matrices in $M(\Lambda)$. It follows from the last paragraph that $GL(\Lambda)$ acts by conjugation on $\mathrm{Idem}(\Lambda)$, and so, we can identify the semi-group $I\mathcal{P}(\Lambda)$ with the semi-group of conjugation orbits $(\mathrm{Idem}(\Lambda))\hat{}$ of the action of $GL(\Lambda)$ on $\mathrm{Idem}(\Lambda)$ where the semi-group operation is induced by $(p, q) \to \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$. $K_0(\Lambda)$ is the Grothendieck group of this semi-group $(\mathrm{Idem}(\Lambda))\hat{}$.

1.4 K_0 of exact categories – definitions and examples

Definition 1.4.1 *An exact category is an additive category $\mathcal{C}$ embeddable as a full subcategory of an Abelian category $\mathcal{A}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of short exact sequences $0 \to M' \to M \to M'' \to 0$ (I) satisfying*

(i) $\mathcal{E}$ is a class of sequences (I) in $\mathcal{C}$ that are exact in $\mathcal{A}$.

(ii) $\mathcal{C}$ is closed under extensions in $\mathcal{A}$, i.e., if (I) is an exact sequence in $\mathcal{A}$ and $M', M'' \in \mathcal{C}$, then $M \in \mathcal{C}$.

Definition 1.4.2 *For a small exact category $\mathcal{C}$, define the Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ as the Abelian group generated by isomorphism classes (C) of $\mathcal{C}$-objects subject to the relation $(C') + (C'') = (C)$ whenever $0 \to C' \to C \to C'' \to 0$ is an exact sequence in $\mathcal{C}$.*

Remarks 1.4.1 (i) $K_0(\mathcal{C}) \simeq \mathcal{F}/\mathcal{R}$ where $\mathcal{F}$ is the free Abelian group on the isomorphism classes (C) of $\mathcal{C}$-objects and $\mathcal{R}$ the subgroup of $\mathcal{F}$ generated by all $(C) - (C') - (C'')$ for each exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$. Denote by $[C]$ the class of (C) in $K_0(\mathcal{C}) = \mathcal{F}/\mathcal{R}$.

(ii) The construction satisfies the following property: If $\chi : \mathcal{C} \to A$ is a map from $\mathcal{C}$ to an Abelian group A given that $\chi(C)$ depends only on the isomorphism class of $\mathcal{C}$ and $\chi(C) = \chi(C') + \chi(C'')$ for any exact sequence $0 \to C' \to C \to C'' \to 0$, then there exists a unique $\chi' : K_0(\mathcal{C}) \to A$ such that $\chi(C) = \chi'([C])$ for any $\mathcal{C}$-object C.

(iii) Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between two exact categories $\mathcal{C}, \mathcal{D}$ (i.e., F is additive and takes short exact sequences in $\mathcal{C}$ to such sequences in $\mathcal{D}$). Then F induces a group homomorphism $K_0(\mathcal{C}) \to K_0(\mathcal{D})$.

(iv) Note that an Abelian category $\mathcal{A}$ is also an exact category and the definition of $K_0(\mathcal{A})$ is the same as in definition 1.4.2.

Examples 1.4.1 (i) Any additive category is an exact category as well as a symmetric monoidal category under '$\oplus$', and $K_0(\mathcal{C})$ is a quotient of the group $K_0^{\oplus}(\mathcal{C})$ defined in 1.3.1.

If every short exact sequence in $\mathcal{C}$ splits, then $K_0(\mathcal{C}) = K_0^{\oplus}(\mathcal{C})$. For example, $K_0(\Lambda) = K_0(\mathcal{P}(\Lambda)) = K_0^{\oplus}(\mathcal{P}(\Lambda))$ for any ring Λ with identity.

(ii) Let Λ be a (left) Noetherian ring. Then the category $\mathcal{M}(\Lambda)$ of finitely generated (left)-Λ-modules is an exact category and we denote $K_0(\mathcal{M}(\Lambda))$ by $G_0(\Lambda)$. The inclusion functor $\mathcal{P}(\Lambda) \to \mathcal{M}(\Lambda)$ induces a map $K_0(\Lambda) \to G_0(\Lambda)$ called the Cartan map. For example, $\Lambda = RG$ (R a Dedekind domain, G a finite group) yields a Cartan map $K_0(RG) \to G_0(RG)$.

If Λ is left Artinian, then $G_0(\Lambda)$ is free Abelian on $[S_1], \dots, [S_r]$ where the $[S_i]$ are distinct classes of simple Λ-modules, while $K_0(\Lambda)$ is free Abelian on $[I_1], \dots, [I_l]$ and the $[I_i]$ are distinct classes of indecomposable projective Λ-modules (see [39]). So, the map $K_0(\Lambda) \to G_0(\Lambda)$ gives a matrix a_{ij} where a_{ij} = the number of times S_j occurs in a composition series for I_i. This matrix is known as the Cartan matrix.

If Λ is left regular (i.e., every finitely generated left Λ-module has finite resolution by finitely generated projective left Λ-modules), then it is well known that the Cartan map is an isomorphism (see [215]).

(iii) Let R be a Dedekind domain with quotient field F. An important example of (ii) above is when Λ is an R-order in a semi-simple F-algebra Σ. Recall (see [39, 174]) that Λ is a subring of Σ such that R is contained in the centre of Λ, Λ is a finitely generated R-module, and $F \otimes_R \Lambda = \Sigma$. For example, if G is any finite group, then the group-ring RG is an R-order in the group algebra FG.

Recall also that a maximal R-order Γ in Σ is an order that is not contained in any other R-order. Note that Γ is regular (see [38, 39]). So, as in (ii) above, we have Cartan maps $K_0(\Lambda) \to G_0(\Lambda)$ and when Λ, is a maximal order, we have $K_0(\Lambda) \simeq G_0(\Lambda)$.

(iv) Let R be a commutative ring with identity, Λ an R-algebra. Let $\mathcal{P}_R(\Lambda)$ be the category of left Λ-lattices, i.e., Λ-modules that are finitely generated and projective as R-modules. Then $\mathcal{P}_R(\Lambda)$ is an exact category and we write $G_0(R,\Lambda)$ for $K_0(\mathcal{P}_R(\Lambda))$. If $\Lambda = RG$, G, a finite group, we write $\mathcal{P}_R(G)$ for$\mathcal{P}_R(RG)$ and also write $G_0(R,G)$ for $G_0(R,RG)$. If $M,N \in \mathcal{P}_R(\Lambda)$, then, so is $(M \otimes_R N)$, and hence the multiplication given in $G_0(R,G)$ by $[M][N] = (M \otimes_R N)$ makes $G_0(R,G)$ a commutative ring with identity.

(v) If R is a commutative regular ring and Λ is an R-algebra that is finitely generated and projective as an R-module (e.g., $\Lambda = RG$, G a finite group or R is a Dedekind domain with quotient field F, and Λ is an R-order in a semi-simple F-algebra), then $G_0(R,\Lambda) \simeq G_0(\Lambda)$.

Sketch of proof. Define a map $\varphi : G_0(R,\Lambda) \to G_0(\Lambda)$ by $\varphi[M] = [M]$. Then φ is a well-defined homomorphism. Now for $M \in \mathcal{M}(\Lambda)$, there exists an exact sequence $0 \to L \to P_{n-1} \xrightarrow{\varphi_{n-1}} P_{n-2} \to \cdots\ P_0 \to M \to 0$ where $P_i \in \mathcal{P}(\Lambda)$, $L \in \mathcal{M}(\Lambda)$. Now, since $\Lambda \in \mathcal{P}(R)$, each $P_i \in \mathcal{P}(R)$ and hence $L \in \mathcal{P}(R)$. So $L \in \mathcal{P}_R(\Lambda)$. Now define $\delta[M] = [P_0] - [P_1] + \cdots + (-1)^{n-1}[P_{n-1}] + (-1)^n[L] \in G_0(R,\Lambda)$. One easily checks that $\delta f = 1 = f\delta$.

(vi) Let G be a finite group, S a G-set, $\underline{S}$ the category associated to S (see 1.1.3), $\mathcal{C}$ an exact category, and $[\underline{S},\mathcal{C}]$ the category of covariant functors $\zeta : \underline{S} \to \mathcal{C}$. We write ζ_s for $\zeta(s)$, $s \in S$. Then, $[\underline{S},\mathcal{C}]$ is an exact category where the sequence $0 \to \zeta' \to \zeta \to \zeta'' \to 0$ in $[\underline{S},\mathcal{C}]$ is defined to be exact if $0 \to \zeta'_s \to \zeta_s \to \zeta''{}_s \to 0$ is exact in $\mathcal{C}$ for all $s \in S$. Denote by $K_0^G(S,\mathcal{C})$ the K_0 of $[\underline{S},\mathcal{C}]$. Then $K_0^G(-,\mathcal{C}) : G\mathcal{S}et \to Ab$ is a functor that can be seen to be a 'Mackey' functor. We shall prove this fact for $K_n^G(-,\mathcal{C}), n \geq 0$ in chapter 10 (see theorem 10.1.2).

As seen earlier in 1.1.5, if $\underline{S} = \underline{G/G}$, the $[\underline{G/G},\mathcal{C}] \simeq \mathcal{C}_G$ in the notation of 1.1.2. Also, constructions analogous to the one above will be done for G, a profinite group, in chapter 11, and compact Lie groups in chapter 12.

Now if R is a commutative Noetherian ring with identity, we have $[\underline{G/G},\mathcal{P}(R)] \simeq \mathcal{P}(R)_G \simeq \mathcal{P}_R(RG)$, and so, $K_0^G((\underline{G/G},\mathcal{P}(R)) \simeq G_0(R,G) \simeq G_0(RG)$. This provides an initial connection between K-theory of representations of G in $\mathcal{P}(R)$ and K-theory of the group ring RG. As observed in (iv) above $G_0(R,G)$ is also a ring.

In particular, when $R = \mathbb{C}$, $\mathcal{P}(\mathbb{C}) = \mathcal{M}(\mathcal{C})$, and $K_0(\mathcal{P}(\mathbb{C})_G \simeq G_0(\mathbb{C},G) = G_0(\mathbb{C}G) =$ the Abelian group of characters, $\chi : G \to \mathbb{C}$ (see [39]), as already observed in this chapter.

If the exact category $\mathcal{C}$ has a pairing $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which is naturally associative and commutative, and there exists $E \in \mathcal{C}$ such that $\langle E, M \rangle =$

$\langle M, E\rangle = M$ for all $M \in \mathcal{C}$, then $K_0^G(-,\mathcal{C})$ is a Green functor (see [52]), and we shall see also in 10.1.6 that for all $n \geq 0$, $K_n^G(-,\mathcal{C})$ is a module over $K_0^G(-,\mathcal{C})$.

(vii) Let k be a field of characteristic p, G a finite group. We write $a(G)$ for $K_0(\mathcal{M}(kG))$. Let H be a subgroup of G.

A sequence $0 \to M' \to M \to M'' \to 0$ (I) of modules in $\mathcal{M}(kG)$ is said to be H-split if upon restriction to H, (I) is a split exact sequence. Write $a(G,H)$ for K_0 of the exact category $\mathcal{M}(kG)$ with respect to the collection of H-split exact sequences. For $X, Y \in \mathcal{M}(kG)$, $X \otimes_R Y \in M(kG)$, with $g(x \otimes y) = gx \otimes gy$, $g \in G$. If $0 \to M' \to M \to M'' \to 0$ is an H-split exact sequence in $\mathcal{M}(kG)$, so is $0 \to M' \otimes X \to M \otimes X \to M'' \otimes X \to 0$. If we put $[M][X] = [M \otimes_R X]$, then $a(G,H)$ is a commutative ring with identity element $[1_G]$. Call $a(G,H)$ the relative Grothendieck ring with respect to H. This ring has been well studied (see [128, 129, 130]).
For example, if $H = 1$, then $a(G,1)$ is $\mathbb{Z}$-free on $[F_1],\ldots,[F_s]$ where $\{[F_i]\}$ is a finite set of non-isomorphic irreducible G-modules. Also $a(G,1) \simeq \sum_{i=1}^{s} \mathbb{Z}\varphi_i$ where $\{\varphi_i\}$ are the irreducible Brauer characters of G relative to k. Also $a(G,1)$ contains no non-zero nilpotent element. If $H = G$, $a(G,G)$ is a free $\mathbb{Z}$-module spanned by the indecomposable modules, and $a(G,G)$ is called the representation ring of kG.

(viii) Let $H \leq G$. A module $N \in \mathcal{M}(kG)$ is (G,H)-projective if every exact sequence $0 \to M' \to M \to N \to 0$ of kG-modules that is H-split is also G-split. Note that N is (G,H)-projective iff N is a direct summand of some induced module $V^G := kG \otimes_{kH} V$ where $V \in \mathcal{M}(kH)$.

- Let $\mathcal{P}_H :=$ category of all (G,H)-projective modules $P \in \mathcal{M}(kG)$, $\mathcal{E}_H :=$ collection of H-split (and hence G-split) sequences in $\mathcal{P}_H$. Let $\underline{p}(G,H)$ be the K_0 of the exact category $\mathcal{P}_H$ with respect to $\mathcal{E}_H$.
  Then, $\underline{p}(G,G) = a(G)$, $\underline{p}(G,H)$ is an ideal of $a(G)$, and $\underline{p}(G,H)$ is $\mathbb{Z}$-free on the indecomposable projective kG-modules. If $i(G,H)$ is the additive subgroup of $a(G)$ generated by all $[M]-[M']-[M'']$ where $0 \to M' \to M \to M'' \to 0$ ranges over all H-split exact sequences of kG-modules, then $i(G,H)$ is an ideal of $a(G)$ and $a(G,H) \simeq a(G)/i(G,H)$.

Also we have the Cartan map $\underline{p}(G,H) \to a(G,H)$ defined by $\underline{p}(G,H) \hookrightarrow a(G) \to a(G)/i(G,H) = a(G,H)$.

(ix) We have the following generalization of (vii) and (viii) above (see [49]). Let G be a finite group, S a G-set, k a field of characteristic p. Then