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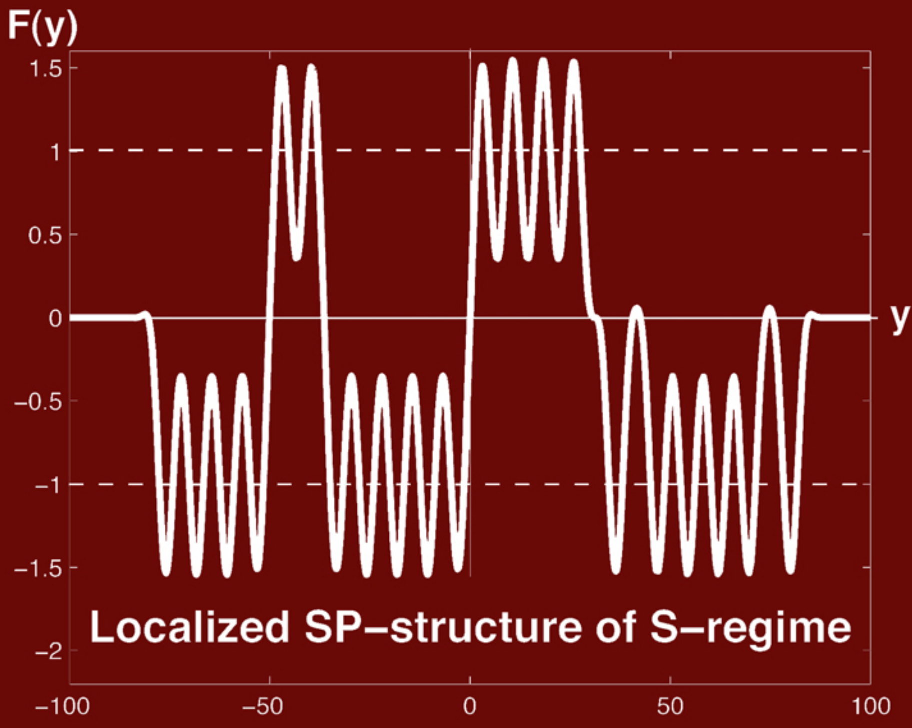

Victor A. Galaktionov
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To our teacher, Sergey Pavlovich Kurdyumov

Contents

Introduction: Nonlinear Partial Differential Equations and Exact Solutions

Exact solutions: history, classical symmetry methods, extensions

One of the crucial problems in the theory of partial differential equations (PDEs) at its early stages in the eighteenth and nineteenth century was finding and studying classes of important equations that were integrable in closed form and, in particular, possessed explicit solutions. It seems that the first general type of explicit solutions were traveling waves in d'Alembert's formula for the linear wave equation. The method of separation of variables was developed by Fourier in the study of heat conduction problems, and was later generalized and extended by Sturm and Liouville in the 1830s. Many famous mathematicians, such as Euler, Lagrange, Liouville, Sturm, Laplace, Darboux, Bäcklund, Lie, Jacobi, Boussinesq, Goursat, and others developed various techniques for obtaining explicit solutions of a variety of linear and nonlinear models from physics and mechanics. Their methods included a number of particular transformations, symmetries, expansions, separation of variables, etc. Similarity solutions appeared in the works by Weierstrass around 1870, and by Bolzman around 1890. After the Blasius construction (1908) of the exact self-similar solution for the two-dimensional (2D) boundary layer equations proposed by Prandtl in 1904, similarity solutions of linear and nonlinear boundary-value problems became more common in the literature. General principles for finding solutions of systems of ODEs and PDEs by symmetry reductions date back to the famous Lie papers [389]–[393] published in the 1880s and 1890s.

In the first half of the twentieth century, the basic priorities in PDE theory were re-evaluated in light of the influence of mathematical physics. As a result of this, and possibly in view of the essential progress achieved in existence-uniqueness-regularity theory for classes of PDEs of different types, explicit solutions gradually began to lose their exceptional role. At that time, many results and techniques on explicit integration were forgotten. On the other hand, in the 1930s, and especially in the 1940s and 1950s, exact solutions and similarity reductions returned to the scene in the asymptotic and singularity analysis of difficult practical problems of gas and hydrodynamics which appeared in many fundamental technological, industrial, and military areas in different countries. In the 1930s, the first basic ideas and results in this area were due to von Mises, von Kármán, Bechert, Guderley, Sedov (in the 1940s), and others, who applied scaling and similarity techniques to the study of complicated nonlinear models and singularity phenomena. These gas and hydrodynamic models included systems of several nonlinear PDEs, for many of which a rigorous mathematical analysis remains elusive, even now. The exact similarity solutions were the only possible way to detect crucial features of nonstationary and singular evolution, such as focusing of spherical waves in gas dynamics and shock-wave phenomena. In light of this, it was no accident that the gas dynamic and hydrodynamic equations became the first applications of new general ideas and methods of the group analysis of the PDEs, which Ovsiannikov began to develop in the 1950s. On the basis of Lie groups, he proposed a general approach to invariant and partially invariant solutions of nonlinear PDEs. A notion of group-invariant solutions, including special cases of traveling waves and similarity patterns, was emphasized by Birkhoff on the basis of hydrodynamic problems in the 1940s.

In the second half of the twentieth century, the increase of interest in exact solutions and exactly solvable models was two-fold. Firstly, the applied areas related to modern physics, mechanics and technology induced more and more complicated models dealing with systems of nonlinear PDEs. In this context, it is worth mentioning the new theory of weak solutions of nonlinear degenerate porous medium equations initiated in the 1950s (uniqueness approaches dated back to classical Holmgren's method, 1901), and self-focusing in nonlinear optics described by blow-up solutions of the nonlinear Schrödinger equation in the beginning of the 1960s. Secondly, the effective development in the 1960s and 1970s of the method for the exact integration of nonlinear PDEs, such as the inverse scattering method and Lax pairs introduced an exceptional class of fully integrable evolution equations possessing countable sets of exact solutions, such as N-solitons.

It seems that the beginning of the twenty-first century may be characterized in a manner similar to the 1950s. At that time, the complexity of many nonlinear PDE models of principal interest rose so high that one could not expect a mathematically rigorous existence-regularity theory to be created soon. For instance, there are many fundamental open problems in the theory of higher-order multi-dimensional quasilinear thin film equations, higher-order KdV-type PDEs with nonlinear dispersion possessing compacton, peakon and cuspon-type solutions, quasilinear degenerate wave equations and systems including equations of general relativity. Modern PDE theory proposes a number of new canonical higher-order models, to which many classical techniques do not apply in principle. In these and other difficult areas of general PDE theory, exact solutions will continue to play a determining role and often serve as basic patterns, exhibiting the correct classes of existence, regularity, uniqueness and specific asymptotics.

The classical method for detecting similarity reductions and associated explicit solutions of various classes of PDEs is the Lie group method of infinitesimal transformations. These approaches and related extensions are explained in a series of monographs by L.V. Ovsiannikov, N.H. Ibragimov, G.W. Bluman and J.D. Cole, P.J. Olver, G.W. Bluman and S. Kumei amongst others. We refer to the "*CRC Handbook of Lie Group Analysis of Differential Equations*" [10] containing a large list of results and references on this subject.

Over the years, many generalizations of the concept of symmetry groups of nonlinear PDEs have been proposed. The first of these go back to Lie himself (contact transformations), to E. Cartan (dynamical symmetries, 1910), and to E. Noether (generalized symmetries, 1918). Other ideas that appeared in this period are discussed in Anderson–Kamran–Olver [11]. Many generalizations can be viewed as extensions of the classical *semi-inverse method* in Continuum Mechanics, which has a natural counterpart in symmetry methods (as was first noted by G. Birkhoff in the 1950s).

During the last fifty years, when more nonlinear models and applied PDEs began to attract the attention of mathematicians, many other fruitful attempts were made to extend the classical apparatus of Lie group symmetries for PDEs. A significant number of new classes of such generalized symmetries and corresponding exact solutions were found. Not pretending to completeness, precise statements, and the correct characterization of such ideas, we include in this list the following (specific power tools for integrable equations are not mentioned):

- the method of nonclassical symmetries (invariant surface conditions);
- the method of partially invariant solutions;
- the Bäcklund transformation method;
- the Baker–Hirota bilinear method;
- the direct and modified method;
- the conditional and generalized conditional symmetry method;
- the non-local symmetry method;
- the truncated Painlevé approach;
- the weak symmetries method;
- the side conditions method;
- the method of linear invariant subspaces for nonlinear operators;
- the method of linear determining equations;
- the method of B-determining equations;
- the nonlinear separation method;
- the functional separation method;
- the method of symmetry-preserving constraints;
- the symmetry-enhancing method;
- the differential constraint method.

We will present descriptions and references concerning most of the methods that are related to the techniques used in our analysis (some of the others can be traced out through use of the Index).

Most of the above methods can be reformulated by using the technicalities of the *method of differential constraints*. Such ideas initially appeared in the theory of first-order PDEs. In particular, Lagrange used differential constraints to determine total integrals of nonlinear equations with two independent variables

$$F(x, y, u, u_x, u_y) = 0.$$

Monge and Ampère proposed the technique of first integrals for solving the second-order PDEs

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

and in 1870, Darboux extended this approach by introducing an extra second-order PDE which is in involution with the original equation (this is what is now called a differential constraint). The history of this analysis and the detailed description of Darboux's method are given in Goursat [260] and Forsyth [196]. General theory of related overdetermined systems is due to many famous names, such as Riquier, Cartan, Ritt, and Spencer, as explained in Pommaret [468].

Systematic approaches to differential constraints, related symmetry and Lie group methods were proposed by Birkhoff in the 1940s (hydrodynamics and fluid dynamics) and by Yanenko in the 1960s (gas dynamics). A formal description of the method is not difficult: consider a PDE for solutions $u = u(x,t)$, with independent variables $(x,t) \in I\!R \times I\!R_+$, where t denotes the time-variable. Given a sufficiently smooth function $F(\cdot)$, consider the evolution PDE

$$F[u] \equiv F(u, Du, D^2u, ...) = 0, \tag{0.1}$$

where $Du = \{u_x, u_t\}$, $D^2u = \{u_{xx}, u_{xt}, u_{tt}\}$, etc. denote vectors of partial derivatives of arbitrary fixed finite orders. To find particular exact solutions consider, instead of the single PDE (0.1), a system of two (or possibly more) equations

$$\begin{cases} F_1[u] = 0, \\ \Phi[u] = 0, \end{cases} \tag{0.2}$$

where the second equation plays the role of an extra *differential constraint*. As usual, one can take $F_1 = F$ in the first equation, but, in general, these operators can be different under the hypothesis that the *consistency* of the system implies that such functions $u(x,t)$ also satisfy the original equation (0.1). For example, if $F[u] = F_1[u] - F_2[u]$, the following system may be considered:

$$\begin{cases} F_1[u] = \Phi[u], \\ \Phi[u] = F_2[u], \end{cases}$$

with an unknown operator Φ to be determined from the consistency condition.

The key ingredient of the differential constraint analysis is to find such suitable operator pairs $\{F_1, \Phi\}$ in (0.2). This is a difficult problem. Indeed, the consistency condition of the system leads to a PDE for Φ, which may be much more complicated than the original one (0.1) for u (to say nothing about PDEs in the multi-dimensional Euclidean space, where $x \in I\!R^N$). Nevertheless, there exists an essential advantage of this constraint analysis: one needs to find a *particular solution* of the compatibility equation.* In the methods listed above, the choice of suitable constraint operators was heavily affected by applying new additional ideas, including some results of classical group-invariant analysis and extensions, or those from neighboring areas of the theory and applications of the PDEs under consideration.

In sufficiently general settings (that do not deal with hard consistency of the PDEs for Φ) the scheme for the differential constraint method looks like using a practically random choice of consistent constraint operators Φ. In a natural sense, such a procedure does not essentially differ from *trial and error* dealing with *a priori* prescribed classes of functions $\{u(x,t,\alpha)\}$ (α is a parameter, possibly functional) to be substituted into the PDE (0.1) to check whether some of the functions by chance satisfy it.

* We do not mention the second important aspect of the method: how to find the solutions, corresponding to a consistent pair $\{F_1, \Phi\}$; this can also be extremely difficult.

The differential constraint may determine the possible class of solutions $\{u(x, t, \alpha)\}$, and, in many cases, this makes the procedure of seeking exact solutions algorithmic, rather than the trivial, random substitution of functions.

It is worth mentioning what is meant here by exact solutions. Indeed, the best opportunity is to detect the *explicit solutions* expressed in terms of elementary or, at least, known functions of mathematical physics (Euler's Gamma, Beta, elliptic, etc.), in terms of quadratures, and so on. But this is not always the case, even for simple semilinear PDEs. Therefore, *exact solutions* will mean those that can be obtained from some ODEs or, in general, from PDEs of *lower order* than the original PDE (0.1). For instance, such an extension of the notion of exact solutions was proposed by A.A. Dorodnitsyn in the middle of the 1960s.

In particular, our goal is to find a reduction of the PDEs to a finite number of ODEs representing a *dynamical system*.

Three-fold role of exact solutions: existence-uniqueness-asymptotics

Exact solutions of nonlinear models have always played a special role in the theory of nonlinear evolution equations. For difficult quasilinear PDEs or systems, exact solutions can often be the only possibility to formally describe the actual behavior of general, more arbitrary solutions. Furthermore, exact solutions are often crucial for developing general existence-uniqueness and asymptotic theory. There are many remarkable examples of important nonlinear models where an appropriate exact solution simultaneously reveals an optimal description of:

(i) local and global existence functional classes;

(ii) uniqueness classes; and,

(iii) classes of correct generic asymptotic behavior.

Actually, **(iii)** is well understood in rigorous or, more often, formal asymptotic analysis of nonlinear PDEs. The first two conclusions **(i)** and **(ii)** are harder to see and difficult to prove, even for reasonably simple evolution PDEs. Moreover, the particular space-time structure of such solutions may also detect useful features of the new methods and tools, which are necessary for studying general solutions. In the theory of parabolic reaction-diffusion equations, there exist seminal examples where the exact solutions determine the correct rescaled variables obtained via nonlinear transformations, in terms of which the Maximum Principle can be applied to extend regularity properties of these particular solutions to more general ones.

More and more often, modern theory of evolution PDEs deals with classes of extremely difficult, strongly nonlinear, higher-order equations with degenerate and singular coefficients. In particular, for at least twenty five years, a permanent source of such models is *thin film theory*, generating various fourth, sixth and higher-order thin film equations with non-monotone and non-divergent operators (essential parts of Chapters 3 and 6 are devoted to such equations). Bearing in mind the multi-dimensional setting in $I\!R^N$ for $N \geq 2$, it is unlikely that a rigorous, mathematically closed existence-uniqueness-regularity and singularity (blow-up) theory for these equations in different free-boundary settings will be developed soon. New exact solutions of thin film models will continue to supply us with a new regularity information that will be used to correct the existing methods in order to create a more general theory.

Linear invariant subspaces for nonlinear operators

As a key idea, we seek exact solutions of (0.1) on linear n-dimensional subspaces which in many cases are *invariant under the nonlinear operators* of the models. A formal general scheme for the approach is easy, though, as often happens, its abstract mathematical formulation leads to rather obscure explanations.

We define a subspace in terms of the linear span denoted by

$$W_n = \mathcal{L}\{f_1(x), ..., f_n(x)\},$$

with n unknown linearly independent basis functions $\{f_j(x)\}$. For instance, these functions are picked to be solutions of a given linear PDE

$$P[f] = 0, \tag{0.3}$$

where $P = P(D_x)$ is the *annihilator* of subspace W_n, in the sense that there holds $P : W_n \to \{0\}$. Then (0.1) is replaced by a system

$$\begin{cases} F[u] = 0, \\ \Phi[u] \in W_n, \end{cases} \tag{0.4}$$

where Φ is another unknown function (or, in general, a nonlinear operator). Using the annihilator (0.3), the second condition in (0.4) is written as a differential constraint

$$P[\Phi[u]] = 0.$$

Here the main difficulty appears: how to choose consistent pairs of operators Φ and P. We next can look for solutions in the form of finite expansions

$$\Phi[u(x,t)] = C_1(t) f_1(x) + ... + C_n(t) f_n(x) \in W_n \quad \text{for } t \in I\!R, \tag{0.5}$$

with unknown coefficients $\{C_j(t)\}$.

Finally, as the crucial step, assuming that the inverse Φ^{-1} exists, we demand the subspace W_n be *invariant* under the superposition of operators,

$$F \circ \Phi^{-1} : W_n \to W_n. \tag{0.6}$$

Then the operator $F \circ \Phi^{-1}$ is said to *preserve* or *admit* the subspace W_n. Substituting the expansion (0.5) into the PDE (0.1), most plausibly, leads to a low-dimensional reduction of the original PDE restricted to this invariant subspace.

In the case of first-order (in t) evolution PDEs with independent variables x and t,

$$u_t = F[u] \equiv F(u, u_x, u_{xx}, ...), \tag{0.7}$$

taking the identity $\Phi = I$ in (0.5), it follows that if W_n is invariant under F, then

$$F[u] = \Psi_1(C_1, ..., C_n) f_1 + ... + \Psi_n(C_1, ..., C_n) f_n \in W_n \quad \text{for } u \in W_n, \tag{0.8}$$

where $\{\Psi_j\}$ denote the expansion coefficients of $F[u]$ on W_n. Hence, (0.7) restricted to the invariant subspace W_n is the n-dimensional dynamical system (DS) for the expansion coefficients $\{C_j(t)\}$ in (0.5),

$$\begin{cases} C_1' = \Psi_1(C_1, ..., C_n), \\ \quad ... \qquad ... \qquad ... \\ C_n' = \Psi_n(C_1, ..., C_n). \end{cases} \tag{0.9}$$

For $n = 1, 2$, or 3, such DSs can often be studied on the phase-plane, or, at least, admit asymptotic analysis of some of their generic, stable orbits.

We will give several examples for which the above approach represents an easy way to predict such a linear structure of exact solutions. For instance, let us observe that, under the same invariance conditions, the second-order evolution equation

$$u_{tt} = F[u] \equiv F(u, u_x, u_{xx}, ...)$$

admits solutions (0.5), where $\Phi = I$, with a harder $2n$th-order DS,

$$\begin{cases} C_1'' = \Psi_1(C_1, ..., C_n), \\ \quad ... \qquad ... \qquad ... \\ C_n'' = \Psi_n(C_1, ..., C_n). \end{cases}$$

As a principal feature, this book can be viewed as a practical guide that introduces a number of techniques for constructing exact solutions of various nonlinear PDEs in $I\!R^N$ for arbitrary dimensions $N \geq 1$. Indeed, several such exact solutions can be obtained by other techniques including differential constraints which have been successfully developed algorithmically on the basis of computer symbolic manipulation techniques. Nevertheless, some other solutions, especially those of higher-order equations in $I\!R^N$, will be difficult to detect by such "purely computational" approaches. The ideas of linear invariant subspaces can play a decisive role in explaining such a geometric origin of invariant manifolds, the corresponding exact solutions, and extensions to other PDEs.

Examples: classic fundamental solutions belong to invariant subspaces

For linear homogeneous PDEs, the three-fold existence-uniqueness-asymptotics nature **(i)–(iii)** of exact solutions is straightforward in view of the classical concept of *fundamental* solutions of linear operators and convolution representations of general solutions. It is remarkable and surprising that, for a number of classical linear and quasilinear models, the *fundamental solutions are associated with linear subspaces invariant under nonlinear operators*.

The heat equation and linear subspace for its fundamental solution

Consider the canonical *heat equation* (HE)

$$u_t = \Delta u \quad \text{in } I\!R^N \times I\!R_+ \quad \Big(\Delta = \textstyle\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}\Big). \tag{0.10}$$

Its *fundamental solution* denoted by $b(x, t)$ is given by the *Gaussian kernel*,

$$b(x, t) = \big(4\pi t\big)^{-\frac{N}{2}} \mathrm{e}^{-\frac{|x|^2}{4t}}, \tag{0.11}$$

and takes Dirac's delta $\delta(x)$ as initial data,

$$\lim_{t\to 0^+} b(x,t) = \delta(x), \tag{0.12}$$

where the convergence is understood in the sense of distributions.

As is well known in parabolic theory (see e.g., Friedman [205]), the structure of the Gaussian kernel in (0.11) illustrates *Tikhonov's uniqueness* (1935) [552] and local *existence* functional class of measurable functions,

$$\mathcal{U} = \left\{ v(x) : \ \exists\ A > 0 \text{ and } a > 0, \ \text{such that } |v(x)| \le A\mathrm{e}^{a|x|^2} \ \text{ in } \ I\!R^N \right\}.$$

Then the Cauchy problem for the HE with initial data $u_0(x) \in \mathcal{U}$ has a unique solution that is local in time and is given by the convolution

$$u(x,t) = b(\cdot,t) * u_0 \equiv \left(4\pi t\right)^{-\frac{N}{2}} \int_{I\!R^N} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, u_0(y)\,\mathrm{d}y. \tag{0.13}$$

By checking the convergence of the integral, it is easy to see that this formula guarantees the existence and uniqueness of the solution locally in time, at least for all $t < \frac{a}{4}$. In order to make the solution global in time, another growth condition should be imposed on the initial data, e.g., assuming that $|u_0(x)| \le A\mathrm{e}^{a|x|^{2-\varepsilon}}$, with an arbitrarily small constant $\varepsilon > 0$. In this case, the integral in (0.13) is finite for all $t > 0$.

The explicit formula (0.13) also determines the *asymptotic behavior* as $t \to \infty$ of global solutions. Namely, if initial data are integrable, $u_0 \in L^1(I\!R^N)$, and have unit mass, $\int u_0(x)\,\mathrm{d}x = 1$, as the fundamental solution does in (0.12), then

$$u(x,t) \approx b(x,t) \quad \text{for } \ t \gg 1. \tag{0.14}$$

It is convenient to express this asymptotic convergence in the rescaled sense by using the time-scaling factor $t^{N/2}$ as in (0.11). Then (0.14) reads

$$t^{\frac{N}{2}} \left| u(x,t) - b(x,t) \right| \to 0 \quad \text{as } \ t \to \infty \tag{0.15}$$

uniformly on expanding sets $\{|x| \le c\sqrt{t}\}$, where $c > 0$ is an arbitrary constant.

Invariant subspaces. The exponential structure of the fundamental solutions (0.11) suggests introducing the logarithmic variable

$$v(x,t) = \ln b(x,t) \equiv -\tfrac{N}{2} \ln(4\pi t) - \tfrac{1}{4t}\, |x|^2,$$

where the right-hand side belongs to the 2D subspace W_2 that is given by the span

$$W_2 = \mathcal{L}\{1, |x|^2\}. \tag{0.16}$$

The new function $v = \ln u$ satisfies the semilinear parabolic equation

$$v_t = \Delta v + |\nabla v|^2 \equiv F[v], \tag{0.17}$$

that contains the quadratic Hamilton–Jacobi operator $|\nabla v|^2$. Thus, the logarithmic change of variables leads to the nonlinear operator F in (0.17) that obviously preserves the subspace W_2. Substituting into (0.17) an arbitrary function

$$v(x,t) = C_0(t) + C_1(t)|x|^2 \in W_2 \quad \text{for } \ t \ge 0, \tag{0.18}$$

we find by calculating $\Delta|x|^2 = 2N$ and $|\nabla|x|^2|^2 = 4|x|^2$ that

$$C_0' + C_1'|x|^2 = F\big[C_0 + C_1|x|^2\big] \equiv 2NC_1 + 4C_1^2|x|^2.$$

This yields the dynamical system

$$\begin{cases} C_0' = 2NC_1, \\ C_1' = 4C_1^2, \end{cases} \tag{0.19}$$

which is easily integrated. The second equation implies that $C_1(t) = -\frac{1}{4t}$ up to translations in t. Therefore, $C_0' = -\frac{N}{2t}$, and this gives the fundamental solution (0.11) in terms of the original variable $u = \mathrm{e}^v$.

This analysis admits some easy and immediate extensions. Firstly, it is evident that the operator in (0.17) admits the $(N{+}1)$-dimensional invariant subspace

$$W_{N+1} = \mathcal{L}\{1, x_1^2, ..., x_N^2\} \implies v(x,t) = C_0(t){+}C_1(t)x_1^2{+}...{+}C_N(t)x_N^2. \tag{0.20}$$

The DS then becomes $(N{+}1)$-dimensional,

$$\begin{cases} C_0' = 2\sum_{(i)} C_i, \\ C_j' = 4C_j^2, \quad j = 1, ..., N, \end{cases}$$

which can also be integrated. Secondly, one can consider the general invariant subspace of arbitrary quadratic polynomials

$$W_M = \mathcal{L}\{1, x_i, x_i x_j, \; i, j, = 1, ..., N\} \tag{0.21}$$

of dimension $M = \frac{N^2+3N+2}{2}$, where the expansion contains more coefficients generating an M-dimensional DS. Clearly, using the orthogonal transformations and translations, the exact solutions on the subspace (0.21) reduce to those on (0.20). But this is not the case for the corresponding second-order *hyperbolic* equation

$$v_{tt} = \Delta v + |\nabla v|^2$$

for which the family of solutions on the subspaces (0.21) and (0.20) differ essentially. In the corresponding DS, we have the second-order derivatives C_j'', and hence, both $\{C_j(0)\}$ and $\{C_j'(0)\}$ should be prescribed as initial data, so, for the subspace (0.21), it is a $2M$-dimensional DS.

The porous medium equation and linear subspace for its fundamental solution

For quasilinear parabolic equations for which convolution and eigenfunction expansion techniques are not applicable the determining features **(i)–(iii)** of exact solutions are not straightforward and demand different and difficult nonlinear mathematics. Consider the classic *porous medium equation* (PME)

$$u_t = \Delta u^m \quad \text{in} \;\; I\!\!R^N \times I\!\!R_+, \tag{0.22}$$

where $m > 1$ is a fixed exponent. By the Maximum Principle, the PME possesses nonnegative solutions $u(x,t)$, so that u^m makes sense for any non-integer value of m. The advanced theory of such degenerate parabolic equations that admit weak (generalized) solutions can be found in a number of monographs on parabolic PDEs;

see [148, 206, 245]. For $m = 1$, (0.22) reduces to the heat equation (0.10), so the PME can be viewed as its nonlinear extension.

Let us see if the quasilinear PME inherits some distinctive evolution properties available for the HE, and, especially, whether it admits a kind of *fundamental solution* to be understood, of course, in a different nonlinear way. The answer is yes, and the PME has the famous *Zel'dovich–Kompaneetz–Barenblatt* (ZKB, 1950) source-type self-similar solution that is again denoted by $b(x,t)$,

$$b(x,t) = t^{-kN} f(y), \quad y = \tfrac{x}{t^k}, \quad \text{where } k = \tfrac{1}{N(m-1)+2}\,. \tag{0.23}$$

The rescaled profile $f(y)$ is given explicitly,

$$f(y) = \big[A_0(a^2 - |y|^2)_+\big]^{\frac{1}{m-1}}, \quad \text{with the constant } A_0 = \tfrac{k(m-1)}{2m}\,, \tag{0.24}$$

where $(\cdot)_+$ denotes the positive part $\max\{(\cdot), 0\}$. The constant $a > 0$ characterizes the preserved total mass of the solution. We want $b(x,t)$ to initially take Dirac's delta, as shown in (0.12). Direct computations yield the unique value of $a = a(m)$ (see e.g., [509, p. 21]),

$$\begin{aligned} 1 = \int_{I\!R^N} f(y)\,\mathrm{d}y \equiv N\,\omega_N \int_0^a z^{N-1}\big[A_0(a^2 - z^2)\big]^{\frac{1}{m-1}}\,\mathrm{d}z \\ \Longrightarrow \quad a^{\frac{2}{m-1}+N} = \pi^{-\frac{N}{2}} A_0^{-\frac{1}{m-1}} \frac{\Gamma(\frac{m}{m-1}+\frac{N}{2})}{\Gamma(\frac{m}{m-1})}\,, \end{aligned} \tag{0.25}$$

where Γ is Euler's Gamma function, and $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ denotes the volume of the unit ball in $I\!R^N$.

Returning to the rescaled fundamental profile (0.24), it follows that, unlike (0.11) for the heat equation, $b(x,t)$ is compactly supported in x for any $t > 0$. This is a striking property of the *finite propagation* for the quasilinear degenerate parabolic equation (0.22). At the free-boundary (interface), where $|y| = a$, the profile $f(y)$ has finite regularity, and $f^{m-1}(y)$ is just Lipschitz continuous.

Thus, it seems that the solutions of the HE and the PME correspond to entirely different functional settings. Nevertheless, a striking continuity with respect to the exponent m can be observed when passing to the limit as $m \to 1^+$ in (0.24). Then, using that, in (0.25), the ratio of Gamma functions is equal to $\big(\frac{m}{m-1}\big)^{N/2} + \ldots$, it is easy to conclude that, uniformly in y,

$$f(y) \to \big(4\pi\big)^{-\frac{N}{2}}\, \mathrm{e}^{-\frac{|y|^2}{4}} \quad \text{as } m \to 1^+,$$

where, on the right-hand side, there appears the rescaled Gaussian kernel of the fundamental solution (0.11). This means a continuous "branching" at $m = 1^+$ of the solution (0.24) from the fundamental solution (0.11) of the linear HE. Once more, this asserts using the term *fundamental solution* of the nonlinear PME.

It turns out that, in PME theory, the ZKB solution plays a similar three-fold role **(i)–(iii)**. Firstly, the inverse parabolic profile of the rescaled kernel $f(y)$ in (0.24) determines the class of uniqueness and local existence,

$$\mathcal{U} = \big\{v(x) \ge 0 : \ \exists\, A > 0 \text{ such that } v(x) \le A(1+|x|^2)^{\frac{1}{m-1}} \text{ in } I\!R^N\big\}.$$

It follows by comparison with the following separate variables blow-up solution:

$$u_*(x,t) = C_*|x|^{\frac{2}{m-1}}(T-t)^{-\frac{1}{m-1}}, \quad \text{with } C_* = \left[\tfrac{k(m-1)}{2m}\right]^{\frac{1}{m-1}},$$

that the weak solution exists, at least for all $t < T \sim A^{1-m}$. For global existence it suffices to restrict the growth rate at infinity, e.g., by assuming that $u_0(x) = O\big(|x|^{\frac{2}{m-1}-\varepsilon}\big)$ as $x \to \infty$ for some arbitrarily small $\varepsilon > 0$.

Secondly, in a similar manner, for nonnegative initial data $u_0 \in L^1(\mathbb{R}^N)$ with unit mass, $\int u_0(x)\,\mathrm{d}x = 1$, (0.14) holds. The asymptotic convergence (0.14) is again to be understood in the rescaled sense (0.15) with the time factor t^{kN}, instead of $t^{N/2}$. Note that $k = \frac{1}{2}$ for $m = 1$. The convergence is uniform on compact sets $\{|x| \le c\,t^k\}$, $c > 0$, corresponding to the new similarity variable y in (0.23).

Invariant subspaces. Though the ZKB-solution (0.23) is a classical example of self-similar solutions induced by a group of scaling transformations, let us now interpret it in terms of the same invariant subspace (0.16). The rescaled inverse parabolic profile (0.24) suggests using the new dependent variable

$$v = u^{m-1},$$

which is known as the *pressure* in the theory of filtration of liquids and gases in porous media. Most of the regularity results for the PME are formulated in terms of the pressure. Substituting $u = v^{\frac{1}{m-1}}$ into the PME yields the *pressure equation*

$$v_t = v\Delta v + \tfrac{1}{m-1}|\nabla v|^2 \equiv F[v]. \tag{0.26}$$

Similar to the transformed HE (0.17), the quadratic operator F in (0.26) preserves the subspace (0.16). Plugging (0.18) yields a slightly different dynamical system for the expansion coefficients,

$$\begin{cases} C_0' = 2NC_0C_1, \\ C_1' = \frac{2}{(m-1)k}\,C_1^2. \end{cases}$$

As in (0.19), the second equation is integrated independently, determining (0.23).

We easily reveal the dynamics on other extended subspaces of F in (0.26): the subspace (0.20) remains invariant, leading to the $(N+1)$-dimensional DS

$$\begin{cases} C_0' = 2C_0\sum_{(i)} C_i, \\ C_j' = 2C_j\sum_{(i)} C_i + \frac{4}{m-1}\,C_j^2, \quad j = 1, ..., N. \end{cases}$$

On the invariant subspace of arbitrary quadratic polynomials (0.21), the PME becomes an M-dimensional DS which is again reduced to that on the subspace (0.20) via rotations and translations. The quasilinear degenerate hyperbolic equation

$$v_{tt} = v\Delta v + \tfrac{1}{m-1}|\nabla v|^2$$

restricted to W_M becomes a $2M$th-order DS.

Elementary extensions to higher-order equations. We formally combine operators in (0.17) and (0.26), add extra operators, and create a fourth-order parabolic equation

$$v_t = F[v] \equiv -\alpha\Delta^2 v + \beta\Delta v + \gamma v\Delta v + \delta|\nabla v|^2 + \mu v + \nu, \tag{0.27}$$

with six arbitrary constants denoted by Greek letters. Such PDEs belong to the class of *Kuramoto–Sivashinsky equations* from flame propagation theory that will be studied in the subsequent chapters. Obviously, the fourth-order term $-\alpha\,\Delta^2 v$ vanishes on the invariant subspace (0.21), so (0.27) on W_M is an M-dimensional DS. The corresponding hyperbolic PDE is a *Boussinesq-type equation* from water-wave theory,

$$v_{tt} = -\alpha\,\Delta^2 v + \beta\,\Delta v + \gamma\, v\,\Delta v + \delta|\nabla v|^2 + \mu v + \nu,$$

which becomes a $2M$th-order DS on the same invariant subspace W_M.

Models, targets, prerequisites

On nonlinear models and PDEs to be considered

The underlying idea of invariant subspaces for nonlinear operators applies here to a large variety of nonlinear PDEs from many areas of mathematics, mechanics, and physics. Exact solutions on invariant subspaces arise in many quasilinear equations and various free-boundary problems from different applications. In this book, we will deal with various PDEs and models that exhibit some common nonlinear invariant features. Beyond this "invariant essence," many of the models have nothing in common and often belong to completely disjoint areas of mathematics.

We begin Chapter 1 with some history and present those classical and more recent examples of interesting solutions on invariant subspaces that were constructed in the twentieth century. In the rest of the book, we develop several techniques for constructing exact solutions that describe singularity behavior for various nonlinear PDEs, including (see Index for details and precise references)

- reaction-diffusion-absorption PDEs and combustion models;
- parabolic and hyperbolic PDEs with the p-Laplacian operators;
- gas dynamics models, including the *Kármán–Fal'kovich–Guderley equation*;
- fourth, sixth, and $2m$th-order *thin film equations*;
- fourth-order *Riabouchinsky–Proudman–Johnson equations*;
- free-boundary problems for the *Navier–Stokes equations* in $\mathbb{R}^2$;
- *Kuramoto–Sivashinsky equations* and extensions;
- *KdV-type equations* with blow-up, nonlinear dispersion PDEs with compactons;
- higher-order extensions of the *Rosenau–Hyman equation*;
- modifications of the *Fuchssteiner–Fokas–Camassa–Holm equations*;
- *Green–Naghdi equations*;
- *Harry Dym-type equations*;
- quasilinear pseudo-parabolic (*magma*) equations;
- quasilinear *wave* equations and dispersive *Boussinesq* models;
- *Zabolotskaya–Khokhlov-type equations*;
- *Zakharov–Kuznetsov equation with nonlinear dispersion*;
- quasilinear parabolic, hyperbolic, and *KdV-type* systems;
- *Maxwell equations* from nonlinear optics;
- *Monge-Ampère-type equations* of second and higher orders;
- *logarithmic Gauss curvature equations*;
- non-integrable PDEs admitting bilinear *Baker–Hirota* representations; etc.

In some cases, using exact solutions, we will describe interesting evolution properties that are related to singularity *blow-up* or *extinction* phenomena, *finite interface* propagation and regularity, with the special attention to *oscillatory, changing sign* behavior of weak solutions near interfaces. For several PDEs, this leads to many mathematical open problems, which we state when necessary. Most of the results are published for the first time.

Main problems and targets

There exist two main fundamental problems in invariant subspace theory:

• **Problem I,** $\mathbf{F} \mapsto \{\mathbf{W_n}\}$: *Given a nonlinear operator F, which invariant subspaces W_n does it preserve?*

• **Problem II,** $\mathbf{W_n} \mapsto \{\mathbf{F}\}$: *Given a subspace W_n, which nonlinear operators F admit it?*

In addition, there are a number of other practical questions, e.g.,

• *Which operators F admit higher-dimensional invariant subspaces as further extensions of the basic, simple invariant subspaces?*

• *Is there a well-defined procedure to detect invariant subspaces and their maximal dimensions* (*i.e., maximal dynamical systems that are restrictions of the PDE to the subspace*)*?*

Problem I is fundamental, and is key for the existence of lower-dimensional reductions of the PDEs. For arbitrary operators F, this does not admit a complete solution, but we will successfully study Problem I for many particular classes of nonlinear differential and discrete operators.

On the contrary, Problem II admits a complete algorithmic solution. It was solved for $N = 1$, i.e., for ordinary differential operators, by the second author of the book [544, 545] in terms of Lie–Bäcklund symmetries of linear ODEs. For general operators in $\mathbb{R}^N$, Problem II was solved in Kamran–Milson–Olver [312] by introducing a new approach to the annihilating differential operators. Nevertheless, as often happens in mathematics, a complete algorithmic solution does not assume easy practical applications of the results. It is said in [312, p. 316] that (for operators in $\mathbb{R}^N$) "The formulae for the affine annihilators and annihilators are often extremely complicated, even for relatively simple subspaces." Bearing in mind the practical aspects of calculations, the geometric concepts of invariant subspaces will continue to play an important role.

The general problem of finding invariant subspaces for wide classes of nonlinear operators in $\mathbb{R}^N$ is not completely solved here. We suspect that such a problem cannot be tackled with sufficient generality. Nevertheless, for quadratic and polynomial operators in $\mathbb{R}$, we present a complete classification of some types of invariant subspaces. We also introduce examples of invariant subspaces and exact solutions for classes of multi-dimensional quasilinear operators in $\mathbb{R}^N$.

Partially invariant subspaces: invariant sets

Another related direction of our analysis is the construction of *invariant sets* $M \subset W_n$ *on a linear subspace* W_n for operator F. This simply means that W_n is *partially invariant*, i.e., $F[W_n] \nsubseteq W_n$, but, for some part M of W_n,

$$F[M] \subseteq W_n. \tag{0.28}$$

The principal difference from the invariant subspaces for which $F : W_n \to W_n$ is that condition (0.28) leads to an *overdetermined* DS for the expansion coefficients $\{C_j(t)\}$ in (0.5).

Let us illustrate this for equation (0.7), assuming that W_n is not invariant under F in the sense of (0.6) with $\Phi = I$. Suppose, for instance, that F maps W_n onto an $(n+s)$-dimensional subspace, so that s new functions appear in the expansion

$$F : W_n \to W_{n+s} = \mathcal{L}\{f_1, ..., f_n, f_{n+1}, ..., f_{n+s}\},$$

and, instead of (0.8),

$$F[u] = \Psi_1(\cdot) f_1 + ... + \Psi_n(\cdot) f_n \\ + \Psi_{n+1}(\cdot) f_{n+1} + ... + \Psi_{n+s}(\cdot) f_{n+s} \in W_{n+s}.$$

This leads to the same DS (0.9) accompanied by s extra algebraic conditions

$$\begin{cases} \Psi_{n+1}(C_1, ..., C_n) = 0, \\ \quad ... \qquad ... \qquad ... \\ \Psi_{n+s}(C_1, ..., C_n) = 0. \end{cases}$$

Such overdetermined DSs are not always consistent and are hard to study. The proof of the existence of the corresponding solutions on M becomes more involved, though we present a number of nonlinear evolution PDEs for which such overdetermined DSs are consistent.

Partial invariance as a manifestation of "partial integrability"

We discuss the principal link to integrable equations which admit countable sets of exact N-soliton and other solutions. We illustrate this by starting with the most classical integrable *Korteweg–de Vries* (KdV) *equation*

$$u_t + 6uu_x + u_{xxx} = 0, \tag{0.29}$$

which has been known since the 1870s and was first derived by J. Boussinesq. Following the standard scheme for integrable PDEs (see Newell [436, Ch. 4]), we apply the change $u = w_x$, yielding the *potential KdV equation*

$$w_t + 3(w_x)^2 + w_{xxx} = 0.$$

Next, setting

$$w = 2(\ln|v|)_x = \tfrac{2v_x}{v}, \quad \text{so that} \quad u = 2(\ln|v|)_{xx}, \tag{0.30}$$

reduces it to the homogeneous quadratic equation

$$F_*[v] \equiv vv_{xt} - v_x v_t + vv_{xxxx} - 4v_x v_{xxx} + 3(v_{xx})^2 = 0. \tag{0.31}$$

As a final step, the *Baker–Hirota bilinear method* † [284] is applied to derive a countable set of N-solitons $\{v_k(x,t)\}$, such that each solution $v_k(x,t)$ of (0.31) belongs to a *linear subspace* of exponential functions. We will use various linear subspaces to illustrate finite-dimensional dynamics, which exist for equation (0.31) and related models and correspond to well-known soliton-type solutions.

1-soliton on subspace $\mathbf{W_2^{exp}}$. This is the simplest *travelling wave* (TW) given by a single exponent,

$$v_1(x,t) = 1 + \mathrm{e}^{\theta_1(x,t)}, \quad \text{where} \;\; \theta_1(x,t) = p_1 x - p_1^3 t \tag{0.32}$$

and $p_1 \ne 0$ is a constant. Clearly, in this case, the 2D linear subspace (a module)

$$W_2^{\rm exp} = \mathcal{L}\{1, \mathrm{e}^{p_1 x}\} \tag{0.33}$$

is invariant under the quadratic operator F_* in (0.31). Indeed, as usual, looking for solutions of (0.31) on $W_2^{\rm exp}$,

$$v(x,t) = C_1(t) + C_2(t)\mathrm{e}^{p_1 x}, \tag{0.34}$$

and plugging it into (0.31) yields a single term, $(C_1C_2' - C_2C_1' + p_1^3 C_1C_2)\mathrm{e}^{p_1 x} = 0$, since the coefficient of the highest-degree exponential $\mathrm{e}^{2p_1 x}$ vanishes, as the integrability demands. Therefore, the PDE (0.31) on $W_2^{\rm exp}$ reduces to the single ODE (an *underdetermined* DS)

$$C_1C_2' - C_2C_1' = -p_1^3 C_1C_2 \quad \Longrightarrow \quad \big(\tfrac{C_2}{C_1}\big)' = -p_1^3\, \tfrac{C_2}{C_1}, \tag{0.35}$$

so, on integration, $C_2(t) = AC_1(t)\mathrm{e}^{-p_1^3 t}$, where A is a constant. Here, $C_1(t) \ne 0$ is an arbitrary smooth function that is eliminated by the differential change (0.30). Thus, up to an arbitrary multiplier $C_1(t)$, (0.34) represents the 1-soliton solution (0.32) belonging to the invariant subspace $W_2^{\rm exp}$. Notice that exact solutions (0.34) on $W_2^{\rm exp}$ can satisfy various PDEs involving operator F_*, e.g.,

$$\begin{aligned} \alpha v_{tt} + \beta v_t = {} & F_*[v] + \mu v + \nu + \sigma v_{xx} + \rho\big[v v_{xx} - (v_x)^2\big] \\ & + \varepsilon(v v_{xxx} - v_x v_{xx}) + \lambda\big[v v_{xxxx} - (v_{xx})^2\big] + \ldots, \end{aligned} \tag{0.36}$$

with some linear and nonlinear operators preserving the subspace (0.33).

2-soliton on $\mathbf{W_4^{exp}}$. The 2-solitons are composed of three exponential patterns

$$v_2(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + C_4\mathrm{e}^{\theta_1+\theta_2}, \tag{0.37}$$

where, as in (0.32), $\theta_1 = p_1 x - p_1^3 t$, $\theta_2 = p_2 x - p_2^3 t$, and $p_1 \ne p_2$. In soliton theory [436, p. 123], applying the Baker–Hirota differential operator to (0.31) yields

$$C_4 = \big(\tfrac{p_1-p_2}{p_1+p_2}\big)^2.$$

As above, we can interpret (0.37) by using the linear subspace (module)

$$W_4^{\rm exp} = \mathcal{L}\{1, \mathrm{e}^{p_1 x}, \mathrm{e}^{p_2 x}, \mathrm{e}^{(p_1+p_2)x}\}.$$

† The bilinear differential operator in (0.31), transformations, such as (0.30), hierarchies of the KdV and KP equations, hyperelliptic representation of periodic solitons, etc., were introduced by H.F. Baker in 1903, [22]; see details in Athorne–Eilbeck–Enolskii [20, p. 275].

For instance, if $W_4^{\exp} = \mathcal{L}\{1, e^x, e^{2x}, e^{3x}\}$ (this assumes more nonlinear interaction between terms than for the standard 2-soliton; see below), looking for

$$v(x,t) = C_1(t) + C_2(t)e^x + C_3(t)e^{2x} + C_4(t)e^{3x} \tag{0.38}$$

and substituting into (0.31) yields

$$\begin{aligned} &\big(C_1C_2' - C_2C_1' + C_1C_2\big)e^x + 2\big(C_1C_3' - C_3C_1' + 8C_1C_3\big)e^{2x} \\ &+\big[C_2C_3' - C_3C_2' + 3\big(C_1C_4' - C_4C_1'\big) + C_2C_3 + 81C_1C_4\big]e^{3x} \\ &+2\big(C_2C_4' - C_4C_2' + 8C_2C_4\big)e^{4x} + \big(C_3C_4' - C_4C_3' + C_3C_4\big)e^{5x} = 0. \end{aligned} \tag{0.39}$$

Hence, for the given module $W_4^{\exp}$, there exists another module $\tilde{W}_5^{\exp}$ such that

$$F_* : W_4^{\exp} \to \tilde{W}_5^{\exp} = \mathcal{L}\{e^x, e^{2x}, e^{3x}, e^{4x}, e^{5x}\}$$

(the coefficients of 1 and e^{6x} vanish). Equating the five coefficients in (0.39) to zero yields the *overdetermined* system of five equations for four functions

$$\begin{cases} C_1C_2' - C_2C_1' = -C_1C_2, \\ C_1C_3' - C_3C_1' = -8C_1C_3, \\ C_2C_3' - C_3C_2' + 3\big(C_1C_4' - C_4C_1'\big) + C_2C_3 + 81C_1C_4 = 0, \\ C_2C_4' - C_4C_2' = -8C_2C_4, \\ C_3C_4' - C_4C_3' = -C_3C_4. \end{cases} \tag{0.40}$$

According to (0.28), the last two ODEs (projections onto e^{4x} and e^{5x}) determine an *invariant set* M on $W_4^{\exp}$, in the sense that $F[u] \in W_4^{\exp}$ for all $u \in M$. Hence, the module $W_4^{\exp}$ is *partially invariant*. Writing all the ODEs (0.40), excluding the third one, in the form of (0.35) and integrating gives

$$C_2(t) = AC_1(t)e^{-t}, \quad C_3(t) = BC_1(t)e^{-8t}, \quad \text{and} \quad C_4(t) = DC_1(t)e^{-9t},$$

where, as above, $C_1(t)$ is arbitrary, and A, B, and D are constants. Plugging these expressions into the long third ODE in (0.40), rewritten in the form of

$$C_2^2\big(\tfrac{C_3}{C_2}\big)' + 3C_1^2\big(\tfrac{C_4}{C_1}\big)' + C_2C_3 + 81C_1C_4 = 0,$$

we obtain a single relation between constants, $AB = 9D$. This gives two exact solutions of 2-soliton type

$$v(x,t) = 1 \pm \big(e^{x-t} + 9be^{2(x-4t)}\big) + be^{3(x-3t)} \quad (b \in \mathbb{R}).$$

A similar interpretation of general N-soliton solutions means that, for the integrable equation (0.31),

$$\exists \text{ solutions on partially invariant modules } W_n^{\exp} \text{ for arbitrarily large } n.$$

In this sense, the fully integrable equations represent an exceptional limit case of evolution PDEs that possess exact solutions belonging to *an infinite number of invariant sets on linear exponential subspaces (modules) of arbitrarily large dimension.*

The invariance under the nonlinear operators can be treated as a kind of a *partial integrability property* (cf. "...*remnants of integrability*" [192, p. 573]), in the sense that we describe classes of nonlinear non-integrable PDEs for which only a *finite* *number* of invariant subspaces W_n, or sets with exact solutions, can be detected. In fact, for any arbitrarily large l, there exists a family of nonlinear non-integrable PDEs possessing at least l different types of solutions (looking like "N-solitons") on linear invariant subspaces W_n, or on sets, with n large enough (see Section 1.5.2). Such PDEs may be treated as *intermediate*, i.e., between general equations with no invariant properties at all, and the rather thin class of fully integrable PDEs.

Trigonometric subspace $\mathbf{W_3^{tr}}$: TWs. We next try solutions

$$v(x,t) = C_1 + C_2 \cos\gamma x + C_3 \sin\gamma x \in W_3^{\rm tr} = \mathcal{L}\{1, \cos\gamma x, \sin\gamma x\}, \tag{0.41}$$

where $\gamma \in I\!R$ is a parameter. $W_3^{\rm tr}$ is *invariant* under F_*, so that restricting the PDE (0.31) to $W_3^{\rm tr}$ yields three ODEs

$$\begin{cases} C_2C_3' - C_3C_2' + 4\gamma^3\big(C_2^2 + C_3^2\big) = 0, \\ C_1C_3' - C_3C_1' + \gamma^3 C_1C_2 = 0, \\ C_2C_1' - C_1C_2' + \gamma^3 C_1C_3 = 0. \end{cases}$$

The matrix of this first-order DS is singular and nontrivial solutions are possible for $C_1(t) \equiv 0$. This gives the TW

$$v(x,t) = \sin(\gamma x + 4\gamma^3 t).$$

In terms of the original function $u = 2(\ln|v|)_{xx}$, such solutions describe moving blow-up singularities with the following behavior near the poles:‡

$$u(x,t) \sim \tfrac{1}{(x-x_0(t))^2}, \quad \text{where} \quad x_0(t) = -4\gamma^2 t + \text{constant}. \tag{0.42}$$

A slight modification of the KdV equation (0.31) produces another interesting evolution on $W_3^{\rm tr}$. For instance,

$$v(x,t) = 1 + \cos(x+t) \equiv 2\cos^2\big[\tfrac{1}{2}(x+t)\big] \quad \text{satisfies} \quad F_*[v] = 4.$$

This function, being extended by zero in $\{(x,t) \,:\, \frac{1}{2}|x+t| \ge \frac{\pi}{2}\}$, becomes a smooth *compacton*. Such compact structures entered nonlinear dispersion theory in the 1980s. We will discuss their mathematical well-posedness in Chapters 3–7.

Polynomial subspace $\mathbf{W_4^p}$: second rational solution. Similarly, equation (0.31) can be considered on the polynomial subspace such as

$$W_4^{\rm p} = \mathcal{L}\{1, x, x^2, x^3\}, \;\; \text{i.e.,} \;\; v(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3, \tag{0.43}$$

which leads to a similar DS. Solving it yields

$$v(x,t) = 36t + b^2x + 3bx^2 + 3x^3 \quad (b \in I\!R),$$

which, by (0.30), gives the second *rational* solution $u_2(x,t)$ of the KdV equation with the singular behavior (0.42) near poles (these are known since 1978, see survey [407]; the first rational solution is elementary, $u_1(x,t) = -\frac{2}{x^2}$).

‡ The study of the Schrödinger operator with the inverse square potential $U(x) \sim (x-x_0)^{-2}$ goes back to Hardy (1920), [280] (Hardy's inequality for embeddings of functional L^2 spaces with singular weights) and Friedrichs (1935), [208].

Polynomial-trigonometric subspace $\mathbf{W_4}^{\mathbf{p,t}}$: positons. Consider next the subspace $W_4^{\rm p,t} = \mathcal{L}\{1, x, \cos x, \sin x\}$, composed of basis functions of subspaces in (0.43) and (0.41). Plugging the expansion on $W_4^{\rm p,t}$ into equation (0.31) gives the solutions

$$v(x,t) = C_1(t) + C_2(t)x + C_3(t)\cos x + C_4(t)\sin x, \quad \text{where}$$
$$\begin{cases} C_1C_2' - C_2C_1' + C_3C_4' - C_4C_3' + 2\big(C_3^2 + C_4^2\big) = 0, \\ C_3C_1' - C_1C_3' + C_1C_4 - 3C_2C_3 = 0, \\ C_1C_4' - C_4C_1' + C_1C_3 + 3C_2C_4 = 0, \\ C_2C_4' - C_4C_2' + C_2C_3 = 0, \\ C_3C_2' - C_2C_3' + C_2C_4 = 0. \end{cases}$$

The first three ODEs are projections of the PDE onto 1, $\cos x$, and $\sin x$ respectively, while the last two represent the expansion coefficients of $x\cos x$ and $x \sin x$ that do not belong to $W_4^{\rm p,t}$. Similar to (0.40), this DS yields two solutions

$$v(x,t) = \pm(3t + x) + \sin(x + t),$$

which are indeed the *positon solutions* of the KdV equation. Such positons, or harmonic breathers, have been recognized since the 1980s, [16, 418]. They exhibit the same type (0.42) of singularity (for continuous integrable models, all known positons have singularities), but a different behavior as $x \to \infty$. Similarly, the polynomial-exponential subspace $W_4^{\rm p,e} = \mathcal{L}\{1, x, \cosh x, \sinh x\}$ leads to the *negatons*, that were first constructed in 1996, [485].

Exponential-trigonometric subspace $\mathbf{W_4}^{\mathbf{e,t}}$: complexitons. We now look for solutions of (0.31) on the trigonometric-exponential subspace,

$$v(x,t) = C_1(t)\cos x + C_2(t)\sin x + C_3(t){\rm e}^{x} + C_4(t){\rm e}^{-x}.$$

Substituting yields five ODEs being the projections of (0.31) onto 1, ${\rm e}^x \sin x$, ${\rm e}^x \cos x$, ${\rm e}^{-x}\cos x$, and ${\rm e}^{-x}\sin x$ respectively,

$$\begin{cases} -C_2C_1' + C_1C_2' + 2C_4C_3' - 2C_3C_4' + 4\big(C_1^2 + C_2^2\big) + 16C_3C_4 = 0, \\ -C_3C_1' - C_3C_2' + (C_1 + C_2)C_3' - 4C_2C_3 = 0, \\ -C_3C_1' + C_3C_2' + (C_1 - C_2)C_3' - 4C_1C_3 = 0, \\ C_4C_1' + C_4C_2' - (C_1 + C_2)C_4' - 4C_1C_4 = 0, \\ -C_4C_1' + C_4C_2' + (C_1 - C_2)C_4' - 4C_2C_4 = 0. \end{cases}$$

These are easily integrated by adding and subtracting two pairs of similar ODEs. Besides TWs, we obtain one more solution

$$v(x,t) = \cos(x - 2t) + \sinh(x + 2t), \tag{0.44}$$

which is determined up to an arbitrary smooth multiplier $C(t)$. This is precisely the *complexiton* solution that was constructed rather recently, [405]. Concerning a perturbed equation, note that $v(x,t) = \sin(x - 4t)$ solves $F_*[v] = 8$, and $v(x,t) = \cos(x - 2t) + \cosh(x + 2t)$ (cf. (0.44)) satisfies the equation $F_*[v] = 12$.

We have illustrated all types of known elementary soliton-type solutions of (0.31). All these solutions of the KdV equation can be constructed by the Wronskian method for integrable equations; a modern description is given in [407]. Similar DS reductions are also key for classes of non-integrable PDEs, though, of course, the consistency of DSs can be tricky and will be established in a few cases.

Sign-invariants for second-order parabolic equations (Chapter 8)

We also aim to emphasize a new interesting aspect of our exact solutions. It turns out that, for *second-order* parabolic equations, many solutions on invariant subspaces W_n may induce so-called *sign-invariants*, which are nonlinear differential operators $\mathcal{H}[u] = H(x, u, Du, D^2u, ...)$ preserving both their signs on evolution orbits. For the Cauchy problem in $I\!R^N \times I\!R_+$ with initial data $u_0(x)$, this means that

$$\mathcal{H}[u_0(x)] \le 0 \ (\ge 0) \text{ in } I\!R^N \implies \mathcal{H}[u(x,t)] \le 0 \ (\ge 0) \text{ in } I\!R^N \text{ for } t > 0. \quad (0.45)$$

Such *partial differential inequalities* are naturally associated with different *barrier techniques* in the theory of parabolic equations, where the Maximal Principle applies to control the operator signs on evolution orbits. Barrier approaches are the cornerstone of regularity and asymptotic theory of linear and nonlinear parabolic PDEs. For instance, classical Schauder and Bernstein estimates, as well as the Nash–Moser technique, are based on the Maximum Principle and use barrier analysis of parabolic differential inequalities. We refer to monographs [206, 245, 442, 472, 550].

In (0.45), the sign-invariant $\mathcal{H}$ preserves both signs, ≥ 0 and ≤ 0, on solutions of the parabolic PDE. The connection with invariant subspaces W_n is as follows:

$$\mathcal{H}[u] = 0 \quad \text{on} \ \ W_n \ \ (\text{or on a set} \ \ M \subset W_n). \quad (0.46)$$

Vice versa, the equality (0.46) can be used to determine the corresponding sign-invariant $\mathcal{H}[u]$. We will show how to reconstruct such operators $\mathcal{H}$ by means of the structure of the invariant (or partially invariant) subspaces W_n. Of course, (0.46) is then a differential constraint generating solutions on W_n. It is important that, unlike just the constraint (0.46), the partial differential inequalities (0.45) characterize evolution properties of wider classes of solutions than simply those on W_n.

Discrete operators: applications to moving mesh methods and lattices (Chapter 9)

We will also deal with *discrete* nonlinear operators F for which we prove some results on the existence of linear invariant subspaces W_n and construct exact solutions of some discrete equations. As a further application, we describe invariant aspects of *moving mesh methods* (MMMs), which have become a powerful tool of numerical solutions of nonlinear PDEs possessing blow-up and other evolution singularities. We also introduce exact solutions on invariant subspaces for some *anharmonic lattices* associated with different evolution PDEs.

Prerequisites: a GUIDE *on models, nonlinear PDEs, and solutions*

The book is meant for advanced graduate level students and does not assume a knowledge of the fundamentals of the mathematical theory of PDEs and functional analysis, except the basics of the Maximum Principle for second-order parabolic equations in the theory of sign-invariants in Chapter 8 (though we have included necessary preliminary information). The knowledge of some standard aspects of ODE theory would be useful for performing some analytical manipulations and phase-plane diagrams. Sometimes our discussions around exact solutions on invariant subspaces include specific aspects of PDE theory. These parts can be omitted without causing any future confusion.

We hope that the present methods for parabolic, hyperbolic, KdV-type, and nonlinear dispersion PDEs, as well as discrete equations, will be useful for the readers with a mathematical background that is not necessarily applied or pure. We expect that several aspects of our analysis can be fruitful for researchers and students specializing in mechanics, physics, engineering, and those working with nonlinear PDEs.

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Sergey R. Svirshchevskii,
Bath–Moscow, November 2004

List of Frequently Used Abbreviations

1D – one-dimensional
AKNS – Ablowitz–Kaup–Newell–Segur
BI – Born–Infeld
CP – Cauchy problem
DS – dynamical system
FW – Fornberg–Whitham
FBP – free-boundary problem
FK – Frenkel–Kontorova
FFCH – Fuchssteiner–Fokas–Camassa–Holm
GSV – generalized separation of variables
GT – Gibbons–Tsarev
GN – Green–Naghdi
HE – heat equation
KP – Kadomtsev–Petviashvili
KFG – Kármán–Fal'kovich–Guderley
KS – Kuramoto–Sivashinsky
KdV – Korteweg–de Vries
KPPF – Kolmogorov–Petrovskii–Piskunov–Fisher
LL – Landau–Lifshitz
LRT – Lin–Reissner–Tsien
MP – Maximum Principle
M-A – Monge-Ampère
MMM – moving mesh method
ODE – ordinary differential equation
PDE – partial differential equation
PDI – partial differential inequality
PBBM – Peregrine–Benjamin–Bona–Mahoney
PME – porous medium equation
RH – Rosenau–Hyman
sdYM – self-dual Yang–Mills
SI – sign-invariant
TFE – thin film equation
TW – traveling wave
ZKB – Zel'dovich–Kompaneetz–Barenblatt

CHAPTER 1

Linear Invariant Subspaces in Quasilinear Equations: Basic Examples and Models

We begin this chapter with a few well-known and even classical examples of exact solutions of various nonlinear PDEs of mathematical physics with quadratic or cubic nonlinearities. We will treat these solutions from the point of view of the linear subspaces invariant under appropriate nonlinear operators. Indeed, ideas of low-dimensional reductions of evolution equations restricted to linear subspaces or manifolds have been known for a long time. Certainly there are other interesting solutions of a similar invariant nature, which we are not aware of. It would be interesting to detect more examples which date back to the first half of the twentieth and, hopefully, to the nineteenth century.

The rest of the chapter is devoted to further examples, in which we introduce empirical tools to study general properties of invariant subspaces, spaces of the corresponding nonlinear ordinary differential operators, and exact solutions. More systematic and advanced mathematics is developed in Chapter 2.

1.1 History: first examples of solutions on invariant subspaces

1.1.1 Five models from gas dynamics

Example 1.1 (**Ovsiannikov solutions**) In 1948, L.V. Ovsiannikov [455] showed that the study of spatial transonic flows of ideal polythropic gas leads to the following quasilinear elliptic-hyperbolic equation* in $I\!R^3$:

$$\boxed{\Delta u \equiv u_{xx} + u_{yy} = [(u-1)^2]_{zz} \equiv F[u],} \tag{1.1}$$

where $u = u(x, y, z)$ is the reduced projection of the flow velocity on the z-axis. Equation (1.1) is hyperbolic in the domain $\{(x, y, z) \in I\!R^3 : \; u(x, y, z) > 1\}$ and is elliptic in $\{u < 1\}$. Ovsiannikov detected its exact solutions in the following form:

$$u(x, y, z) = 1 + u_0(x, y) + u_1(x, y)z + \tfrac{1}{12}\, u_2(x, y)z^2. \tag{1.2}$$

Substituting this expression into (1.1) and equating the coefficients of 1, z, and z^2 (projections onto these functions) to zero yields that the functions $u_0(x, y)$, $u_1(x, y)$, and $u_2(x, y)$ satisfy the following system of elliptic PDEs in $I\!R^2$:

$$\begin{cases} \Delta u_0 = \frac{1}{3} u_2 u_0 + 2u_1^2, \\ \Delta u_1 = u_2 u_1, \\ \Delta u_2 = u_2^2. \end{cases} \tag{1.3}$$

Actually, existence of such solutions as (1.2), (1.3) reflects the straightforward fact

* We put boxes around the main PDEs possessing solutions on invariant subspaces.

that the linear subspace defined by the span $W_3 = \mathcal{L}\{1, z, z^2\}$ is *invariant* under the nonlinear operator F in (1.1), in the sense that

$$\text{for any } u \in W_3, \quad F[u] \in W_3 \quad (\text{or} \quad F[W_3] \subseteq W_3).$$

This invariance of W_3 under F is understood, as in standard linear algebra.

Of course, the system of three PDEs (1.3) is not easy to study in general, but it is a low-dimensional system for three functions defined in $I\!R^2$, unlike the original PDE (1.1) that is posed in $I\!R^3$. Moreover, the last equation for u_2 is independent of the others and can be studied separately.† Once this has been solved and a suitable function $u_2(x, y)$ has been determined, the rest of (1.3) yields a system of linear elliptic PDEs for u_0 and u_1 that can be studied by standard techniques.

This class of *Ovsiannikov's solutions*, as well as applied problems of analytical fluid mechanics [29] and other important applications in combustion theory [594], stimulated mathematical interests to such canonical semilinear elliptic PDEs

$$\Delta u = f(u), \tag{1.4}$$

with a given nonlinear function $f(u)$. The typical power nonlinearity is $f(u) = \pm u^p$, with the exponent $p > 1$, or $f(u) = |u|^{p-1}u$ for solutions u of changing sign. In elliptic theory, two classes of problems were most popular:

(i) the Dirichlet problem in a bounded domain $\Omega \subset I\!R^N$, with $u = 0$ on the boundary $\partial\Omega$, and

(ii) the problem in the whole space $I\!R^N$.

In the former case, for $f(u) = -u^p$, the famous *Sobolev critical exponent* occurs

$$p_S = \tfrac{N+2}{N-2} \quad \text{for } N \geq 3 \quad (p_S = \infty \ \text{ for } N = 1, \text{ or } 2).$$

The global and local properties of solutions are completely different in the subcritical, $p < p_S$, and the supercritical, $p > p_S$, ranges. In the critical case, $p = p_S$, there exists the explicit *Loewner–Nirenberg solution* [400]

$$u(x) = \left[\frac{N(N-2)\lambda^{\frac{2}{N-2}}}{N(N-2)+\lambda^{\frac{4}{N-2}}|x|^2} \right]^{(N-2)/2},$$

where $\lambda > 0$ is arbitrary. The questions of existence, nonexistence, and multiplicity of solutions for equation (1.4) have been actively studied in general elliptic theory during the last forty years. We refer to classical papers [331, 464] and to Mitidieri–Pohozaev [425] for history, references, and a systematic treatment of the nonexistence problem via the nonlinear capacity approach.

Example 1.2 (**Von Mises solutions**) Consider the potential equation of the 1D flow of a compressible gas

$$\boxed{\Phi_{tt} + 2\Phi_x\Phi_{xt} + a\Phi_{xx} + b\Phi_t\Phi_{xx} + c(\Phi_x)^2\Phi_{xx} = 0,} \tag{1.5}$$

where a, b, and c are constants. R. von Mises introduced the following class of exact

† L.V. Ovsiannikov was the supervisor, who proposed the equation $\Delta u = u^2$ in a bounded domain to S.I. Pohozaev in 1958 [467], that led to "Pohozaev's Identities" (1965) [464] in elliptic theory.

solutions of (1.5) (this is mentioned in Titov [553], we have not succeeded in tracing out the original von Mises work):

$$\Phi(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2. \tag{1.6}$$

Plugging (1.6) into (1.5) yields an ODE system for the expansion coefficients $\{C_i\}$,

$$\begin{cases} C_1'' = -2C_2C_2' - 2aC_3 - 2bC_3C_1' - 2cC_2^2C_3, \\ C_2'' = -4(C_2C_3)' - 2bC_3C_2' - 8cC_2C_3^2, \\ C_3'' = -2(4+b)C_3C_3' - 8cC_3^3. \end{cases} \tag{1.7}$$

Similar to the example above, the finite expansion (1.6) indicates that the operator on the left-hand side of (1.5) composed of linear, quadratic, and cubic terms preserves the 3D subspace

$$W_3 = \mathcal{L}\{1, x, x^2\}.$$

The last equation for C_3 in (1.7) can be solved independently in terms of Jacobi elliptic functions.

In view of differential manipulations with expansion coefficients in square products on the right-hand side of (1.7), it is relevant to call W_3 an *invariant module*, which in Algebra [373, Ch. III] is used as a generalization of linear vector spaces with a field replaced by a ring; see Section 2.8. For simplicity, we sometimes keep using the term subspace if no confusion is likely.

Example 1.3 (**Guderley solutions**) Consider the potential equation for transonic flow written as

$$\boxed{\Phi_{yy} + \tfrac{N-1}{y}\,\Phi_y = (\gamma+1)\Phi_x\Phi_{xx} \quad \text{in } \{x>0,\ y>0\},} \tag{1.8}$$

where $N = 1$ or 2 and $\gamma = \frac{c_p}{c_v} > 1$ is the fixed constant, called the *adiabatic exponent*. From Guderley's book [267, p. 65]: "The solution of the exact potential equation of the flow in the throat of De Laval nozzle has been obtained by Meyer [422] in the form of a series expansion. We shall show that the first term of this expansion represents the exact solution of the equation for transonic flow." K.G. Guderley presented two explicit solutions of (1.8),

$$\Phi(x,y) = \tfrac{c}{2}\,x^2 + \tfrac{c^2}{2}\,(\gamma+1)xy^2 + \tfrac{c^3}{24}(\gamma+1)^2y^4 \quad \text{for} \quad N=1,$$

$$\Phi(x,y) = \tfrac{c}{2}\,x^2 + \tfrac{c^2}{4}(\gamma+1)xy^2 + \tfrac{c^3}{64}(\gamma+1)^2y^4 \quad \text{for} \quad N=2;$$

see [267, p. 66, 69] (we keep the original notation).

These explicit *Guderley's solutions* belong to the subspace $W_3 = \mathcal{L}\{1, x, x^2\}$ which is invariant under the quadratic operator $F[\Phi] = \Phi_x\Phi_{xx}$ given on the right-hand side of (1.8). In addition, Guderley described properties of solutions

$$\Phi(x,y) = x^3 f(y)$$

belonging to the 1D invariant subspace $\mathcal{L}\{x^3\}$ of F. "The exponent of x could then be chosen such that the powers of x would cancel out from the equation," [267, p. 69]. Such solutions were also studied by H. Görtler [259].

These results altogether are expressed by saying that operator F admits the 4D invariant subspace

$$W_4 = \mathcal{L}\{1, x, x^2, x^3\}$$

with exact solutions

$$\Phi(x, y) = C_1(y) + C_2(y)x + C_3(y)x^2 + C_4(y)x^3$$

governed by the eighth-order DS

$$\begin{cases} C_1'' + \frac{N-1}{y}\, C_1' = 2(\gamma + 1)C_2C_3, \\ C_2'' + \frac{N-1}{y}\, C_2' = 2(\gamma + 1)\big(2C_3^2 + 3C_2C_4\big), \\ C_3'' + \frac{N-1}{y}\, C_3' = 18(\gamma + 1)C_3C_4, \\ C_4'' + \frac{N-1}{y}\, C_4' = 18(\gamma + 1)C_4^2. \end{cases}$$

Guderley's solutions correspond to $C_4(y) \equiv 0$. The last equation is the radial version of the quadratic elliptic PDE (1.4) with $f(u) = 18(\gamma + 1)u^2$.

Example 1.4 (**Titov's solutions**) It was shown by S.S. Titov [555] that the same quadratic operator $F[\Phi] = \Phi_x \Phi_{xx}$ admits another 3D subspace

$$W_3 = \mathcal{L}\big\{1, x^{\frac{3}{2}}, x^3\big\}.$$

This gives *Titov's solutions* of (1.8)

$$\Phi(x, y) = C_1(y) + C_2(y)x^{\frac{3}{2}} + C_3(y)x^3 \in W_3,$$

where the coefficients of the expansion satisfy the following ODE system:

$$\begin{cases} C_1'' + \frac{N-1}{y}\, C_1' = \frac{9}{8}(\gamma + 1)C_2^2, \\ C_2'' + \frac{N-1}{y}\, C_2' = \frac{45}{4}(\gamma + 1)C_2C_3, \\ C_3'' + \frac{N-1}{y}\, C_3' = 18(\gamma + 1)C_3^2. \end{cases}$$

Example 1.5 (**Ryzhov–Shefter solutions**) The *Lin–Reissner–Tsien* (LRT) *equation*

$$-\varphi_x\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - 2\varphi_{xt} = 0 \quad \text{in } I\!R^3 \times I\!R \tag{1.9}$$

was discovered in 1948 [395] as a model for oscillation of a thin profile in transonic flow. O.S. Ryzhov and G.M. Shefter derived this equation later "...for the investigation of nonstationary processes in the vicinity of the surface of transition through the speed of sound in Laval nozzles when the dimensions and form of the critical cross section change with time sufficiently rapidly," [505, p. 939]. In cylindrical coordinates

$$\begin{cases} z = r\cos\vartheta, \\ y = r\sin\vartheta, \end{cases}$$

(1.9) takes the form

$$\boxed{-\varphi_x\varphi_{xx} + \varphi_{rr} + \frac{1}{r}\,\varphi_r + \frac{1}{r^2}\,\varphi_{\vartheta\vartheta} - 2\varphi_{xt} = 0,} \tag{1.10}$$

and admits the following exact *Ryzhov–Shefter solutions*, 1959 (we keep the original notation from [505]):

$$\varphi = \lambda(t)x + \tfrac{1}{2}\,A(t)[x - \Delta(t)]^2 + h_1(\vartheta, t)[x - \Delta(t)]r^2 + h_2(\vartheta, t)r^4. \tag{1.11}$$

The expansion coefficients solve the following *PDE system*:

$$\begin{cases} \lambda_t + \frac{1}{2} A\lambda = A\Delta_t, \\ 2A_t + A^2 = h_{1\vartheta\vartheta} + 4h_1, \\ 2h_{1t} + h_1 A = h_{2\vartheta\vartheta} + 16h_2. \end{cases}$$

As shown in Example 1.3, solutions (1.11) are associated with the invariant subspace $W_3 = \mathcal{L}\{1, x, x^2\}$ of the operator $\varphi_x\varphi_{xx}$. There exists its 4D invariant extension $W_4 = \mathcal{L}\{1, x, x^2, x^3\}$. There are other more detailed invariant interpretations. For instance, taking the subspace $W_6 = \mathcal{L}\{1, x, r^2, x^2, xr^2, r^4\}$ and hence solutions

$$\varphi(x, r, \vartheta, t) = C_1 + C_2 x + C_3 x^2 + C_4 r^2 + C_5 x r^2 + C_6 r^4$$

yields the following system of PDEs for the coefficients $\{C_i(\vartheta, t)\}$:

$$\begin{cases} 2C_2C_3 = 4C_4 + C_{4\vartheta\vartheta} - 2C_{2t}, \\ 4C_3^2 = 4C_5 + C_{5\vartheta\vartheta} - 4C_{3t}, \\ 2C_3C_5 = 16C_6 + C_{6\vartheta\vartheta} - 2C_{5t}, \\ C_{1\vartheta\vartheta} = 0, \;\; C_{2\vartheta\vartheta} = 0, \;\; C_{3\vartheta\vartheta} = 0. \end{cases} \tag{1.12}$$

Solutions (1.11) then correspond to

$$C_1 = \tfrac{1}{2} A\Delta^2, \;\; C_2 = \lambda - A\Delta, \;\; C_3 = \tfrac{1}{2} A, \;\; C_4 = -h_1\Delta, \;\; C_5 = h_1, \;\; C_6 = h_2.$$

The general solution of (1.12) is as follows:

$$C_1 = a_1(t)\vartheta + b_1(t), \;\; C_2 = a_2(t)\vartheta + b_2(t), \;\; C_3 = a_3(t)\vartheta + b_3(t),$$

$$\begin{aligned} C_4 &= K_1 \cos 2\vartheta + K_2 \sin 2\vartheta + \alpha\vartheta^2 + \beta\vartheta + \gamma, \\ C_5 &= \tilde{K}_1 \cos 2\vartheta + \tilde{K}_2 \sin 2\vartheta + \tilde{\alpha}\vartheta^2 + \tilde{\beta}\vartheta + \tilde{\gamma}, \\ C_6 &= (\mu_1 + \nu_1\vartheta)\cos 2\vartheta + (\mu_2 + \nu_2\vartheta)\sin 2\vartheta, \end{aligned}$$

where $K_{1,2}(t)$, $\tilde{K}_{1,2}(t)$ are arbitrary functions, and other coefficients $\alpha(t)$, $\beta(t)$, ... are expressed by functions $\{a_i(t), b_i(t)\}$ by substituting into the PDE (1.10). Other PDE systems occur by studying (1.10) on the 3D invariant subspace $\mathcal{L}\{1, r^2, r^4\}$.

1.1.2 Nonlinear wave equation

Example 1.6 (**Quadratic wave equation**) Ovsiannikov [456, p. 286] performed a classification of group-invariant solutions of the following system:

$$\begin{cases} u_y = v_x, \\ uu_x = v_y, \end{cases}$$

which also describes transonic gas flows. This is equivalent to the *quadratic wave equation* $u_{yy} = (uu_x)_x$, or, replacing $y \mapsto t$,

$$\boxed{u_{tt} = F[u] \equiv \tfrac{1}{2}(u^2)_{xx} \quad \text{in } I\!R \times I\!R.} \tag{1.13}$$

Olver and Rosenau introduced the following explicit solution of (1.13):

$$u(x, t) = \alpha t^2 + at + b \pm \sqrt{2\alpha}\, x, \quad \text{where } \alpha > 0, \tag{1.14}$$

that is "...not obtainable by partial invariance by appending the second order side condition

$$u_{tt} = 2\alpha, \tag{1.15}$$

where α is a constant," [448, p. 112].

These solutions belong to the 3D invariant subspace $W_3 = \mathcal{L}\{1, x, x^2\}$ preserved by operator F in (1.13). Plugging

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 \in W_3 \tag{1.16}$$

into the PDE yields the following DS:

$$\begin{cases} C_1'' = C_2^2 + 2C_1C_3, \\ C_2'' = 6C_2C_3, \\ C_3'' = 6C_3^2. \end{cases} \tag{1.17}$$

The solutions (1.14) then correspond to the particular case $C_3(t) \equiv 0$, where the second ODE is $C_2'' = 0$. Choosing $C_2(t) = \pm\sqrt{2\alpha}$ yields $C_1'' = 2\alpha$, whence come solutions (1.14). On the other hand, taking $C_2(t) = \alpha t$ ($\alpha \neq 0$) leads to the new polynomial solution

$$u(x,t) = \tfrac{\alpha^2}{12}\, t^4 + at + b + \alpha t x.$$

Fixing now a nontrivial solution $C_3(t) = \frac{1}{t^2}$ of (1.17) yields Euler's ODE for C_2,

$$t^2 C_2'' = 6C_2 \quad \Longrightarrow \quad C_2(t) = At^3 + \tfrac{B}{t^2}, \tag{1.18}$$

where A and B are arbitrary constants of integration. Finally, solving the first ODE yields a more general family of solutions on W_3 ($D,\ E \in I\!R$),

$$u(x,t) = \tfrac{A^2t^8}{54} + \tfrac{ABt^3}{2} + \tfrac{B^2}{4t^2} + Dt^2 + \tfrac{E}{t} + \left(At^3 + \tfrac{B}{t^2}\right)x + \tfrac{1}{t^2}\,x^2.$$

For $\alpha = 0$ in the side condition (1.15), the explicit solution [448, p. 112] is

$$u(x,t) = \pm(t+a)\sqrt{x+b},$$

which, after translation, belongs to the 1D invariant subspace $W_1 = \mathcal{L}\{\sqrt{x}\}$. The dynamics on W_1 with solutions $u(x,t) = C(t)\sqrt{x}$ is described by the ODE

$$C'' = 0.$$

1.1.3 Quadratic Boussinesq-type equations

Example 1.7 (**Olver–Rosenau solution**) In 1986, Olver and Rosenau [448] considered the following *Boussinesq-type equation*:

$$\boxed{u_{tt} = F[u] \equiv u_{xx} + \beta(u^2)_{xx} + \gamma\, u_{xxtt} \quad \text{in } I\!R \times I\!R,} \tag{1.19}$$

which was introduced by Boussinesq in 1871 [74] for studying long waves in shallow water. This equation also describes longitudinal waves in solid rods with effects of lateral inertia included. In [448], the following *Olver–Rosenau solution* of (1.19) was constructed:

$$u(x,t) = -\tfrac{1}{2\beta} + \tfrac{3\gamma}{2\beta t^2} + \tfrac{1}{2\beta t^2}\,x^2, \tag{1.20}$$

where parameters of translations in x and t are not included. Therefore, this is a two-parameter family of solutions.

Such a simple solution initiated a discussion on general invariant group origins of exact solutions. Written in the form of

$$u(x,t) = -\tfrac{1}{2\beta} + \varphi(t)\psi(x), \quad \text{with} \;\; \varphi(t) = \tfrac{1}{2\beta t^2} \;\; \text{and} \;\; \psi(x) = x^2 + 3\gamma,$$

the solution looks like a standard affine version of a separable solution $\big($i.e., becoming separable after shifting in u by $-\frac{1}{2\beta}\big)$, and hence is expected to be obtained by local group approaches dealing with groups of scaling or other non-classical methods. Nevertheless, it was proved that "...the entire two-parameter family could not have come from a single local group," [448, p. 111].

Concerning the invariant subspace treatment of (1.20), it is easy to observe the subspace $W_2 = \mathcal{L}\{1, x^2\}$ that is invariant under the quadratic operator F in (1.19). As done in Example 1.6, we take solutions (1.16) on the extended subspace W_3, and, on substitution into the PDE, obtain the system

$$\begin{cases} C_1'' = 2C_3 + 2\beta C_2^2 + 4\beta C_1 C_3 + 2\gamma\, C_3'', \\ C_2'' = 12\beta C_2 C_3, \\ C_3'' = 12\beta C_3^2. \end{cases}$$

As far as explicit solutions are concerned, the last equation gives

$$C_3(t) = \tfrac{1}{2\beta t^2}.$$

Substituting into the second ODE yields Euler's equation (1.18). Finally, the following solutions of the Boussinesq-type equation (1.19) are obtained:

$$\begin{aligned} u(x,t) = {} & -\tfrac{1}{2\beta} + \tfrac{\beta A^2 t^8}{27} + \beta A B t^3 + \big(\tfrac{3\gamma}{2\beta} + \tfrac{\beta B^2}{2}\big)\tfrac{1}{t^2} \\ & + D t^2 + \tfrac{E}{t} + \big(A t^3 + \tfrac{B}{t^2}\big)x + \tfrac{1}{2\beta t^2}x^2. \end{aligned}$$

Bearing in mind translations in x and t, this is a six-parameter family of solutions which, for $A = B = D = E = 0$, gives the Olver–Rosenau solution (1.20).

1.1.4 Examples from reaction-diffusion-absorption theory

We next turn the attention to nonlinear reaction-diffusion-absorption PDEs which have given a record number of various exact solutions, including those on invariant subspaces. The basic nonlinear diffusion operator in such parabolic equations was already derived by J. Boussinesq [77] , who, in 1904, studied non-stationary flows of soil water under the presence of *free surface*, and derived the PDE

$$u_t = \gamma\,(u u_x)_x. \tag{1.21}$$

Here, $\gamma = \frac{k}{m}$ is a positive constant, where k is the *filtration coefficient* and m is the *porosity* of soil. The function $u = u(x,t)$ is the pressure of the ground water. Here, (1.21) is the quadratic *porous medium equation* (PME). Boussinesq also derived the exact solution of the PME (1.21) in separate variables

$$u(x,t) = X(x)T(t).$$

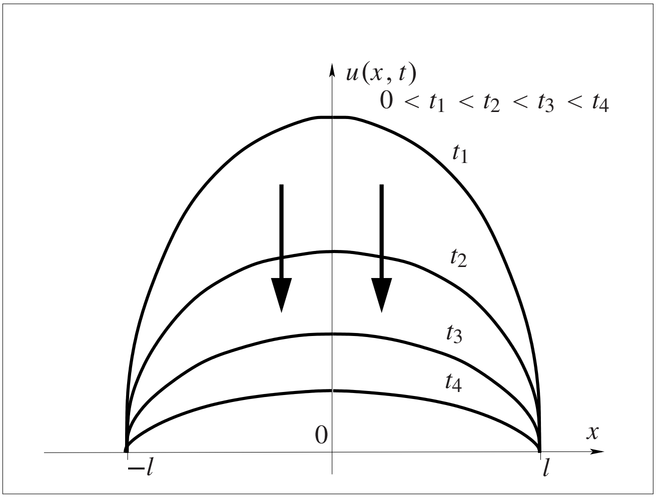

Figure 1.1 Evolution described by the Boussinesq solution (1.22).

Plugging into (1.21) yields two independent ODEs for functions $T(t)$ and $X(x)$,

$$\frac{T'}{T^2} = \gamma \, \frac{(XX')'}{X} = -\lambda,$$

where $\lambda > 0$ is the parameter of separation. Solving the first equation leads to the so-called *Boussinesq solution*

$$u(x,t) = \frac{X(x)}{\lambda t}, \tag{1.22}$$

where $X \geq 0$ is a solution of the ODE

$$\gamma \, (XX')' = -\lambda X.$$

Solving this ODE on a bounded interval $x \in (-l, l)$ with the zero Dirichlet boundary conditions

$$X(-l) = X(l) = 0$$

yields the *Boussinesq ordered regime* that describes the time decay of solutions of the initial-boundary value problem for the PME on a bounded interval. See Figure 1.1. The fact that the Boussinesq solution (1.22) is asymptotically stable and that the corresponding decay rate $O\left(\frac{1}{t}\right)$ for $t \gg 1$ is correct for general solutions of the PME for arbitrary bounded initial data $u(x,0) = u_0(x) \geq 0$ was proved much later in the 1970s; see details and references in [245, Ch. 2].

For the PME in the whole space, i.e., for $x \in I\!R$ (the Cauchy problem), the famous *Zel'dovich–Kompaneetz–Barenblatt* (ZKB) *solution* is key for stability analysis as $t \to \infty$. We have discussed the ZKB solution in the Introduction (see (0.23)) and refer to a great amount of literature in [245] concerning the foundation of PME theory.

More complicated spatio-temporal patterns can occur for the PME with extra low-order operators, such as reaction or absorption ones. There are many models of this type. For instance, the PME with a nonlinear *convection* term

$$u_t = \gamma (uu_x)_x + \beta uu_x,$$

also known as the *diffusion-convection Boussinesq equation*, occurs in the various fields of petroleum technology and ground water hydrology. Let us begin with another example, where the interesting exact solutions on invariant subspaces arise.

Example 1.8 (**PME with absorption: Kersner's solution**) Consider the exact solution constructed by R. Kersner in the middle of the 1970s; see references in [333, 334]. At that time, Kersner was a PhD student supervised by A.S. Kalashnikov, who performed in the 1960s-70s the pioneering research of localization-extinction phenomena for nonlinear degenerate parabolic PDEs, including equations from diffusion-absorption theory. Key results are reflected in his fundamental survey [309]. Among Kalashnikov's other PDE models, there is a famous diffusion-absorption equation with the *critical* absorption exponent

$$v_t = \big(v^\sigma v_x\big)_x - v^{1-\sigma}, \tag{1.23}$$

where $\sigma > 0$ is a parameter. In filtration theory, according to G.I. Barenblatt, absorption power-like terms $-v^p$ describe the phenomenon of seepage on a permeable bed. The Cauchy problem for equation (1.23) admits weak nonnegative compactly supported solutions. The first explicit localized solutions of such diffusion-absorption equations were constructed by L.K. Martinson and K.B. Pavlov in 1972; see details and references in [509, p. 21].

Let us derive explicit solution of (1.23) using the invariant subspaces. Introducing the *pressure* variable from filtration theory, $u = v^\sigma$, yields a PDE with the quadratic differential operator and a constant sink,

$$\boxed{u_t = F[u] \equiv uu_{xx} + \tfrac{1}{\sigma}(u_x)^2 - \sigma.} \tag{1.24}$$

Clearly, operator $F[u]$ preserves the 2D subspace $W_2 = \mathcal{L}\{1, x^2\}$, since

$$F[C_1 + C_2x^2] = 2C_1C_2 - \sigma + 2\big(1 + \tfrac{2}{\sigma}\big)C_2^2x^2 \in W_2.$$

Therefore, (1.24) admits solutions

$$u(x,t) = C_1(t) + C_2(t)x^2, \tag{1.25}$$

with the expansion coefficients $C_1(t)$ and $C_2(t)$ satisfying the dynamical system

$$\begin{cases} C_1' = 2C_1C_2 - \sigma, \\ C_2' = 2\big(1 + \frac{2}{\sigma}\big)C_2^2. \end{cases} \tag{1.26}$$

Integrating the uncoupled second ODE and substituting

$$C_2(t) = -\tfrac{\sigma}{2(\sigma+2)t}$$

into the first equation yields *Kersner's solution* (1976)

$$u(x,t) = \big[A_0 t^{-\frac{\sigma}{\sigma+2}} - \tfrac{\sigma(\sigma+2)}{2(\sigma+1)}\, t - \tfrac{\sigma}{2(\sigma+2)t}\, x^2\big]_+,$$

where A_0 is an arbitrary constant. Despite its elementary structure, the solution is

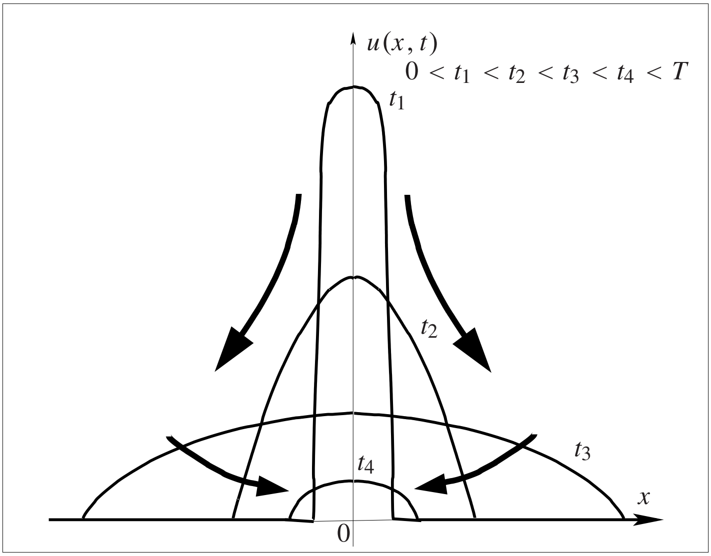

Figure 1.2 Finite-time extinction for the PME with absorption (1.23) described by Kersner's solutions (1.25); T is the extinction time, so $u(x, T) \equiv 0$.

not group-invariant if $A_0 \neq 0$. The positive part $[\cdot]_+$ determines weak solutions of (1.24) with finite interfaces, so they describe interesting and principal phenomena of non-Darcy interface propagation with turning points, extinction patterns, quenching, etc. Figure 1.2 shows this unusual extinction behavior. Similar explicit solutions also exist for the multi-dimensional PME with absorption in $I\!R^N \times I\!R_+$ (Martinson, 1979, [414])

$$u_t = \nabla \cdot (u^\sigma \nabla u) - u^{1-\sigma},$$

and for other extended PME-type models, see [509, p. 103].

Example 1.9 (**Oron–Rosenau solution**) In 1986, A. Oron and P. Rosenau considered the following fast diffusion equation with absorption [453]:

$$v_t = (\sqrt{v}\,)_{xx} - \sqrt{v}, \tag{1.27}$$

which, in plasma physics, describes energy diffusion in a strong magnetic field in the presence of energy sinks due to plasma radiation. It was shown that (1.27) admits the *Oron–Rosenau solution*

$$v(x,t) = B^2(x)\big(C_0 \textstyle\int \frac{\mathrm{d}x}{B^2(x)} - t\big)^2, \tag{1.28}$$

where C_0 is a constant and $B(x)$ satisfies the ODE

$$B'' + 2B^2 - B = 0.$$

Bearing in mind the idea of invariant subspaces, we derive the quadratic version of (1.27) by setting $v = u^2$, to obtain the PDE

$$F[u] \equiv 2uu_t = u_{xx} - u. \tag{1.29}$$

In the space of smooth functions of the time-variable t, operator F in (1.29) admits the 2D subspace

$$W_2 = \mathcal{L}\{1, t\}.$$

Since

$$F[C_1 + C_2 t] = 2C_1 C_2 + 2C_2^2 t \in W_2,$$

there exist the corresponding solutions

$$u(x,t) = C_1(x) + C_2(x) t \in W_2. \tag{1.30}$$

On substitution into (1.29), we obtain the following fourth-order DS:

$$\begin{cases} C_1'' - C_1 = 2C_1 C_2, \\ C_2'' - C_2 = 2C_2^2. \end{cases}$$

Since $C_2 C_1'' = C_1 C_2''$, on integration, we have

$$C_2 C_1' = C_1 C_2' + C_0,$$

with a constant C_0. Integrating again yields

$$C_1(x) = C_0 C_2(x) \int \frac{\mathrm{d}x}{C_2^2(x)},$$

which yields the solution (1.28) with $B = -C_2$.

Example 1.10 (**Dyson–Newman solution**) In 1980, W.I. Newman [437] considered the following quasilinear parabolic equation:

$$\boxed{u_t = F[u] \equiv \tfrac{1}{2}(u u_x)_x + u(1-u).} \tag{1.31}$$

It is a quasilinear extension of the *Kolmogorov–Petrovskii–Piskunov–Fisher* (KPPF) *equation* of population genetics,

$$u_t = \tfrac{1}{2} u_{xx} + u(1-u),$$

which, since the 1930s, induced several fundamental directions in mathematical theory of nonlinear parabolic PDEs. The original KPP-paper (1937) [353] contains a number of famous mathematical ideas and results.

As stated in [437], using the idea from a personal communication with F. Dyson (1978), Newman looked for solutions composed of the hyperbolic cosine. To be precise, in terms of invariant subspaces, solutions take the form

$$u(x,t) = C_1(t) + C_2(t) \cosh x, \tag{1.32}$$

belonging to the subspace $W_2 = \mathcal{L}\{1, \cosh x\}$ which is invariant under the quadratic operator F in (1.31). Then the expansion coefficients satisfy the DS

$$\begin{cases} C_1' = -C_1^2 - \tfrac{1}{2} C_2^2 + C_1, \\ C_2' = -\tfrac{3}{2} C_1 C_2 + C_2. \end{cases} \tag{1.33}$$

Unlike a simpler quadratic DS (1.26), system (1.33) cannot be solved explicitly, but is integrated in quadratures, giving interesting properties of finite-front propagation and evolution to traveling waves in such nonlinear media. In particular, this

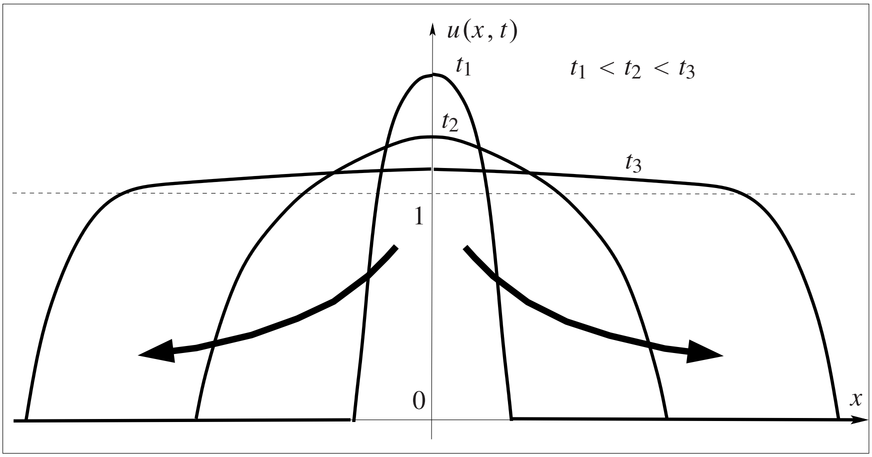

Figure 1.3 Formation of a traveling wave in the quasilinear model (1.31) described by Dyson–Newman's solution (1.32).

Dyson–Newman's solution propagates for $t \gg 1$ with the asymptotic speed $\frac{1}{2}$. See Figure 1.3. There are other applications of such solutions in the theory of reaction-absorption PDEs; see [509, p. 106] and references therein.

Example 1.11 (**Blow-up: Galaktionov's solution**) The semilinear heat equation

$$\boxed{u_t = F[u] \equiv u_{xx} + (u_x)^2 + u^2 \quad (u > 0),} \tag{1.34}$$

which was introduced to PDE theory in 1979 (see [245, Ch. 9] for history), plays a decisive role in blow-up combustion problems. This is the only *semilinear* reaction-diffusion equation of the second order that generates the *regional blow-up* (*S-regime*) for which bell-shaped solutions blow up on spatial intervals of the length 2π, [509, p. 294]. The change $u = \ln v$ transforms (1.34) into a semilinear heat equation,

$$v_t = v_{xx} + v \ln^2 v, \tag{1.35}$$

where the reaction term, $q(v) = v \ln^2 v$, is "almost" linear as $v \to +\infty$, but, nevertheless, satisfies the *Osgood criterion* of blow-up,

$$\int^{\infty} \frac{\mathrm{d}s}{q(s)} < \infty.$$

Therefore, solutions of (1.35) with sufficiently large initial data blow-up in finite time creating unusual localized blow-up patterns. Mathematical analysis of such blow-up localization phenomena uses specific stability techniques from singular perturbation theory and exact solutions; see details in books [509, Ch. 4] and [245, Ch. 9].

Operator $F[u]$ in (1.34) preserves the 2D subspace $W_2 = \mathcal{L}\{1, \cos x\}$ [232, 217]. Thus, for arbitrary C_1 and C_2,

$$F[C_1 + C_2 \cos x] = C_1^2 + C_2^2 + (2C_1 - 1)C_2 \cos x \in W_2.$$

This gives the exact solutions of (1.34) of the form

$$u(x,t) = C_1(t) + C_2(t) \cos x, \tag{1.36}$$

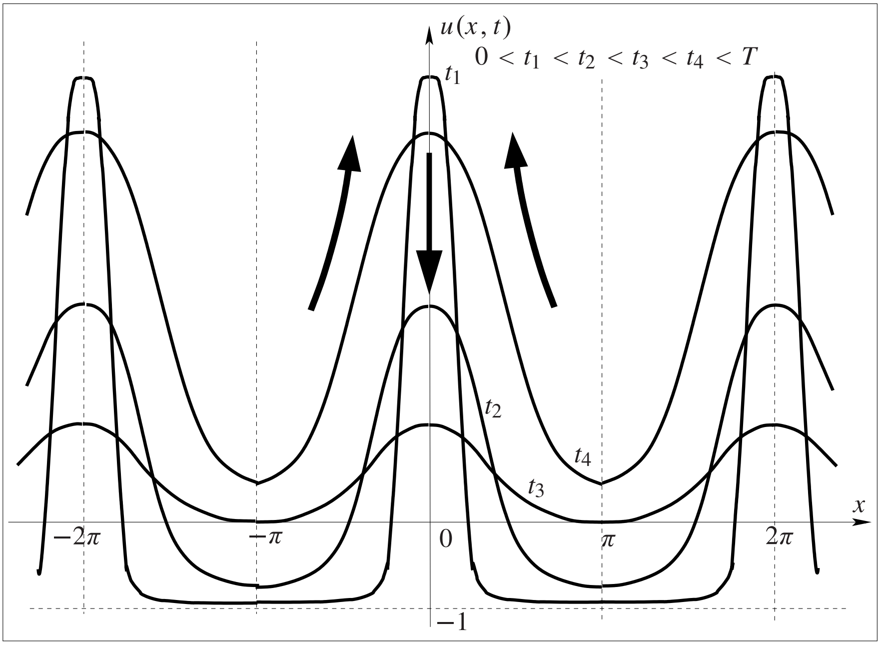

Figure 1.4 Non-monotone blow-up evolution of the invariant solutions (1.36), (1.37).

where the coefficients $C_1(t)$ and $C_2(t)$ satisfy the DS

$$\begin{cases} C_1' = C_1^2 + C_2^2, \\ C_2' = (2C_1 - 1)C_2. \end{cases} \tag{1.37}$$

This is not integrated explicitly and is studied on the phase-plane. In Figure 1.4 the non-monotone with time behavior of such explicit solutions is shown. These describe two singularities: the initial collapse of Dirac's delta-type initial data posed at points $\pm 2\pi k$, and finite-time blow-up afterwards. It is curious that this exact 2π-periodic (in x) *Galaktionov's solution* (1.36), (1.37) [217, 232] is not localized and blow-up globally as $t \to T^-$ at any point $x \in I\!R$. The blow-up rate is strikingly *non-uniform* [245, p. 242]: as $t \to T^-$, at maxima $x = 0$ and minima points $x = \pm\pi$, respectively,

$$u(0,t) = \tfrac{1}{T-t}\,(1 + o(1)) \to +\infty \qquad \text{and}$$

$$u(\pm\pi, t) = \tfrac{1}{2}\,|\ln(T-t)|(1 + o(1)) \to +\infty.$$

Nevertheless, the intersection comparison with such exact solutions guarantees that any bell-shaped blow-up solution of (1.34) is spatially *effectively* localized as $t \to T^-$ on intervals of length 2π, [245, p. 258].

Example 1.12 (**Parabolic system: King's first solution**) The following system of two second-order PDEs:

$$\boxed{\begin{cases} v_t = (w v_x - v w_x)_x, \\ w_t = v_{xx}\,, \end{cases}} \tag{1.38}$$

is a simple model for the solid-state diffusion of a substitutional impurity by a vacancy mechanism; see King [340] and references therein. In this paper, among other results on explicit and similarity solutions, it was shown that (1.38) admits exact polynomial *King's first solution*

$$\begin{aligned} v(x,t) &= C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3, \\ w(x,t) &= D_1(t) + D_2(t)x + D_3(t)x^2 + D_4(t)x^3, \end{aligned}$$

where the expansion coefficients solve the DS

$$\begin{cases} C_1' = 2(D_1C_3 - C_1D_3), \\ C_2' = 2(D_2C_3 - C_2D_3) + 6(D_1C_4 - C_1D_4), \\ C_3' = 6(D_2C_4 - C_2D_4), \\ C_4' = 4(D_3C_4 - C_3D_4), \\ D_1' = 2C_3, \quad D_2' = 6C_4, D_3' = 0, \quad D_4' = 0. \end{cases}$$

Here, operators in the right-hand sides of (1.38) preserve the 4D subspace $W_4 = \mathcal{L}\{1, x, x^2, x^3\}$. For the operator $F_1[v,w] = (wv_x - vw_x)_x$ from the first equation, this means that $F_1 : W_4 \times W_4 \to W_4$.

The second polynomial expansion detected in [340] is as follows:

$$\begin{aligned} v(x,t) &= C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3 + C_5(t)x^4, \\ w(x,t) &= D_1(t) + D_2(t)x + D_3(t)x^2, \end{aligned}$$

with the resulting DS

$$\begin{cases} C_1' = 2(D_1C_3 - C_1D_3), \\ C_2' = 2(D_2C_3 - C_2D_3) + 6D_1C_4, \\ C_3' = 6D_2C_4 + 12D_1C_5, \\ C_4' = 4D_3C_4 + 12D_2C_5, \\ C_5' = 10D_3C_5, \quad D_1' = 2C_3, \\ D_2' = 6C_4, \quad D_3' = 12C_5. \end{cases}$$

Note that components v and w belong to different subspaces,

$$v \in W_5 = \mathcal{L}\{1, x, x^2, x^3, x^4\}, \quad w \in W_3 = \mathcal{L}\{1, x, x^2\}, \quad \text{so} \quad F_1 : W_5 \times W_3 \to W_5.$$

Example 1.13 (**Fast diffusion equation: King's second solution**) The following construction is also due to King [342]. Using in the fast diffusion equation

$$v_t = \left(v^{-\frac{3}{2}} v_x\right)_x$$

the pressure transformation $u = v^{-3/2}$ reduces it to the equation with quadratic nonlinearities

$$\boxed{u_t = F[u] \equiv uu_{xx} - \tfrac{2}{3}(u_x)^2.} \tag{1.39}$$

This possesses exact *King's second solution*

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3.$$

Plugging this into the PDE yields the DS

$$\begin{cases} C_1' = 2C_3C_1 - \frac{2}{3}\,C_2^2, \\ C_2' = 6C_1C_4 - \frac{2}{3}\,C_2C_3, \\ C_3' = 2C_2C_4 - \frac{2}{3}\,C_3^2, \\ C_4' = 0. \end{cases}$$

This means that the quadratic operator F in (1.39) admits the 4D subspace

$$W_4 = \mathcal{L}\{1, x, x^2, x^3\} \quad \big(\text{and } \; F : W_4 \to W_3 = \mathcal{L}\{1, x, x^2\}\big).$$

The final two examples represent some remarkable invariant subspaces of the *maximal* dimension (a crucial theoretical aspect to be studied in the next chapter).

Example 1.14 (**Reaction-diffusion equation: 5D polynomial subspace**) Consider now the quasilinear equation with the negative exponent $\sigma = -\frac{4}{3}$, corresponding to the case of fast diffusion and a specific superlinear reaction term:

$$v_t = \big(v^{-\frac{4}{3}} v_x\big)_x + v^{\frac{7}{3}}. \tag{1.40}$$

Using the pressure transformation $u = v^{-4/3}$ yields the quadratic PDE

$$\boxed{u_t = F[u] \equiv uu_{xx} - \tfrac{3}{4}(u_x)^2 - \tfrac{4}{3}.} \tag{1.41}$$

It was shown in Galaktionov [220] that operator F preserves the 5D subspace

$$W_5 = \mathcal{L}\{1, x, x^2, x^3, x^4\},$$

so (1.41) admits the solution

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3 + C_5(t)x^4,$$

with the coefficients $\{C_i(t)\}$ satisfying the DS

$$\begin{cases} C_1' = 2C_1C_3 - \frac{3}{4}\,C_2^2 - \frac{4}{3}, \\ C_2' = 6C_1C_4 - C_2C_3, \\ C_3' = 12C_1C_5 + \frac{3}{2}\,C_2C_4 - C_3^2, \\ C_4' = 6C_2C_5 - C_3C_5, \\ C_5' = 2C_3C_5 - \frac{3}{4}\,C_4^2. \end{cases}$$

This fifth-order DS is not easy to study, but some particular features of such exact solutions can be obtained and used for comparison with general solutions of (1.40). The equation (1.40) admits single point blow-up, and the exact solutions describe interesting generic blow-up patterns.

Example 1.15 (**Reaction-absorption equation: 5D trigonometric subspace**) Consider an equation with the same fast diffusion and a different absorption term,

$$v_t = \big(v^{-\frac{4}{3}} v_x\big)_x - v^{-\frac{1}{3}}. \tag{1.42}$$

The pressure transformation $u = v^{-4/3}$ now yields the quadratic PDE

$$\boxed{u_t = F[u] \equiv uu_{xx} - \tfrac{3}{4}(u_x)^2 + \tfrac{4}{3}\,u^2.} \tag{1.43}$$

Here, F admits the 5D subspace spanned by trigonometric functions,

$$W_5 = \mathcal{L}\big\{1, \cos(\lambda x), \sin(\lambda x), \cos(\tfrac{\lambda x}{2}), \sin(\tfrac{\lambda x}{2})\big\}, \quad \text{where } \lambda = \tfrac{4}{\sqrt{3}}.$$

Therefore, the PDE (1.43) has exact solutions on W_5 [220]

$$u(x,t) = C_1 + C_2 \cos(\lambda x) + C_3 \sin(\lambda x) + C_4 \cos(\tfrac{\lambda x}{2}) + C_5 \sin(\tfrac{\lambda x}{2}),$$

where the coefficients $\{C_i(t)\}$ solve the DS

$$\begin{cases} C_1' = \frac{4}{3}\, C_1^2 - 4\big(C_2^2 + C_3^2\big) - \frac{1}{2}\, C_3^2, \\ C_2' = -\frac{8}{3}\, C_1 C_2 + \frac{1}{2}\big(C_4^2 - C_5^2\big), \\ C_3' = -\frac{8}{3}\, C_1 C_3 + C_4 C_5, \\ C_4' = \frac{4}{3}\, C_1 C_4 - 4(C_3 C_5 + C_2 C_4), \\ C_5' = \frac{4}{3}\, C_1 C_5 - 4(C_2 C_5 - C_3 C_4). \end{cases}$$

This DS is more difficult, though some key asymptotic properties of orbits can be detected that describe interface and extinction phenomena for (1.42).

1.2 Basic ideas: invariant subspaces and generalized separation of variables

1.2.1 Invariant subspaces

Following the above examples, consider a general first-order evolution PDE

$$u_t = F[u], \tag{1.44}$$

where F is a kth-order ordinary differential operator,

$$F[u] \equiv F\big(x, u, u_x, ..., D_x^k u\big).$$

Here, $F(\cdot)$ is a given sufficiently smooth function and D_x denotes $\frac{\partial}{\partial x}$.

Let $\{f_i(x),\ i = 1, ..., n\}$ be a finite set of $n \geq 1$ linearly independent functions, and let W_n denote their linear span,

$$W_n = \mathcal{L}\{f_1(x), \ ..., f_n(x)\}.$$

W_n is an n-dimensional linear subspace consisting of their linear combinations with real coefficients,

$$u = \sum_{i=1}^{n} C_i f_i, \quad \text{for any vector } \mathbf{C} = \{C_i\} \in I\!R^n.$$

The subspace W_n is said to be *invariant* under the given operator F if

$$F[W_n] \subseteq W_n,$$

and then F is said to *preserve* or *admit* W_n. As in linear algebra, this means

$$F\Big[\sum_{i=1}^{n} C_i f_i(x)\Big] = \sum_{i=1}^{n} \Psi_i(C_1, \ ..., C_n) f_i(x) \quad \text{for any } \mathbf{C} \in I\!R^n,$$

where $\{\Psi_i\}$ are the expansion coefficients of $F[u] \in W_n$ in the basis $\{f_i\}$.

It follows that *if the linear subspace W_n is invariant under F, then equation* (1.44) *has solutions of the form*

$$u(x,t) = \sum_{i=1}^{n} C_i(t) f_i(x), \tag{1.45}$$

where the coefficients $\{C_i(t)\}$ satisfy the dynamical system

$$C_i'(t) = \Psi_i(C_1(t), \dots, C_n(t)), \quad i = 1, \dots, n.$$

The PDE (1.44), which is an infinite-dimensional DS, being restricted to the invariant subspace W_n becomes an n-dimensional dynamical system.

1.2.2 First extension: second-order hyperbolic equations

A first extension is obvious: for the second-order evolution PDE

$$u_{tt} = F[u], \tag{1.46}$$

there exist solutions (1.45) governed by the $2n$th-order DS

$$C_i''(t) = \Psi_i(C_1(t), \dots, C_n(t)), \quad i = 1, \dots, n. \tag{1.47}$$

There are other easy generalizations to higher-order PDEs. For instance, if operator P is a linear *annihilator* of the subspace W_n, i.e., $P : W_n \to \{0\}$, then, for arbitrary operators F_1, the PDE

$$u_{tt} = F[u] + (P[u])\, F_1[u]$$

on W_n reduces to the same DS (1.47).

1.2.3 Second extension: invariant subspaces for delay-PDEs

If, for a given operator F, an invariant subspace W_n has been detected, one can find other types of equations of differential, integral, or functional types, which can be restricted to W_n. Another simple extension is to consider the functional *delay-PDE* corresponding to (1.44),

$$u_t(t) = F[u(t-1)], \tag{1.48}$$

where the right-hand side is defined for the solution $u(\cdot, t-1)$ with the 1-retarded time-argument. The discrete evolution mechanism of such equations is well-suited for various applications. Differential delay models appear in population genetics, bioscience problems, control theory, electrical networks with lossless transmission lines, etc.; see Remarks. Theory of functional delay-ODEs, to say nothing of the delay-PDEs, is not as advanced as that of standard differential equations. In particular, the questions of symmetries, constraints, reductions, and exact solutions are less developed, and, in many cases, it is not clear how to translate related notions to non-local-in-time functional operators.

In the present case, we arrive at the same invariant conclusion: if $F : W_n \to W_n$, then (1.48) admits exact solutions (1.45) for which the expansion coefficients solve the following system of delay-ODEs:

$$C_i'(t) = \Psi_i(C_1(t-1), \dots, C_n(t-1)), \quad i = 1, \dots, n.$$

Delay-ODEs are infinite-dimensional DSs, which are difficult to study, but are simpler than delay-PDEs (1.48).

1.2.4 Generalized separation of variables: first simple example

Let us present next an example explaining some features of the main problem of determining invariant subspaces for a given nonlinear operator. Consider the standard quadratic ordinary differential operator from reaction-diffusion theory

$$F[u] = \alpha(u_{xx})^2 + \beta u u_{xx} + \gamma\,(u_x)^2 + \delta u^2, \tag{1.49}$$

with arbitrary real parameters α, β, γ, and δ. Such operators occur in several applications that will be discussed later on. Consider a 2D subspace

$$W_2 = \mathcal{L}\{1, f(x)\}, \tag{1.50}$$

where the first basic function is constant 1, and the set $\{1, f(x)\}$ is assumed to be linearly independent. Obviously, the simplest 1D subspace $W_1 = \mathcal{L}\{1\}$ is invariant under F, since

$$F[1] = \delta \in W_1.$$

Therefore, we need to determine a single second function $f(x)$ from the invariance condition

$$F[W_2] \subseteq W_2. \tag{1.51}$$

Substituting into (1.49)

$$u = C_1 + C_2\, f \in W_2,$$

where C_1 and C_2 are arbitrary constants, yields

$$F[u](x) = \delta C_1^2 + 2\delta C_1 C_2 f(x) + \beta C_1 C_2 f''(x) + C_2^2 F[f](x).$$

The first two terms belong to W_2. Consider the last two terms. Since C_1C_2 and C_2^2 are independent, (1.51) is valid iff there exist parameters $\mu_{1,2}$ and $\nu_{1,2}$ such that f satisfies the following *overdetermined* system of ODEs:

$$\begin{cases} f'' = \mu_1 + \nu_1 f, \\ F[f] = \mu_2 + \nu_2 f. \end{cases} \tag{1.52}$$

The second equation implies that $\hat{W}_1 = \mathcal{L}\{f\}$ is also invariant if such an f exists for $\mu_2 = 0$ (and $\nu_2 \neq 0$). If $\mu_2\nu_2 \neq 0$, then $F : \hat{W}_1 \to W_2$ and, in a natural sense, the element f *generates* the 2D invariant subspace (1.50).

This procedure of determining admissible basis functions $f(x)$ from an overdetermined system of ODEs with several free parameters is called the *generalized separation of variables* (GSV). In the present case, the GSV can be performed easily, since the first equation is linear and, clearly, for various values of parameters, there are six types of functions,

$$f(x) \in \{x, x^2, \cos\lambda x, \sin\lambda x, \cosh\lambda x, \sinh\lambda x\}, \quad \text{with } \lambda = \text{constant} \neq 0. \tag{1.53}$$

Substituting each of the functions f into the second equation in (1.52), we obtain the set (a linear subspace) of quadratic operators preserving subspace (1.50). We do not do this here; however, we do present the results of more general computations in the next section.

For 3D and multi-dimensional subspaces, the GSV leads to complicated overdetermined systems of ODEs that do not admit a simple treatment. Even for a general 2D subspace $\mathcal{L}\{f_1, f_2\}$ with two unknown basis functions, the GSV becomes essentially more involved. In our further study of invariant subspaces in Chapter 2, we will use another approach associated with Lie–Bäcklund symmetries of linear ODEs, and will return to the general theory of GSV in Section 7.3.

The above GSV reveals typical basis functions (1.53) of the invariant subspaces (1.50) for quadratic operators. These are:

(i) polynomial,
(ii) trigonometric, and
(iii) exponential subspaces,

which will be studied later on.

On related aspects of finite commutative rings. Consider the operator F in (1.49) in the linear space K of real analytic functions of the single variable x. The quadratic polynomial structure of (1.49) suggests introducing the *commutative product*

$$u * v = \alpha u_{xx} v_{xx} + \tfrac{\beta}{2}\big(u v_{xx} + v u_{xx}\big) + \gamma\, u_x v_x + \delta u v \tag{1.54}$$

for any $u, v \in K$. In this case, K becomes a *commutative ring* with the product (1.54), which is not associative in general.

It is interesting to interpret *nilpotents* and *idempotents* of this ring. To this end, for instance, consider the corresponding hyperbolic PDE (1.46). Then a nilpotent $\varepsilon(x)$ satisfying

$$\varepsilon * \varepsilon = 0, \quad \text{i.e.,} \ \ F[\varepsilon] = 0,$$

is indeed a stationary solution of (1.46). On the other hand, any idempotent $e(x)$ satisfying

$$e * e = e, \quad \text{i.e.,} \ \ F[e] = e,$$

is associated with the separate variables solution

$$u(x,t) = \varphi(t) e(x), \quad \text{where} \ \ \varphi''(t) = \varphi^2(t).$$

For instance, the blow-up function $\varphi(t) = 6(T-t)^{-2}$ can be chosen.

We are now looking for 2D subrings A of K, and will describe where a link to overdetermined systems of ODEs is coming from. Assume that, in a subring A, there exists a generating element p such that p and $p*p$ are linearly independent. Actually, it can be shown that this is the case for any subring; see references in Remarks. This implies that p satisfies the system of two ODEs

$$\begin{cases} p * (p * p) = \mu_1 + \nu_1 (p * p), \\ (p * p) * (p * p) = \mu_2 + \nu_2 (p * p), \end{cases}$$

with four free parameters, as above. It is a system of two fourth-order nonlinear ODEs for p, which is difficult to study for general quadratic operators F.

1.3 More examples: polynomial subspaces

In the last three sections we present further examples of PDEs with quadratic, cubic, and other polynomial operators preserving linear subspaces of various dimensions. These results are introductory to more advanced theory developed in the subsequent chapters.

1.3.1 Classification and first examples of polynomial subspaces

We study second-order ($k = 2$) quadratic and cubic operators admitting subspaces that are composed of polynomials of the fixed order n,

$$W_n = \mathcal{L}\{1, x, ..., x^{n-1}\}, \quad \text{with } n \geq 2. \tag{1.55}$$

Operators preserving such a given subspace form a linear space. In the next propositions, the bases of such linear spaces of nonlinear operators are described.

Proposition 1.16 *Subspace* (1.55) *is invariant under the general quadratic operator of the second order*

$$F[u] = b_6(u_{xx})^2 + b_5 u_x u_{xx} + b_4 u u_{xx} + b_3 (u_x)^2 + b_2 u u_x + b_1 u^2 \tag{1.56}$$

only in the following four cases:

(i) $\mathbf{n = 2}$ *with a 5D space spanned by operators*

$$F_1[u] = (u_{xx})^2, \;\; F_2[u] = u_x u_{xx}\,,$$
$$F_3[u] = u u_{xx}\,, \;\; F_4[u] = (u_x)^2, \;\; F_5[u] = u u_x,$$

i.e., $b_1 = 0$ *in* (1.56)*;*

(ii) $\mathbf{n = 3}$ *with a 4D space spanned by*

$$F_1[u] = (u_{xx})^2, \;\; F_2[u] = u_x u_{xx}\,, \;\; F_3[u] = u u_{xx}\,, \;\; F_4[u] = (u_x)^2,$$

i.e., $b_1 = b_2 = 0$*;*

(iii) $\mathbf{n = 4}$ *with a 3D space spanned by*

$$F_1[u] = (u_{xx})^2, \;\; F_2[u] = u_x u_{xx}\,, \;\; F_3[u] = u u_{xx} - \tfrac{2}{3}(u_x)^2,$$

i.e., $b_1 = b_2 = 0$ *and* $b_3 = -\frac{2}{3}\, b_4$*;*

(iv) $\mathbf{n = 5}$ *with a 2D space spanned by*

$$F_1[u] = (u_{xx})^2 \quad \text{and} \quad F_2[u] = u u_{xx} - \tfrac{3}{4}(u_x)^2,$$

i.e., $b_1 = b_2 = b_5 = 0$ *and* $b_3 = -\frac{3}{4}\, b_4$.

For $\mathbf{n \geq 6}$, *no nontrivial operators* (1.56) *preserving subspace* (1.55) *exist.*

Proof. For $n \leq 5$, the proof is straightforward by plugging the finite sum expansion

$$u = C_1 + C_2 x + ... + C_n x^{n-1}$$

into operator (1.56) and equating the coefficients of the expansion of $F[u]$, corresponding to higher-degree terms x^l with $l \geq n$, to zero. Any computer codes on algebraic manipulations in Maple, Matematica, MatLab, or Reduce, etc., are suitable for this analysis. The last negative statement for $n \ge 6$ will follow from a more general result to be proved in Section 2.2 (Theorem 2.8). $\square$

A similar approach applies to other propositions presented below for various operators and subspaces.

Proposition 1.17 *Subspace* (1.55) *is invariant under the general cubic operator of the second order*

$$\begin{aligned} F[u] = b_{10}(u_{xx})^3 + b_9(u_{xx})^2u_x + b_8(u_{xx})^2u + b_7u_{xx}(u_x)^2 \\ + b_6u_{xx}u_xu + b_5u_{xx}u^2 + b_4(u_x)^3 + b_3(u_x)^2u + b_2u_xu^2 + b_1u^3 \end{aligned} \tag{1.57}$$

only for the following three cases:

(i) $\mathbf{n = 2}$ *with an 8D space spanned by*

$$\begin{gathered} F_1[u] = (u_{xx})^3, \quad F_2[u] = u_x(u_{xx})^2, \quad F_3[u] = u(u_{xx})^2, \\ F_4[u] = (u_x)^2u_{xx}\,, \quad F_5[u] = uu_xu_{xx}\,, \quad F_6[u] = u^2u_{xx}\,, \\ F_7[u] = (u_x)^3, \quad F_8[u] = u(u_x)^2; \end{gathered}$$

(ii) $\mathbf{n = 3}$ *with a 6D space spanned by*

$$\begin{gathered} F_1[u] = (u_{xx})^3, \quad F_2[u] = u_x(u_{xx})^2, \\ F_3[u] = u(u_{xx})^2, \quad F_4[u] = (u_x)^2u_{xx}\,, \\ F_5[u] = u_x[2uu_{xx} - (u_x)^2], \quad F_6[u] = u[2uu_{xx} - (u_x)^2]; \end{gathered}$$

(iii) $\mathbf{n = 4}$ *with a 2D space spanned by*

$$F_1[u] = (u_{xx})^3 \quad \text{and} \quad F_2[u] = u_{xx}\big[uu_{xx} - \tfrac{2}{3}(u_x)^2\big].$$

For $\mathbf{n \ge 5}$, *no nontrivial cubic operators* (1.57) *preserving subspace* (1.55) *exist.*

Example 1.18 (**Quadratic PDEs**) As an illustration of case (iii) in Proposition 1.16, we consider a fully nonlinear PDE

$$u_t = \alpha(u_{xx})^2 + \beta u_xu_{xx} + \gamma\big[uu_{xx} - \tfrac{2}{3}(u_x)^2\big]. \tag{1.58}$$

Nonlinearities $(u_{xx})^2$ and $(u_{xx})^3$ are typical for the *dual porous medium equations* in filtration theory; see references in [49]. In this case, (1.58) has solutions

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3,$$

$$\begin{cases} C_1' = 2\gamma\, C_1C_3 - \frac{2}{3}\,\gamma\, C_2^2 + 2\beta C_2C_3 + 4\alpha C_3^2, \\ C_2' = 6\gamma\, C_1C_4 + 6\beta C_2C_4 - \frac{2}{3}\,\gamma\, C_2C_3 + 4\beta C_3^2 + 24\alpha C_3C_4, \\ C_3' = 2\gamma\, C_2C_4 - \frac{2}{3}\,\gamma\, C_3^2 + 18\beta C_3C_4 + 36\alpha C_4^2, \\ C_4' = 18\beta C_4^2. \end{cases}$$

The last equation with $\beta > 0$ and $C_4(0) > 0$ implies finite-time blow-up,

$$C_4(t) = \tfrac{C_4(0)}{1-18\beta C_4(0)t} \to +\infty \quad \text{as } \ t \to T^-, \tag{1.59}$$

where $T = [18\beta C_4(0)]^{-1}$. If $C_4(0) < 0$, then

$$C_4(t) = -\tfrac{|C_4(0)|}{1+18\beta|C_4(0)|t} \to 0 \quad \text{as } \ t \to +\infty$$