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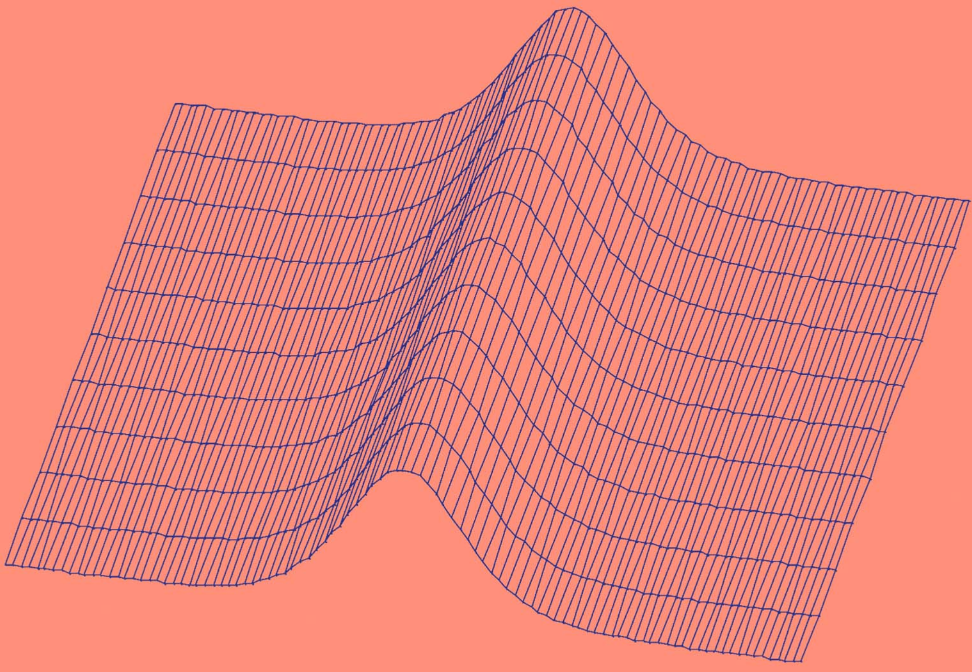
Introduction to non-Kerr Law Optical Solitons



Anjan Biswas
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Preface

Recent years have witnessed an explosion of research activities in the field of soliton propagation in nonlinear optical media. These activities are motivated by the fact that optical solitons, both temporal and spatial variety, do have practical relevance in the latest communication technology based on generation and transportation of localized optical pulses. To date, these optical pulses have already made tera bit/s transmission through optical fibers feasible in the laboratory. In addition, they also play significant roles in several other technologically relevant aspects, such as optical switching and signal processing.

Early investigations in these areas began with Kerr law nonlinearity in which the refractive index of the medium is proportional to the light intensity. With further development of the subject, several forms of nonlinearity have come under investigation. Notable among these forms of nonlinearity are parabolic or cubic-quintic, power law, dual-power law, and saturating nonlinearities. These nonlinearities reveal many new and interesting behaviors hitherto unknown in Kerr law of nonlinearity. However, in spite of tremendous progress in these areas in the last 2 decades, and despite the fact that several important and extremely well-written books are now available that deal with soliton propagation in optical communication systems and allied areas, no textbook of worth deals exclusively with soliton propagation in media that possess non-Kerr law nonlinearities. Thus, to bridge the gap between availability and nonavailability, we felt the need to bring out a book exclusively dealing with optical soliton propagation in non-Kerr law media.

This book is organized as follows: Chapter 1 presents an introduction to the field of fiber optics and basic features of fiber-optic communications. The nonlinear Schrödinger's equation (NLSE) has been introduced and mathematical aspects, including conserved quantities, have been outlined in chapter 2. In this chapter, we have also introduced the perturbation to the NLSE. Adiabatic dynamics of soliton parameters have also been introduced. Finally, we have discussed the concept of quasi-stationary solitons and their influences in this chapter. In chapter 3, we have derived the NLSE for Kerr law nonlinearity from basic principle. The inverse scattering transform has been outlined and, using this principle, the 1-soliton solution has been obtained in this chapter. In addition, we have explained the variational principle and Lie transform, which are used to integrate the NLSE with Hamiltonian-type perturbation. The non-Kerr law solitons have been discussed in chapters 4 through 7. Chapters 4, 5, 6, and 7 are respectively devoted to the study of solitons with power law, parabolic law, dual-power law, and saturable law nonlinearities. In each case, we have developed the soliton dynamics, evaluated integrals of motion, and devoted enough space to develop adiabatic dynamics of perturbed quantities based on multiple-scale perturbation theory. In addition, the existence of bistable soliton is discussed in chapter 7. Chapter 8 is devoted to intrachannel collision of optical solitons in the presence of perturbation terms. Both Hamiltonian as well as non-Hamiltonian type perturbations have been considered. The nonlinearities that are studied in this chapter are Kerr, power, parabolic, and dual-power laws. In chapter 9, the stochastic perturbation of optical solitons has been studied. The corresponding Langevin equations are derived and analyzed for each of the laws of nonlinearity, namely Kerr, power, parabolic, and dual-power laws. Optical couplers are introduced in chapter 10. Twin core and multiple-core couplers have been discussed. At the end of this chapter, we have briefly discussed solitons in magneto-optic waveguides. The book concludes with chapter 11, which treats an introduction to optical bullets.

This book is intended for graduate students at the master's and doctoral levels in applied mathematics, physics, and engineering. Undergraduate students with senior standing in applied mathematics, physics, and engineering will also benefit from this book. The prerequisite of this book is a knowledge of partial differential equations, perturbation theory, and elementary physics.

Anjan Biswas, is extremely thankful to Dr. Michael Busby, director of the Center of Excellence in Information Systems Engineering and Management of Tennessee State University in Nashville, Tennessee, with which this author was previously affiliated. Without constant encouragement and financial support from Dr. Busby, this project would not have been possible. The first author is also extremely thankful and grateful to Dr. Tommy Frederick, vice provost of research at Delaware State University, with which this author is presently affiliated, for his constant encouragement. Without these two persons' blessings, this project would not have been possible. Finally, the author is extremely grateful to his parents for all their unconditional love in his upbringing, blessings, education, support, encouragement, and sacrifices throughout his life, till today. This author is deeply saddened by the sudden death of his mother after a massive heart attack in Calcutta, India, which occurred during the course of writing this book.

Swapan Konar is grateful to Prof. H. C. Pande, vice chancellor emeritus, Birla Institute of Technology, India; Prof. S. K. Mukherjee, vice chancellor, Birla Institute of Technology, and Prof. P. K. Barhai, head of the Department of Applied Physics, Birla Institute of Technology, for encouragement and constant support. Finally, he sincerely thanks his wife Tapati for her tolerance and encouragement and his little son Argho for sacrificing his playtime.

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Contents

1

Introduction

This introductory chapter is intended to provide a general overview of fiber optics. It starts with a history of and current developments in fiber optics in Section 1.1. Section 1.2 provides a brief account of types of optical waveguides and the issues of fiber-optic communications.

1.1 History

The propagation of optical pulses, or solitons, through optical fibers has been a major area of study given its potential applicability in optical communication systems. The field of telecommunications has undergone a substantial evolution in the last couple of decades due to the impressive progress in the development of optical fibers, optical amplifiers, and transmitters and receivers. In a modern optical communication system, the transmission link is composed of optical fibers and amplifiers that replace the electrical regenerators. However, the amplifiers introduce some noise and signal distortion that limit the system capacity. Presently, the optical systems that show the best characteristics in terms of simplicity, cost, and robustness against the degrading effects of a link are those based on intensity modulation with direct detection (IM-DD). Conventional IM-DD systems are based on the non-return-to-zero (NRZ) format, but for soliton-based transmission at higher data rates the return-to-zero (RZ) format is used. Soliton-based transmission allows the exploitation of the fiber capacity much more. [9].

The performance of optical system is limited by several effects that are present in optical fibers and amplifiers. Signal propagation through optical fibers can be affected by group velocity dispersion (GVD), polarization mode dispersion (PMD), and nonlinear effects. The chromatic dispersion that is essentially the GVD when waveguide dispersion is negligible is a linear effect that introduces pulse broadening and generates intersymbol interference. The PMD arises due to the fact that optical fibers for telecommunications have two polarization modes, in spite of the fact that they are called *monomode fibers*. These modes have two different group velocities that induce pulse broadening depending on the input signal state of polarization. The transmission impairment due to PMD looks similar to that caused by GVD. However, PMD is random whereas GVD is a deterministic process. So, PMD cannot be controlled at the receiver. Newly installed optical fibers have quite low values of PMD that are about 0.1 $ps/\sqrt{km}$.

The main nonlinear effects that arise in monomode fibers are Brillouin scattering, Raman scattering, and the Kerr effect. Brillouin is a backward scattering that arises from acoustic waves and can generate noise at the receiver. Raman scattering takes place in both forward and backward directions from silica molecules. The Raman gain response is characterized by low gain and wide bandwidth, namely about 30 THz. The Raman threshold in conventional fibers is of the order of 500 mW for copolarized pump and Stokes' wave (that is about 1 W for random polarization), thus making Raman effect negligible for a single-channel signal. However, it becomes important for multichannel wavelength-division-multiplexed (WDM) signals due to an extremely wide band of wide gain curve.

The Kerr effect of nonlinearity is due to the dependence of the fiber refractive index on the field intensity. The intensity dependence of the refractive index leads to a larger number of interesting nonlinear effects. Notable among them, which have been studied widely, are self-phase modulation (SPM) and cross-phase modulation (XPM). SPM refers to the self-induced nonlinear phase shift experienced by an optical field during its propagation through an optical fiber. SPM is responsible for spectral broadening. The SPM-induced chirp combines with the linear chirp generated by the chromatic dispersion. If the fiber dispersion coefficient is positive, namely in the normal dispersion regime, linear and nonlinear chirps have the same sign, whereas in an anomalous dispersion regime, they are of opposite signs. In the former case, pulse broadening is enhanced by SPM, while in the later case it is reduced. In the anomalous dispersion case, the Kerr nonlinearity induces a chirp that can compensate the degradation induced by GVD. Such a compensation is total if soliton signals are used.

If multichannel WDM signals are considered, the Kerr effect can be more degrading since it induces nonlinear cross-talk among the channels that are known as XPM. In addition, WDM generates new frequencies called four-wave mixing (FWM). The other issue in the WDM system is the collision-induced timing jitter that is introduced due to the collision of solitons in different channels. The XPM causes further nonlinear chirp that interacts with the fiber GVD as in the case of SPM. The FWM is a parametric interaction among waves that satisfies a particular relationship called *phase-matching* that leads to power transfer among different channels.

To limit the FWM effect in a WDM, it is preferable to operate with a local high GVD that is periodically compensated by devices having an opposite GVD sign. One such device is a simple optical fiber with appropriate GVD, and the method is commonly known as *dispersion management*. With this approach, the accumulated GVD can be very low and, at the same time, FWM is strongly limited. Through dispersion management it is possible to achieve the highest capacity for both RZ and NRZ signals. In that case, the overall link dispersion has to be kept very close to zero, while a small amount of chromatic anomalous dispersion is useful for the efficient propagation of a soliton signal. It has been demonstrated with soliton signals that dispersion management is very useful since it reduces collision-induced timing jitter and also pulse interactions. It thus permits the achievement of higher capacities than those allowed by the link having constant chromatic dispersion.

1.2 Optical Waveguides

One of the most promising applications of soliton theory is in the field of optical communications. In optical communications systems, information is encoded into light pulses and transmitted through optical fibers over long distances. Commercial systems have been in operation since 1977 and a transatlantic undersea optical cable has been developed. In 1973, Hasegawa and Tappert [183, 184] proposed that soliton pulses could be used in optical communications. However, the technology was not available until 7 years later, at which time researchers at Bell Laboratories had experimentally demonstrated the propagation of solitons in optical fibers.

Rapid developments in communications technology have occurred—for example, the change from the use of wires to send signals (wire telegraphy) to wireless or radio telegraphy—leading to enormously increased communication rates, measured by bits per second by a factor of 1 billion. The latest in this series of advances is the optical fiber system in which large amounts of information, coded as light pulses, pass along silica fibers. The first transoceanic links, namely along the Atlantic and Pacific oceans, have been established. As marvelous as these advances have been, the present system still uses only a tiny fraction of the information-carrying capacity of optical fibers.

Taking a look at waveguides, in particular an optical fiber, one can see how solitons promise to revolutionize the field of telecommunications. The main idea of a waveguide is to guide a beam of light by employing a variation of refractive index in the transverse direction so as to cause the light to travel along a well-defined channel. The dependence of refractive index on the transverse direction, the direction perpendicular to that in which the wave propagates, can be continous or discontinous. The essential feature, however, is that the refractive index is maximal in the channel along which one wishes the light to be guided.

Figures 1.1(a) and 1.1(b) show the cross-section of an optical fiber. The inner core consists of a special form of silica glass with very low absorption and is between 10 to 60 μm in diameter. This core is surrounded by a glass cladding whose refractive index, n_2, is very close to but slightly less than n_1, the linear refractive index of the inner core. This ensures that the wave is

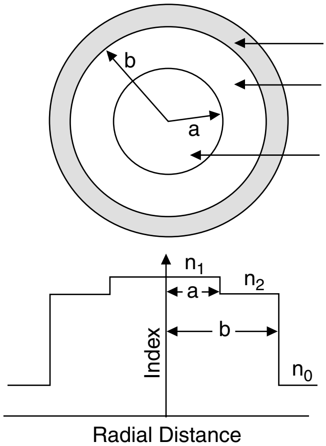

FIGURE 1.1
(a) & (b) Cross section of an optical fiber.

guided, namely its intensity is largely confined, to the inner core by virtue of total internal reflection.

Two parameters characterize an optical fiber, namely the core-cladding index difference (Δ) that is defined as:

$$\Delta = \frac{n_1 - n_2}{n_1} \tag{1.1}$$

and the normalized frequency (V) that is defined as:

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2} \tag{1.2}$$

where a is the radius of the fiber core as shown in Figure 1.2 and λ is the wavelength of light. The parameter V determines the number of modes supported by the fiber. For a V less than 2.405 the fibers support a single mode and so the fibers that are designed to satisfy such conditions are known as

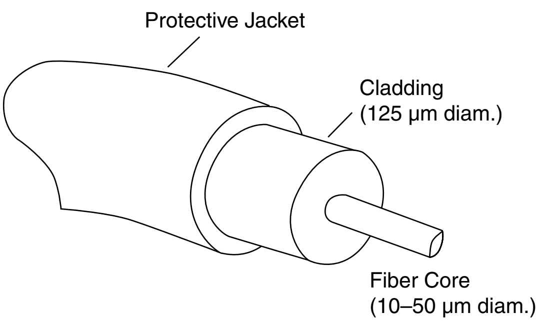

FIGURE 1.2
Structure of an optical fiber.

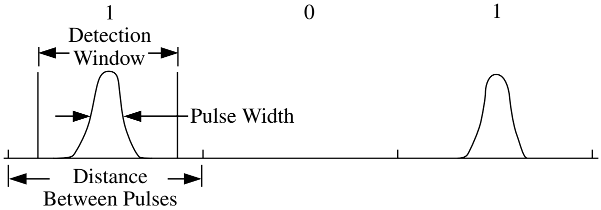

FIGURE 1.3
Soliton detection window and soliton train.

single-mode fibers. A typical multimode fiber would have the core radius as $a = 25$–$30\ \mu$m. However, for a single-mode fiber, a typical value of Δ is $\sim 3 \times 10^{-3}$ and requires a to be in the range of 2–4 μm. The value of the outer radius b is less critical as long as it is large enough to confine the modes entirely. Typically, $b = 50$–$60\ \mu$m for both single-mode and multimode fibers (Figure 1.2).

The basic idea of using optical fibers for communications is relatively simple. The message is coded in binary by representing one as a pulse-like modulation of a carrier wave whose wavelength is in the micrometer (10^{-6} m) range and whose frequency is in the terahertz (10^{14} Hz) range and representing zero by the absence of such a pulse. The arrangement is shown in Figure 1.3. The pulses are approximately 10–25 picoseconds (10^{-12} s) wide and the average distance between them is four times that amount. Experimentally, fibers have managed effective transmission rates in the gigabit range (10^9 bits/s).

1.2.1 Types of Optical Fibers

Based on the refractive index profile there are two types of optical fibers:

1. *Step index fiber*: In a step index fiber, the refractive index of the core is uniform throughout and undergoes an abrupt or a step change at the core-cladding boundary.
2. *Graded index fiber*: In a graded index fiber, the refractive index of the core is made to vary in a parabolic manner such that the maximum value of the refractive index is at the center of the core (Figure 1.4).

Propagating rays in the fiber can be classified as meridional and skew rays. Meridional rays are confined to the meridional plane of the fiber, which are planes that contain the axis of symmetry of the fiber. Skew rays are not confined to a single plane. They propagate along the fiber.

1.2.2 Advantages of Fiber-Optic Communications

The various advantages of soliton communication through optical fibers are enumerated here:

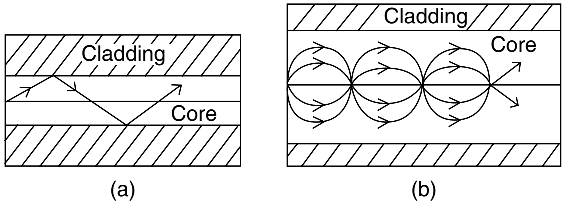

FIGURE 1.4
(a) Step-index fiber and (b) graded-index fiber.

1. *Wider bandwidth:* The information-carrying capacity of a transmission system is directly proportional to the carrier frequency of the transmitted signals. The optical carrier frequency is in the range of 10^{13}–10^{15} Hz while the radio wave frequency is about 10^6 Hz and the microwave frequency is about 10^{10} Hz. Thus, the optical fiber yields greater transmission bandwidth than conventional communication systems and the data rate or number of bits per second is increased to a greater extent in the optical fiber communication system.
2. *Low transmission loss:* Due to the usage of ultra-low-loss fibers and erbium-doped silica fibers as optical amplifiers, one can achieve almost lossless transmission. In modern optical fiber telecommunication systems, the fibers having a transmission loss of 0.2 dB/km are used. Furthermore, using erbium-doped silica fibers over a short length in transmission path selective points, appropiate optical amplification can be achieved. Thus, the repeater spacing is more than 100 km. Since the amplification is done in the optical domain itself, the distortion produced during the strengthening of the signal is almost negligible.
3. *Dielectric waveguide:* Optical fibers are made from silica, which is an electrical insulator. Therefore, they do not pick up any electromagnetic waves or any high-current lightning. They are also suitable in explosive environments. Furthermore, the optical fibers are not affected by any interference originating from power cables, railway power lines, and radio waves. There is no cross-talk between the fibers in a cable because of the absence of optical interference between the fibers.
4. *Signal security:* Signals transmitted through the fibers do not radiate. In addition, signals cannot be tapped from a fiber in an easy manner. Therefore, optical communication provides 100% signal security.
5. *Small size and weight:* Fiber-optic cables are developed with small radii and are flexible, compact, and lightweight. They can be bent or twisted without any damage. Optical fiber cables are superior to copper cables in terms of storage, handling, installation, and transportation, maintaining comparable strength and durability.

2

The Nonlinear Schrödinger's Equation

This chapter will talk about the mathematical aspects of the nonlinear Schrödinger's equation (NLSE) that governs the propagation of solitons through an optical fiber. Section 2.1 is an introduction to NLSE. In Section 2.2, the conserved quantities of the NLSE will be derived. In Section 2.3 the soliton parameters will be introduced and the formulae for the adiabatic dynamics of these parameters in the presence of the perturbation terms will be given. Finally, in Section 2.4, the concept of quasi-stationarity will be introduced.

2.1 Introduction

The NLSE plays a vital role in various areas of physical, biological, and engineering sciences. It appears in many applied fields, including fluid dynamics, nonlinear optics, plasma physics, and protein chemistry. The NLSE that is going to be studied in this book is given by [86, 108, 399]

$$iq_t + \frac{1}{2}q_{xx} + F(|q|^2)q = 0 \tag{2.1}$$

In (2.1), F is a real-valued algebraic function and one needs to have the smoothness of the complex function $F(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, it can be said that the function $F(|q|^2)q$ is k times continuously differentiable so that one can write

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k((-n, n) \times (-m, m); R^2)$$

In equation (2.1), q is the dependent variable, x and t are the independent variables, and the subscripts represent the partial derivative of q with respect to that variable. So, q_t stands for $\partial q/\partial t$ while q_{xx} stands for $\partial^2 q/\partial x^2$. The first term in (2.1) represents the time evolution term, while the second term is due to the group velocity dispersion and the third term accounts for nonlinearity. Thus, equations of these types are sometimes known as *nonlinear evolution*

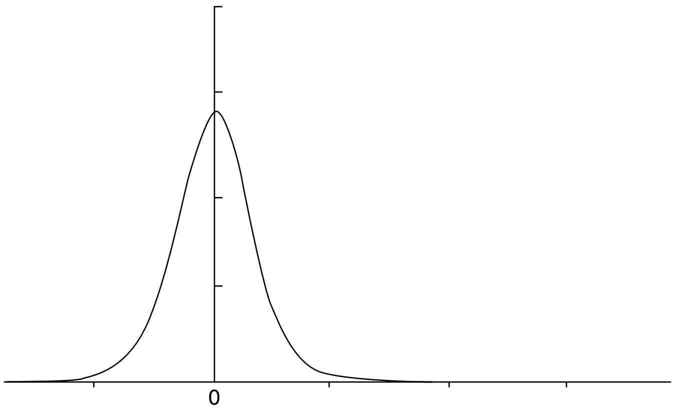

FIGURE 2.1
Profile of a soliton.

equations. This is a nonlinear partial differential equation that is not integrable, in general. The nonintegrability is not necessarily related to the nonlinear term in (2.1). Higher order dispersion, for example, can also make the system nonintegrable while it still remains Hamiltonian.

Equation (2.1) has been shown to govern the evolution of a wave packet in a weakly nonlinear and dispersive medium and has thus arisen insuch diverse fields as water waves, plasma physics, and nonlinear optics. One other application of this equation is in pattern formation, where it has been used to model some nonequilibrium pattern forming systems. In particular, this equation is now widely used in the optics field as a good model for optical pulse propagation in nonlinear fibers. Equation (2.1) is known to support solitons or soliton solutions for various kinds of nonlinearity that will be discussed in the upcoming chapters. The term *soliton* refers to a nonlinear wave that propagates without changing properties and is stable against mutual collisions with other solitons that retain their identities. Solitons have been studied extensively in various areas of mathematical physics. In the context of optical fibers, solitons are not only of fundamental interest but also have potential applications in the field of optical fiber communications. Figure 2.1 shows an illustration of a soliton. This text is devoted to the study of the propagation of such solitons through optical fibers with emphasis on the various kinds of the function $F(s)$ in equation (2.1).

2.1.1 Nonlinearity Classification

There are various kinds of nonlinearities of the function F in (2.1) that are known so far. They are as follows:

1. *Kerr law*: $F(s) = s$
   This is also known as *cubic nonlinearity* and is the simplest known form of the law of nonlinearity. In this case, the NLSE is integrable by a method called the Inverse Scattering Transform. This method will

be discussed in the next chapter. Most optical fibers that are commercially available nowadays obey this Kerr law of nonlinearity.

2. *Power law*: $F(s) = s^p$
   In this case, it is necessary to have $0 < p < 2$ to avoid wave collapse. In fact, it is mandatory that $p \neq 2$ to avoid self-focusing singularity. This law of nonlinearity arises in nonlinear plasmas and solves the problem of small K-condensation in weak turbulence theory. It also arises in the context of nonlinear optics. Physically, various materials, including semiconductors, exhibit power law nonlinearities. This case of nonlinearity has been studied, including the perturbation term by multiple-scale analysis. The case where $p = \frac{1}{2}$ is studied in the context of soliton turbulence.
3. *Parabolic law*: $F(s) = s + \nu s^2$
   This law is commonly known as the *cubic-quintic nonlinearity*. The second term is large for the case of p-toluene sulfonate crystals. This law arises in the nonlinear interaction between Langmuir waves and electrons. It describes the nonlinear interaction between the high-frequency Langmuir waves and the ion-acoustic waves by pondermotive forces. This case of cubic-quintic nonlinearity was also studied by multiple-scale analysis.
4. *Dual-power law*: $F(s) = s^p + \xi s^{2p}$
   This model is used to describe the saturation of the nonlinear refractive index, and its exact soliton solutions are known. The effective GNLSE with this dual-power law nonlinearity serves as a basic model to describe spatial solitons in photovoltaic-photorefractive materials such as lithium niobate. Optical nonlinearities in many organic and polymer materials can be modelled using this form of nonlinearity. The solitons of this model become unstable and decay in the unstable region $1 \leq p < 2$, while for $p \geq 2$, the solitons collapse in a finite time.
5. *Saturating law*: $F(s) = \frac{\lambda s}{1+\lambda s}$
   This law with $\lambda > 0$ accurately describes the variation of the dielectric constant of gas vapors through which a laser beam propagates [30]. Optical nonlinearity saturates at a finite value of optical intensity in most materials. $F(s)$ in those materials can be modeled using the above form, which is known as the *saturating form* of nonlinearity. In semiconductor-doped fibers, the soliton propagation has been modeled using saturable nonlinearity rather than the usual Kerr nonlinearity. The main motivation behind such attempts is the observation of such nonlinearity at not too high intensities in semiconductor-doped glass and other composite materials. This case was studied numerically.
6. *Exponential law*: $F(s) = \frac{1}{2\lambda}(1 - e^{-2\lambda s})$
   This case of exponential nonlinearity serves as useful model in homogenous, unmagnetized plasmas and laser-produced plasmas.