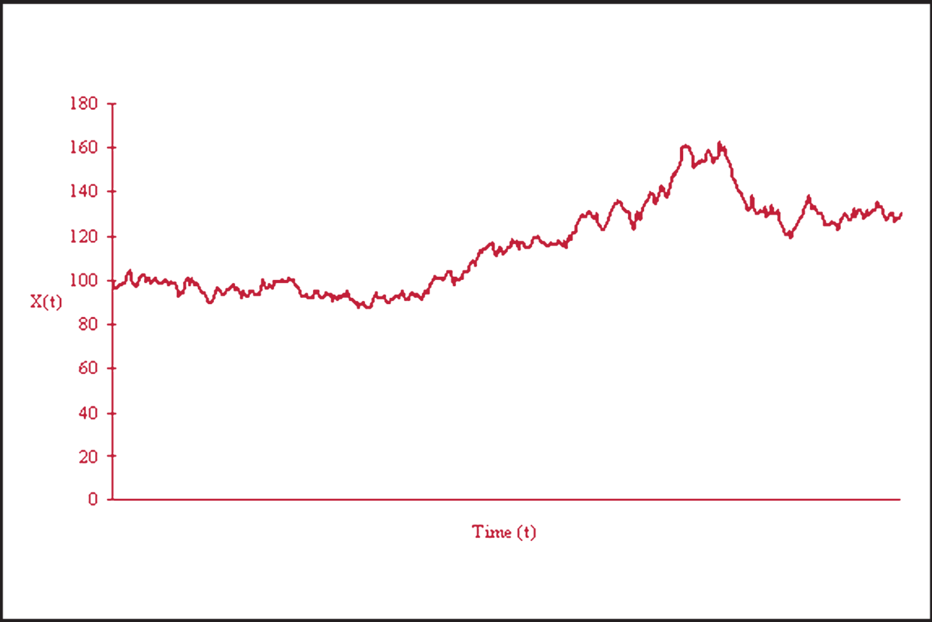
Damien Lamberton
Bernard Lapeyre

Introduction to Stochastic Calculus Applied to Finance

Second Edition



Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES

Introduction to Stochastic Calculus Applied to Finance

Second Edition

CHAPMAN & HALL/CRC
Financial Mathematics Series

Aims and scope:
The field of financial mathematics forms an ever-expanding slice of the financial sector. This series aims to capture new developments and summarize what is known over the whole spectrum of this field. It will include a broad range of textbooks, reference works and handbooks that are meant to appeal to both academics and practitioners. The inclusion of numerical code and concrete real-world examples is highly encouraged.

Series Editors

M.A.H. Dempster
Centre for Financial Research
Judge Business School
University of Cambridge

Dilip B. Madan
Robert H. Smith School of Business
University of Maryland

Rama Cont
Center for Financial Engineering
Columbia University
New York

Published Titles

American-Style Derivatives; Valuation and Computation, *Jerome Detemple*

Engineering BGM, *Alan Brace*

Financial Modelling with Jump Processes, *Rama Cont and Peter Tankov*

An Introduction to Credit Risk Modeling, *Christian Bluhm, Ludger Overbeck, and Christoph Wagner*

Introduction to Stochastic Calculus Applied to Finance, Second Edition, *Damien Lamberton and Bernard Lapeyre*

Numerical Methods for Finance, *John A. D. Appleby, David C. Edelman, and John J. H. Miller*

Portfolio Optimization and Performance Analysis, *Jean-Luc Prigent*

Robust Libor Modelling and Pricing of Derivative Products, *John Schoenmakers*

Structured Credit Portfolio Analysis, Baskets & CDOs, *Christian Bluhm and Ludger Overbeck*

Understanding Risk: The Theory and Practice of Financial Risk Management, *David Murphy*

Proposals for the series should be submitted to one of the series editors above or directly to:
CRC Press, Taylor and Francis Group
Albert House, 4th floor
1-4 Singer Street
London EC2A 4BQ
UK

CHAPMAN & HALL/CRC FINANCIAL MATHEMATICS SERIES

Introduction to Stochastic Calculus Applied to Finance

Second Edition

Damien Lamberton
Bernard Lapeyre

Chapman & Hall/CRC
Taylor & Francis Group
Boca Raton London New York

Chapman & Hall/CRC is an imprint of the
Taylor & Francis Group, an **informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20110725

International Standard Book Number-13: 978-1-4200-0994-1 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Preface to the second edition

The topic of mathematical finance has been growing rapidly since the first edition of this book. For this new edition, we have not tried to be exhaustive on all new developments but to select some techniques or concepts that could be incorporated at reasonable cost in terms of length and mathematical sophistication. This was partly done by adding new exercises. The main addition concern:

- complements on discrete models (Rogers' approach to the Fundamental Theorem of Asset Pricing, super-replication in incomplete markets, see chapter 1 exercises 1 and 2),
- local volatility and Dupire's formula (see Chapter 4),
- change of numéraire techniques and forward measures (see Chapter 1 and Chapter 6),
- the forward libor model (BGM model, see Chapter 6),
- a new chapter on credit risk modelling,
- an extension of the chapter dealing with simulation with numerical experiments illustrating variance reduction techniques, hedging strategies and so on.

We are indebted, in addition to those cited in the introduction, to a number of colleagues whose suggestions have been helpful for this new edition. In particular we are grateful to Marie-Claire Quenez, Benjamin Jourdain, Philip Protter and, for the chapter on credit risk, to Monique Jeanblanc and Rama Cont (whose lectures introduced us to this new area) and to Aurélien Alfonsi.

Contents

Introduction

The objective of this book is to give an introduction to the probabilistic techniques required to understand the most widely used financial models. In the last few years, financial quantitative analysts have used more sophisticated mathematical concepts, such as martingales or stochastic integration, in order to describe the behavior of markets or to derive computing methods.

In fact, the appearance of probability theory in financial modeling is not recent. At the beginning of this century, Bachelier (1900), in trying to build up a "Theory of Speculation", discovered what is now called Brownian motion. From 1973, the publications by Black and Scholes (1973) and Merton (1973) on option pricing and hedging gave a new dimension to the use of probability theory in finance. Since then, as the option markets have evolved, Black-Scholes and Merton results have developed to become clearer, more general and mathematically more rigorous. The theory seems to be advanced enough to attempt to make it accessible to students.

Options

Our presentation concentrates on options, because they have been the main motivation in the construction of the theory and still are the most spectacular example of the relevance of applying stochastic calculus to finance. An option gives its holder the right, but *not the obligation*, to buy or sell a certain amount of a financial asset, by a certain date, for a certain strike price.

The writer of the option needs to specify:

1. the type of option: the option to buy is called a *call* while the option to sell is a *put*;

2. the underlying asset: typically, it can be a stock, a bond, a currency and so on;

3. the amount of an underlying asset to be purchased or sold;

4. the expiration date; if the option can be exercised at any time before maturity, it is called an *American* option but, if it can only be exercised at maturity, it is called a *European* option;

5. the exercise price which is the price at which the transaction is done if the option is exercised.

The price of the option is the *premium*. When the option is traded on an organised market, the premium is quoted by the market. Otherwise, the problem is to price the option. Also, even if the option is traded on an organized market, it can be interesting to detect some possible abnormalities in the market.

Let us examine the case of a European call option on a stock, whose price at time t is denoted by S_t. Let us call T the expiration date and K the exercise price. Obviously, if K is greater than S_T, the holder of the option has no interest whatsoever in exercising the option. But, if $S_T > K$, the holder makes a profit of $S_T - K$ by exercising the option, i.e., buying the stock for K and selling it back on the market at S_T. Therefore, the value of the call at maturity is given by

$$(S_T - K)_+ = \max(S_T - K, 0).$$

If the option is exercised, the writer must be able to deliver a stock at price K. It means that he or she must generate an amount $(S_T - K)_+$ at maturity. At the time of writing the option, which will be considered as the origin of time, S_T is unknown and therefore two questions have to be asked:

1. How much should the buyer pay for the option? In other words, how should we price at time $t=0$ an asset worth $(S_T - K)_+$ at time T? That is the problem of *pricing* the option.

2. How should the writer, who earns the premium initially, generate an amount $(S_T - K_+$ at time T? That is the problem of *hedging* the option.

Arbitrage and put/call parity

Answers the above questions require some modelling. The basic one, which is commonly accepted in every model, is the absence of arbitrage opportunity in liquid financial markets, i.e. there is no riskless profit available in the market. We will translate that into mathematical terms in the first chapter. At this point, we will only show how we can derive formulae relating European put and call prices from the no arbitrage assumption. Consider a put and a call with the same maturity T and exercise price K, on the same underlying asset which is worth S_t at time t. We shall assume that it is possible to borrow or invest money at a constant rate r.

Let us denote by C_t and P_t respectively the prices of the call and the put at time t. Because of the absence of arbitrage opportunity, the following equation called *put/call parity* is true for all $t < T$

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

To understand the notion of arbitrage, let us show how we could make a riskless profit if, for instance,

$$C_t - P_t > S_t - Ke^{-r(T-t)}.$$

At time t, we purchase a share of stock and a put, and sell a call. The net value of the operation is

$$C_t - P_t - S_t.$$

If this amount is positive, we invest it at rate r until time T, whereas if it is negative we borrow it at the same rate. At time T, two outcomes are possible:

- $S_T > K$: the call is exercised, we deliver the stock, receive the amount K and clear the cash account to end up with a wealth $K + e^r(T-t)(C_t - P_t - S_t) > 0$.
- $S_T \le K$: we exercise the put and clear our bank account as before to finish with wealth $K + e^{T-t}(C_t - P_t - S_t > 0$.

In both cases, we locked in a positive profit without making any initial endowment: this is an example of an arbitrage strategy.

There are many similar examples in the book by Cox and Rubinstein (1985). We will not review all these formulae, but we shall characterize mathematically the notion of a *financial market without arbitrage opportunity.*

Black-Scholes model and its extensions

Even though no-arbitrage arguments lead to many interesting equations, they are not sufficient in themselves for deriving pricing formulae. To achieve this, we need to model stock prices more precisely. Black and Scholes were the first to suggest a model whereby we can derive an explicit price for a European call on a stock that pays no dividend. According to their model, the writer of the option can hedge himself perfectly, and actually the call premium is the amount of money needed at time 0 to replicate exactly the payoff $(S_T - K)_+$ by following their dynamic hedging strategy until maturity. Moreover, the formula depends on only one non-directly observable parameter, the so-called *volatility.*

It is by expressing the profit and loss resulting from a certain trading strategy as a stochastic integral that we can use stochastic calculus and, particularly, Itô formula, to obtain closed form results. In the last few years, many extensions of the Black-Scholes approach has been considered. From a thorough study of the Black-Scholes model, we will attempt to give to the reader the means to understand those extensions.

Contents of the book

The first two chapters are devoted to the study of discrete time models. The link between the mathematical concept of martingale and the economic notion of arbitrage is brought to light. Also, the definition of complete markets and the pricing of options in these markets are given. We have decided to adopt the formalism of Harrison and Pliska (1981) and most of their results are stated in the first chapter, taking the Cox, Ross and Rubinstein model as an example. The second chapter deals with American options. Thanks to the theory of optimal stopping in a discrete time set-up, which uses quite elementary methods, we introduce the reader to all the ideas that can be developed in continuous time.

Chapter 3 is an introduction to the main results in stochastic calculus that we will use in Chapter 4 to study the Black-Scholes model. As far as European options are concerned, this model leads to explicit formulae. But, in order to analyze American options or to perform computations within more sophisticated models, we need numerical methods based on the connection between option pricing and partial differential equations. These questions are addressed in Chapter 5.

Chapter 6 is a relatively quick introduction to the main interest rate models and Chapter 7 looks at the problems of option pricing and hedging when the price of the underlying asset follows a simple jump process.

In these latter cases, perfect hedging no longer possible and we must define a criterion to achieve optimal hedging. These models are rather less optimistic than the Black-Scholes model and seem to be closer to reality. However, their mathematical treatment is still a matter of research, in the framework of so-called *incomplete markets.*

Finally, in order to help the student to gain a practical understanding, we have included a chapter dealing with the simulation of financial models and the use of computers in the pricing and hedging of options. Also, a few exercises and longer questions are listed at the end of each chapter.

This book is only an introduction to a field that has already benefited from considerable research. Bibliographical notes are given in some chapters to help the reader to find complementary information. We would also like to warn the reader that some important questions in financial mathematics are not tackled. Amongst them are the problems of optimization and the questions of equilibrium for which the reader might like to consult the book by Duffie (1988).

A good level in probability theory is assumed to read this book. The reader is referred to Dudley (2002)) and Williams (1991) for prerequisites. However, some basic results are also proved in the Appendix.

Acknowledgments

This book is based on the lecture notes of a course taught at *l'Ecole des Ponts* since 1988. The organisation of this lecture series would not have been possible without the encouragement of N. Bouleau. Thanks to his dynamism, CERMA (Applied Mathematics Institute of ENPC) started working on financial modeling as early as 1987, sponsored by *Banque Indosuez* and subsequently by *Banque Internationale de Placement.*

Since then, we have benefited from many stimulating discussions with G. Pagès and other academics at CERMA, particularly O. Chateau and G.

Caplain. A few people kindly read the earlier draft of our book and helped us with their remarks. Amongst them are S. Cohen, O. Faure, C. Philoche, M. Jeanblanc and X. Zhang. Finally, we thank our colleagues at the university and at INRIA for their advice and their motivating comment: N. El Karoui, T. Jeulin, J.F. Le Gall and D. Talay.

Chapter 1

Discrete-time models

The objective of this chapter is to present the main ideas related to option theory within the very simple mathematical framework of discrete-time models. Essentially, we are following the first part of the paper by Harrison and Pliska (1981). Cox, Ross and Rubinstein's model is detailed at the end of the chapter in the form of a problem with its solution.

1.1 Discrete-time formalism

1.1.1 Assets

A discrete-time financial model is built on a finite probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration, i.e. an increasing sequence of σ-algebras included in $\mathscr{F}$: $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_N$. The σ-algebra $\mathscr{F}_n$ can be seen as the information available at time n and is sometimes called the *σ-algebra of events up to time n*. The *horizon* N will often correspond to the maturity of the options. From now on, we will assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathscr{F}_N = \mathscr{F} = \mathscr{P}(\Omega)$, where $\mathscr{P}(\Omega)$ denotes the collection of all subsets of the finite sample space Ω, and we also assume that $\mathbb{P}(\{\omega\}) > 0$, for $\omega \in \Omega$. Working with a finite probability space avoids some technicalities: for instance, all real-valued random variables are integrable.

The market consists of $(d+1)$ financial assets, whose prices at time n are given by the positive random variables $S_n^0, S_n^1, \ldots, S_n^d$, which are measurable with respect to $\mathscr{F}_n$ (investors know past and present prices but obviously not the future ones). The vector $S_n = (S_n^0, S_n^1, \ldots, S_n^d)$ is the vector of prices at time n. The asset indexed by 0 is the *riskless asset* and we set $S_0^0 = 1$. If the return of the riskless asset over one period is constant and equal to r, we will obtain $S_n^0 = (1+r)^n$. The coefficient $\beta_n = 1/S_n^0$ is interpreted as the discount factor (from time n to time 0): if an amount β_n is invested in the riskless asset at time 0, then one dollar will be available at time n. The assets indexed by $i = 1 \ldots d$ are called *risky assets*.

1.1.2 Strategies

A *trading strategy* is defined as a stochastic process (i.e. a sequence in the discrete case)

$$\phi = ((\phi_n^0, \phi_n^1, \ldots, \phi_n^d))_{0 \le n \le N}$$

in $\mathbb{R}^{d+1}$, where ϕ_n^i denotes the number of shares of asset i held in the portfolio at time n. The sequence ϕ is assumed to be *predictable*, i.e.

$$\forall i \in \{0, 1, \ldots, d\} \begin{cases} \phi_0^i \text{ is } \mathscr{F}_0\text{-measurable} \\ \text{and, for } n \ge 1, \quad \phi_n^i \text{ is } \mathscr{F}_{n-1}\text{-measurable.} \end{cases}$$

This assumption means that the positions in the portfolio at time n, namely ϕ_n^0, ϕ_n^1,..., ϕ_n^d, are decided with respect to the information available at time $(n-1)$ and kept until time n, when new quotations are available.

The *value of the portfolio* at time n is the scalar product

$$V_n(\phi) = \phi_n.S_n = \sum_{i=0}^{d} \phi_n^i S_n^i.$$

Its *discounted value* is

$$\tilde{V}_n(\phi) = \beta_n(\phi_n.S_n) = \phi_n.\tilde{S}_n,$$

where $\beta_n = 1/S_n^0$ and $\tilde{S}_n = (1, \beta_n S_n^1, \ldots, \beta_n S_n^d)$ is the vector of *discounted prices*. By considering discounted prices, we take the price of the non-risky asset as a monetary unit or *numéraire* (see Exercise 3 for an introduction to change of numéraire techniques).

A strategy is called *self-financing* if the following equation is satisfied for all $n \in \{0, 1, \ldots, N-1\}$:

$$\phi_n.S_n = \phi_{n+1}.S_n.$$

The interpretation is the following: at time n, once the new prices $S_n^0, \cdots, S_n^d$ are quoted, the investor readjusts his positions from ϕ_n to ϕ_{n+1} without bringing or consuming any wealth.

Remark 1.1.1. The equality $\phi_n.S_n = \phi_{n+1}.S_n$ is obviously equivalent to

$$\phi_{n+1}.(S_{n+1} - S_n) = \phi_{n+1}.S_{n+1} - \phi_n.S_n,$$

or to

$$V_{n+1}(\phi) - V_n(\phi) = \phi_{n+1}.(S_{n+1} - S_n).$$

At time $n+1$, the portfolio is worth $\phi_{n+1}.S_{n+1}$ and $\phi_{n+1}.S_{n+1} - \phi_{n+1}.S_n$ is the net gain caused by the price changes between times n and $n+1$. Hence, the profit or loss realized by following a self-financing strategy is only due to the price moves.

The following proposition makes this clear in terms of discounted prices.

Proposition 1.1.2. *The following are equivalent:*

(i) *The strategy* ϕ *is self-financing.*

(ii) *For any* $n \in \{1, \dots, N\}$,

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j,$$

where ΔS_j *is the vector* $S_j - S_{j-1}$.

(iii) *For any* $n \in \{1, \dots, N\}$,

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta \tilde{S}_j,$$

where $\Delta \tilde{S}_j$ *is the vector* $\tilde{S}_j - \tilde{S}_{j-1} = \beta_j S_j - \beta_{j-1} S_{j-1}$.

Proof. The equivalence between (i) and (ii) results from Remark 1.1.1. The equivalence between (i) and (iii) follows from the fact that $\phi_n.S_n = \phi_{n+1}.S_n$ if and only if $\phi_n.\tilde{S}_n = \phi_{n+1}.\tilde{S}_n$. □

This proposition shows that, if an investor follows a self-financing strategy, the discounted value of his portfolio, and hence its value, are completely defined by the initial wealth and the strategy $(\phi_n^1, \dots, \phi_n^d)_{0 \le n \le N}$ (this is only justified because $\Delta \tilde{S}_j^0 = 0$).More precisely, we can prove the following proposition.

Proposition 1.1.3. *For any predictable process* $((\phi_n^1, \dots, \phi_n^d))_{0 \le n \le N}$ *and for any* $\mathscr{F}_0$*-measurable variable* V_0*, there exists a unique predictable process* $(\phi_n^0)_{0 \le n \le N}$ *such that the strategy* $\phi = (\phi^0, \phi^1, \dots, \phi^d)$ *is self-financing and its initial value is* V_0.

Proof. The self-financing condition implies

$$\begin{aligned}\tilde{V}_n(\phi) &= \phi_n^0 + \phi_n^1 \tilde{S}_n^1 + \cdots + \phi_n^d \tilde{S}_n^d \\ &= V_0 + \sum_{j=1}^{n} \left(\phi_j^1 \Delta \tilde{S}_j^1 + \cdots + \phi_j^d \Delta \tilde{S}_j^d \right),\end{aligned}$$

which defines ϕ_n^0. We just have to check that ϕ^0 is predictable, but this is obvious if we consider the equation

$$\begin{aligned}\phi_n^0 = V_0 + \textstyle\sum_{j=1}^{n-1} &\left(\phi_j^1 \Delta \tilde{S}_j^1 + \cdots + \phi_j^d \Delta \tilde{S}_j^d \right) \\ &+ \left(\phi_n^1 \left(-\tilde{S}_{n-1}^1 \right) + \cdots + \phi_n^d \left(-\tilde{S}_{n-1}^d \right) \right).\end{aligned}$$

□

1.1.3 Admissible strategies and arbitrage

We did not make any assumption on the sign of the quantities ϕ_n^i. If $\phi_n^0 < 0$, we have borrowed the amount $|\phi_n^0|$ in the riskless asset. If $\phi_n^i < 0$ for $i \geq 1$, we say that we are *short* a number ϕ_n^i of asset i. In this model, short-selling and borrowing are allowed, but, by the following *admisibility* condition, the value of the portfolio must remain non-negative at all times.

Definition 1.1.4. A strategy ϕ is admissible if it is self-financing and if $V_n(\phi) \geq 0$ for any $n \in \{0, 1, \ldots, N\}$.

The investor must be able to pay back his debts (in the riskless or the risky assets) at any time. The notion of *arbitrage* (possibility of a riskless profit) can be formalised as follows:

Definition 1.1.5. An arbitrage strategy is an admissible strategy with zero initial value and non-zero final value.

In other words, an arbitrage starts with a zero initial value and achieves a nonnegative value at all times, with strictly positive probability of the final value being positive. Most models exclude any arbitrage opportunity, and the objective of the next section is to characterize these models with the notion of martingale.

1.2 Martingales and arbitrage opportunities

In order to analyze the connections between martingales and arbitrage, we must first define a *martingale* on a finite probability space. The conditional expectation plays a central role in this definition, and the reader can refer to the appendix for a quick review of its properties.

1.2.1 Martingales and martingale transforms

In this section, we consider a finite probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\mathscr{F} = \mathscr{P}(\Omega)$ and $\forall \omega \in \Omega, \mathbb{P}(\{\omega\}) > 0$, equipped with a filtration $(\mathscr{F}_n)_{0 \leq n \leq N}$ (without necessarily assuming that $\mathscr{F}_N = \mathscr{F}$, nor $\mathscr{F}_0 = \{\phi, \Omega\}$). A sequence $(X_n)_{0 \leq n \leq N}$ of random variables is adapted to the filtration if, for any n, X_n is $\mathscr{F}_n$-measurable.

Definition 1.2.1. An adapted sequence $(M_n)_{0 \leq n \leq N}$ of real-valued random variables is

- a martingale if $\mathbb{E}(M_{n+1} | \mathscr{F}_n) = M_n$ for all $n \leq N - 1$;
- a supermartingale if $\mathbb{E}(M_{n+1} | \mathscr{F}_n) \leq M_n$ for all $n \leq N - 1$;
- a submartingale if $\mathbb{E}(M_{n+1} | \mathscr{F}_n) \geq M_n$ for all $n \leq N - 1$.

These definitions can be extended to the multidimensional case: for instance, a sequence $(M_n)_{0\leq n\leq N}$ of $\mathbb{R}^d$-valued random variables is a martingale if each component is a real-valued martingale.

In a financial context, saying that the price $(S^i_n)_{0\leq n\leq N}$ of the asset i is a martingale implies that, at each time n, the best estimate (in the least-square sense) of S^i_{n+1} is given by S^i_n.

The following properties are easily derived from the previous definition and stand as a good exercise to get used to the concept of conditional expectation:

1. $(M_n)_{0\leq n\leq N}$ is a martingale if and only if
$$\mathbb{E}(M_{n+j}|\mathscr{F}_n) = M_n \quad \forall j \geq 0.$$
2. If $(M_n)_{n\geq 0}$ is a martingale, then for any n: $\mathbb{E}(M_n) = \mathbb{E}(M_0)$.
3. The sum of two martingales is a martingale.
4. Obviously, similar properties can be shown for supermartingales and submartingales.

Definition 1.2.2. An adapted sequence $(H_n)_{0\leq n\leq N}$ of random variables is predictable if, for all $n \geq 1$, H_n is $\mathscr{F}_{n-1}$-measurable.

Proposition 1.2.3. *Let $(M_n)_{0\leq n\leq N}$ be a martingale and $(H_n)_{0\leq n\leq N}$ a predictable sequence with respect to the filtration $(\mathscr{F}_n)_{0\leq n\leq N}$. Denote $\Delta M_n = M_n - M_{n-1}$. The sequence $(X_n)_{0\leq n\leq N}$ defined by*
$$\begin{aligned} X_0 &= H_0 M_0 \\ X_n &= H_0 M_0 + H_1 \Delta M_1 + \cdots + H_n \Delta M_n \quad \text{for } n \geq 1 \end{aligned}$$
is a martingale with respect to $(\mathscr{F}_n)_{0\leq n\leq N}$.

(X_n) is sometimes called the *martingale transform* of (M_n) by (H_n). A consequence of this proposition and Proposition 1.1.2 is that if the discounted prices of the assets are martingales, the *expected value* of the wealth generated by following a self-financing strategy is equal to the initial wealth.

Proof. Clearly, (X_n) is an adapted sequence. Moreover, for $n \geq 0$,
$$\begin{aligned} &\mathbb{E}(X_{n+1} - X_n|\mathscr{F}_n) \\ &= \mathbb{E}(H_{n+1}(M_{n+1} - M_n)|\mathscr{F}_n) \\ &= H_{n+1}\mathbb{E}(M_{n+1} - M_n|\mathscr{F}_n) \text{ since } H_{n+1} \text{ is } \mathscr{F}_n\text{-measurable} \\ &= 0. \end{aligned}$$
Hence
$$\mathbb{E}(X_{n+1}|\mathscr{F}_n) = \mathbb{E}(X_n|\mathscr{F}_n) = X_n,$$
which shows that (X_n) is a martingale. □

The following proposition is a very useful characterization of martingales.

Proposition 1.2.4. *An adapted sequence of real-valued random variables (M_n) is a martingale if and only if for any predictable sequence (H_n), we have*

$$\mathbb{E}\left(\sum_{n=1}^{N} H_n \Delta M_n\right) = 0.$$

Proof. If (M_n) is a martingale, the sequence (X_n) defined by $X_0 = 0$ and, for $n \geq 1$, $X_n = \sum_{j=1}^{n} H_j \Delta M_j$ for any predictable process (H_n) is also a martingale, by Proposition 1.2.3. Hence, $\mathbb{E}(X_N) = \mathbb{E}(X_0) = 0$. Conversely, we notice that if $j \in \{1, \ldots, N\}$, we can associate the sequence (H_n) defined by $H_n = 0$ for $n \neq j+1$ and $H_{j+1} = \mathbf{1}_A$, for any $\mathscr{F}_j$-measurable A. Clearly, (H_n) is predictable and $\mathbb{E}\left(\sum_{n=1}^{N} H_n \Delta M_n\right) = 0$ becomes

$$\mathbb{E}(\mathbf{1}_A (M_{j+1} - M_j)) = 0.$$

Therefore $\mathbb{E}(M_{j+1} | \mathscr{F}_j) = M_j$. □

1.2.2 Viable financial markets

Let us get back to the discrete-time models introduced in the first section.

Definition 1.2.5. A market is viable if there is no arbitrage opportunity.

The following result is sometimes referred to as the *Fundamental Theorem of Asset Pricing*.

Theorem 1.2.6. *The market is viable if and only if there exists a probability measure $\mathbb{P}^*$ equivalent[1] to $\mathbb{P}$ such that the discounted prices of assets are $\mathbb{P}^*$-martingales.*

Proof. (a) Let us assume that there exists a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which discounted prices are martingales. Then, for any self-financing strategy (ϕ_n), Proposition 1.1.2 implies

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j . \Delta \tilde{S}_j.$$

Thus, by Proposition 1.2.3, $(\tilde{V}_n(\phi))$ is a $\mathbb{P}^*$-martingale. Therefore, $\tilde{V}_N(\phi)$ and $V_0(\phi)$ have the same expectation under $\mathbb{P}^*$:

$$\mathbb{E}^*(\tilde{V}_N(\phi)) = \mathbb{E}^*(\tilde{V}_0(\phi)).$$

[1] Recall that two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent if and only if for any event A, $\mathbb{P}_1(A) = 0 \Leftrightarrow \mathbb{P}_2(A) = 0$. Here, $\mathbb{P}^*$ equivalent to $\mathbb{P}$ means that, for any $\omega \in \Omega$, $\mathbb{P}^*(\{\omega\}) > 0$.

If the strategy is admissible and its initial value is zero, then $\mathbb{E}^*(\tilde{V}_N(\phi))=0$, with $\tilde{V}_N(\phi)\geq 0$. Hence $\tilde{V}_N(\phi)=0$ since $\mathbb{P}^*(\{\omega\})>0$, for all $\omega\in\Omega$.

(b) The proof of the converse implication is more tricky. Denote by Γ the set of all non-negative random variables X such that $\mathbb{P}(X>0)>0$. Clearly, Γ is a convex cone in the vector space of real-valued random variables. The market is viable if and only if for any admissible strategy ϕ, $V_0(\phi)=0\Rightarrow\tilde{V}_N(\phi)\notin\Gamma$.

(b1) To any admissible process $(\phi_n^1,\ldots,\phi_n^d)$ we associate the process defined by

$$\tilde{G}_n(\phi)=\sum_{j=1}^{n}\left(\phi_j^1\Delta\tilde{S}_j^1+\cdots+\phi_j^d\Delta\tilde{S}_j^d\right),$$

which is the cumulative discounted gain realised by following the self-financing strategy $\phi_n^1,\ldots,\phi_n^d$. According to Proposition 1.1.3, there exists a (unique) process (ϕ_n^0) such that the strategy $((\phi_n^0,\phi_n^1,\ldots,\phi_n^d))$ is self-financing with zero initial value. Note that $\tilde{G}_n(\phi)$ is the discounted value of this strategy at time n, and, because the market is viable, the fact that this value is non-negative at any time, i.e $\tilde{G}_n(\phi)\geq 0$ for $n=1,\ldots,N$, implies that $\tilde{G}_N(\phi)=0$. The following lemma shows that even if we do not assume that all the $\tilde{G}_n(\phi)$'s are non-negative, we still have $\tilde{G}_N(\phi)\notin\Gamma$.

Lemma 1.2.7. *If a market is viable, any predictable process $(\phi^1,\ldots,\phi^d)$ satisfies*

$$\tilde{G}_N(\phi)\notin\Gamma.$$

Proof. Let us assume that $\tilde{G}_N(\phi)\in\Gamma$. First, if $\tilde{G}_n(\phi)\geq 0$ for all $n\in\{0,\ldots,N\}$, the market is obviously not viable. Second, if the $\tilde{G}_n(\phi)$'s are not all non-negative, we define $n=\sup\{k|\mathbb{P}(\tilde{G}_k(\phi)<0)>0\}$. It follows from the definition of n that

$$n\leq N-1,\quad \mathbb{P}(\tilde{G}_n(\phi)<0)>0 \text{ and } \forall m>n,\quad \tilde{G}_m(\phi)\geq 0.$$

We can now introduce a new process ψ:

$$\psi_j(\omega)=\begin{cases}0 & \text{if } j\leq n\\ \mathbf{1}_A(\omega)\phi_j(\omega) & \text{if } j>n,\end{cases}$$

where A is the event $\{\tilde{G}_n(\phi)<0\}$. Because ϕ is predictable and A is $\mathscr{F}_n$-measurable, ψ is also predictable. Moreover,

$$\tilde{G}_j(\psi)=\begin{cases}0 & \text{if } j\leq n\\ \mathbf{1}_A(\tilde{G}_j(\phi)-\tilde{G}_n(\phi)) & \text{if } j>n;\end{cases}$$

thus, $\tilde{G}_j(\psi)\geq 0$ for all $j\in\{0,\ldots,N\}$ and $\tilde{G}_N(\psi)>0$ on A. That contradicts the assumption of market viability and completes the proof of the lemma. □

(b2) The set $\mathscr{V}$ of random variables $\tilde{G}_N(\phi)$, with ϕ a predictable process in $\mathbb{R}^d$, is clearly a vector subspace of $\mathbb{R}^\Omega$ (where $\mathbb{R}^\Omega$ is the set of real-valued random variables defined on Ω). According to Lemma 1.2.7, the subspace $\mathscr{V}$

does not intersect Γ. Therefore, it does not intersect the convex compact set $K = \{X \in \Gamma | \sum_\omega X(\omega) = 1\}$, which is included in Γ. As a result of the convex sets separation theorem (see the Appendix), there exists $(\lambda(\omega))_{\omega \in \Omega}$ such that:

1. $\forall X \in K, \quad \displaystyle\sum_\omega \lambda(\omega) X(\omega) > 0.$

2. For any predictable ϕ,
$$\sum_\omega \lambda(\omega) \tilde{G}_N(\phi)(\omega) = 0.$$

$>$From Property 1, we deduce that $\lambda(\omega) > 0$ for all $\omega \in \Omega$, so that the probability $\mathbb{P}^*$ defined by
$$\mathbb{P}^*(\{\omega\}) = \frac{\lambda(\omega)}{\sum_{\omega' \in \Omega} \lambda(\omega')}$$
is equivalent to $\mathbb{P}$.

Moreover, if we denote by $\mathbb{E}^*$ the expectation under measure $\mathbb{P}^*$, Property 2 means that, for any predictable process (ϕ_n) in $\mathbb{R}^d$,
$$\mathbb{E}^*\left(\sum_{j=1}^N \phi_j \Delta \tilde{S}_j\right) = 0.$$

It follows that for all $i \in \{1, \ldots, d\}$ and any predictable sequence (ϕ_n^i) in $\mathbb{R}$, we have
$$\mathbb{E}^*\left(\sum_{j=1}^n \phi_j^i \Delta \tilde{S}_j^i\right) = 0.$$

Therefore, according to Proposition 1.2.4, we conclude that the discounted prices $(\tilde{S}_n^1), \ldots, (\tilde{S}_n^d)$ are $\mathbb{P}^*$-martingales.

1.3 Complete markets and option pricing

1.3.1 Complete markets

A *European option*[2] with maturity N can be characterized by its payoff h, which is a non-negative $\mathscr{F}_N$-measurable random variable. For instance, a *call* on the underlying S^1 with strike price K will be defined by setting $h = \left(S_N^1 - K\right)_+$. A *put* on the same underlying asset with the same strike price K will be defined by $h = \left(K - S_N^1\right)_+$. In these two examples, which are actually the two most important in practice, h is a function of S_N only. There are some options dependent on the whole path of the underlying asset, i.e. h is a function of $S_0, S_1, \ldots, S_N$. That is the case of the so-called *Asian options*,

[2] Or, more generally, a contingent claim.

where the strike price is equal to the average of the stock prices observed during a certain period of time before maturity.

Definition 1.3.1. The contingent claim defined by h is attainable if there exists an admissible strategy worth h at time N.

Remark 1.3.2. In a viable financial market, we just need to find a *self-financing* strategy worth h at maturity to say that h is attainable. Indeed, if ϕ is a self-financing strategy and if $\mathbb{P}^*$ is a probability measure equivalent to $\mathbb{P}$ under which discounted prices are martingales, then $(\tilde{V}_n(\phi))$ is also a $\mathbb{P}^*$-martingale, being a martingale transform. Hence, for $n \in \{0, \ldots, N\}$, $\tilde{V}_n(\phi) = \mathbb{E}^*(\tilde{V}_N(\phi)|\mathscr{F}_n)$. Clearly, if $\tilde{V}_N(\phi) \geq 0$ (in particular if $V_N(\phi) = h \geq 0$), the strategy ϕ is admissible.

Definition 1.3.3. The market is complete if every contingent claim is attainable.

To assume that a financial market is complete is a rather restrictive assumption that does not have such a clear economic justification as the no-arbitrage assumption. The interest of complete markets is that it allows us to derive a simple theory of contingent claim pricing and hedging. The Cox-Ross-Rubinstein model, which we shall study in the next section, is a very simple example of a complete market model. The following theorem gives a precise characterization of complete, viable financial markets.

Theorem 1.3.4. *A viable market is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, under which discounted prices are martingales.*

The probability $\mathbb{P}^*$ will appear to be the *computing tool* whereby we can derive closed-form pricing formulae and hedging strategies.

Proof. (a) Let us assume that the market is viable and complete. Then, any non-negative, $\mathscr{F}_N$-measurable random variable h can be written as $h = V_N(\phi)$, where ϕ is an admissible strategy that replicates the contingent claim h. Since ϕ is self-financing, we know that

$$\frac{h}{S_N^0} = \tilde{V}_N(\phi) = V_0(\phi) + \sum_{j=1}^{N} \phi_j . \Delta\tilde{S}_j.$$

Thus, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are two probability measures under which discounted prices are martingales, $(\tilde{V}_n(\phi))_{0\leq n\leq N}$ is a martingale under both $\mathbb{P}_1$ and $\mathbb{P}_2$. It follows that, for $i = 1, 2$,

$$\mathbb{E}_i(\tilde{V}_N(\phi)) = \mathbb{E}_i(V_0(\phi)) = V_0(\phi),$$

the last equality coming from the fact that $\mathscr{F}_0=\{\emptyset,\Omega\}$. Therefore,

$$\mathbb{E}_1\left(\frac{h}{S_N^0}\right)=\mathbb{E}_2\left(\frac{h}{S_N^0}\right)$$

and, since h is arbitrary, $\mathbb{P}_1=\mathbb{P}_2$ on the whole σ-algebra $\mathscr{F}_N$, which is assumed to be equal to $\mathscr{F}$.

(b) Let us assume that the market is viable and incomplete. Then, there exists a random variable $h\geq 0$ that is not attainable. We call $\tilde{\mathscr{V}}$ the set of random variables of the form

$$U_0+\sum_{n=1}^{N}\phi_n.\Delta\tilde{S}_n, \tag{1.1}$$

where U_0 is $\mathscr{F}_0$-measurable and $\left((\phi_n^1,\ldots,\phi_n^d)\right)_{0\leq n\leq N}$ is an $\mathbb{R}^d$-valued predictable process.

It follows from Proposition 1.1.3 and Remark 1.3.2 that the variable h/S_N^0 does not belong to $\tilde{\mathscr{V}}$. Hence, $\tilde{\mathscr{V}}$ is a strict subset of the set of all random variables on $(\Omega,\mathscr{F})$. Therefore, if $\mathbb{P}^*$ is a probability equivalent to $\mathbb{P}$ under which discounted prices are martingales, and if we define the following scalar product on the set of random variables $(X,Y)\mapsto\mathbb{E}^*(XY)$, we notice that there exists a non-zero random variable X orthogonal to $\tilde{\mathscr{V}}$. We now define

$$\mathbb{P}^{**}(\{\omega\})=\left(1+\frac{X(\omega)}{2\|X\|_\infty}\right)\mathbb{P}^*(\{\omega\})$$

with $\|X\|_\infty=\sup_{\omega\in\Omega}|X(\omega)|$. Because $\mathbb{E}^*(X)=0$, that defines a new probability measure equivalent to $\mathbb{P}$, and different from $\mathbb{P}^*$. Moreover,

$$\mathbb{E}^{**}\left(\sum_{n=1}^{N}\phi_n.\Delta\tilde{S}_n\right)=0$$

for any predictable process $\left((\phi_n^1,\ldots,\phi_n^d)\right)_{0\leq n\leq N}$. It follows from Proposition 1.2.4 that $(\tilde{S}_n)_{0\leq n\leq N}$ is a $\mathbb{P}^{**}$-martingale. □

1.3.2 Pricing and hedging contingent claims in complete markets

The market is assumed to be viable and complete and we denote by $\mathbb{P}^*$ the unique probability measure under which the discounted prices of financial assets are martingales. Let h be an $\mathscr{F}_N$-measurable, non-negative random variable and ϕ be an admissible strategy replicating the contingent claim hence defined, i.e.

$$V_N(\phi)=h.$$

The sequence $(\tilde{V}_n)_{0\leq n\leq N}$ is a $\mathbb{P}^*$-martingale, and consequently

$$V_0(\phi)=\mathbb{E}^*(\tilde{V}_N(\phi)),$$